ANNA O. GRAEBER AND DINA TIROSH

INSIGHTS FOURTH AND FIFTH GRADERS BRING TO MULTIPLICATION AND DIVISION WITH DECIMALS

ABSTRACT. This study was designed to gain information about the understandings children in Israel and the United States have about multiplication and division of whole numbers that may be useful in building accurate understandings of these operations with decimals and the extent to which they hold conceptions about these operations that may interfere with their work with decimals. Data from interviews of the fourth and fifth graders indicate that students of this age already hold misconceptions such as "multiplication always makes bigger." However they also hold conceptions that are prerequisite to understanding the area model of multiplication and the measurement model of division. These early conceptions might be used to build understanding of multiplication and division by decimals. Implications for the content and sequencing of instructional activities are presented.

INTRODUCTION

Studies have shown that children's, adolescents' and preservice elementary teachers' performance in solving multiplication and division word problems is influenced by misconceptions such as "multiplication always makes bigger," "division always makes smaller," and "you can not divide a smaller number by a larger number" (Bell, Fischbein, and Greer, 1984; Brown, 1981; Fischbein et al., 1985; Graeber, Tirosh, and Glover, 1989). Investigations of the justifications and sources of support people have for these misconceptions (Bell, 1982; Graeber, Tirosh and Glover, 1989) suggest that people frequently overgeneralize rules from the domain of whole numbers to the domain of rational numbers. Further, their rote understanding of the algorithms of multiplication and division with decimal and common fractions is sometimes a source of support for these misbeliefs. For example, the procedure of "moving the decimal point" in calculating $.5\overline{)6}$ and performing the equivalent calculation $5\overline{)60}$ has been cited in support of the fact that the divisor must be a whole number and that the quotient, i.e., 12, must be smaller than the dividend, 60. Others cite the "invert and multiply" rule for dividing common fractions as the reason that "you can't divide by a fraction" (Tirosh and Graeber, 1989). Since helping students overcome misconceptions is widely recognized as a difficult task, it seems logical to direct some efforts toward "preventive" instruction. That is to find ways of tailoring instruction for students who

Educational Studies in Mathematics **21**: 565–588, 1990.

are just beginning to expand the definitions of multiplication and division from the domain of whole numbers to the rational numbers in ways such that misbeliefs are, if not prevented, at least not reinforced.

One commonly accepted approach to instructional design suggests that children be introduced to new ideas in a manner that exploits their existing conceptual understandings (e.g., Case, 1985; Steffe, in press). A clear understanding of the initial reasoning processes children use in approaching multiplication and division tasks would seem to be helpful in designing such instruction. What understandings do fourth and fifth graders have about multiplication and division that can be transferred to work with nonnegative rationals? Do these students have conceptions of the operations that will likely interfere with their work with decimals?

Unfortunately some of the skills and concepts that teachers might especially want students to be able to utilize at this point are frequently not in the students' repertoires. Past studies in England, Australia, and the United States [studies R. Davis (1984) would describe as "disaster studies"] suggest that

- many students' understanding of decimal notation is relatively low. (See, for example: Bell, Swan, and Taylor, 1981; Carpenter et al., 1981; Hiebert and Wearne, 1983.)
- students rarely use the common fraction skills they have to attack decimal computations. (See, for example: Ekenstam, 1977; Hiebert and Wearne, 1983.)
- The common fraction form, a/b, is rarely viewed as a notation of indicated division. (See, for example: Hart, 1981.)

Do these students have other conceptions of the operations that will likely interfere with their work with decimals? This study explores the students' conceptions of multiplication and division that may be enabling or disabling as students attempt to make sense of multiplication and division by decimals.

METHOD

Sample

Students interviewed in the United States were from one elementary (K-5) school in a county near Washington, D.C. The study includes data from 30 (14 male, 16 female) fourth graders and 30 (14 male, 16 female) fifth graders. The interviewees were from different math classes and represented a range of achievement.

The Israeli interviewees were students who had just entered the fourth and fifth grades in one of two schools in Tel Aviv. All the children were native Hebrew speakers from the middle and upper middle class. The study includes data from 14 fourth graders (five male, nine female) and 16 fifth graders (seven male, nine female).

Instruments

Two interview protocols were written, one for multiplication and one for division. The multiplication protocol is reproduced here as Figure 1. The protocol for each operation included tasks in five major areas: (1) terminology and notation; (2) definitions and properties; (3) models in the domain of whole numbers; (4) facility with decimals; (5) writing expressions for word problems. The tasks were constructed by considering the models usually used in developing the concepts of the whole number operations, including those that have potential for carry over into the domain of rational numbers. Thus, there was specific attention to facility with the area model of multiplication and measurement interpretation of division. Draft protocols were used in interviews with ten fourth graders and ten fifth graders from a cooperating school in the United States. The protocols were then revised and translated into Hebrew.

Procedure

Each author conducted 20–25 minute, individual interviews with each of the students in her country. The interviews were conducted in the respective countries over two months time (April and May 1987 in the U.S., October and November 1987 in Israel). A simple student work sheet, an interviewer's record sheet, and an audio tape recording were kept for each student.

In the United States each student was given the protocols for only one of the operations. In Israel, each child was given both protocols. However, for a given protocol the authors attempted to follow a fairly standard administration and to cover all items whenever it seemed reasonable. Word problems were omitted if students experienced great difficulties with or took excessive amounts of time solving them.

Comparison between countries was not a major emphasis of the study. The topics, sequence and approaches for studying the operations with whole numbers and the introduction of operations with decimals is quite similar. In both countries instruction in multiplication and division with decimals typically begins in the fifth grade. Two differences are relevant. In

Warm-up: (Student has counting blocks or squares and is shown the sentence $2 + 7 = 9$.)

How would you use the blocks to show me, or a second grader, that $2 + 7 = 9$?

(If the student is successful, go to Task 1.)
If not, say here are 2 blocks and here are 7 blocks: Now can you show $2 + 7$?
If still unsuccessful, demonstrate and then go on.

Task 1: Show student card with the sentence $4 \times 5 = 20$.

What does this sign (point to $\times$) mean? What does multiplication mean? What does this sign (point to $=$) mean?
Do you remember what we call these numbers (point to 4 and 5) in a multiplication sentence? (Establish term "factors.")
What do you call this number (20)? (Establish term "answer" or "product.")

Task 2: (Student given cubes or blocks all of one color and shown the sentence $7 \times 2 = 14$.)

Can you use these cubes to show me $7 \times 2 = 14$?
Can you show me another way?
Another way? (looking for sets and arrays)

Task 3: (Student given a worksheet with cm square grid and shown the phrase 3×4.)

What is 3×4?
Can you use this (point to the grid) to show me 3×4?
Can you show it another way?

Task 4: (Student shown the number sentence $2 + 4 = 6$, the written story problem, and the number sentence $5 \times 6 = 30$.).

If I showed you $2 + 4 = 6$ and asked you to tell me a story problem about it you might say: "Joe had 2 cookies in the afternoon and 4 cookies after dinner. How many did he have altogether?" This (point) is an addition sentence and an addition story. Six (point) is the answer to the question.

We have been working on multiplication. This (point) is a multiplication sentence.

Can you tell me a multiplication story problem for $5 \times 6 = 30$? You can write it or just tell me, whatever is easier. Take your time.

Task 5: (Student is shown, one at a time, the three cards each with one of the number sentences listed below. Interviewer refers to the ? as the hidden number. Ask relevant questions from the list below.)

Can you tell me what the hidden number is? Can you tell me anything about the hidden number? Why do you think so?
When is the product zero? When is the product the same as one of the factors?

$7 \times 6 = 6 \times ?$ $7 \times ? = 7$ $9 \times ? = 0$

Figure continued on next page

Task 6: (Student is shown the symbols .1 and .5.)

There are two numbers on this card. What is the number at the top?
Do you know what .1 means? Is it more or less than 0, 1?
Do you know what .5 means? Is it more or less than 0?, 1, .1?

Task 7A: (If the student was reasonably successful with task 6, use tasks 7A and 7B. Students less successful with task 6, go to task 7C. For 7A show student card with number sentence. If student does not know how to respond, suggest that one of the ways of picturing multiplication used in tasks 3 or 4 might help.)

$15 \times .6 = ?$

Can you tell me what the hidden number is? Can you tell me anything about it? Why do you think so?

Task 7B: (Show student the card with the number sentence.)

$.1 \times 10 = ?$

Can you tell me what the hidden number is? Can you tell me anything about it? Why do you think so?

Task 7C: (Show student the card with the number sentence.)

$15 \times .6 = 9$

Some adults think that this number sentence is strange, even though it is correct. Can you guess why they might think it is strange? Do you think it is strange?

Task 8: (In any remaining time the student will be asked to write how she/he would solve word problems. Word problems are given to the student one at a time.)

Now, for the last set of questions there are some word problems like you might find in your math book. We have been working on multiplication, but these problems might be division or multiplication or addition or subtraction. They will not all be multiplication problems.

Read this problem aloud for me and then tell me how you would solve it. You only need to tell me what operation you would use and what numbers. You do not have to do the actual computation.

If a probe is needed to establish the expression, ask "What number do you (add, subtract, multiply, divide) to/from/by what number?"

After stating or writing an expression the student will be asked, "Why did you choose to (divide, multiply, add, or subtract)?"

Fig. 1. Multiplication protocol.

the Israeli schools instruction in the one system of measurement used in the country, the metric system, may influence students' familiarity with decimal notation. Second it appears that Israeli students are more frequently asked to write word problems for given expressions than are their American counterparts. Data are given by country only when obvious differences appeared between the apparent concepts the students held.

MULTIPLICATION TASK RESULTS

Definitions and Properties

Students were shown the sentence $4 \times 5 = 20$ and asked to tell what the "$\times$" sign meant without using the word "multiply" or "times." The categories into which responses were tallied, examples of responses in each category, and the number of students who gave definitions in each category are shown in Table I. The primitive repeated addition model was clearly a favorite.

Students were asked to complete statements illustrating various properties of multiplication. The statement, $7 \times 6 = 6 \times ?$, was answered correctly by 96% of the students. Over 75% of the responses were rapid (within 3 seconds). When asked how they could respond so quickly, almost all of the students, even those who took slightly longer, made some statement conveying the notion that if the order of the factors was reversed the product was the same.

Concrete, Pictorial, and Word Problem Representations

The students were quite successful with the tasks of producing pictorial and concrete models for multiplication expressions. Given the expression $7 \times 2 = 14$, 57 of the 60 students produced at least one grouping of like-colored wooden blocks (either two groups of seven or seven groups of two) and 48 produced both groupings for the expression.

Students were asked to state the product for the expression 3×4 and illustrate the fact on a 7-by-7 centimeter grid composed of centimeter squares. Thirty-eight of the successful illustrations used 3×4 or 4×3 arrays, fourteen of the illustrations involved 12 squares not in an array, and two students simply recorded three tallies in each of four squares or four tallies in each of three squares.

Each student was asked to write a word problem for the expression $5 \times 6 = 30$. Students were told that they were to write a word problem for the multiplication phrase, five times six, and that the answer to the question would need to be 30. Table II illustrates the categories into which the word problems were grouped and shows how students fared on this task. If the first four categories in the table are considered to indicate "success," approximately 60% of the U.S. students and 90% of the Israeli students met this standard. Each of the problems written reflected either the union of equivalent groups or an example of repeated addition.

TABLE I

Number of students offering various meanings for " × " in the sentence $4 \times 5 = 20$.

Categories Examples	Country Israel	 US
Repeated addition/counting on You add, you add 5 four times (USM-10)* To repeat the specific number the other number of times in the addition operation (I-12) $4 + 4 + 4 \ldots$ five times (I-11) Like if you say count by 4 five times you would get twenty (USM-2)	28	11
Vague relationship to addition It is like plus. Like we use 5. You make 5 into 4, I mean not 5 into 4 but how many times does 5 and 4. Like 5 and 4 equals twenty. [Interviewer asks – 5 AND 4?]** Five into four. No 5 times 4 equals 20. It's like twice what that is. Well it is sort of unexplained. (USM-30) The opposite of adding. [Interviewer asks – it is not *like* adding?] No response. (USM-38)		5
Vague relationship to groups Groups. That's it. 4 and 5. There are like 4 people and five more. There are four groups and five more coming so that altogether there will be twenty. (USM-26) I don't know. Uhm. Groups? I don't know. (USM-56)		3
Way to check division It is a way to check division. [Interviewer – anything else?] No. (I-28)	1	
Incoherent definition or no definition It means you have 4 of them and you ... I don't know. (USM-4) Two times 4 times 5 (USM-14) It just means "times." (I-27)	1	11

* I indicates Israeli student, USM indicates American student.

** [ ] signals remarks by interviewer.

TABLE II

Responses to the task of writing a word problem for the fact $5 \times 6 = 30$.

	Grade/Country			
	4th		5th	
Responses	Israel	US	Israel	US
Stated a problem quickly, no prompts needed or blocks used	5	1	7	3
Gave a correct problem statement, but needed to be asked what question went with it	2	1	5	2
Started out incorrectly, realized their own error – may have needed a prompt to formulate question	2	3	1	2
Faulty or long delayed start. At interviewer's suggestion set out blocks to show 5×6 and then formulated problem on their own	2	2	3	4
Faulty or delayed start. Needed to set up blocks or were given cues or prompts and eventually formulated a problem	2	2	0	1
Faulty or delayed start. Needed to set up blocks and were given cues or prompts but did not successfully formulate a problem	1	6	0	3

Facility with Decimals

About one-half of the students who were given the multiplication tasks claimed to have seen numbers such as .1 and .5; however, less than half of these students recalled names for these numbers. Students who did not use the names of the decimals were told that .1 was another way of writing 1/10 and .5 another way of writing 5/10. Those whose responses suggested some familiarity with decimals were asked to complete four comparisons stated orally: Which is larger .1 or 0?, .1 or 1?, .5 or 0, .5 or 1? Of the 41 who were asked to make such comparisons, 17 answered all four questions correctly and five completed three questions correctly. Israeli students comprised 15 of the 22 students who completed at least three of the comparisons correctly.

Eighteen of the students who correctly answered three or more of the comparison items were then asked to tell what number was hidden in the sentence $15 \times .6 = \Delta$. Sixteen of them used a rule they knew about multiplication with decimals or thought of .6 as 6/10 and multiplied with common fractions to reach the answer of 9. One student guessed the answer would be 60. Another student claimed she had no idea and refused to guess citing the task as unfair because it had not yet been taught. Fifteen of the 16 students successful with the $15 \times .6$ task were also successful with the task of $.1 \times 10$. Eventually, all 60 students were shown the sentence $15 \times .6 = 9$, told that many people including adults thought the sentence was strange, or weird, or odd, even though it really, honestly was correct. Could the student guess why it was considered strange? Seventeen students expressed the idea that there was nothing wrong with the answer (this group was comprised of those who had earlier calculated the answer to be nine). However even some of these students acknowledged that the answer was nevertheless somewhat strange. The majority of the students reacted with surprise to the sentence by raising their eyebrows, asking for the sentence to be read, or rereading the sentence aloud (one even demanding to know how it could be true). Sixteen of the students indicated that what made the sentence strange was the appearance of a decimal. Fourteen of the students, about 25%, indicated some specific displeasure with the size of the product, saying "the answer should be bigger than 9," "the answer should be like fifteen," "the answer should be higher." Eleven students could only express their uneasiness vaguely saying "It just doesn't look true", "It looks wrong," or "it's no stranger than most." Data from the grid, cubes and create-a-word-problem tasks, suggest that U.S. students were most apt to interpret the second factor as the number of sets. Thus this sentence would represent .6 sets of 15. Their focus on the decimal point as the "strange thing" about the sentence may be more than a mere reflection of their lack of exposure to such a sentence. Responses from several Israeli students suggested that the standard form 0.6 used in their materials evoked its own set of difficulties. (For example, some argued that one would multiply 15×0 and then add .6.)

Expressions for Word Problems

The students were given cards with word problems, asked to read the problem aloud, and tell what operation and numbers were needed to solve the problem. The interviewer stressed that any of the operations, addition, subtraction, multiplication or division, might be needed. In order to emphasize the shift away from multiplication, the first word problem presented was always an addition or subtraction problem. All students who answered this first item incorrectly were given feedback, led to the correct expression, and reminded that any of the operations might be needed to solve the remaining problems. Students were given at least two multiplication and one division word problems, not necessarily in that order.

Tables III–IV indicate that the students' performance in solving word problems appears to be influenced by the role a decimal factor plays. Although most students selected the correct operation in cases where the operator was a whole number and the operand was a decimal less than one, many who succeed with this type of problem choose division rather than

TABLE III

Performance of students given the two multiplication word problems requiring the expressions $.8 \times 5$ and $6 \times .8$.

		Phil spent $.80 for a pound of jelly beans. What is the cost of 5 pounds of these jelly beans?	
		Correct response	Incorrect response
Jane spent 6 dollars for 1 pound of cheese. What is the cost of .8 pounds of cheese?	Correct response	19	2
	Incorrect response	12	2

$n = 35$.

TABLE IV

Performance of students given the two multiplication word problems requiring the expressions $.75 \times 3$ and $3 \times .75$

		Mrs. Johnson's fudge recipe calls for .75 pounds of sugar. How much sugar does she need to make the recipe 3 times?	
		Correct response	Incorrect response
The recipe for a large cake calls for 3 ounces of chocolate. How much chocolate is needed to make .75 of the recipe?	Correct response	17	0
	Incorrect response	8	0

$n = 25$.

multiplication when the operator was a decimal less than one. Thus, their behavior in this task was consistent with dominance of the repeated addition model of multiplication.

Students included in the count of the lower left hand cell in either Table III or IV are those who responded correctly when the operator in a multiplication word problem was a whole number but who responded incorrectly when the operator was a decimal less than one. A typical response to the more difficult problem in each set was, respectively, ".8 divided by six" and "divide three into .75." Asked why she did not choose multiplication for the problem about cheese, the student said, "because you would get an answer bigger than six. Oh no, you wouldn't!" Asked why she would divide for the cake recipe problem she said, "Because you will need less than 3 ounces of chocolate." Interestingly, this same student correctly completed the computation of the product $15 \times .6$ using a rule about decimal points. Nevertheless in the face of word problems, she succumbed (at least temporarily) to behaving as if multiplication always makes larger and division always makes smaller. This student's performance was not unusual. At least half of the students who were not successful on the more difficult word problem tasks had correctly calculated $15 \times .6$. It is clear that success in computation with decimals is not a sufficient condition for success in selecting an operation for word problems.

Tables III and IV indicate that the percent of those missing the more difficult item while succeeding on the easier item is in the range of 25% to 30%. Although this seems to match well with the percent who expressed the opinion that the product of $15 \times .6$ should be larger than nine, readers are reminded that far fewer than 60 students are included in Tables III and IV. Thus more students may be affected implicitly by the belief, multiplication always makes bigger, than express this belief explicitly.

DIVISION TASK RESULTS

Notation

The interviewer read a division fact as many times as the student wished, and all the student needed to do was write it down however he or she liked best. Once students had written an expression, they were asked if they knew a second or third way to write it. The division "house" notation, $6\overline{)24}$, and its Israeli counterpart, $\overline{24}(6$, was clearly the students' favorite. It was used by all but one of the students. All of the Israeli students and 25 U.S. students produced the number sentence form. Clearly the least favored and least known format was the "fraction" form, a/b. Only five students, all Israelis, included this form among written expressions. When shown the fraction form, well over half of each of the populations claimed that they had either never seen this form used to indicate division or did not know that it could be used to express division.

Definitions

Fifteen of the 30 Israeli students produced only a partitive definition of division, five produced only a repeated subtraction definition, and five produced only a measurement definition of division. The Israeli interviewer pressed her students for a second definition and five were able to give both a partitive and measurement interpretation.

The U.S. students' definitions included six successful partitive and five successful measurement definitions. Four other students claimed division had some ill defined relationship with subtraction (e.g., "Like if you have 28 boxes you take away seven is four."). Eight of the students could say only that division undid, or was the opposite of, multiplication; and seven of the students were unable to give any verbal definition.

Overall, the partitive model dominated students' definitions, but a substantial number of students were able to provide a measurement interpretation.

Concrete, Pictorial, and Word Problem Representations

As was the case with multiplication, Israeli and U.S. children were quite successful with the tasks of producing concrete and pictorial representations. Given the expression $16 \div 8 = 2$, 58 of the 60 students produced at least one model with cubes (either 2 sets of 8 or 8 sets of 2) and 32 children produced both. The "two sets of 8" format was clearly favored, but not all of these models can be classified as measurement interpretations as students' verbal descriptions sometimes indicated that what they produced was a partitive interpretation of $16 \div 2$. Clearly two groups of eight is also the least cumbersome construction.

Students were also presented with a grid, the expression $12 \div 4$, and asked if they could state the answer and then show the fact on the grid. All but three of the students were successful in producing at least one example. The "three groups of four" representation, used by 45 students, was a clear favorite as a first representation. Slightly over half of the students who were asked to produce a second drawing could do so.

Most students were then shown a number line and asked if they could illustrate the same $12 \div 4$ fact on it. About one-third of the students were successful, one-half said they had no idea how to proceed, and the rest attempted the task but were not successful. Although the number line representation is connected most naturally with a measurement or repeated subtraction definition of division, there seemed to be little correlation between success with the number line representation and type of definition produced. Nine of the 18 students successful with the number line task had given only a measurement or repeated subtraction definition, one had given both a measurement and partitive definition, four had given only a partitive definition, and four had given no verbal definition.

The final task included in this representations category was to write a word problem for the expression $30 \div 5$. Some students had to be asked to construct the question at the end of their problem, and some were prompted to act out the sentence with blocks and then make up a problem (Table V). Twenty-seven of the Israeli students constructed a partitive

TABLE V

Responses to the task of writing a word problem for the fact $30 \div 5 = 6$.

	Grade/Country			
	4th		5th	
Responses	Israel	US	Israel	US
Stated a problem quickly, no prompts needed or blocks used	7	3	12	6
Gave a correct problem statement, but needed to be asked what question went with it	4	2	2	3
Started out incorrectly, realized their own error – may have needed a prompt to formulate question	0	0	2	1
Faulty or long delayed start. At interviewer's suggestion, set out blocks to show $30 \div 5$ and then formulated problem on their own	2	3	0	1
Faulty or delayed start. Needed to set up blocks or were given cues or prompts and eventually formulated a problem	1	3	0	1
Faulty or delayed start. Needed to set up blocks and were given cues or prompts but did not successfully formulate a problem	0	4	0	3

word problem first and three (all fourth graders) constructed a measurement problem first. Eighteen of those who first wrote a partitive problem were asked if they could write a second story in which division had another meaning; nine managed to produce a measurement problem. This word problem construction task was more difficult for the U.S. students. Thirteen of the U.S. students successfully constructed a partitive problem and 10 constructed a measurement problem. Seven of the U.S. students were

TABLE VI

Sample of students' responses to the task of giving a definition for division and writing a word problem for $30 \div 5 = 6$.

Excerpts from an interview with I-16, a 4th grade girl in Israel.

I*: What does division mean to you?
S: To share with friends. [sharing, i.e., partitive]
I: Anything else?
S: No.

I: Can you write a word problem for $30 \div 5$?
S: I had 30 sweets. For my birthday, I gave to each child 5. How many girls did I give sweets to? [measurement]

Excerpts from an interview with I-4, a 4th grade boy in Israel.

I: What is division?
S: I have 4 biscuits and 2 children and I divide the biscuits equally among them. [partitive]

I: Can you write a word problem for $30 \div 5$?
S: Yossi had 30 marbles. He gave 5 to his mother, 5 to his father, 5 to his friend, 5 to another friend, and 5 to another friend. How many did he give marbles to? [measurement]

Excerpts from an interview with USD-32, a 5th grade boy in the U.S.

I: What is division?
S: Like if you have 28 groups and divide them, put them in groups of 7, how many groups you have. [measurement]

I: Can you write a word problem for $30 \div 5$?
S: OK. There are five people who want apples from the apple tree. There are only 30 left. How would you give them so they each have an equal amount? [partitive]

Excerpts from an interview with USD-33, a 5th grade girl in the U.S.

I: What is division?
S: Equals, sometimes.
I: Twenty-eight equals 7 equals 4?
S: Well, if you go backwards. Four times seven equals 28. [inverse of multiplication.]
I: Is there any other way you think about division?
S: No.

I: Can you write a word problem for $30 \div 5$?
S: I have 30 cookies and five of my friends wanted some. How many did I give each? [partitive]

* I indicates interviewer, S indicates student.

unable to structure a problem setting on their own even with blocks. In looking at the data by student, it appeared that the division model reflected in the word problem was not related to the model used in the verbal definitions (some examples are shown in Table VI). Overall students were more apt to write partitive than measurement problems, although the dominance was more evident in the Israeli students than in U.S. students.

Facility with Decimals

Although the U.S. students who responded to the division protocol were a different group than the group that responded to the multiplication protocol, the groups were very similar with respect to their knowledge of decimals. The same procedures for inquiring about decimals were followed. And again approximately half of all the students given the division tasks (30 U.S. students and all 30 Israeli students) claimed to have recognized decimals, and somewhat less than half of the students could name them. Of the 42 students who were asked to make comparisons between 0 and .5, 1 and .5, 0 and .1, and 1 and .1, 27 responded correctly to three or four of the comparisons.

Twenty-one students were asked to tell what number would complete the sentence $12 \div .6 = ?$ Popular answers for this quotient were 2 and .2. Other answers included .4, 1.4, 11.4, 5, 6/12, and .02. Only one student responded "a number greater than 12." Four Israeli students given the clue, "Think of how many times the divisor is contained in the dividend?," were able to reason that the answer should be greater than 12. One Israeli student used common fractions to calculate the correct answer.

Eventually, fifty-nine of the students were asked for their reaction to the statement $12 \div .6 = 20$. Twenty-two of them said they thought this example was strange because the answer should be less than 12, or because 20 was greater than 12. Eleven said that what was strange was the decimal point; four said it was strange but could supply no reason, five maintained the equation was wrong, five either gave no reaction or simply demanded to know how this could be true. Twelve of the students, including the five who had previously calculated the quotient and five who reasoned the sensibility of the answer out on their own, said it was not strange.

Written Expressions for Word Problems

During the interview, students were given word problems. The procedures were the same as those noted in the section on multiplication word problems. The word problems given included one partitive problem ($15 \div 5$) and one measurement problem ($10 \div 2$). [It should be noted that if students responded "Multiply, $2 \times ? = 20$," their response was coded as a correct division response.] Each of the students interviewed with the division protocol also responded to one partitive problem with a divisor greater than the dividend ($5 \div 15$).

The problems were all straight forward and involved simple facts. The measurement problem was: The Scouts are selling salt water taffy 2 pounds to a box. How many boxes can he fill with 20 pounds of taffy? In the Hebrew version, the Scouts became a merchant packing soap powder 2 kilograms to a box. The partitive problem read: Five friends bought 15 pounds of cookies. If the cookies were equally shared, how many pounds of cookies did each person get? In the Hebrew version the pounds of cookies were kilograms of almonds. The students had an 90% success rate on the partitive problem and an 80% success rate on the measurement problem.

The students' success on both of these problems is spectacular when compared to the success rate (10 of 60) on the problem with a divisor greater than the dividend. The U.S. version of the problem was "Five pounds of trail mix was equally shared by 15 friends. How many pounds of trail mix did each person get? The Hebrew version presented a different context: "Fifteen friends bought five meters of rope. If they shared it equally, how many meters of rope would each get?" As shown in Table VII, all but three of the students first responded with some division expression. "Fifteen divided by five" was the first answer given by one-half of the Israeli students and by all but two of the U.S. students.

A review of the interview transcripts indicates that the interviewers proceeded slightly differently in administering this task. Once the students were aware that answer was less than one, the Israeli interviewer pressed the students for a mathematical expression. What was of interest in this case was that 6 of the children were convinced that while $5 \div 15$ could be done with rope in the real world, it could not be done mathematically. Division of a smaller number by a larger number was avoided. One student rewrote the problem as 500 cm divided by 15. Other students tried other operations, rejecting $15 \div 5$ and 15×5 as giving inappropriate answers. One said "perhaps we will do $15 - 5 = 10$, and then we write 0.1, that will be greater than 0, and maybe 0.1 says what we need." A few stuck to their opinion that it was a division problem. Two students suggested that perhaps the three from $15 \div 5$ could be interpreted as .3 meters, 3 centimeters, 3 decimeters or 30 decimeters. Several said, "it must be

TABLE VII

Students responses, by country, to the task of writing an expression for a word problem with a divisor (15) greater than the dividend (5).

Response	Country	
	Israel	U.S.
Immediately responded with $5 \div 15$	9	1
Responded immediately by saying the answer was less than one	0	1
Said $15 \div 5$, realized by themselves that this was incorrect and produced $5 \div 15$	2	3
Said $15 \div 5$, after a prompt, they state that the answer must be less than 1	13	6
Said $15 \div 5$, after a question about the size of the answer, they say it is less than 1	–	19
After a prompt, decide expression must be $5 \div 15$, but claim this cannot be done in mathematics	4	–
Other incorrect response	2	–

division, but 5 divided by 15 is impossible." One student claiming that division by a larger number was impossible, said, "to a great scientist, maybe there is an answer. We didn't study how to do this." Another student said "I know it must be one-third meter, but I don't know how to do it with five and fifteen."

The American interviewer probed for an answer – allowing students to reason out an answer from concrete materials and not pressing them for a mathematical expression. Most of the U.S. students ended in the same state as the student quoted above; they knew the answer but did not know how to use the numbers to calculate it. Even those shown 5/15 and asked (or told) about renaming it as 1/3 had a difficult time connecting this back to the problem or to a division example. This is consistent with the fact that none of the U.S. students had used fractional notation to indicate division.

DISCUSSION

This study was designed to gain information about the extent to which fourth and fifth graders hold conceptions about multiplication and division that may impede their work with decimals and about understandings they have that may be useful in building accurate conceptions of multiplication and division with decimals. The beliefs that were targets of investigation were "multiplication always makes bigger, division always makes smaller," and "the divisor must be smaller than the dividend."

Existence of Misleading Conceptions

The study confirms earlier studies of students' lack of understanding of decimals, their lack of linkages between decimal and fractional knowledge, and their difficulty in seeing a/b as a statement of division. Two other observations about the U.S. students are noteworthy.

- almost 25% of the U.S. students could provide no definition of division other than that of the inverse of multiplication. While this definition may support learning of division basic facts, it does not seem particularly helpful in characterizing instances in which division is used in problem solving.
- writing word problems for multiplication and division expressions was not an easy task for the U.S. students. Many students volunteered that they did not do this task in school.

The study also provides evidence that the beliefs "multiplication always makes bigger" and "division always makes smaller" are held by students as early as fourth and fifth grade. Students non-calculated answers and reactions to the results of $15 \times .6$ and $12 \div .6$ are evidence of this. These data indicate that the misconception about division is more commonly held than the corresponding misconception about multiplication. One-half of the students interviewed were relatively unfamiliar with decimal numerals, and 20–25% of all the students indicated that what was strange or unusual about these sentences was the decimal point. It is possible that these students hold these same misconceptions but focus on the use of decimals as the unusual aspect of the sentence. One might anticipate a, hopefully temporary, rise in the percent of students expressing the common misconceptions as students become more accustomed to seeing the decimal notation.

Students given a word problem involving multiplication of a whole number by a decimal less than one did more poorly on it than they did on a similar word problem where the role of the factors was reversed. This is generally interpreted as showing the influence of the primitive model of multiplication and the accompanying belief, "multiplication always makes bigger." The definitions the students produced for multiplication and the multiplication word problems they constructed also suggest that the vast majority of them view multiplication as repeated addition.

Resistance to writing an expression that involves division by a larger number seems clear. The majority of initial responses to the word problem with solution $5 \div 15$ was $15 \div 5$. The students could reason out an answer, but they rejected the notation $15\overline{)5}$, and even when students were shown the statement $5/15 = 1/3$ (which they recognized as correct), they were unable to see what connection this had to the problem. For these students the belief, "you can't divide by a larger number" seemed to be more a manifestation of an inability to symbolize this operation than it does of inability to conceptualize the operation.

In reviewing students' responses to the different division tasks, it is apparent that the model of division evoked varied with the context. For example, some students who used only a measurement model when asked to define division, wrote only partitive problems when asked to write word problems for the sentence $30 \div 5 = 6$. The reverse situation also occurred.

Students' comments also revealed some attitudes that might interfere with their performance on word problems. A number of students in both countries did not find "real world" possibilities reflected in mathematics. Transcripts included examples of the problem solving strategy test-all-operations and pick the one with answer that seems to fit best (see Sowder, 1986). There were also a number of students who expressed the "given up on understanding" attitude. For example, one student said: "Most of them [number sentences] are strange, and this is no stranger than the rest."

Potentially Useful Knowledge

Multiplication knowledge. However, one task we set, was to identify knowledge the students bring that may be helpful for building understanding of multiplication and division with decimals.

The only popular model for illustrating multiplication of two rationals less than one is the "area" model. Logically the progression from a union of like sets, to an array, and finally to an area model does not seem insurmountable. While it is not anticipated that the transition from integral array/area models to rational area models is necessarily easy, the students seem to possess a number of the prerequisite understandings and skills. For example, about 80% of the students verbalized a somewhat reasonable definition for multiplication in terms of repeated addition or collection of like-sized group and performed well in exemplifying a multiplication sentence or expression with cubes (95% successful) or diagrams (90% successful). When given a large grid, approximately 60% of the U.S. students and 90% of the Israeli students produced drawings of at least one compact array model (area model) for the expression 3×4.

Initially, simple multiplication expressions with a whole number multiplier can also be interpreted as the union of an integral number of "like-sized parts of sets." In fact a few students managed this interpretation for the expression $.1 \times 10$. This extension can likely be made accessible to other students. Further, students responses to the $7 \times 6 = 6 \times ?$ sentence and their discussion of block groupings and grid drawings indicated strong belief in the commutative property for whole numbers. If this property is tested on "easy" decimal examples or on decimals in fraction form and then extended to work with decimals, the numerical equivalence of $.6 \times 15$ and $15 \times .6$ can be argued. We are not suggesting that this somewhat abstract argument be the only argument, but rather that it might serve as an additional means for helping some students make sense of a decimal multiplier. At the same time this approach could be used to evoke discussion about the conditions under which multiplication does or does not "make bigger."

Division knowledge. An understanding of division by a decimal less than one almost demands the application of the measurement model. Fischbein et al. (1985) examined their data on fifth graders' ability to write expressions needed to solve word problems and were "led to conjecture that initially, there is only one intuitive primitive model for division problems – the partitive model" (p. 14). For the two samples of students involved in this study, we would not be convinced of the need to drop the measurement model from consideration as a primitive model. On the write-a-word-problem and give-a-definition tasks, students used the partitive model somewhat more frequently than the measurement model. However, substantial numbers of students made use of the measurement model, if not in the word problem settings than in their representations with blocks or diagrams. Also, students who struggled with the interpretation of $12 \div .6 = 20$ and were prompted to consider the measurement interpretation, "How many .6's are there in 12?," were at least able to recognize that the answer would be greater than 12.

IMPLICATIONS

In both Israel and the U.S., multiplication by decimals and division by decimals is typically introduced at the fifth grade level. Clearly, the students in the study possessed some knowledge that would be helpful to building understanding of operations with decimals. Further the reasoning exhibited by some students, suggests that given certain knowledge and attitudes some students at this age are capable of making reasonable interpretations of at least simple expressions involving multiplication and division of a whole number by a decimal. We do not, however, mean to imply that the skills should be a priority at these grade levels. It seems essential that before students deal with such computations they should (1) exhibit a good understanding of the meaning of decimal notation, (2) be able to interpret division phrases using both the partitive and measurement model, (3) understand that *a/b* notation can signal division, and (4) be facile in translating between common fractions and decimal fractions. Perhaps this is one area in which the "Underachieving" curriculum's (McKnight et al., 1987) circularity deserves adjustment.

Whenever the skills for multiplication and division by decimals are taught, it seems that the instruction should be designed to assure that understandings of the operation with whole numbers and decimal notation are accessible to students, so that understanding can be brought to the decimal tasks. Possible instructional activities for achieving these goals are:

- introduce students in the early grades to the process of estimating whether the answer to a division word problem is greater than, less than, or equal to one.
- provide opportunity for students to compare and contrast their varied definitions, models, and understandings of multiplication and division.
- relate decimal notation to concrete embodiments and to currency notation. Some student textbooks treat decimal notation in monetary amounts as a topic totally separate from the decimal numeration system. This seems to be totally inappropriate.
- use both decimal notation and common fraction notation to perform the same calculation. Compare results.
- use the area model for multiplication with whole numbers, (perhaps linking this back to an array and repeated addition model) prior to using it to illustrate the multiplication of decimals greater than and less than one.
- introduce multiplication involving decimals less than one in a word problem setting with a whole number multiplier. (Repeated addition might even be used to justify answers perhaps using a number line to illustrate).
- explore the extension of the commutative property to indicated products involving a decimal and a whole number.

- introduce word problems with a decimal operator. Discuss the problem structure and estimate the anticipated answer. Consider why the operation will result in an answer compatible with the estimate. Debate about the results of the computation may lead to discussion of the misbeliefs.
- contrast the results of multiplication or division by a decimal less than one with the results of the operation with factors greater than one. Study patterns of answers when operating with one; with decimals less than one; and, for multiplication, with zero.
- use the measurement model for division with whole numbers. Have students write measurement word problems and interpret division expressions for whole numbers in a measurement manner. Extend such translations to simple expressions involving decimal divisors that are easily solved when the measurement interpretation is applied (For example, $.5 \div .25$, $.8 \div .1$, $.25 \div .05$.) Use manipulatives such as decimal squares and currency to model such examples.
- reinforce the use of a/b notation for division; connect indicated division such as $a \div b$ with the common fraction a/b.

Since computational facility with whole numbers does exist alongside lack of such understanding of the operations, tasks such as writing word problems and writing expressions to solve word problems are likely necessary diagnostic tasks. Tasks such as those described above can be used both to diagnose the students' understanding with whole numbers and to probe for the development of understanding of the operations with decimals.

As Fischbein et al. (1985), Thorndike (1921) and others have suggested, it is probably not realistic to consider prevention of the beliefs multiplication makes bigger and division makes smaller. They seem to be a natural outcome of years of work with whole numbers. The middle school teacher will undoubtedly always be in the position of needing to help students control the influence of these beliefs. The "conflict teaching" approach described by Swan (1983) and the contrast of operations in the whole number and rational domains suggested by Semadeni (1984) are likely candidates for approaches that should be tested. Teachers must also consider Fischbein's (1987) concerns about helping each student cope with the idea that "while being absolutely convinced about the truth of an idea," such as division makes bigger, "he was in fact wrong" (p. 37). The suggestions above can easily be incorporated into such treatments.

If students are to gain an understanding of multiplication and division with decimals, then teachers and textbooks need to reflect these understandings. Although the teacher's guides for some recent textbooks mention the misconceptions, they do not provide suggestions for alleviating their effects. Further the texts rarely provide meaning for counter-intuitive cases, e.g., when "multiplication makes smaller." Early findings from a review of several texts in both Israel and the U.S. also indicate that texts contain many computational exercises that multiplication of two decimals less than one, but far fewer word problems that require such an operation (Graeber and Baker, in press; Tirosh, 1989).

It is clear that teachers will need support in providing instruction on these topics. Eighty-five percent of a sample of 136 preservice elementary teachers correctly argued that the statement, "In multiplication problem, the product is greater than either factor," was false, yet only 50% selected the correct operation for a word problem with a decimal factor less than one. Fifty-five percent incorrectly agreed with the statement "In a division problem, the quotient must be less than the dividend" (Tirosh and Graeber, 1989). Other researchers' (e.g., Greer and Mangan, 1986) findings suggest that this in not a uniquely American situation. Continued investigations in this area will have implications not only for the elementary and middle school curriculum but also for teacher education.

NOTE

An earlier version of this paper was presented at the Annual Meeting of the American Educational Research Association, New Orleans, LA, April 1988. The research described in this article was supported in part by a Summer Research Grant from the General Research Board of the Graduate School, University of Maryland. This is Working Paper Number 71 of the Tel-Aviv University Unit of Human Development and Education.

REFERENCES

Bell, A. W.: 1982, 'Diagnosing students misconceptions', *Australian Mathematics Teacher* **38**, 6–10.

Bell, A. W., E. Fischbein, and B. Greer: 1984, 'Choice of operation in verbal arithmetic problems: The effects of number size, problem structure and content', *Educational Studies in Mathematics* **15**, 129–147.

Bell, A., M. Swan, and G. Taylor: 1981, 'Choice of operation in verbal problems with decimal numbers', *Educational Studies in Mathematics* **12**, 399–420.

Brown, M.: 1981, 'Place value and operations', in K. M. Hart (ed.), *Children's Understanding of Mathematics: 11–16*, pp. 48–65, John Murray, London.

Carpenter, T. C., M. K. Corbitt, H. S. Kepner, M. M. Lindquist, and R. E. Reys: 1981, 'Decimals: Results and implications from national assessment', *Arithmetic Teacher* **28**(8), 34–37.

Case, R.: 1985, *Intellectual Development: Birth to Adulthood*, Plenum, New York.

Davis, R.: 1984, *Learning Mathematics: The Cognitive Science Approach to Mathematics Education*, Ablex Publishing, Norwood, NJ.

Ekenstam, A.: 1977, 'On children's quantitative understanding of numbers', *Educational Studies in Mathematics* **8**, 317–332.

Fischbein, E., M. Deri, M. S. Nello, and M. S. Marino: 1985, 'The role of implicit models in solving problems in multiplication and division', *Journal for Research in Mathematics Education* **16**, 3–17.

Fischbein, E.: 1987, *Intuition in science and mathematics: An educational approach*, Kluwer Academic, Boston.

Graeber, A. O. and K. Baker: (in press), 'Curriculum materials and misconceptions concerning multiplication and division', *Focus on Learning Problems in Mathematics.*

Graeber, A. O., D. Tirosh, and R. Glover: 1989, 'Preservice teachers' misconceptions in solving verbal problems in multiplication and division', *Journal for Research in Mathematics Education* **20**(1), 95–102.

Greer, B. and C. Mangan: 1986, 'Choice of operations: From 10-year-olds to student teachers', in *Proceedings of the Tenth International Conference Psychology of Mathematics Education*, pp. 25–30, University of London, Institute of Education, London, England.

Hart, K. (ed.): 1981, *Children's Understanding of Mathematics: 11–16*, John Murray, London.

Hiebert, J. and D. Wearne: 1983, April, 'Students' conceptions of decimal numbers', Paper presented at the annual meeting of the American Educational Research Association, Montreal.

Kieran, T. E.: 1980, 'Knowing rational numbers: Ideas and symbols', in M. M. Lindquist (ed.), *Selected Issues in Mathematics Education*, pp. 69–81, McCutchan, Berkeley, CA.

McKnight, C., J. Crosswhite, J. Dossey, E. Kifer, J. Swafford, K. Travers, and T. Cooney: 1987, *The Underachieving Curriculum: Assessing U.S. School Mathematics from an International Perspective*, Stipes Publishing, Champaign, IL.

Semadeni, Z.: 1984, 'A principle of concretization permanence for the formation of arithmetical concepts', *Educational Studies in Mathematics* **15**, 379–395.

Sowder, L.: 1986, 'Strategies children use in solving problems', in *Proceedings of the Tenth International Conference Psychology of Mathematics Education*, pp. 469–474, University of London, Institute of Education, London, England.

Steffe, L.: (in press), 'Children's construction of meaning for arithmetical words: A curriculum problem', in D. Tirosh (ed.), *Implicit and Explicit Knowledge: An Educational Approach.*

Swan, M.: 1983, Teaching decimal place value: A comparative study of 'conflict' and 'positive only' approaches, Shell Centre for Mathematical Education, University of Nottingham, Nottingham, England.

Thorndike, E.: 1921, *The New Methods in Arithmetic*, Rand McNally, New York.

Tirosh, D.: 1989, 'Division and multiplication: What is in textbooks?' (unpublished manuscript), Tel-Aviv University (in Hebrew), Tel-Aviv.

Tirosh, D. and A. O. Graeber: 1989, 'Preservice elementary teachers' explicit beliefs about multiplication and division', *Educational Studies in Mathematics*, **20**(1), 79–96.

College of Education,
University of Maryland,
College Park Campus,
Room 2311,
HRW Benjamin Building,
Maryland 20742, U.S.A.