CAROLYN KIERAN

CONCEPTS ASSOCIATED WITH THE EQUALITY SYMBOL[1]

ABSTRACT. This paper looks at recent research dealing with uses of the equal sign and underlying notions of equivalence or non-equivalence among preschoolers (their intuitive notions of equality), elementary and secondary school children, and college students. The idea that the equal sign is a "do something signal"[2] (an operator symbol) persists throughout elementary school and even into junior high school. High schoolers' use of the equal sign in algebraic equations as a symbol for equivalence may be concealing a fairly tenuous grasp of the underlying relationship between the equal sign and the notion of equivalence, as indicated by some of the "shortcut" errors they make when solving equations.

1. INTRODUCTION

On the topic of equivalence, Gattegno (1974) has stated:

> We can see that *identity* is a very restrictive kind of relationship concerned with actual sameness, that *equality* points at an attribute which does not change, and that *equivalence* is concerned with a wider relationship where one agrees that for certain purposes it is possible to replace one item by another. Equivalence being the most comprehensive relationship it will also be the most flexible, and therefore the most useful (p. 83).

However, the symbol which is used to show equivalence, the equal sign, is not always interpreted in terms of equivalence by the learner. In fact, as will be seen, an equivalence interpretation of the equal sign does not seem to come easily or quickly to many students.

2. PRESCHOOL CONCEPTS OF EQUALITY

Once preschoolers begin to tag objects with number words in a systematic and consistent fashion, they seem, at some time after that, to be able to determine the numerical equality of two sets. Gelman (Gelman & Gallistel, 1978) has found evidence that the preschool child's (3- to 5-years-old) judgment of whether two sets are numerically equal ordinarily rests on whether they yield the same cardinal numeral when counted (see also, Siegel, 1978). In other words, to determine whether two different and not necessarily homogeneous sets, A and B, are numerically equal, the preschooler's normal principle is "Count them and see". His/her statement that "they're both the same" would correspond symbolically to "card. (A) = card. (B)". This can be considered a comparative notion of equality – based on the child's ability to count two different sets and to compare their numerosities.

Educational Studies in Mathematics **12** (1981) 317–326. 0013–1954/81/0123–0317$01.00

Preschoolers' acquisition of the ability to count two distinct sets is generally followed by the acquisition of the ability to *put together* two distinct sets and to count the number of elements in their "union" (Brush, 1978; Fuson, 1979; Gelman & Gallistel, 1978). This counting of the number of elements in the combined set leads to an operator notion of equality which emphasizes the result of the arithmetic operation. This corresponds symbolically to "card. (A) + card. (B) = card. (A ∪ B)" where A and B are disjoint sets.

Thus we can distinguish two intuitive meanings of equality among preschoolers: one involving a comparison between two sets where the child counts the elements of the two different sets and, on the basis of the same cardinality, establishes their equality; the other involving the addition of two sets where the child (s/he may or may not count the items in the individual sets first) *combines* the two sets and then counts the elements in the resulting set. As will be seen, it is the latter, the operator notion of equality, which is appealed to when the equal sign is introduced in school.

3. THE USE OF THE EQUAL SIGN IN ELEMENTARY SCHOOL

By the time most children enter school, they have sound intuitions about addition and subtraction (Ginsburg, 1977). It is these intuitions which are first symbolized in their school arithmetic. Many children learn fairly quickly to read and write the elementary written symbolism of simple arithmetic, but do not necessarily understand it the same way we do. Whereas we can look upon equality sentences as equivalence relations (involving the reflexive, symmetric, and transitive properties), primary school children, Ginsburg points out, interpret + and = in terms of actions to be performed. One of his second grade subjects typically said that in $3 + 4 = \square$, "the equal sign means what it adds up to". Others read "$3 + 5 = 8$" as "3 and 5 *make* 8". Many first and second graders, when asked how to read $\square = 3 + 4$, would answer, "Blank equals 3 plus 4", but then add: "It's backwards! Do you read backwards?" and then change it to $4 + 3 = \square$. As can be seen, one consequence of the child's interpretation of equalities in terms of actions is that he finds it difficult to read arithmetic sentences that do not reflect the order of his calculations. Ginsburg also found that many children cannot read sentences that express relationships like $3 = 3$.

It can be argued that these notions reflect the kind of instruction that these children have received. One might then assume that later exposure to equality sentences involving the commutative and associative properties might broaden the elementary school child's notion of the equal sign. However, this does not seem to be the case. Behr, Erlwanger, and Nichols (1976) did a study with children in grades 1 to 6 which investigated their view of equality sentences. When Behr et al. asked sixth graders about the meaning of "3 = 3", a typical response was: "This could mean 6 − 3 = 3 or 7 − 4 = 3". Behr and his colleagues emphasized that there was no evidence to suggest that children changed in their thinking about equality as they progressed to upper grades; in fact, even sixth graders seemed to view the equal sign as a "*do something signal*".

Another type of sentence which Behr presented to his subjects involved equalities such as 4 + 5 = 3 + 6. A common response to this was: "After '=' should be your answer. It's the end, not another problem", followed by the transformation of the initial equality into two sentences – "4 + 5 = 9" and "3 + 6 = 9" – and the comparison of the results of those statements. This kind of reaction brings into question the psychopedagogical validity of the "name for a number" approach, that is, that "4 + 5 = 3 + 6" because both 4 + 5 and 3 + 6 are *other names for* 9. The "name for a number" idea, advocated approximately twenty years ago by the School Mathematics Study Group (SMSG) and adopted by most current elementary school mathematics text books, assumes that the young learner can assimilate an equivalence view of the equality symbol; for, indeed, "is another name for" is an equivalence relationship defined on ordered pairs of numbers $[(a, b)\ \mathscr{R}\ (c, d)$ iff $a + b = c + d]$. However, as both Denmark (1976) and Collis (1974) have shown, these assumptions are not warranted.

Denmark designed a teaching experiment to teach the concept of equality as an equivalence relation to a group of first grade students (before they encountered the + and = signs in school). This study showed that students were able (by means of activities with a balance and the corresponding written equations) to acquire some flexibility in accepting the use of the equal sign in a variety of sentence structures (e.g., 3 = 3, 3 + 2 = 4 + 1, 5 = 4 + 1); however, the equal sign was still viewed primarily as an operator, not as a relational symbol. Furthermore, the data do not support the conjecture that if students (first graders) are provided with appropriate instructional experiences in which they encounter the use of the equal sign in a variety of sentence forms, they will acquire a conceptualization of equality as a relation between two names for the same number.

Collis has also provided evidence of children's inability to deal with the "two names for the same number" approach. Based on his observations of children's mathematical behavior, he has distinguished various levels of capability with respect to their "Acceptance of Lack of Closure", that is, their ability to hold unevaluated operations in suspension. Collis has found that primary school children who are between the ages of approximately 6 years and 10 years require that two elements connected by an operation be *actually* *replaced* by a third element. Quite clearly, according to Collis, they cannot handle equations such as $4 + 5 = 3 + 6$. The child needs literally to be able "to see" a unique result before the operations on numbers mean anything to him, that is, $4 + 5 = 3 + 6$ must be written $4 + 5 = 9$. Later (between approximately 10 years and 13 years), the learner may regard the outcome as unique, but may not need to make the actual replacement to guarantee this. There is less reliance on "seeing" the uniqueness in the results of his operations, though he is still bound by empirical evidence. After about age 13, Collis points out, the learner is willing to infer beyond physical models and to use specific cases for forming adequate generalizations.

4. THE USE OF THE EQUAL SIGN IN HIGH SCHOOL AND COLLEGE

For many 13-year-olds, however, this is a transition period – transition between requiring the answer after the equal sign and accepting the equal sign as a symbol for equivalence. During this transition period, a fair amount of confusion exists, as shown by the research of Vergnaud and his colleagues (1979). They point to the type of errors made by students whose written work merely encodes their procedures, thus yielding false equalities, such as,

$$1063 + 217 \;=\; 1280 - 425 \;=\; 1063.$$

This string of equalities illustrates the written work involved in solving the problem: "In an existing forest 425 new trees were planted. A few years later, the 217 oldest trees were cut. The forest then contains 1063 trees. How many trees were there before the new trees were planted?" An analysis of this type of answer leads to several observations. The two operations shown in this string of equalities clearly indicate that the student understands the problem. Thus, just as Ginsburg found with younger children, the difficulty is at the symbolic level. The statement, $1063 + 217 = 1280 - 425$, reflects a well-known pattern (Kieran, 1979), that of writing down the operations in the order in which they are being thought and that of keeping a running-total. The last statement, $1280 - 425 = 1063$, indicates the need that the student has to somehow relate his calculations back to the initial problem. This kind of evidence illustrates that perhaps one of the difficulties in using word problems to give meaning to mathematical symbolism is precisely this issue of relationships versus procedures.

An alternate approach, avoiding the translation difficulties implied in the use of word problems, has been successfully attempted in a teaching experiment, involving six case studies, which investigated the construction of meaning for non-trivial algebraic equations (Herscovics & Kieran, 1980; Kieran, 1979, 1980). Subjects (12- to 14-years-old) were first asked what the equal sign meant to them, followed by the request for an example showing the use of the equal sign. It is telling that most of them described the equal sign in terms of the *answer* and limited themselves to examples involving an operation on the left side and the result on the right. The ensuing teaching sessions focussed on extending subjects' use of the equal sign to include multiple operations on both sides. This was done by having the students construct arithmetic equalities, initially with one operation on each side, e.g.,

$$2 \times 6 = 4 \times 3 \text{ (the same operation)}$$

and then

$$2 \times 6 = 10 + 2 \text{ (different operations).}$$

They then went on to constructing equalities with two operations on each side, and then to multiple operations on each side, e.g.,

$$7 \times 2 + 3 - 2 = 5 \times 2 - 1 + 6.$$

These were given the name "arithmetic identities" in order to reserve the term "equation" for use in the algebraic sense. Despite initial insistence from one subject on writing $4 + 3 = 6 + 1$ as $4 + 3 = 7$, and from another on inserting the "answer" between both sides (i.e., $5 \times 3 = 15 = 10 + 5$), subjects seemed in general to be quite comfortable with equality statements containing multiple operations on both sides. They justified them in terms of both sides being equal because they had the same value. The comparisons that subjects were eventually able to make between left and right sides of the equal sign suggest that the equality symbol was being seen at this stage more as a relational symbol than as a "do something signal". The right side, by this time, did not have to contain the answer, but rather could be some expression that had the same value as the left side.

The reason for extending the notion of the equal sign to include multiple operations on both sides was to provide a foundation for the later construction of meaning for non-trivial algebraic equations (which have multiple operations on both sides). If this expansion were not done first, the student would be bringing with him into the study of algebraic equations the idea that the result is always on the right side of the equal sign. Thus, equations such as $3x + 5 = 26$ might fit in with his existing notions, but $3x + 5 = 2x + 12$ would not. Not only would the presence of this multiple operation on the right side be foreign to him, but also seeing it for the first time within the context of an algebraic equation would add to the cognitive strain. Therefore, extending the notion of the equal sign within the framework of arithmetic equalities *prior* to the introduction of algebraic equations was considered essential in the construction of meaning for non-trivial equations.

The next step, introducing the concept of equation, involved taking one of the student's arithmetic identities, e.g.,

$$7 \times 2 - 3 = 5 \times 2 + 1$$

and hiding any one of the numbers. The hiding was done at first by a finger:

$7 \times$  $- 3 = 5 \times 2 + 1,$

then by a box:

$$7 \times \square - 3 = 5 \times 2 + 1,$$

and finally by a letter:

$$7 \times a - 3 = 5 \times 2 + 1.$$

Thus an equation was defined as

AN ARITHMETIC IDENTITY WITH A HIDDEN NUMBER.

By following these three modes of representation (which parallel Bruner's enactive, iconic, and symbolic stages), the student could acquire an intuitive understanding of the meaning of equation and then gradually transform this to an understanding of the form of an algebraic equation. Thus, his algebra would be anchored in his arithmetic. The letter hiding the number was called an unknown – a term which corresponds closely to the idea of a hidden number.

At this stage the student constructed several equations from his or her own arithmetic identities. It was important for him or her to realize that a given arithmetic identity could lead to the construction of many different equations: e.g., $3 \times 7 + 3 = 25 - 1$ yielding

$$a \times 7 + 3 = 25 - 1,$$

$$3 \times b + 3 = 25 - 1,$$

$$3 \times 7 + 3 = c - 1 \text{ etc.}$$

This kind of variety in building equations prevented students from imposing unnecessary restrictions on the concept of equation.

The question of hiding more than one number came up soon afterward. Since students were not solving equations at this point, but were still constructing them, there seemed to be no reason to restrict them to equations with just one letter. Therefore, when they began to hide more than one number, they were told of the convention that the same letter could be used in an equation more than once, as long as it was used to hide the same number; otherwise one would have to use two different letters. Thus, in hiding the same number twice – one occurrence on the left side of the equal sign and the other occurrence on the right – the student was constructing non-trivial algebraic equations, e.g., $2 \times 3 + 7 = 5 \times 3 - 2$ leading to $2 \times c + 7 = 5 \times c - 2$. Just as with arithmetic identities, the right side of an algebraic equation did not have to contain the answer, but rather could be some expression that had the same value as the left side. For example, equations such as $2x + 3 = 4x + 1$ were described by subjects: "If you know what number x is, then 2 times that number plus 3 has the same value as 4 times that number plus 1".

However, the ability to consider an algebraic equation as an expression of equivalence because both sides have the same value does not seem to be sufficient for an adequate conceptualization of the equation-solving process. For not only does equation solving involve a grasp of the notion that right and left sides of the equation are equivalent expressions, but also that each equation can be replaced by an equivalent equation (i.e., one having the same solution set). Unfortunately, very little research has addressed itself to the question of how this concept is acquired by high schoolers. That both these notions of equivalence may be quite fuzzy for many students is reflected in the procedures they use to solve equations. Byers and Herscovics (1977) have provided examples from the written work of some students, reminiscent of Vergnaud's subjects, which yield evidence that the equal sign is not being used consistently as a symbol for equivalence:

$$
\begin{aligned}
\text{Solve for } x: \quad 2x + 3 &= 5 + x \\
2x + 3 - 3 &= 5 + x - 3 \\
2x &= 5 + x - x - 3 \\
2x - x &= 5 - 3 \\
x &= 2
\end{aligned}
$$

$$
\begin{aligned}
\text{And:} \quad x + 3 &= 7 \\
&= 7 - 3 \\
&= 4
\end{aligned}
$$

It may be argued that these students possess an underlying equivalence notion of the equal sign and that they are just using shortcuts in their procedural work. However, it can also be contended that their written work reflects a basic lack of awareness of the equivalence role of the equal sign. These "short-cut" errors suggest that many high school students are still interpreting the equal sign as an operator symbol – as a signal to do something in order to get the "answer". Even college students taking calculus, when asked to find the derivative of a function, frequently seem to be using the equal sign merely as a link between steps:

$$\begin{aligned} f(x) &= \sqrt{x^2+1} \\ &= (x^2+1)^{1/2} \\ &= \tfrac{1}{2}(x^2+1)^{-1/2}\, D_x(x^2+1) \\ &= \tfrac{1}{2}(x^2+1)^{-1/2}\,(2x) \\ &= x(x^2+1)^{-1/2} \\ &= \frac{x}{\sqrt{x^2+1}} \qquad \text{(Clement, 1980, p. 7)} \end{aligned}$$

Whether these errors are due to taking shortcuts or to the application of the operator principle or both is an open question.

The importance of the equal sign in high school algebra cannot be overestimated. Most children identify algebra with equations. In fact the absence of the equal sign seems to create huge conceptual problems. In an ongoing research project on the learning of algebra (Kieran, in progress), 12- and 13-year-olds could not assign any meaning to indeterminate forms, such as, $3a$, $a + 3$, $3a + 5a$ ("a means nothing . . . there is no equal sign with a number after it"). Perhaps this explains why students have such difficulty in dealing with polynomials later in high school when they are introduced as indeterminate forms.

5. SUMMARY

Mathematics makes no distinction between the left and right sides of expressions such as $3 + 2 = 5$, $3 + 5 = 7 + 1$, or even $3x + 3 = 4x + 2$. They are all considered examples of equivalence relations. However, as has been pointed out in this paper, the concept of equivalence is an elusive one not only for elementary school students but for high schoolers as well. That the equal sign is a "do something signal" is a thread which seems to run through the interpretation of equality sentences throughout elementary school, high school, and even college. Early elementary school children, despite efforts to teach them otherwise, view the equal sign as a symbol which separates a problem and its answer. This thinking remains as children get older and advance to the upper elementary grades. Junior high school students are able to relax some of the constraints placed earlier on the interpretation of the equal sign to make room for arithmetic equalities (with multiple operations on both sides) where both sides "yield the same value". It is not clear, however, whether or even how these beginnings toward interpreting the equal sign in terms of an equivalence relation develop into an awareness of the notion of equivalent equations in high school algebra and college calculus. The procedures used by students to solve equations and to find the derivative of a function would seem to indicate that high school and college students may also tend to interpret the equal sign in terms of an operator symbol, albeit at a more sophisticated level, rather than as a symbol for an equivalence relation.

Mathematics Department
Concordia University, Montreal

NOTES

[1] This article is based on a paper presented at Fourth International Conference of the International Group for the Psychology of Mathematics Education, Berkeley, California, August 1980.

[2] This expression was first coined by Behr, Erlwanger and Nichols in their 1976 PMDC Technical Report (S. Erlwanger, personal communication, June 1980).

REFERENCES

Behr, M., Erlwanger, S. and Nichols, E.: 1976, 'How children view equality sentences', PMDC Technical Report No. 3, Florida State University. (ERIC Document Reproduction Service No. ED144802).

Brush, L. R.: 1978, 'Preschool children's knowledge of addition and subtraction', *Journal for Research in Mathematics Education* 9, 44–54.

Byers, V. and Herscovics, N.: 1977, 'Understanding school mathematics', *Mathematics Teaching* 81, 24–27.

Clement, J.: 1980, 'Algebra word problem solutions: Analysis of a common misconception', Paper presented at annual meeting of American Educational Research Association, Boston.

Collis, K. F.: 1974, 'Cognitive development and mathematics learning', Paper presented at Psychology of Mathematics Education Workshop, Centre for Science Education, Chelsea College, London.

Denmark, T., Barco, E. and Voran, J.: 1976, 'Final report: A teaching experiment on equality', PMDC Technical Report No. 6, Florida State University. (ERIC Document Reproduction Service No. ED144805).

Fuson, K. C.: 1979, 'Counting solution procedures in addition and subtraction', Paper presented at Wingspread Conference on the Initial Learning of Addition and Subtraction Skills, Racine, Wisconsin.

Gattegno, C.: 1974, *The Common Sense of Teaching Mathematics*, Educational Solutions, New York.

Gelman, R. and Gallistel, C. R.: 1978, *The Child's Understanding of Number*, Harvard University Press, Cambridge.

Ginsburg, H.: 1977, *Children's Arithmetic,* Van Nostrand, New York.

Herscovics, N. and Kieran, C.: 1980, 'Constructing meaning for the concept of equation', *The Mathematics Teacher* 73, 572–580.

Kieran, C.: 1979, 'Children's operational thinking within the context of bracketing and the order of operations', in Tall, D. (ed.), *Proceedings of the Third International Conference for the Psychology of Mathematics Education,* Mathematics Education Research Centre, Warwick University, Coventry, England.

Kieran, C.: 1980, 'Constructing meaning for non-trivial equations', Paper presented at annual meeting of American Educational Research Association, Boston. (ERIC Document Reproduction Service No. ED184899).

Siegel, L. S.: 1978, 'The relationship of language and thought in the preoperational child: A reconsideration of nonverbal alternatives to Piagetian tasks', in Siegel, L. S. and Brainerd, C. J. (eds.), *Alternatives to Piaget,* Academic Press, New York.

Vergnaud, G., Benhadj, J. et Dussouet, A.: 1979, *La Coordination de l'Enseignement des Mathématiques entre le cours moyen 2e année et la classe de sixième,* Institut National de Recherche Pédagogique, France.