LEEN STREEFLAND

SEARCH FOR THE ROOTS OF RATIO: SOME THOUGHTS ON THE LONG TERM LEARNING PROCESS (TOWARDS . . . A THEORY)

Part I: *Reflections on a Teaching Experiment*

ABSTRACT. The main subject is a ratio theme for third graders. The instructional text belonging to it was first submitted to an international panel of experts. The reactions of the panel are reported and tested against classroom experiences.

1. INTRODUCTION AND OVERVIEW

In the present paper a few ideas are exposed which might serve as building blocks for a teaching-learning theory on ratio. The subjects dealt with are:

Part I:
- a concise description of an instructional theme (Section 2);
- the response to questions that were submitted to an (inter)national panel with respect to this theme and their own vision of ratio (Section 3);
- comparing five particular questions in the responses with real experience of instructional practice (Section 4).

Part II:
- building blocks for a theory (Section 5);
- conclusion (Section 6).

Views of children play an important part in the intended theory building although this did not (and could not) happen too often in the way that they expressed their views. Theory building has also been influenced by ideas of colleagues found in their publications.

Finally we processed the views of the members of an (inter)national panel as far as they were determined by similar views on mathematics and mathematics instruction. We did so in order to acquire external arguments that might serve to strengthen the objectivity of a theory depending on a vision.

With respect to theory displayed in the literature we could not neglect sources which are only available in Dutch and thus less accessible to many readers, who are asked to accept our apology.

2. WITH THE GIANT'S REGARDS

In the theme 'With the giant's regards' ratio as a phenomenon is explored by third-graders (8–9 years olds) by investigating among other things the effect

Educational Studies in Mathematics 15 (1984) 327–348. 0013–1954/84/0154–0327$02.20.

of linear enlargement on length and area. The activities particularly aimed at are:

- estimating and determining more precisely the giant's length, compared with that of a human being, given a print of his hand;
- comparing a number of objects from the giant's world (sole of his shoe, newspaper, handkerchief) with those of the human world, estimating and determining more precisely the dimensions of the giant's objects;
- exploring and composing cake recipes;
- comparing the steps (and connected to it the speed) of giant and man;
- comparing numbers of oranges and prices as related to each other;
- developing visualisations such as the number line (also multi-scaled) and the sector diagram as models for ratio invariance;
- developing the ratio table as a schematisation of ratio

man's steps	5	10	15	....	....			
giant's steps	1	2	3	....	....			

The story

Giants play an important part in stories that each generation tells the next. Ellert and Brammert were two well-known vicious giants, whose frightening action were remembered for centuries.

Egbert, however, is a good-natured giant, a noble character fallen into oblivion. This injustice should be redressed, and in order to know more about him some data are required.

Fortunately the teacher acquired a trace of his hand. It appears to be four times as long as hers. Measuring and reasoning was required to state this fact.

The sole of the shoe

If all parts of Egbert are four times those of the teacher, his footing must be firm. What size should be the sole of his shoe? Also four times? Or will all the children reason in a way to abandon linearity?

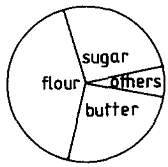
Four times in the length, this is obvious, but would not this make the giant's sole much too narrow?

First they estimate the number of human soles in a Giant sole, then a construction is made (drawing). Will the insight be established that the increase works in two dimensions, that is four times four human soles in a giant sole?

Buttercake with ginger

The sector diagram

Giant Egbert gets familiar with the baker, who bakes a buttercake for him. What about the ingredients? Flour, butter, sugar and a few other things. The shape of the cake is modelled by a drawing where the contribution of each ingredient remains visible: a circle diagram.



For more dough – let us say four times – how much is needed of each ingredient? The children seem to have enough experience to understand that from the point of view of baking, the big cake should be as thick as the little one.

What about the diagram? 'The same, Miss, though larger'. Obviously the children identify the cake and the diagram. Will this eventually lead to model forming?

Density

Afterwards the baker puts pieces of ginger on the cake. Which one had more pieces, the ordinary or the four times larger one, or did they contain the same number? First estimating, then counting (by groups) and finally summarising the arithmetic activity in rules like $12 \times 5 = 60$ and $24 \times 5 = 120$ for the numbers of piece of ginger. Does this agree with the former expectations?

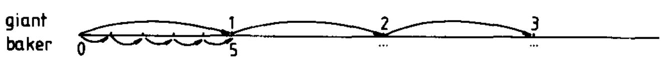
In fact the big cake has the bigger number of ginger pieces, but does it also taste more of ginger? If you took a bite of the little cake and the same of the big one, what do you think?

The fixed ginger pieces are considered in relation to the area. If the little cake were like the big one it would contain a mere 30 pieces. What looked more might be the less. It depends on how one looks at it. And that is what matters in ratio.

Setting out together

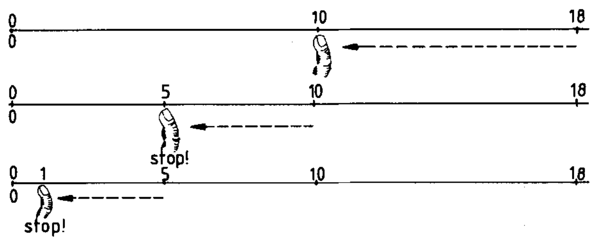
The giant has a son. Sometimes they take a walk with the baker. This is not easy. For three steps of the giant the baker needs fifteen.

How to compare their steps? It is done along a little path:

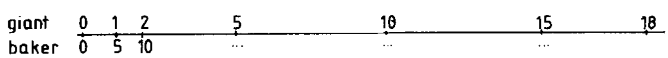


It is not easy to walk together if the lengths of the steps differ so much. Take the distance from the giant's dwelling place to the baker's house. The giant can do it in 18 steps. What about the baker?

The teacher moves her finger along the number line from 18 towards 0 and asks, where the giant has done 10 steps, 5 steps, 1 step.



Once this has been done the walk can be built up:



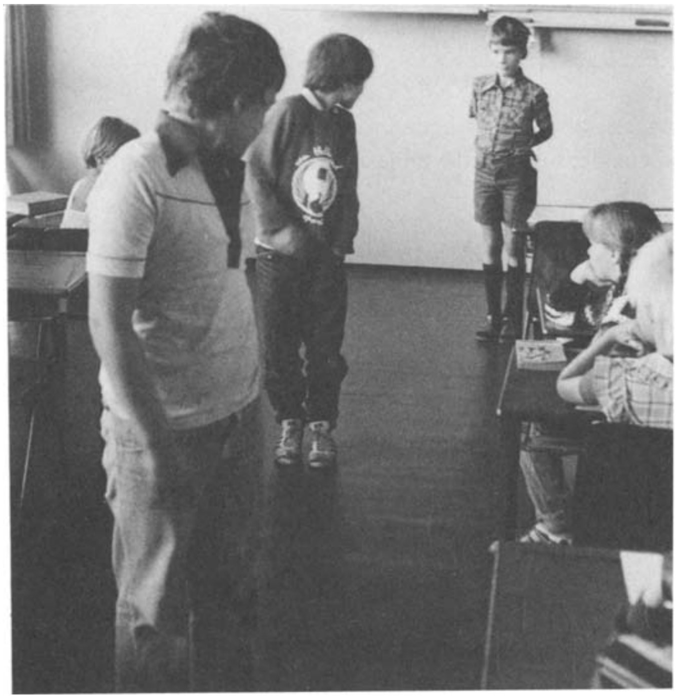
or in a table:

giant's steps	1								18	
man's steps	5								....	

The open approach and the gradual built-up with number line and table can provoke supplementary efforts: interpolations, extensions of number line and table beyond the numerical limits of the problem.

Some children might not need the proposed support by number line and table. They can do it as an arithmetic problem. 'The baker needs 90 steps to cross. I did $20 \times 5 = 100$, then 2×5 off, makes it $100 - 2 \times 5 = 90$. It is obvious.'

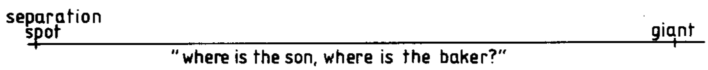
One checks what happens if the baker, the giant's son and the giant proceed by stepping simultaneously. After a few steps their distance is quite large. Who is the fastest?



If one pays attention to the speed of the steps, then . . .. But if the covered distance is considered, things are different. Giant, son and baker decide to walk a distance separately and then to wait for each other. Indeed, the baker cannot keep up with the giant.

When the giant is at the meeting point, where is his son, where is the baker? The giant is twice as fast as his son. The giant is . . . as fast as the baker.

Where is the son, where is the baker?



The progress of the giant, the son and the baker is numerically described, for which the number line and the table are helpful. In the same time, where:

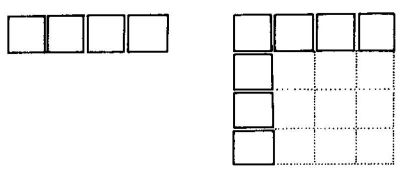
the giant walks:	100m	... m	...	...				
the son walks:	....m	... m	...	...				

Handkerchief
When the baker reaches the meeting point, he is warm. Egbert passes him his handkerchief so he can wipe off his forehead. The baker bursts into laughter: 'Oh dear, what a handkerchief.'

> How large is the handkerchief?

Earlier the children solved the soles problem. They also discussed how large might be a giant newspaper. With these experiences they will get quite a way. Their own handkerchief is a model. Estimates and similarity (of squares!) are steering the thought in two directions.

The linear factor 4 has become as it were axiomatic truth. While laying down or drawing handkerchiefs they will reason: this (in the width) there should also be four, since a handkerchief is a square:



Estimates can be dispensed with. They make four rows of four handkerchiefs: 16 baker's handkerchiefs in one of the giant. It is the same as with the sole but the newspaper and the handkerchief fit better.

Hungry
Walkers like refreshments. Twice they buy a bag of oranges. The first time 6 oranges for 3 florins:

oranges				6			
florins				3			

Next time 8 oranges for 6 florins. Which bag was cheaper?

oranges				8				
florins				6				

Of course one cannot rely on numbers. The children know it. Quality and size count too, but if they were the same, then . . ..

The table invites the child to fill in the prices for some other quantities by obeying the given number–price relations so that ratios remain invariant:

oranges		2	4	6	8	12		
florins		1	2	3	4	6		

oranges		2	4		8			
florins		1½	3		6			

The first bag appears to be the least expensive. Some children saw it and in this way convinced the others.

End

Giant Egbert, his son and the baker are left to their fate. It was a first acquaintance with Egbert (which might be followed by more). Anyway the baker received 'our regards to the giant'.

3. (INTER)-NATIONAL RESPONSES WHAT DO OTHERS THINK ABOUT RATIO?

3.1. *Introduction*

The IOWO-theme 'With the giant's regards' (De Jong, 1978, pp. 23–43) was originally centered around area as a basis for ratio. In the new version it has been tried to approach ratio in a more variegated way. More justice had to be done to the phenomenon 'ratio' such as it appears in the world of and for children. It was my intention to submit the theme to an (international) panel of expert colleagues in order to learn what they think about ratio and to confront their views with my own (5). The group that had been asked to cooperate numbered 68 (40 from abroad, 28 from my country). 32 reacted (from each group 16). Three reactions from abroad were, with respect to the given information, irrelavent for the present research, so that 29 responses could be used.

The 29 respondents were:

- 3 mathematicians
- 1 didactician for chemistry
- 3 teacher trainers

- 14 researchers (including 6 psychologists)
- 5 didacticians for mathematics (including two curriculum developers)
- 1 mathematics teacher (secondary education)
- 2 didacticians for physics

A text containing a description of the 'giant' theme, close to the actual instruction and illustrated by pictures and drawings had been submitted to the panel. A lesson plan had been added, from which the various objectives could be globally derived.

For instance:

- ratio and measuring length (lesson 1);
- measuring and increasing in two directions (lesson 5), specified by descriptions of lessons and suggestions.

The panel was informed that the lessons were meant for third-graders (8–9 years old). The working sheets for the pupils were part of the material submitted to the panel.

The panel were asked the following questions:

1. Do you judge that in the theme 'With the giant's regards' full justice has been done to the phenomenon 'ratio'?
2. Does ratio in all its aspects occur in the proposed activities?
3. What is lacking? What extensions do you propose?
4. What is in your eyes irrelevant with respect to ratio and should be omitted?
5. Other remarks.

3.2. *Responses Received*[1]

3.2.1. ***Preliminary remarks.*** Let me first make two remarks with respect to the data-processing:

1. I did not include the responses in an appendix, because of their extension and their diffusion caused by the wide range of aspects touched on by the respondents. This diffusion also influenced the description of the results.
2. A respondent has only been mentioned by name in cases where he or she expressed a 'strong' opinion. In general my main concern was the phenomenon of ratio, so there was no need for individual treatment of responses.

The queries caused as it were an outburst of reactions, which both with respect to extension and profundity surpassed the expectation. Among other things the respondents contributed precious suggestions for extensions, enrichment and didactical sophistication. With a certain regret I am obliged to disregard the suggestions at the moment. In fact I was concerned with the vision of the respondents on 'ratio and proportion'. This intention has determined the categorisation of the responses. Problems related to the macro-structure and micro-elaboration of a learning path for ratio will be considered in their mutual connection. Successively I will deal with the question of:

- (in)completeness of the phenomenon 'ratio' in the submitted theme (3.2.2);
- the significance of the context (the story of the giant) as a general feature and of the physical context (similarity and human shapes; taste and linearity) as a more specific feature (3.2.3);
- the application of visual models and schemes (3.2.4).

Views on learning theory for complex processes of concept attainment such as that of ratio were an undertone of most of the reactions. This will be dealt with in subsection 3.2.5.

Before passing to the description of the responses according to these categories, it should be mentioned that the vision on ratio and proportion, as put forward, was not unanimous.

One group judged the theme on the basis of what might be characterised as a logical and rational analysis of 'ratio' (cf. Resnick and Ford, 1981; Davydov, 1975). For instance, it was posited that pupils should first learn to distinguish constant sums, differences, products, and ratio before the concept 'ratio' is to be explored in a more general sense. Obviously (according to this view) the mathematical structure was the starting-point, moreover this point of view led to the objection that the theme was *not at all related to ratio* but rather restricted to multiple counting.

Another group favoured the present approach – *construction of ratios by a mathematical activity.* In a general way it is the view that mathematics is an activity – a human activity – thus a dynamic process. Mathematics as a human activity elicits concepts, operations, structures by problems from the real world (of and for children). This dynamical process of problem solving generates mathematics. It is this activity that justifies the mathematical product (Treffers, 1982). Further we agree with Hilton,[2] asserting that only by the dynamics of mathematics as a process – embedded in a mathematical activity – the essentials of the learning process can, as studied in (psychological) research, be discovered and understood. In other words: good research may not show mathematical didactical deficiencies.

3.2.2. *Aspects and manifestations of ratio.* The respondents had been informed that the theme had arisen from a mathematical-didactical analysis of the phenomenon 'ratio' and that the text was an attempt to do justice to its result as much as possible. Seven respondents explicitly dealt with the first question. With one exception (mentioned earlier) all of them judged that *justice had been done to the phenomenon 'ratio'* albeit within a restricted context. What matters in ratio is the 'relativating' view. What this relativity means, appears from suggestions such as the following:

Compare the giant with a number of pupils. Does he have a variety of lengths? How are we seen by the giant? Or: if John were the giant, how tall would the other children in the class be?

Ten respondents were intrigued by the problem of the area of the giant's shoe sole. Seven appreciated it in a positive sense. Some of them even suggested that we should deal with the problem of increasing in three dimensions and its consequences: the food of a man – the food of a giant. Four times, 64 times? Others voiced objections such as the following. 'This is promoting sham precision', or: 'How can you prove mathematically that the giant's sole measures just 4 × 4 a man's sole – or can you produce a proper cover?' Or: 'Better start with rectangles.' Five respondents warned us to be careful with the similarity model of the man–giant relation. Don't the (internal) ratios change in the development from baby to adult?

Moreover remarks were made about the importance of certain aspects of ratio, and certain manifestations of ratio were stressed. For instance:

- comparing magnitudes not only of the same kind, but also of different kinds, such as length and number (four respondents);
- considering mixtures of continuous quantities;
- recipes as a separate theme (children of this age like cooking);
- distinguishing internal and external ratios (that is ratio within and between magnitudes, respectively; for instance, in a uniform motion, paths (s_1, s_2) are to each other as times (t_1, t_2)) $s_1 : s_2 = t_1 : t_2$ – internal; the ratio path-time is constant $s_1 : t_1 = s_2 : t_2$ – external; (cf. Freudenthal, 1983);[3]
- stressing the ratio in the operator (two respondents).

s (m)	0	5	10	...	...	5x	×5	...	...	y	×1/5
t (sec)	0	1	2	...	...	x		...	...	$\frac{1}{5}$y	

Finally we mention that four respondents judged the theme somewhat artificial as a consequence of the abundance of ratio problems within a quite restricted series of lessons. Four respondents warned against the ambiguity of such terms as 'big', 'bigger' and so on.

3.2.3. *Context.* For twelve respondents the non-mathematical context around the mathematical activity was an important issue, judging by their reactions. Ten of them appreciated the theme, while three of them had certain reservations. The two remaining respondents asked of using the rationale of this fairy tale story while there are real contexts that can achieve the same.

Five respondents (one negative and four positive, though with reservations) made remarks about the part played by contexts in learning mathematics. They boiled down to the following: Ratio is important in many contexts. In order to arrive at a sound abstraction, one should apply more contexts than the chosen one. What matters is the functionality of what has been learned, the ability to generalise and to apply it beyond the context in which it has been learned and from which the children should detach themselves.

At this opportunity one respondent warned against exaggerated use of context in mathematics instruction. Although the theme as a local phenomenon was positively appreciated in the present case, the danger of a wrong idea of mathematics in the children's mind was imminent. They might be led to believe that the mathematical reality is one of imagination and fairy tales.

Two respondents referred to Jonathan Swift's 'Gulliver's travels' (cf. Freudenthal (e.a.) (eds.), 1976, subsection 2.6) as a context where area and volume play an important part – indeed; Gulliver visited worlds of giants and dwarfs.

Summarizing it may be concluded that embedding mathematical activities into contexts is judged to be essential for learning mathematics with a view to a manysided approach and the applicability of the learned subject matter (cf. subsection 3.2.5).

Finally some remarks should be mentioned with regard to the physical aspect of the context.

Four respondents considered the part on distance, time and speed as being too difficult for third graders.

There were five warnings against the arbitrary hypothesis of mathematical similarity of men and giants. Another objection concerned wrong ideas about linear models that could be created. For instance, physiology has no use for linear models of taste and smell.

3.2.4. *Visual models and schemes.* Among the various remarks on this subject 'sector diagrams' had the highest score (7 respondents). With one exception, the remarks on sector diagrams were critical. The single positively reacting respondent appreciated the sector diagrams as an important tool for visualising the invariance principle. The critical remarks were the following:

- the misleading 'little piece' in the sector diagram;
- the premature occurrence in the learning process;
- the forced introduction in the learning process, from the point of view of abstraction and model building (3 respondents), in particular the danger of identifying the cake with the sector diagram as a model;
- the choice of the model which could have been better as a rectangle.

The ratio table, whether connected with a double-scaled number line or not, elicited six reactions.

The table was considered as an important means to elicit abstraction from the giant context, to promote the numerical approach to reality, and to introduce ratio in an operator that transforms one magnitude into another. As regards other models, some respondents pleaded for adding graphs to the tables (inclination as a measure for ratio) and for replacing the double-scaled number line by line segments or for adding strips to the number line as representing intervals.

In general the use of means of visualisation and tables was approved but only in those cases mentioned earlier was the function of such mathematical tools in the long run learning process considered.

3.2.5. *The learning process for ratio. Concept attainment.* Ratio is a complex concept, which demands a long-lasting learning process.

Ten respondents broached this question.

Five of them asked questions concerning the place and the intentions of the theme within a broader teaching context. The other five made remarks related to views about the macro-structure of the ratio curriculum.

The suggestions, remarks and proposals were as follows:

- a sketch of a longitudinal curriculum line that extends into the algebra of proportions;
- ratio as an important concept, and a basis for fractions, percentages and decimal numbers;
- aspects of ratio should be regularly resumed (spiral curriculum);
- the essential aspect of ratio *at this stage* is geometrical;
- the theme should be considered as a beacon in a longitudinal line.

In a long term learning process one can distinguish stages that are characterised by the degree of consciousness with respect to notions and intuitions, underlying the mathematical activities or elicited or sharpened by them. This was, amongst other things, for instance expressed by '. . . left at this stage as beginnings of the idea (of ratio) only'.

Other indications in this direction were the following:

- the positive appreciation of concept building from semi-qualitative and global to increasing schematic while raising the complexity of the problems;
- the occurrence of às many aspects of the phenomenon of ratio as possible and (as a consequence) fundamental notions, which are brought to the child's attention, made conscious and consequently developed;
- precedence of the explicitation of the common elements in the ratio problems for the formation of the ratio concept;
- the contribution of the theme to 'an intuitive knowledge of the concept of ratio';
- the necessity for the pupils to reflect on what they have learned;
- the *growth* of concepts if viewed from another system.

Obviously concept attainment is seen here as a process of increasing consciousness and mental growth, in which intuitive ideas are developed to become fully-grown (operational) concepts.

Consciousness is promoted by reflection both on the learning process and the learned subject. Another possibility mentioned in this connection by three respondents, is creating conflicts in the learning process, for instance by the experience and the consciouness of proportion and disproportion.

Conflicts were thus proposed both as instigators of consciousness and as means of confining the concept.

Six respondents pleaded for a many-sided approach to ratio and proportion in terms of the applicability of the subject matter. Children should be confronted with as many aspects of ratio as possible, initially coupled to as many phenomena as possible (length, time, weight). If full justice is to be done to 'ratio', one should explore a rich variety of examples in a rich variety of contexts. Indeed, ratio is important in many different contexts and must be approached from many angles.

The many-sided long-term approach involves considering the connection with other subject areas and mathematics should not be isolated from these other areas. Within mathematics the connection with fractions was already mentioned and the relation with similarity was also stressed. Properties like invariance of angles should also be included in dealing with similarities. The experience of invariance is a significant part of the mathematical learning process. For this reason the introduction of sector diagrams at the given age, and taste as a label for constant ratio (invariant density of pieces of ginger in the cake) was positively appreciated.

If the totality of responses is considered, a fair majority concluded that ratio requires a long term learning process, that concept building moves from intuitive ideas to operational applications, and that conflict and reflection contribute to the development of consciousness.

The approach is to be many-sided in various respects, that is a variety of context is needed as well as phenomena where ratio plays a part (internal, external ratio).

4. RESPONSE CONFRONTED WITH CLASSROOM EXPERIENCE

4.1. *Introduction*

Five questions were raised in the analysed reactions which were caused by among other things, doubts with respect to the feasibility of the theme and with respect to the mathematical-didactical choice.

The questions will be answered by testing the responses against experiences acquired during the try-out in a systematically controlled instruction experiment. Again: what matters is the legitimation of the approach to ratio rather than answering the question who is right. The testing we have in view aims at the elaboration of the objectives for ratio to be pursued in mathematics education. Both the views of the members of the consulted panel and the reactions of children in classroom experiences can serve as such; especially when confronted with one another.

The queries are related to:

- the area of the giant's sole;
- the difficulties third-graders could have with the physical context of distance, time and speed;
- recipes and the possible interest of children of that age in cooking and baking;
- forced model building (cake and sector diagram);
- the arbitrary hypothesis with respect to similarity of giants and men.

4.2. *The Area of the Giant's Sole*

The strongest argument in this question was advanced by Josette Adda (translated). 'In the situation such as presented, it is clear that each dimension must be multiplied by 4, but how to *prove* this result in a multplication by 16 since paving by this shape is impossible. On the contrary the situation of the story of the newspapers is well adapted to the notion. (How big would the giant's newspaper be?)'.

This objection is answered by the protocol of the following lesson.

Teacher: Try to imagine the giant's sole. Estimate how many soles of my shoe would fit into the giant's.

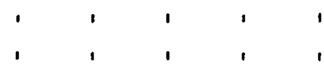
The children estimate: 6, 4, 4, no idea, 8, 8 . . ..

T: Why do you guess 8, Rik?

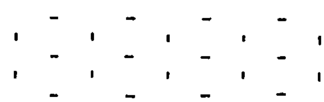
Rik: Well, two might fit alongside each other, that is two series of four that fit, and that is 8.

Frank reasoned the same way. The class agrees. Frank and Rik are allowed to draw a giant's foot on the blackboard. The two boys stake out four times the length of the teacher's shoe:

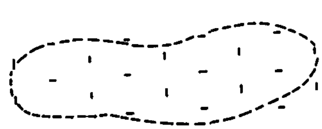
Thereupon they put down the marks for two lines of soles:



Then they add marks for the width which they double in order to account for two rows of soles:



And finally they draw an approximately fitting giant's sole around:



While Frank and Rik are busy with their drawing, the teacher asks a question.

T: What are they doing?

Agniska: Drawing the width.

When Frank and Rik have finished,

T: I see how you got the length of the shoe but what about the width?

Frank: It may also be four times as wide.

T: What do you think, Rik?

Rik: Twice is good. But four is also good, but there is no need.

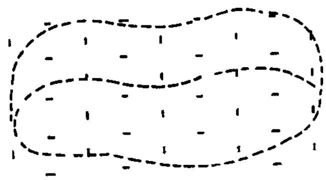
T (to the class): Who agrees and who does not?

Monique: I would think it is three times as wide.

T: Why do you think that?

Monique: If it is that long, it may be somewhat wider.

Agniska, however, judges that in the width two (soles of the teacher's shoe) should be added: she adds them on the blackboard:



Meanwhile the children are asked to think about the question and to write down how many teacher's soles fit into the giant's sole. The teacher repeats the former estimates.

T: We just estimated: 4, 4, 8, 6, 8, ... Aren't you surprised by what you have written down?

Rebecca is allowed to explain precisely her new insight.

Rebecca: In the length you can take four, and that way too (she shows to the width) and thus they must be sixteen. (She first counted, as she said: two, four, six, eight, . .).

One can conclude that Adda's (and others') objections are entirely confirmed by the course of this lesson. The result sixteen is not obtained and verified by paving. It remains hypothetical in so far as one is satisfied by imposing the 4 x 4-model for the sole problem without a sound argumentation. There are indeed no mathematical means available at this level for a conclusive and acceptable proof.

On the other hand this course of thought can also be appreciated in a positive way.

'Length' is a striking magnitude. Young children reason easily on the basis of linearity. The problem of the size of the giant's sole aims primarily at making conscious the fact that increasing operates in two dimensions. Estimates are very appropriate in order to prepare this idea (Streefland, 1982). By estimating, the structure of the problem is anticipated. This is confirmed by the protocol. Many children take the second dimension into account consciously without a clear idea of the mathematical impact of the increase. In the end they accept without a proof that it is 4 x 4.

The goal of the activity however, is realised: breaking the pattern of linearity and making them aware of the more-dimensional impact of increase. Later on the correctness for rectangles and squares is precisely proved by means of construction. Then the soles problem can be resumed.

Designing long term learning processes requires patience and feedback. There is no need to harvest all the mathematics underlying a certain problem once and for all. Moreover it may be questioned whether paving the giant's sole precisely with the teacher's sole – even if it were possible – would add that much to the acquired insight. What matters is the idea that the linear increase acts in two directions. Once this is settled, the conviction that the area is increased by a factor 4 × 4, is a mere consequence.

4.3. *The Context of Distance, Time and Speed*

As mentioned earlier, four respondents doubted the feasibility of this part of the theme with 8–9 years olds.

Without going into details, it must be recognised that these doubts were well-founded. The try-out proved that certain parts of this subject did not function satisfactorily.

The most important cause probably was the planning of too fast a learning process. The physical context of distance, time and speed in their mutual connection was not sufficiently accounted for.

In the experiment this was revised (see subsections 1 and 2) in the way that the concepts 'time' and 'speed' were included at a mainly qualitative level. Moreover the number of problems on this subject was reduced. Anyway the doubts appeared to be justified.

4.4. *Children Like Recipes*

One of the respondents suggested developing a separate theme on recipes. The argument is that children of that age like cooking and baking.

From the instruction experiment (and also in a small scale experiment with 11–12 years olds) this suggestion was strongly supported. Not only did quite a few children bring out their own recipes, they even applied them in the theme and went as far as baking butter cakes, which they took along to school as a treat.

Some respondents warned against the circular model. They doubted whether children's experience went as far as understanding that in reality the ingredients are mixed rather than covering circle sectors. Contrary to this opinion the experiment proved that the children had no difficulty with this problem.

In this connection we must deal with the criticism brought forward on applying the sector diagram. This is expressed in the next subtitle.

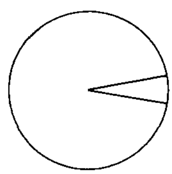
4.5. *Forced Modeling (Sector Diagram)*

Some respondents judged the model as misleading because of the little sector, its premature occurrence in the learning process and a preference for squares and rectangles.

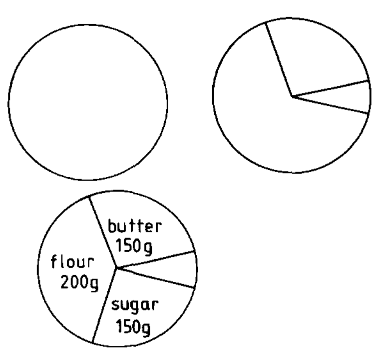
The most important objection aimed at the forced model building was that the children might identify the sector diagram with the cake.

The answer to this criticism is introduced by the protocol of a lesson.

The children attentively listened when the teacher built a cake recipe into a table.
On the right side of the blackboard she writes: Sector diagram.
Teacher: A sector is such a piece (she draws a little wedge).



Spontaneously the children enumerate what is to be put in: milk, lemon-peel, salt, egg(s).
Teacher: How many more wedges should we draw for all those things in the recipe and how big should they be taken?
The children try it first on their own working sheet. Very few need some help. Most of the children manage it very well.
Sandra is allowed to reproduce her drawing on the blackboard. Nobody objects about the size of Sandra's drawing, which on the blackboard is much larger than on the paper.



Questions like twice or four times the different ingredients are discussed. The recipe for 'twice' is first contrived and put into the table. The corresponding sector diagram had to increase too according to the children. They identify sector diagram and cake.

Teacher: Now we will take a look at a cake, which consists of four times the recipe. I am going to make a sector diagram for it.

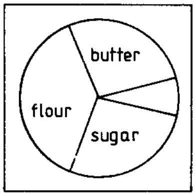
The teacher draws a big circle on the righthand blackboard.

Agniska (protesting): I find it too big.

Teacher: If we continue this way we are getting out of the blackboard. What can we do?

Frank: Just as the first.

He may go to the blackboard and draws the sectors in the circle:



Teacher: Isn't it smart that Frank can make the drawing before knowing the numbers? How is it possible?

Monique: It is exactly the same but larger.

Richard, however, believes that Frank had retained the numbers in his mind. Rik saves the situation.

Rik: You may still *draw* it *small* size and then *think* that it is larger.

Teacher: That is a good idea. With a small drawing you can still show that it is bigger, isn't it?

Rik: Writing down how much there is in it of all kinds.

It is of a primary importance that ill-balanced looking distributions, such as in recipes, are explored to counterbalance the influence of fractions and fraction arithmetic when the starting idea is fair sharing, which is well-balanced (equal units, equal parts, exhausting, everybody gets the same share).

Moreover from the protocol it is clear as daylight that the keypoint was *developing* the sector diagram as a thought model with a heavy accent on *constructions by the children themselves and fed by their own ideas.* The identification of cake (recipe) and sector diagram was no drawback. On the contrary, it should be considered as a favourable side-effect because of the conflict it provoked. Soon it became clear that it was always the same distribution at issue, although not all could detach themselves equally easily from the idea that knowledge of the numbers was indispensible for drawing the diagram. Subsequently they had to learn to disregard the size of the cakes, and this happened spontaneously when the diagram threatened to grow out of proportion. By settling this conflict as described, the idea of the sector diagram as a model had taken shape in the children's mind. This was also proved by their spontaneous coming to grips with measuring the little sector. No great precision was aspired at in the technique of dividing by drawing. Indeed, this would have seriously impeded the learning process. Exaggerated requirements with respect to the technique of dividing distract the attention; moreover more often than not the children's technique lagged behind their understanding (cf. Streefland, 1978). Stress on precision would not have contributed to the development of the sector diagram as a thought model and would even have proved counterproductive.

4.6. *Is the Similarity of Humans and Giants Obvious?*

The children falsify the expectations of those respondents who fear that the similarity of man and giant would be accepted too lightheartedly. When in the first lesson of the theme the giant's stature was discussed, some of the pupils concluded from the factor '4' for the length of the giant's hand compared with that of the teacher that the giant should also be four times as tall as the teacher. But others protested, arguing that if his hand is four times hers, his body could still be five or six times hers. After this it was simply settled that hencefore the scale factor four would be assumed.

4.7. *Retrospection*

The facts we brought forward in the preceding subsections admit various conclusions. For instance:

- Even for judges of instructional texts who advocate realistic mathematics education with the stress on mathematical *activity* it is not easy not to think from the viewpoint of mathematical *products*, to eliminate their own knowledge and to think in terms of the envisaged learning processes of children who still have to learn to create the greater part of their mathematical reality.
  (Cf. Freudenthal, 1981, p. 149.)

Or:

- Realistic problems call on the common sense of children, who for that matter as human children do not differ that much from adults.

On the other hand one might state that it is quite difficult to observe enough patience and reserve to think in terms of long learning processes rather than trying to get the complete mathematical harvest out of each mathematical activity, and to delay certain activities until all preconditions are fulfilled to harvest the related mathematics in its totality.

Is it too farfetched to assume that the attitude behind those temptations is also influential in our theories on mathematics education, or for that matter, in our teaching of ratio?

In the second part of this paper I will try to explain my view on it. The preceding exposition plays a part in it. Some building blocks for a theory on learning and teaching ratio seem to be on hand thanks to the ideas put forward by the (inter)national panel and thanks to the contributions of the children in the instructional experiment.

Three special subjects will be dealt with in Part II – all of them related to the structuring of a long term learning process. They are:

- a consideration aiming at anticipating activities (in particular about ratio) from the perspective of the macro-structure of the learning process (5.2);
- creation and functions of visual models in the long term learning process (5.3);
- the ratio table as a schematising tool in the long term learning process (5.4).

NOTES

[1] The present paper could not have been written without the help of colleagues who explained to the author their vision of ratio in mathematics instruction. These data became available through their responses on a few queries connected to a text on ratio. I express my gratefulness to all that cooperated – in an alphabetic order: Josette Adda, Frederik van der Blij, Lesley Booth, Jan van den Brink, Kees Buijs, Kevin Collis, Gerard Doevedans, Joop van Dormolen, Willem Faes, Ruma Falk, Hans Freudenthal, Fred Goffree, Kathleen Hart, Klaus Hasemann, David Hawkins, Claude Janvier, Onno de Jong, Sieb Kemme, Piet Lijnse, Hartwig Meissner, Arthur Morley†, Jim Moser, Ed de Moor, André Rouchier, George Schoemaker, Jan Sloff, Anne van Streun, Rob Tempelaars and Gérard Vergnaud.

In particular I extend my thanks to Hans Freudenthal and Adri Treffers, who constructively contributed to the plan and composition of the final text of the paper and the translation.

[2] The reader is referred to Hilton's contribution to the Sixth International Conference for the Psychology of Mathematics Education in Antwerp, 1982, 'Creative Thoughts in Mathematics'.

[3] As well as internal and external ratios, Freudenthal also distinguished expositions, for instance population density, and compositions, for instance mixtures or recipes.

For internal and external ratios also other terminology has been applied in the literature, for instance 'within–between' (Karplus).

REFERENCES

Daydovv, V. V.: 1975, 'Logical and psychological problems of elementary mathematics as an academic subject', in J. Kilpatrick, J. Wirszup, E. G. Begle, and J. W. Wilson (eds.), *Soviet Studies in the Psychology of Learning and Teaching Mathematics*, vol. VII, School Mathematics Study Group, Stanford University, Calif.

Freudenthal, H. *et al.* (eds.): 1976, 'Five years IOWO. On H. Freudenthal's retirement from the directorship of IOWO', *Educational Studies in Mathematics* 7, 189–367.

Freudenthal, H.: 1981, 'Major problems of mathematics education', *Educational Studies in Mathematics* **12**, 133–150.

Freudenthal, H.: 1983, *Didactical Phenomenology of Mathematical Structures*, D. Reidel, Dordrecht, Boston, Lancaster, Chapter 6: 'Ratio and proportionality'.

Jong, R. de (ed.): 1978, 'Oppervlakte (2)', *IOWO-leerplanpublikatie* 9, Utrecht, handleiding, 22–43.

Resnick, L. B. and Ford, W. W.: 1981, *The Psychology of Mathematics for Instruction*, Lawrence Erlbaum Assoc., Hillsdale/New Jersey.

Streefland, L.: 1978, 'Some observational results concerning the mental constitution of the concept of fraction', *Educational Studies in Mathematics* 9, 51–73.

Streefland, L.: 1982, 'The role of rought estimation in learning ratio and proportion – an exploratory research', Vermandel, A. (ed.): *Proceedings of the Sixth International Conference for the Psychology of Mathematics Education (PME)*, 193–200.

Treffers, A.: 1982, *Mathematikunterricht Klasse 1 bis 6. Eine Methode der Lernzielbeschreibung des Mathematiksunterrichts nach Wiskobas*, Paderborn.

OW & OC,
Tiberdreef 4,
Utrecht,
The Netherlands