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A CONSTRUCTIVIST APPROACH TO NUMERATION IN PRIMARY SCHOOL: RESULTS OF A THREE YEAR INTERVENTION WITH THE SAME GROUP OF CHILDREN

ABSTRACT. This article is a follow-up to an earlier article (Bednarz and Janvier, 1982) which presented the results of a research study on the understanding of numeration by primary school children. That study pointed out the main difficulties children meet in learning numeration, presented a theoretical framework that made explicit a conception of numeration different from the one considered in current mathematics teaching, and also proposed a reference framework utilizable for learning and evaluating this notion. The experimentation in a classroom announced at the end of the article, was undertaken from 1980 to 1983 with the same group of children from the time they were in first grade (6–7 years old) to the third grade (8–9 years). The theoretical and reference frameworks developed in the former research study proved to be effective for developing a constructivist approach leading children to build a meaningful and efficient symbolism of number.

This article is mainly devoted to presenting the results of this three year longitudinal study (part C). At first, we shall characterize briefly our intervention based on a constructivist approach of numeration (part A). Also we shall describe the conditions under which the experimentation was carried out, and the means used to evaluate the impact on the pupils (part B).

INTRODUCTION

In the previous research project on numeration, we aimed at answering several questions we considered essential in order to diagnose childrens' understanding of numeration. In order to have an over-view of our conception of numeration underlying these questions, we shall restate some of the questions presented in the first article.

Do children see that conventional writing is related to collections of elements "structured in *groupings*"? (collections reorganized to make subsets appear that have the same number of elements).

Do children use grouping as a strategy to communicate information or operate on a collection, when that proves to be an efficient strategy?

What meaning do children attribute to *representations* of numbers that are submitted to them?

Can children carry out operations involving *doing* and *undoing groupings*?

Do concrete materials offer any support to children so that they can build on them and refer to them, particularly when they have to operate on collections (addition, subtraction, sharing, comparison . . .)

Can children work with *two groupings of different order* at the same time?

The analysis of answers to items related to each question pointed out

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several misconceptions and difficulties developed by children in the current teaching of numeration. We summarize these here:-

- For most children, a number is an alignment of digits. Words like hundreds, tens, units are not taken at all into consideration, or they are associated with *a fragmentation, an order* of writing. This leads us to conclude that few children give a true interpretation of the digit position in terms of groupings.
- Many children *don't see the usefulness (relevance) of grouping* in a task where that proves to be an efficient strategy. Some children form groupings but only in order to count the collection (by ten, by five, . . .). In these cases, we could observe that the task asked was not performed by reflecting on the groupings they made, but rather they applied learned algorithms or mental calculation facts. Few children see the validity of doing groupings to perform a task as a whole, and see that writing is a code which springs directly from these groupings.
- Few children can really operate on groupings when they have to do or undo them. Rather they try to re-transcribe with conventional writing what they do in the algorithmic procedure. We observed that children cannot illustrate or explain with any material the operations performed with conventional number writing. We find then different *erroneous interpretations of carrying and borrowing* in operations.
- *Few* children *can work simultaneously* in a task with *groupings of different order*. Even if they see groupings, which is not always the case, children have difficulty in co-ordinating them. In their reasoning, they confound the two different groupings.

We could pursue the analysis of children's answers but their understanding is well enough characterized to convince us that the skills developed by school mathematics teaching do not transfer to any of the numeration tasks proposed to them. In fact a lot of children's misconceptions can even be explained by analyzing the perception of numeration found in school text books, curriculum . . . . This point is discussed more fully in Bednarz and Janvier (1984).

By reading over the previous questions it may be striking to note the insistence we put on collections structured in groupings and on representations of numbers. Indeed for us the symbolic conventional writing is not questioned in itself, but in regard to its meaning in terms of groupings and in terms of the transformations made on these groupings when we operate. When we work on numeration, we work on the process of representation of number. The teaching of numeration should promote this process and take into consideration the operations on collections reorganized in groupings.

We undertook such a teaching strategy over a period of three years, putting these recommendations into practice.

AIMS OF THE RESEARCH

The main aims of the present research can be stated as follows:

(1) to develop a didactical strategy based on a constructivist approach to the concept of numeration and its learning, leading children to build progressively a meaningful and efficient representation for numbers.

(2) to make precise, in the application of this strategy, how the theoretical and reference frameworks developed in the former research are utilizable.

(3) to evaluate the effects of this constructivist approach on the understanding of numeration by children and on the main difficulties and conceptions encountered by them.

(4) to point out the procedures and representations developed by children in the situations and their evolution.

In this article, we shall present essentially the results of this intervention. This is based on a constructivist approach of numeration to be described briefly in a following section. Our didactical strategy, already partly published in Bednarz *et al.* (1984, 1985) will be detailed in another article. During this intervention, we used the theoretical and reference frameworks we had developed in the first research as curriculum guide. However, classroom experimentation enabled us to refine and improve the previous frameworks.

A. A CONSTRUCTIVIST APPROACH OF NUMERATION: CHARACTERIZATION OF OUR INTERVENTION

Our didactical study is based on a constructivist conception of learning. By placing the study in a Piagetian and post-Piagetian perspective on the development of knowledge, the child is viewed as the elaborator of new concepts. Our didactical strategy is further inspired by socio-constructivism putting clearly into evidence the role of interactions and communications between children in the evolution of cognitive processes. These theoretical bases however furnish few explicit directives concerning the organization of constructivist learning situations or the nature of the mathematical knowledge constructed by the child. Recent studies in mathematics education, inspired by the constructivist perspective, provide some information on children's thinking in mathematics. Concerning this aspect we may refer to Ginsburg (1983), Carpenter *et al.* (1982), Cobb and Steffe (1983), Schoenfeld (1987), and many others. Nevertheless a critical question remains: how can learning situations be organized to take into account the mathematical thinking of the child? Our research is an effort to address this question.

In order to elaborate any situation or intervention, the researcher must not only have a model of the children's present conceptual structures, but also an analytical model of the conceptualizations to serve as goals towards which we desire the child to evolve. Hence the approach developed required a knowledge of children's thinking (difficulties, conceptions encountered by them), and continuous analysis of procedures and representations used by them in learning situations. This approach also required a reference framework that made explicit a particular conception of numeration. Based on this framework, the learning of numeration developed places the child in situations that invoke a process of representation of number and leads him to evolve toward a significant and efficient symbolization.

These situations force the child to operate on collections in which grouping rules are defined. The *operations are essential* in our strategy because in addition to giving intentions for carrying out transformations on groupings, they inject meaning for those transformations (to make groupings, to "unmake" them, . . .). Furthermore these situations necessitate that the children themselves develop means to keep track or to communicate information on the transformations operated on collections and on the regrouped collections that result. In doing this, the child is obliged to establish relations between the actions made on groupings and the means that he uses (drawings, oral language, written notations . . .) to symbolise those relations. Hence the representations he uses become more and more meaningful for him. Finally, the solicitation to become more effective in operating on collections and in giving information on these forces the child to have recourse to written representations and to refine them.

In summary, our strategy should be such that the child can develop his own representations (representations of collections as representations of transformations applied to collections) and be able to improve them. *We need,* therefore, *to select our situations* in such a way that the child can construct the concept at a level accessible to him right from the beginning and all through his learning of numeration. It is in this area that the *reference framework* developed in the previous research, and refined in the present study, turned out to be very useful.

We find there in fact the *three elements* that we identified in previous research as determinants of the complexity of a situation:

(1) the pathways in the diagram (Fig. 1),
(2) the skills to use,
(3) the representation brought into play.

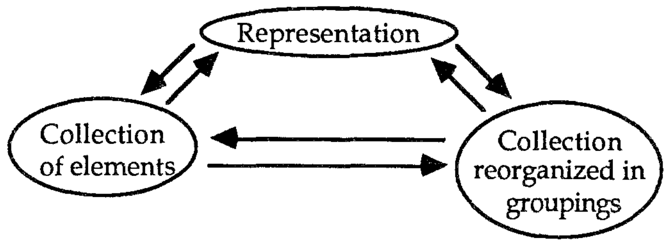


Fig. 1.

We have presented in the first article (Bednarz and Janvier, 1982), a number of examples of situations illustrating the complexity of the pathways and skills brought into play. Concerning the third point, it is especially here that the intervention in class made us refine our reference framework, realizing in effect how *the complexity of the task depends on the "means"* (oral language, drawings, conventional writing, symbolic written notations . . .) used to communicate information on a collection or to operate in a given situation. We have identified *three criteria determinating the accessibility of the representation* brought into play in a given situation:-

(a) the possibility of identifying, more or less rapidly, the number of distinct groupings present, as well as the order of precedence of these groupings one in relation to the other,
(b) the possibility of deducing, more or less directly, the relation between the groupings,
(c) the possibility of operating, more or less directly, on the groupings without needing recourse to intermediaries (for example, exchanging).

Thus, in a situation bringing into play the conventional writing system of representation of number, the groupings are not identifiable and the relation between the groupings is not at all visible; it is a convention that determines the relation. Furthermore, if we wish to operate on collections via this writing notation we call-up rules connected to this convention (that determines the relation between the groupings).

In contrast, in the material "cereal boxes" (Fig. 2) used in our intervention, it is possible to identify rapidly the two groupings and to see that the "basket" is the largest grouping, followed by the grouping "case", followed by the elements "box".

Furthermore, the relation between the different groups is clearly apparent. The written representation, for example 2 1 2 , refers to material where it is possible to operate directly on the groupings: to make or un-make directly the cases and boxes when it is necessary to operate on regrouped collections. This written representation (referring to material cereal boxes)

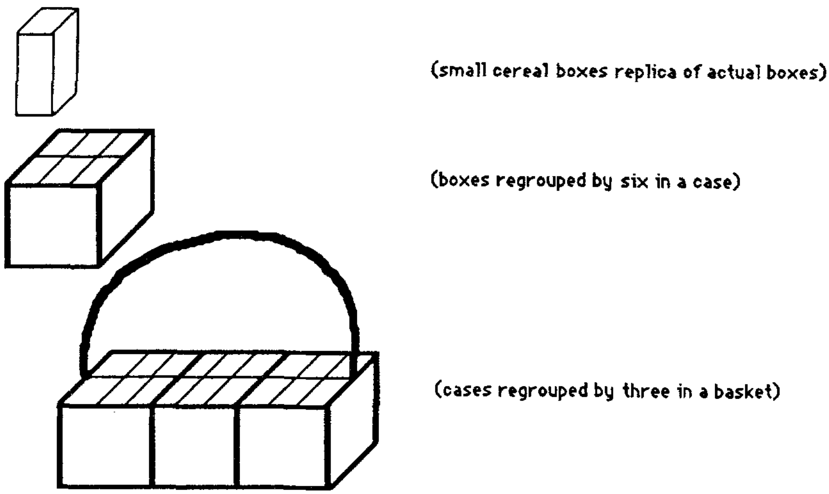


Fig. 2.

could however become just as symbolic as conventional writing (for example if we use 1 ▭ 2 ⊔ 1 □).

In this way, in a situation "the means" used to communicate information on a collection or to operate on it is determinant: the groupings can be more or less evident and the relation between the groupings more or less explicit, thereby rendering the task more or less accessible to the child.

The learning situations in our intervention are constructed on selected materials chosen on the basis of the three preceding criteria of accessibility. These situations become more and more complex as much on the level of

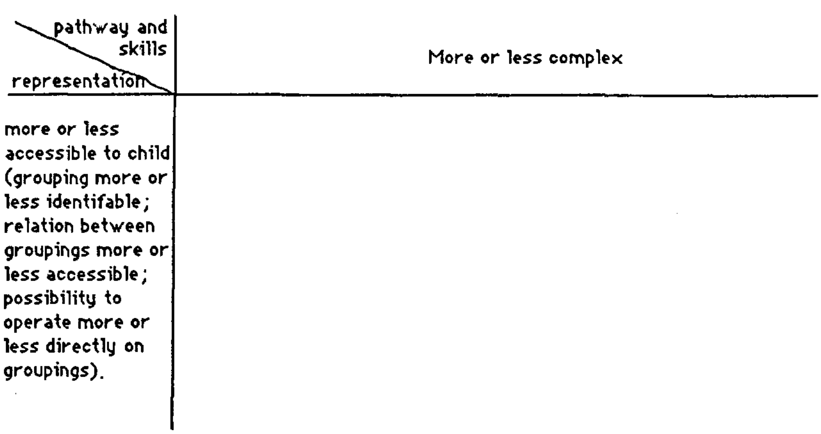


Fig. 3. Reference framework.

pathways and skills brought into play as on the necessities of communication or treatment of collections. This is presented more fully in Bednarz and Janvier (1984). Figure 3 serves to illustrate the reference framework underlying this progressive choice of situations.

The constructivist approach developed during three years is based on this reference framework and on a constant analysis of procedures and representations used by children in the situations. We shall now describe how the experiment was conducted and the means used to evaluate the impact on the pupils.

B. THE EXPERIMENT: METHODOLOGY

1. *Carrying-Out of the Experiment; the Constructivist Group*

The experiment started in January 1981 with two first year classes (39 children) in an average socio-economic region of Montreal. We taught mathematics to both classes until June 1981. At the end of the school year 1981, the principal and the two teachers who were directly involved with those grade one children, divided the group of 39 children in two grade two classes. They made the division as evenly as possible to insure a fair division of good, medium and poor students in each class. New children became part of each class. The grade two class of 22 children assigned to us (17 from first year experiment, 5 new children) was similar to any grade two class in the same school board. We were to be in charge of the math curriculum with this same group of children for the following two years.

The diagram (Fig. 4) illustrates how the "constructivist" population

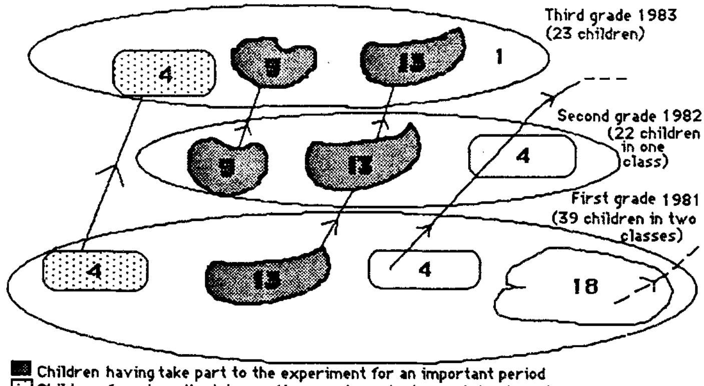


Fig. 4. "Constructivist" group during the three years (constructivist approach).

evolved from the beginning (January 1981) with 39 children to the end of the experimentation with 23 children (June 1983).

This project was carried out within the usual school constraints, that is, those of the school program and of examinations required by the school board, the operational rules of the school, the expectations of the director, the teachers and the parents in relation to the school. Mathematics classes, lasting 50 minutes, were held four days a week. Nadine Bednarz was the teacher for two of these periods and Bernadette Janvier the teacher for the two other periods. Co-ordination meetings were held each week to analyse the children's work, as well as the tasks already experimented, and to elaborate other learning tasks.

2. *Evaluation of the Effects of This Constructivist Approach on Children*

For each year *a set of items, based on* the previously developed *reference framework*, was constructed. Some of the items contained *conventional elements related to the school program* (significance of conventional writing, role of position, significance accorded to calculation algorithms); other items dealt with *non-familiar situations* forcing a reflection on groupings.

In Table I we see, for each set, where the major part of the items is concentrated. We can observe that from first to third year, more and more complex tasks were proposed to children in terms of pathways, skills brought into play and representations used to communicate information or operate on collections.

These items were experimented with in interviews by other persons than the researchers at the end of each of the years 81, 82, 83. These were used with each child in the constructivist group as well as with each child in a group of another school (*current teaching group*) of the same school board: a group that had received the regular teaching of their classroom teacher. This group will be used as *comparison group* at the end of each year.

The Current Teaching of Numeration (*Comparison Group*)

The teaching of numeration occupies an important place in elementary school mathematics programs, particularly in the first three grades. We will point out aspects that characterize this teaching. The traditional teaching of numeration is linked to the capacity to read and write numbers, and to the ability to point out place values in any given number. We find here a type of

TABLE I

Concentration of items

Complexity of the Task (pathway and skills) / Representation (relation between groupings)	Easy	Average	Difficult
Apparent			
Hidden			
Given by a convention			

Major part of items at the end of 1^{st} year

Major part of items at the end of 2^{nd} year

Major part of items at the end of 3^{rd} year

teaching that from a very early stage is focused on conventional writing and on their syntactical rules. To get children to understand these rules, strong emphasis is put on the transition from conventional writing to other symbolizations (unit, tens, hundreds . . .) or to images of material, or to handling of material. Work in different bases has also the same goal. Children have to group and regroup objects according to different bases. However, these exercises precede coding-decoding work, and many children don't see the usefulness of grouping in different bases. Finally, numeration is treated as a prerequisite to the study of operations on numbers greater than two digits and not as something that evolves simultaneously along with calculation procedures. When the children are judged sufficiently advanced in their learning of numeration, teachers go on to operations using conventional vocabulary, or images of material, or concrete materials (Dienes blocks, for example). This current teaching of numeration was the object of analysis in Bednarz and Janvier (1984) and the difficulties experienced by the children were pointed out.

Our Data

The two groups (constructivist and current teaching), belonging to the same schoolboard, were expected to follow the same program objectives and take the same examinations of the schoolboard. Some items, used in interviews with these two groups had been used in the preceding research (1980) with 160 children from the first to fourth grade in three different schools of another schoolboard (that we identify by A).

We therefore have data coming from three sources (see Table II).

TABLE II

Constructivist group and current teaching group

	Interviews conducted from 1981 to 1983 Schoolboard B		Interviews conducted in 1980 Schoolboard A
	Constructivist approach	Current teaching	Current teaching
(age 6–7) end 1st year	39	25	40
(age 7–8) end 2nd year	27 *	23 (2 children left the school)	--
(age 8–9) end 3rd year	23 *	26(23 + 3 new)	75
(age 9–10) end 4th year	--	--	45

*See Figure 4

Underlying Questions to This Evaluation

The interviews conducted at the end of each year with the constructivist group and with the current teaching group were not undertaken in order to compare the two approaches as such, and the study is not comparative research.

The facts from these different interviews were used to help answer a series of precise questions. In particular:

(a) How do children from the group following a constructivist approach perform on the fundamental aspects presented in the introduction: the interpretation given to conventional writing, the ability to operate on groupings, the ability to co-ordinate two groupings of different order, the pertinence to regroup.... Do the children develop the same difficulties, the same underlying misconceptions? Particularly, how do they perform *relative to the usual school requirements*: role of position, ability to treat the four operations...?
(b) How to the children of the constructivist group evolve in their understanding of numeration during this period of three years? Is there *transfer* of acquired skills to situations putting into play systems of representation unknown or less accessible to the children?
(c) How do the children having followed the constructivist approach perform in relation to attitudes developed, procedures and representations used?

C. RESULTS

We shall first see how children in the constructivist group are situated at the end of the longitudinal study in reference to difficulties and conceptions encountered (question a). Then we shall examine how they evolve in their understanding of numeration over the three year period (question b). Observations related to point c will be inserted in passing. Finally, to make even more precise where the children are at the end of three years in their acquisition of numeration, we shall examine to what extent there is transfer of acquired skills to other situations bringing into play unknown or less accessible representations (point b).

I. *Situation of Children (Constructivist Approach) Related to Important Difficulties Encountered in the Learning of Numeration*

In all the items presented to the children, *the material* and *the representations used were not familiar to them and in no way corresponded to previous specific learning*. In order to illustrate the understanding of numeration by the children, the strategies that they used and the difficulties encountered, we will select items used in the interviews and will present the results.

(a) *On the Pertinence of Regrouping and the Significance of Associated Writing*

Let us examine the results obtained on an experimental item at the end of the third year (8–9 years old), that is to say, at the end of the longitudinal study.

Description of the item. We present to the child a sheet of paper where a large number of rods are drawn (see Fig. 5). The sheet is left before the child a short time and presented in a manner that he cannot touch it. We ask him:
"Can you tell me quickly how many rods there are drawn there?"
"I am going to do the same thing later with a friend who will be here after you. Could you do something (we give him the sheet) so that, when I shall show him the sheet, he will be able to tell me very quickly how many rods there are?"
Interventions
When the child has finished
"What did you do?"
"Now can you tell me quickly how many rods there are?" (we show him his sheet of paper).
"How do you know that?"
"Look at what the friend who came before you did" (we present him the sheet of paper of another child who made, for example, a grouping of groupings. "What do you think of it?"
"Can we see quickly how many rods there are?"

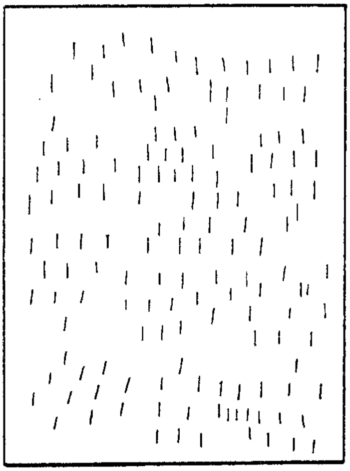


Fig. 5.

THE RESULTS (AT THE END OF THE THIRD YEAR) (SEE TABLE III)

TABLE III

Pertinence of regrouping

	Constructivist approach June 1983	Current teaching/same school board (B) June 1983	Current teaching/other school board (A) June 1980
	% of group (23)	% of group (26)	% of group (75)
Some children *do not see the pertinence of regrouping*. They rely on appearance and say an approximate number. or They have recourse to counting one by one, using different organization to ensure that they count an object a single time. Of course, once they have finished they can only write the number of sticks in all (something we did not want).	4%	38.5%	41%
Others *regroup*, but in order to count quickly the collection (by 10, by 5 . . .). Thus when we asked them after regrouping to tell us quickly how many rods there are, they recount the collection. Faced with two collections to compare they re-count each of the collections before being able to say in which collection there is more. They do not see that writing is a code which is directly associated with the regrouped collections.	4%	34.5%	33%
Others see the pertinence of regrouping and see that writing is a code that is directly associated with these groupings. They can directly deduce the number associated from the re-grouped collection, for example "14 bundles of 10, 140, 144". Some of the children in this group *use* only *one grouping* order.	62.5%	27%	26%
Others *use a grouping of groupings*.	29.5%	0%	0%

Commentary

– Related to this task, we have found, starting at the second grade (7–8 years of age) that the children of the constructivist approach, even if in a classroom situation they could identify groupings and use them when they had to make or un-make some, did not spontaneously have recourse to regrouping even when this appeared to be an efficient strategy to communicate information on a collection or to deal with it. This ability was developed progressively in situations that allowed pointing out the efficiency of this strategy. Similarly, the procedures used by the children to regroup the objects and the "means" (written notations) developed by them to communicate information on these regrouped collections were gradually refined.

For this item we see that almost all children of the constructivist group realize the pertinence of the recourse to regrouping, judging that this was a good means of rapidly communicating information on the cardinality of the collection: only 4% of children don't see the pertinence of regrouping. We can observe otherwise that populations A and B (current teaching) have almost the same performance and difficulties after an interval of three years.

– We need to emphasize here that the children were brought to organize by themselves to regroup and code, no explicit direction of production having been given, even less a coding instrument (table of columns, for example). In this organization, we can find a richness of productions.

By way of illustration we present a few of the ways used by the children in the constructivist group to organize, permitting them to identify the regrouped collection without risk of errors and subsequently to visualize more or less quickly the associated number. Some children point out each of the elements up to ten by barring them, then encircle these elements and mark 10 on each set (the task being to organize the collection so that another child can see quickly how many rods there are) (see Fig. 6). Others number the elements up to ten, then encircle . . . (Fig. 7). Still others colour each element up to ten, encircle them by a line of the same colour then change the colour for the next grouping (Fig. 8). Some children count ten elements, encircle these and identify each grouping so formed by a digit: 1, 2, 3, . . . (Fig. 9). Others number each element of the same grouping in the same way: 1, 1 . . . , 1; 2, 2 . . . , 2 (identifying in this way ten elements) and then encircle (Fig. 10). For those having recourse to a grouping of groupings, the same procedures reappear with, in addition, the use of a different colour or a larger marking when they encircle the grouping of groupings, or the identification of the grouping of groupings by a number: 100 or 1.

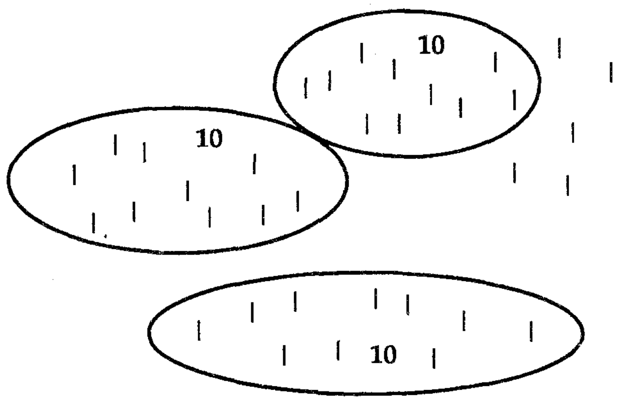


Fig. 6.

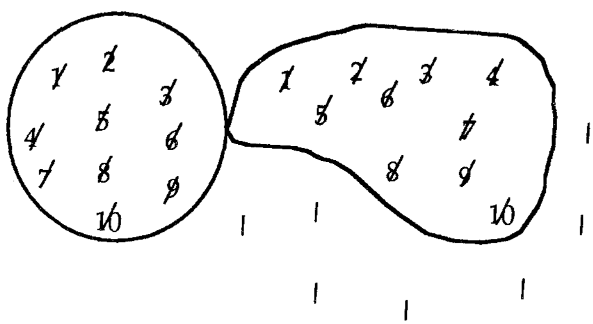


Fig. 7.

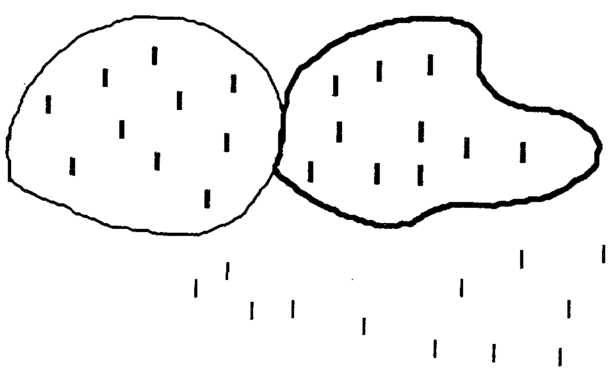


Fig. 8.

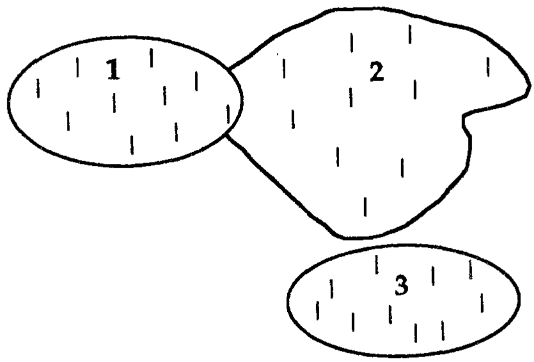


Fig. 9.

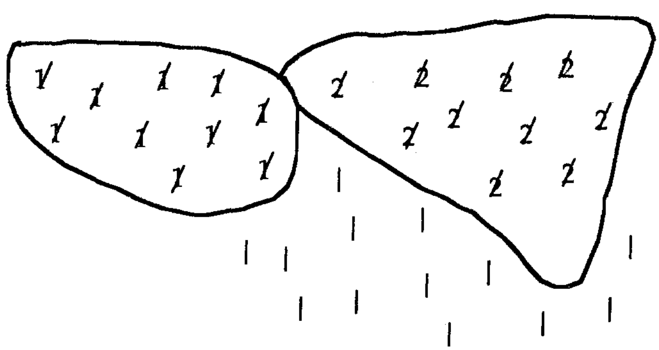


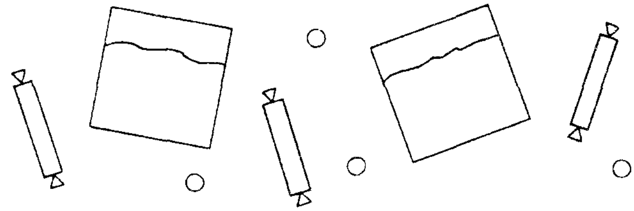
Fig. 10.

(b) *On the Ability to Operate With Groupings when It Is Necessary to Make, Un-Make Them*

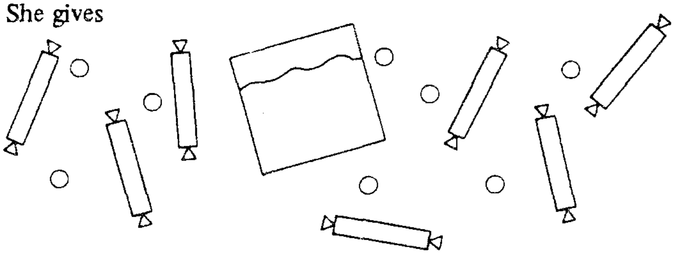
This item is taken from the experimental diagnostic situations used in previous research in third and fourth years, and whose results were presented in Bednarz and Janvier (1982). How do the children of the constructivist approach perform on this item?

Description of the item. "A 'mom' buys peppermints for a birthday party. She wraps them in rolls like this one (showing sample of a roll on the desk) and put the rolls in bags like this one (showing sample of a bag on the desk) in order to give some to the children. She has some peppermints left over".

"Mom has (*this time drawings are shown*)



She gives



Make a picture of what's left."

Fig. 11.

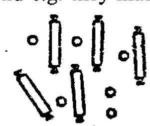
No mention is made of the number of peppermints in a roll or of the number of rolls in a bag. This number was not visible but was accessible by examining the sample or by asking a question (Fig. 11). This item was selected in order to see what meaning children give to borrowing in subtraction. Do they "see groupings, groupings of groupings? Can they operate with them?

The results. In this item, it must be pointed out that the answer must be in the form of a picture of the peppermints remaining and not a number in conventional symbolic form. A necessary (but not sufficient) condition to operate on the task posed is the need to know the number of peppermints in a roll and the content of a bag. This need was evidenced through relevant questions from the children or by their decision to invent missing numbers (see Table IV).

Commentary. At the end of the third year, almost all of the children of the constructivist approach (92%) could identify the groupings and the relations between them, however in a context where they were relatively accessible (relation between hidden groupings) and can operate on them (un-make and co-ordinate them). The only problems still not resolved for some children are problems of organization. In effect, the child must, in the treatment of this item and in the communication of the result, undertake his own organization.

TABLE IV

Ability to operate with groupings

	Constructivist approach	Current teaching school board B	Current teaching school board A	
	3rd	3rd	3rd	4th
	% of (23)	% of (26)	% of (75)	% of (45)
1. The pupils who felt such a need adopted two strategies:				
(a) "*COLLECTION strategy*: They associated a number to each grouping (rolls and bags) of peppermints, did computations and drew the picture associated to the number they found e.g. they make 2 3 4 and drew $\begin{array}{r} -178 \\ \hline 56 \end{array}$ 	0%	15%	17%	42%
(b) "*GROUPING*" *strategy*: they proceeded from the picture and mentally "opened" a bag and a roll (such a process is identified as "unmaking" a grouping).	92%	39%	13%	19%

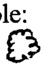
e.g. "I have two bags, I gave him one bag, I have one bag left . . . then I must give him 7 rolls, I give him 3 rolls, I opened my bag and I give him 4 more rolls, there will be 6 left . . . . Then I also have to give him 8 candies, I give him 4, I will take another roll and take 4 other candies from this roll . . . ."				
2. A few who used (a) or (b) were unsuccessful; they "see" groupings but can't operate with them, groupings were confused. For example, at a certain point in his reasoning, the child assigned the number 10 to a bag and to a roll. He counts 10, 20, 30, 40, 54, subtracts 10 from 54 to account for the bag and then attempts to subtract 70 to account for the rolls.	4%	15.5%	10%	7%
3. The others did not even feel the need to inquire about the content of the roll and the bag. Either they considered the problem impossible or they carried out a subtraction taking away the smaller number of articles from the bigger one.	4%	30.5%	60%	32%

The mode of organization has recourse to a memorization and co-ordination of all the facts. For some children however, who see the groupings, and who can un-make and co-ordinate two groupings of different orders, the mode of organization can sometimes lead to forgetting and errors. For example, "I have two bags, I gave him one bag, I have one bag left. I must give him 7 rolls, I opened my bag and I give him 7 rolls, there will be 3 left. Then I also have to give him 8 candies, I shall take another roll and take 8 candies from this roll. I have now 3 rolls, 2 candies". The child here forgets the 3 rolls and 4 candies of the initial collection, and he also forgets to take away the unmade roll. Thus, the treatment of the operation will become sure only from the moment where the child will have succeeded at becoming organized in an efficient manner.

(c) *On the Ability to Work Simultaneously with Two Groupings of Different Order*

We have found this important difficulty encountered by children in current teaching at the elementary school. Even if they see the groupings and can operate on them, they still have trouble in co-ordinating them, and we find in their reasoning a confusion of groupings of different order. In the item described above this difficulty was present (strategy 2). However, we saw only a single child in the constructivist group (4%) who had this difficulty. To illustrate further the evolution of the group (constructivist approach) in reference to this fundamental ability, we will take an example of an item used at the end of the second year. This item has been retained due to the complexity of the representation brought into play, making it still difficult for the child to co-ordinate the two groupings in the required task.

Description of the Item

Material: We have available:
Flat sheets of crepe paper. 
Some flowers (made of ten superimposed sheets); once the flower is constructed, we can no longer clearly distinguish the presence of the ten sheets of paper.
A bouquet (made of ten flowers). The ten flowers are visible in the bouquet (they are simply attached together by a cord to form the bouquet).

Presentation of the Situation

"Let us suppose we are at a florist and we make flowers to sell. Now watch carefully, you are going to help me make one. We'll make one together" (We construct one flower with the child in order to make clear how it is constructed.) "Now when we have ten flowers like this we can make a bouquet". (Here again we construct before the child and attach the flowers into a bouquet.) The child has before him a sample of a bouquet, several flowers and several separate sheets.

"I want to prepare the following display" (We write on his sheet of paper: 4 bouquets, 4 flowers and 5 sheets). "How could I make this? Could you indicate me with the following labels?" The child has before him the following labels in mixed order (Fig. 12).

45 flat sheets | 4 flat sheets | 2 flat sheets | 5 flat sheets

10 flat sheets | 12 flat sheets

1 flower | 2 flowers | 3 flowers | 4 flowers

41 flowers | 42 flowers | 14 flowers | 43 flowers | 3 bouquets | 5 bouquets

Fig. 12.

Interventions (when the child has finished to indicate how he will make the desired display) What display can we make with this? Why? Fig. 13).

a) 41 flowers | 12 flat sheets | 3 flowers

b) 10 flat sheets | 4 flowers | 2 flat sheets

c) 42 flowers

d) 3 flowers | 12 flat sheets

Fig. 13.

THE RESULTS (SAME SCHOOL BOARD (B)–AT THE END OF THE 2ND YEAR) (TABLE V)

TABLE V

Ability to work simultaneously with two groupings of different order.

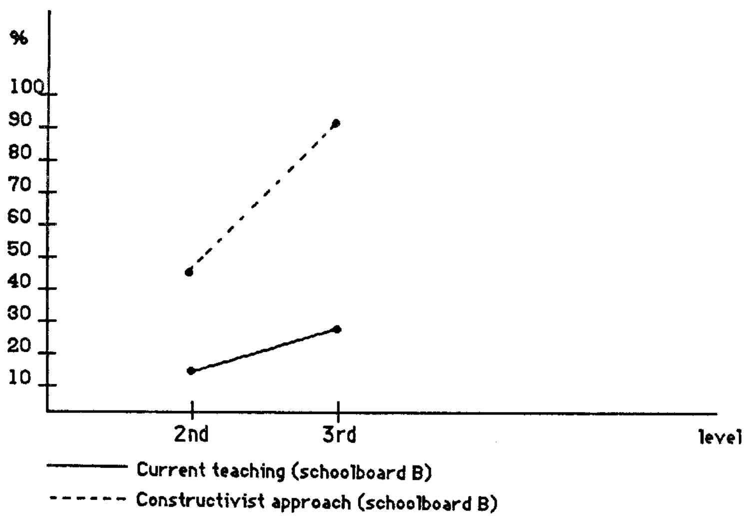
	Constructivist approach	Current teaching
	% of group (27)	% of group (23)
Some children *do not see grouping* in the representation flower, bouquet ... To form the display 4 bouquets, 4 flowers, 5 sheets, they search 4, 4, 5 no matter what is written on the labels; they take for example 4 flowers 4 sheets 5 bouquets.	11%	22%
Other children *retain only the grouping rule*: "We group by 10". For them the bouquet can just as well be 10 flowers as 10 sheets. For example, They succeed in forming the display: 4 bouquets 2 flowers from 42 flowers, but for 3 flowers 12 sheets they associate 1 bouquet 3 flowers 2 sheets.	11%	39%
Other children *operate on* only *one grouping*. They succeed on the item when a single grouping is brought into play. They can either pass to the *1st grouping* (from the sheet to the flower).	15%	22%
or They can pass only from the 1st to the *2nd grouping* (from flower to bouquet).	7%	4%
Others can operate with two different groupings, but have difficulty in co-ordinating them. We find in their reasoning a *confusion of the two groupings*.	19%	0%
Others *completely operate* simultaneously *with* two different *groupings*.	37%	13%

Commentary. In this task the grouping rule attached to the first grouping is given by a convention (it is difficult in effect, when the flower is constructed, to see in this a grouping). Furthermore, the flower does not suggest to the mind of the child a grouping but an entity, a unit; this is also the case, but to a less degree, for the bouquet. One of the difficulties, in the representation proposed, is to struggle against the conception of the bouquet that suggests a grouping of ten flowers and not a collection of sheets. When the two groupings are present it is then very difficult to see them to be able to co-ordinate them.

For these reasons, it is particularly interesting to observe, in this complex situation, that the children who have had the constructivist approach have less tendency to "see 10" in the bouquet (39% of the children in current teaching have this attitude). The flower seems to be more taken into consideration as a grouping, and their performance in being able to operate simultaneously with two groupings of different order appears good (37%).

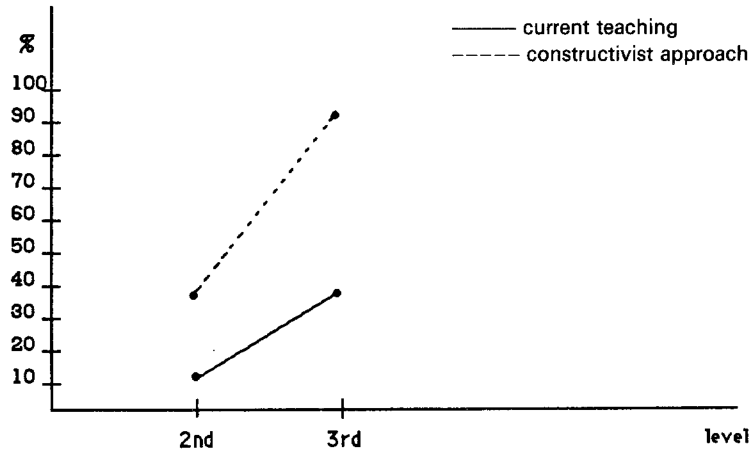
II. *Evolution of the Constructivist Group During the Three Years Relative to the Understanding of Numeration*

The graphs (Figs. 14–17) indicate some of the results of the constructivist approach group. The items referred to by these graphs from first to third



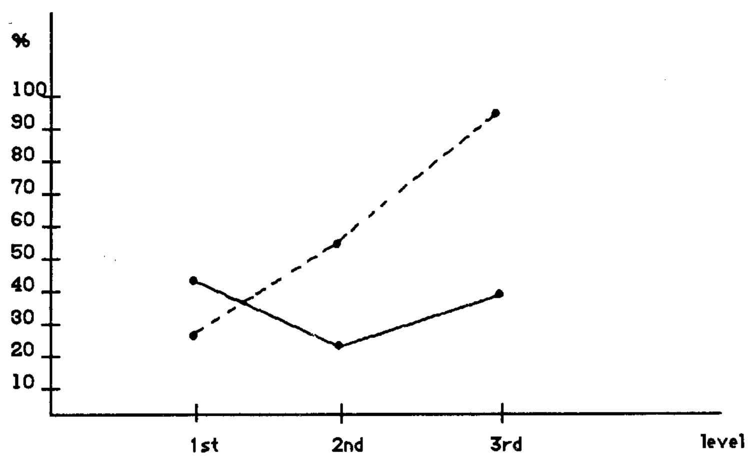
Percent of children who see the *pertinence of regrouping* and the *significance of the associated writing* (ref. item example 1-a).
Note: In the first year, no item with this aim was used.

Fig. 14.



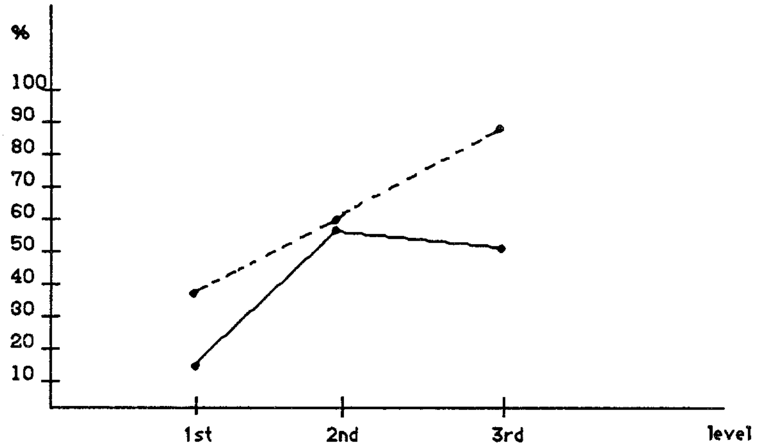
Percent of children who can completely operate with *two groupings of different order* (ref. item example 1-c).
Note: In items used in the first year, two groupings of different order were present but the child had to operate on one grouping only.

Fig. 15.



Percent of children who *operate on groupings* (ref. item example 1-b).
Note: In the 1st year, children had to undo only one grouping. In the 2nd year, two groupings of different order were present but children had to undo only one grouping. In the 3rd year children had to undo two groupings of different order.

Fig. 16.



Significance accorded to conventional writing (items more tied to the curriculum).

Fig. 17.

year are more and more complex in terms of the required pathways and skills, and bring into play unfamiliar representations where the relation between groupings is less and less accessible (reference framework, Fig. 3).

Commentary. By examining the different graphs, we may observe an evolution by the children of the constructivist group relative to the significance they accord to groupings, the ability to operate with these in more and more complex tasks, and to the pertinence they accord to regrouping in situations where this strategy proves to be efficient. At the end of the third year, almost all of the children of this group can see groupings and can operate with them when the representation submitted to them is relatively complex. The difficulty that was pointed out in the introduction, when children had to work simultaneously with two groupings of different order, had practically disappeared; almost all the children can operate with two groupings without confounding them. Furthermore, in items more related to the curriculum, the performance of children appears good (significance given to writing, to procedures of calculating . . .)

This performance of the group is even more interesting to consider for the learning developed with children since the results of the comparison group (current teaching, schoolboard B) confirm those obtained in the 1980 study (schoolboard A): the same important difficulties are shown in the two populations and persist at the end of the third grade.

TABLE VI

Ability to operate with groupings on the abacus

	Constructivist approach	Current teaching school board B	Current teaching school board A	
	3rd	3rd	3rd	4rd
	% of (23)	% of (26)	% of (75)	% of (45)
Some children accord *no significance to the position in terms of groupings*. They see only beads all having the same value. Hence, they start to subtract 128 by removing one bead at the hundred position, two beads at the ten position. Then, children seeing that they cannot remove 8, say "It can't be done, there aren't enough". others remove 2 beads and stop there, signifying that they have accomplished the task requested. others arrange themselves to remove 8 beads at whatever cost. They therefore remove 8 beads no matter from what post.	25%	70%	55%	54%

Some children *try to re-transcribe the rules* they used for subtraction in the conventional *algorithm*.	16%	0%	21%	19%
$\begin{array}{r} 3\,1\,\overset{4}{\cancel{5}}\,\overset{1}{2} \\ -\quad 1\,2\,8 \\ \hline 3\,0\,2\,4 \end{array}$				
We therefore find different erroneous interpretations of borrowing.				
Some children calculate mentally and use the abacus only to reproduce the result.	0%	4%	12%	0%
Finally others carry out the task asked and demonstrate that they *give a significance in terms of groupings to the beads on the abacus*. We find here two behaviours.				
some children do the exchange on the abacus.	50%	0%	0%	5%
others remove one bead from the ten post, calculate mentally 12 – 8 and add two beads to the unit's post to make 4.	9%	26%	12%	22%

III. *Transfer of Acquired Skill*

During all the intervention, children were brought to work in learning situations referring to material where the relation between the groupings was relatively accessible. They developed, in these situations, procedures, representations, ways of organizing . . .

We shall now examine the *transfer* by the child of the acquired skills to other situations involving on one hand unknown and *less accessible representations* (relation between groupings given by a convention), *and* on the other hand *conventional writing*.

(1) *Transfer of the Acquired Skills to Unknown Less Accessible Representations*

Description of the item. We present to the child an abacus  having beads of the same colour on the posts. (The children had never worked on an abacus.)

1st question: "Can you represent that for me on the abacus (we write 3152 on a sheet of paper).

2nd question: "Can you take off one hundred and twenty eight using the abacus?"

"Explain to me."

"What does that give?" (see Table VI).

In this example involving a subtraction, we see that 59% of the children in the constructivist group can transfer the acquired skills to a non-familiar situation where the representation is less accessible, and can make correspond borrowing to an effective action. Similar results were obtained for items involving a division.

(b) *Transfer*: *Global Results*

The following tables (VII and VIII) present the percentage of children who, at the end of the third year, *can operate on groupings* in context of subtraction and division, *giving a significance in terms of groupings* to the rules that they used.

62% (59%) of children in the constructivist group give a significance to borrowing and can transfer the acquired skills to an unknown and less accessible representation.

TABLE VII

Transfer: can operate, giving a significance in terms of groupings (end of 3rd year)

	Representation relatively accessible Subtraction (ex.: item 1-b)	Unknown representation less accessible Subtraction (ex.: abacus)	Division	Conventional writing (ex.: 3276 ÷ 2 =)
Constructivist group	92%	59%	59%	66%
Current teaching	At most 54%	26%	4%	At most 46%

TABLE VIII

Transfer: to unknown representation (less accessible) and conventional writing

	Relation between groupings given by a convention		
	Unknown representation		Conventional writing 3276 ÷ 2 =
	Colour beads (3276 ÷ 2)	Abacus (3276 ÷ 2)	
Constructivist group	62%	59%	66%
Current teaching	19%	4%	At most 46%

We shall now examine how the children of the group perform relative to the treatment of conventional writing (calculation procedures) (see Table IX).

(2) *Transfer to Conventional Writing* (*Item Related to Curriculum*)

What is the performance of children when given a calculation to do? What understanding do they have of the rules they use in these procedures: do they accord to them a significance?

Commentary. In the case of subtraction (division) only 61% (46%) of the children at the end of the 3rd year (current teaching) can do the calculation

TABLE IX

Performance in calculating, and significance at end of 3rd year

	Constructivist approach	Current teaching
Subtraction 234 – 178 =		
– can solve	96%	61%
Among these • operate in terms of groupings according them a significance.	(96%)	(23%)
• use other procedures (collection strategy), according them a significance.	(0%)	(19%)
Division 156 ÷ 2 =		
– can solve	96%	77%
Among these • use a "grouping" strategy.	(88%)	(4%)
• use a "collection" strategy.	(8%)	(73%)
3276 ÷ 2 =		
– can solve	66%	46%
Among these • use a "grouping" strategy.	(66%)	(0%)
• use a "collection" strategy.	(0%)	(46%)

when presented "horizontally". For the other children we find the usual errors encountered in subtraction (or division). Among the children that solve the problem, some children distinguish the two borrowings in terms of words only (19% in subtraction) and others (19%) use a collection procedure (for example, 234 – 100 = 134, 134 – 70 = 64, 64 – 8 = 56).

In the constructivist group, 96% of the children can operate with conventional writing for subtraction (66% for division) in attributing a meaning to what they do; for them borrowing corresponds to effective action carried out on groupings. It is important here to remember that never, in the constructivist group, did children learn the conventional algorithms. (This *was* the case in current teaching group.)

We find in the constructivist group different procedures. For example, the following procedure developed by these children will be used.

$$\begin{array}{r} \scriptstyle 10\,10 \\ 3276 \\ \div \quad \underline{\quad 2} \\ 1638 \end{array}$$

"3 thousand shared in two, 1 thousand and there remains 1 thousand that we undo into 10 hundreds; 12 hundred shared in two, 6 hundreds; 7 tens shared in two, 3 tens it remains 1 ten that we undo into 10 units; 16 shared in two, 8 each one .... Each one will have one thousand six hundreds and thirty eight."

CONCLUSION

By a long term application in a classroom we have been able to organize a constructivist approach based on children's mathematical thinking.

(1) This strategy has helped the children to progressively construct a significant and operational system for the representation of number. We may, in effect, observe as the results of our intervention indicate, an important evolution of the children from 1st to 3rd grade relative to the significance they attribute to writing and to the skill of operating with groupings in more and more complex tasks. Furthermore, we may observe a transfer of these abilities to items more closely related to the curriculum: treatment of operations, of conventional writing and of the symbolism hundred, tens and units ... observations that are very surprising when we know the importance that is given to this symbolism in current teaching, in contrast with our approach.

However only a little more than half of the children in the constructivist group could transfer the acquired skills in situations where the representation is less accessible. This analysis shows that this transfer was not complete, the difficulties and the obstacles developed by children in the learning of numeration were partly taken into account by our intervention. It would have been desirable to continue for a longer period in order to intervene in situations where the relations between the groupings are given by a convention. Recently a new evaluation was undertaken, three years after this study was conducted. This new study reveals that the abilities and procedures developed by the children are still present. Our results however show the conflicts provoked by the confrontation of the two approaches (constructivist and current teaching), which illustrates how a constructivist approach in mathematics necessitates a long term intervention in a classroom.

(2) Our longitudinal study has allowed us to understand better the conceptions, procedures and representations developed by children and to see how these evolve. From descriptive representations, attempting to reproduce all the objects of a collection, and as faithfully as possible, the actions carried out on these collections, they evolve very slowly toward abstract and efficient representations of the collection. This evolution is closely related to the communication of information on collections and to the treatment of these collections.

(3) In this continuous interaction with the children we were brought to refine the ideas developed in our previous research. In this way our theoretical framework and our reference framework were put to the test and we had to make more precise the special role played by the situations, the material and the contextual environments.

The situations reassign to the operational character of numeration the place that it has played in history, and allow provoking and bringing out the process of construction of a system of representation of numbers. The material makes this task accessible to the child as much by the actions the child undertakes on these (make, unmake groupings, exchange . . .) as by the representations he makes of the elements, the collections and the actions on them. Each contextual environment is a take-off point in the construction of the concept of numeration. It permits the child progressively to detach himself from the elements of each one of the environments to evolve towards a system ever more abstract and efficient.

(4) This process seems to us more than a pedagogical variation; it totally brings into question the usual conception we have of numeration and its learning. In the actual teaching of numeration we confound the teaching of the significant and the signified by working essentially on symbolic writing and on the rules of forming these (syntax). Our process in contrast constitutes an effort to regive a meaning to the significant via situations where the operational character of numeration is important. In situations of communication and of treatment of information on collections, a process is provoked. It implies procedures, representations and an evolution within these. The child gradually constructs a significative and efficient representation for numbers.

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