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# MATHEMATICAL INDUCTION: A PEDAGOGICAL DISCUSSION

ABSTRACT. It is observed that many students have difficulty in producing correct proofs by the method of mathematical induction. The notion of a correct proof by this method is analysed mathematically. Subsequently, the topic is analysed into behavioural skills and subjected to a conceptual analysis. Common misconceptions of induction are considered, with recommendations for their remediation. Finally, criteria for the analysis and evaluation of textbook treatments of induction are evolved and applied to a selection of texts.

## 1. INTRODUCTION

The post World War Two years have seen a tremendous growth in the analysis of the mathematics taught to younger school children and their understanding of it. This has been followed by an analysis of the mathematics taught to secondary school children and by the development of new curricula for them. The literature on the teaching of mathematics to these groups of children has grown to vast proportions. In all this growth, there is an area to which comparatively far less attention has been directed. This is the teaching of mathematics to senior school students in sixth forms or high schools. The teaching of some of the topics for students at this level is beginning to receive more attention. For example the teaching of calculus is the subject of the recent book by Neill and Shuard (1982). Many other topics are still too rarely discussed from a pedagogical viewpoint. One such topic is mathematical induction, the subject of the present article. Although not entirely neglected (for an early discussion of some of the pedagogical issues involved see Young (1908)) this topic has been chosen because it needs attention. There are unresolved problems concerning the teaching of mathematical induction which should benefit from a careful analysis.

Students pursuing the academic study of mathematics to the tertiary or pre-tertiary levels will generally meet the method of proof by mathematical induction. Many of these students find the method difficult to master. A graphic account of the confusion that can arise in the mind of a student whilst being taught the method is given by Shaw (1978). Other authors write of the difficulties that arise in teaching mathematical induction, for example:

> I can honestly say that I have almost never encountered a college freshman in a run-of-the-mill calculus class who really understood the concept of proof by mathematical induction. (Brumfiel 1974, p. 616.)

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Wheeler (1981, p. 275) begins his discussion of induction thus: ". . . an important question to ask first is: why are so many students unhappy about mathematical induction?" Both these authors continue by giving their particular teaching responses to students' difficulties in the learning of induction. There is, however, to my knowledge, no systematic account in print of the teaching of mathematical induction, of the problems that arise, of the deeper issues involved or of the treatments given by text books. This article aims to fill this gap.

Before dealing with the more pedagogical issues it seems appropriate to precisely establish the nature of the method of mathematical induction, and what constitutes a correct proof by this method.

## 2. MATHEMATICAL ANALYSIS OF INDUCTION

The Principle of Mathematical Induction can be presented to students in a variety of forms. It can be expressed set-theoretically in terms of the set of all natural numbers which have some property. More commonly, it is expressed simply in terms of a property of natural numbers. Of these two similar forms the latter seems preferable on the grounds of simplicity, for the set of natural numbers satisfying a given property is secondary to the given property.

A further variation depends on the choice of first number on which to base the induction. Both 0 and 1 are commonly used. Some authors, including Woodall (1975), use a parameter $a_0$ which can take any non-negative integral value. In fact, an inductive proof can begin with any integer, positive or negative. However, for pedagogic purposes, in my view it is best to choose a concrete starting point, and following most British school texts I shall take 1 as the starting number.

Bearing in mind the choices that have been made, the Principle can now be stated:

### *The Principle of Mathematical Induction (PMI)*

If 1 has property P, and if any $n$ having property P implies that $n + 1$ has property P, then every $n$ has property P.

In this formulation the variable $n$ ranges over the set of natural numbers (taken to begin with 1), as the choice of letter signifies, and P is a fixed but arbitrary property of natural numbers.

The principle may be expressed more briefly: If P(1) and if for all $n$ P($n$) implies P($n + 1$), then for all $n$, P($n$).

Both these forms of PMI are expressed informally, as is appropriate for introducing PMI to students. PMI can be expressed more formally either as an axiom or as a rule of inference.

*Axiomatic Form of PMI*

$$[\mathrm{P}(1) \wedge \forall n[\mathrm{P}(n) \rightarrow \mathrm{P}(n+1)]] \rightarrow \forall n\, \mathrm{P}(n)$$

It should be noted that this statement, which is (a reformulation of) one of Peano's axioms for the natural numbers, can be derived as a theorem in certain other systems, for example in Zermelo–Fraenkel set theory.

*Rule of Inference Form of PMI*

$$\frac{\mathrm{P}(1) \quad \forall n[\mathrm{P}(n) \rightarrow \mathrm{P}(n+1)]}{\forall n\, \mathrm{P}(n)}$$

As is common in the presentation of rules of inference, the premises are shown above the bar (solidus) and the conclusion which may be drawn is shown underneath. This symbolism probably derives by analogy from arithmetic sums, for example $\begin{array}{r}2\\+3\\\hline 5\end{array}$, in which inputs 2 and 3 are combined through the operation of addition to give the output, or answer, 5. In both school mathematics and in higher mathematics PMI is usually used as a rule of inference, but informally. A correct usage of PMI, also known as a proof by the *method of mathematical induction*, has the following form. There are four components:

(1) The statement of the theorem to be proved: $\forall n \mathrm{P}(n)$;

(2) An explicit invocation of PMI, and two subordinate proofs;

(3) Verification of P(1), the fact that the number one has the required property, known as the *basis of the induction*,

(4) Proof of the universally quantified implication statement:

$\forall n[\mathrm{P}(n) \rightarrow \mathrm{P}(n+1)]$, known as the *induction step*. The proof of the induction step is usually carried out in the simplest possible way, first by adopting the assumption $\mathrm{P}(n)$ known as the *inductive hypothesis*. This is followed by the derivation of $\mathrm{P}(n+1)$. This permits the assertion of $\mathrm{P}(n) \rightarrow \mathrm{P}(n+1)$ and, finally, of $\forall n[\mathrm{P}(n) \rightarrow \mathrm{P}(n+1)]$, provided that the variable $n$ first occurs freely in the inductive hypothesis. This last step, rather a logical nicety, is almost always taken for granted and is theoretically dispensible in a free variable form of PMI. To summarise, a correct proof by mathematical induction consists of:

(1) Statement of theorem $\forall n \mathrm{P}(n)$;

(2) Invocation of PMI;

(3) Basis of induction (Proof of P(1));

(4) Induction step, usually with derivation of $P(n+1)$ from inductive hypothesis $P(n)$.
These four components may be presented in any order, although some orders are evidently more sensible than others. A proof in which either (1) or (2), or both, are omitted can be regarded as an abbreviated, but still correct, proof. If either (3) or (4) is missing, then the proof is not correct. The presentation of a typical proof by PMI is as follows:
*Theorem*: $\forall n\, P(n)$.
*Proof*: by mathematical induction.
*Basis*: proof of $P(1)$.
*Inductive hypothesis*: assume $P(n)$.
*Induction step*: proof of $P(n+1)$ from inductive hypothesis.
□ or other marker signifying end of proof.

Having considered the nature and the form of a correct proof by mathematical induction, we move in the next section to a consideration of the production of such proofs by students.

## 3. ANALYSIS INTO BEHAVIOURAL SKILLS

The question addressed in this section is: what skills does a student need to have in order to construct correct proofs by induction? The answer to this will vary with the context and, in particular, depends on the theorems the student is expected to prove. Assuming that we are only concerned with the proof of routine theorems, that is theorems closely resembling results the student has already encountered, the question can be answered. For specificity we further assume that the routine results to be proved are algebraic identities. This assumption is justified on the grounds that, although problems may originate in many areas including finite and infinite series, number theory (especially divisibility) and geometry, the formulation of these results is usually algebraic.

Given these qualifications the question is: what behavioural skills does a student need in order to construct proofs by mathematical induction? A correct proof by mathematical induction has two subsidiary proofs as components, the basis of the induction and the induction step, so a student needs the ability to construct these subsidiary proofs. In addition, he must be able to incorporate these two components into an inductive proof, with explicit reference to PMI. Thus, the ability to construct a proof of a routine problem expressed algebraically by MI, can be analysed into the following three behavioural skills:

(a) The ability to prove the basis of the induction. This consists of the ability to verify that fixed numerical properties hold for particular numbers.

Under the restrictive conditions considered, this depends on the ability to perform substitution into algebraic expressions in a single variable.

(b) The ability to prove the induction step. This depends on the ability to prove an implication statement by deducing a conclusion from a hypothesis. Under the restrictive conditions considered, this consists of the ability to make deductions from algebraic identities, which in turn depends on the ability to manipulate algebraic expressions and identities.

(c) The ability to present a proof by mathematical induction in the correct form. This is manifested in the ability to communicate the knowledge of the correct form of a proof by mathematical induction in some way – be it verbal, written or diagrammatic.

These, then, constitute the behavioural skills necessary for the production of proofs of routine algebraic results. The analysis is shown as a flow diagram in Figure 1.

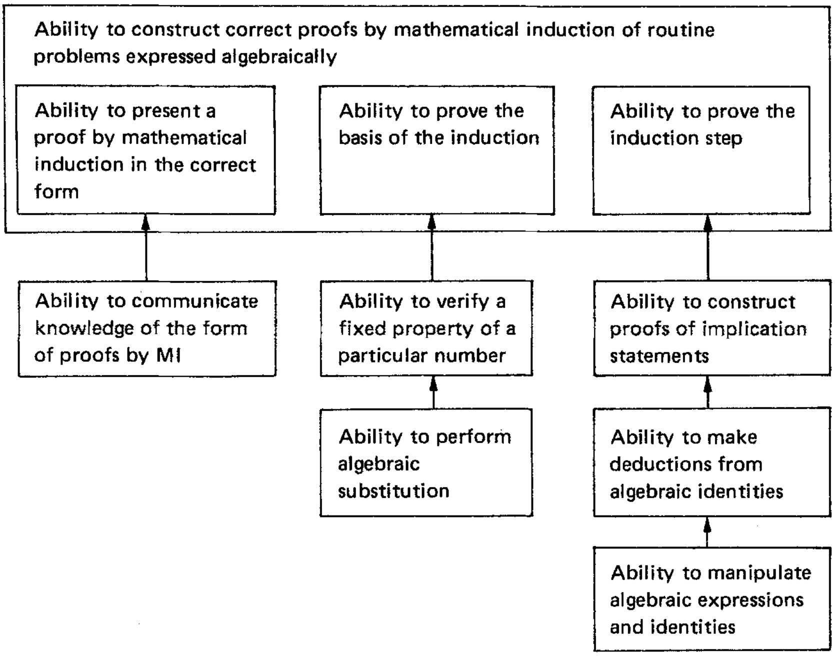


Fig. 1. The analysis of induction into behavioural skills.

The above analysis is quite useful in that it reveals the behavioural skills inherent in the ability to construct correct proofs by induction (of routine algebraic problems). Teaching which follows this behavioural analysis represents a considerable improvement on some of the teaching and texts of the past in which the method of MI is

> some strange jiggery-pokery that has to be used to prove the binomial theorem .... This, quite literally, is the sole experience that some pupils are given of this exciting and powerful technique. [Wheeler (1981, p. 275).]

Teaching which respects the behavioural analysis is not fully open to this criticism because both the statement of PMI and the form of a proof by induction are explicitly taught. In addition, the required subsidiary skills: algebraic substitution and manipulation, the proof of identities and, in particular, of implication statements, are developed. Teaching guided by behavioural analysis should be successful with many pupils in teaching them how to construct proofs by mathematical induction of the restrictive problem types considered. Using the terminology of Skemp (1976) such teaching might give pupils an *instrumental* understanding of the method of proof by mathematical induction. However the behavioural analysis gives no indication of ways in which the method might be explained, rendered plausible or related to previously acquired knowledge or concepts. Teaching with these additional goals aims at giving students a deeper *relational* understanding of the method. This requires a deeper conceptual analysis of the method, which is the subject of the next section.

## 4. A CONCEPTUAL ANALYSIS OF INDUCTION

The aim of this section is, through an analysis of induction, to differentiate between the different conceptual areas it relates to and presupposes.

The analysis of induction into behavioural skills provides a starting point for conceptual analysis, for the concepts underlying these skills are necessarily contributors to the conceptual network underlying mathematical induction. The first concept to be separated from induction is that of *implication*. Both the concept of implication as a binary sentential connective and the concept of the proof of an implication statement are entailed. In addition to the proof of an implication, the concept of an *elementary proof* in general is required for an understanding of the method of MI, since it is a particular method of proof itself. This in turn rests on experience of deductions involving algebraic identities as a major source of elementary proofs. Algebraic identities themselves also contribute directly to the subordinate proofs required by the method of MI. The understanding of the manipulation of algebraic identities and other elementary algebraic skills rests on knowledge of the conventions of algebraic usage and of the underlying laws (associative, distributive, commutative, etc.).

However, since these elements are of a level of complexity and abstraction considerably below that of the method of induction, they are presupposed without further ado. We have thus exhausted the conceptual areas which underlie the behavioural skills. However, there remain further concepts underlying the method. Mathematical induction also presupposes the concept of a *defined property of natural numbers*, for induction ranges over those numbers which have a fixed property. Defined properties of natural numbers arise from algebraic identities, but they also depend on the concept of a *function*, as does the concept of an algebraic identity. In most school contexts a function is represented by an explicit expression in a single variable taking on various values for different values of the variable. Analogously, a property of natural numbers is an expression (a sentence or identity in fact) in a single variable which takes on different truth values for different values of the variable. Out of the concept of a function arises a particular type of function with direct links to mathematical induction, the *inductively defined function*. This concept both aids the development of mathematical induction, and is itself developed by the link. The notion of an inductively defined function interrelates with another concept which is a direct contributor to mathematical induction, namely the concept of *recurrence*. This concept lies at the heart of mathematical induction. It is the key feature of the induction step and of the successive generation of the natural numbers upon which this step depends. The concept of recurrence can be built on the notions of *iteration and flow diagram*, which also aid the development of the concept of inductively defined function. Finally, the concepts of flow diagram, iteration and inductively defined function all arise from the *ordering of the natural numbers* which is one of the major contributors to mathematical induction.

The analysis of induction into a network of interrelated concepts is displayed as a network diagram in Figure 2.

The conceptual analysis of induction brings to light conceptual areas important to the understanding of induction which receive no attention in a purely behavioural analysis. The two major conceptual clusters thus revealed are:

(i) Defined properties of natural numbers and functions.

(ii) Recurrence and the ordering of the natural numbers.

The first of these areas is crucial, since proofs by mathematical induction requires recognition, understanding and the use of properties of natural numbers. The second of these areas contributes to the justificational foundation of the method. The justification for the method of MI, and the mechanism by which it works, are to be found in the well ordering of the natural numbers and their construction by recurring succession.

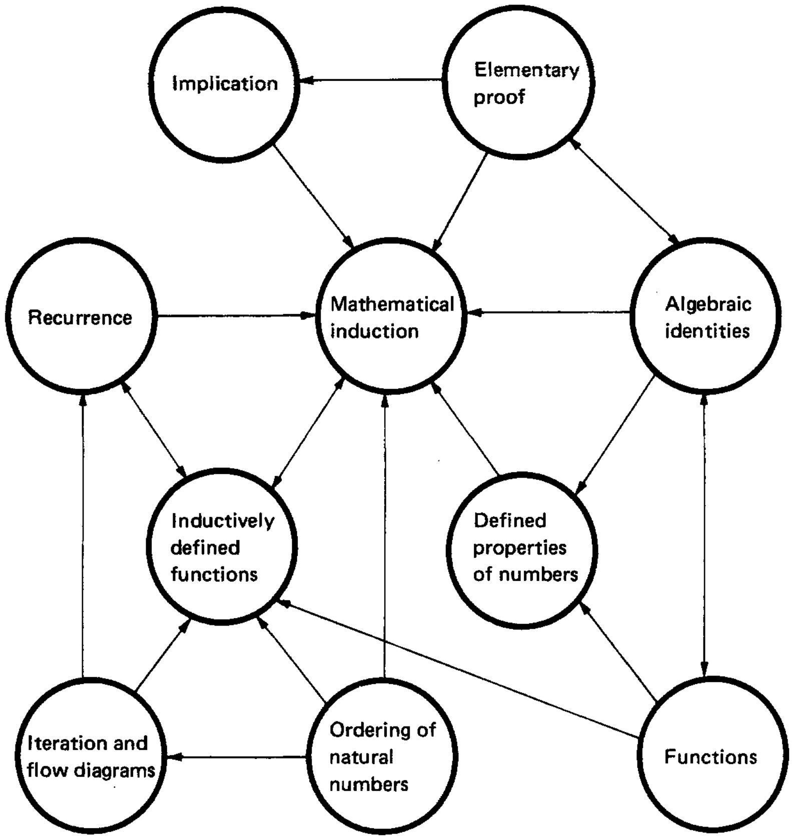


Fig. 2. Network diagram of a conceptual analysis of mathematical induction

The conceptual analysis of mathematical induction provides a theoretical foundation on which the teaching of the method can be erected. However, in building a teaching sequence attention must also be directed to the specific misconceptions and conceptual difficulties which arise in the minds of students learning the method. This constitutes the subject matter of the following section.

## 5. MISCONCEPTIONS IN THE LEARNING OF INDUCTION

Many of the problems that students encounter whilst learning the method of mathematical induction are due to specific misconceptions or conceptual difficulties in the minds of the students. In this section six such sources of error and difficulty are considered.[1]

(i) There is an unfortunate ambiguity in the word 'induction'. On the one hand, the inductive method is a heuristic method for arriving at a conjectured generality describing a finite sequence of examples. On the other hand, mathematical induction is a rigorous form of deductive proof. Moreover, the fruits of the inductive method are often vouchsafed by the method of mathematical induction. That is, these two distinct methods may often concern different aspects of the same example. This ambiguity can cause a great deal of confusion in the minds of students who are not informed of, or fail to grasp the distinction between, these two types of induction. Shaw (1978) illustrates what this confusion can be like. To avoid this difficulty it is recommended that the difference between the heuristic inductive method and the proof method in mathematical induction be carefully explained, and that the former method be referred to as the method of generalization or some name other than induction.

(ii) One of the most common misconceptions among students is that mathematical induction is "the method in which you assume what you have to prove, and then prove it" [Baxandall (1978, p. 85)] and that "it has a suspicious likeness to assuming what you have to prove!" [Mathematical Association (1957, p. 13)]. This is a very reasonable objection. When using the method in free variable form we assume the inductive hypothesis $P(n)$, go through a complicated procedure and end up having proved $P(n)$. As the student suspects, in other elementary deductive proofs such a procedure is illegitimate. However, raising this objection demonstrates that the student has no understanding of the structure of an inductive proof and, in particular, does not understand implication or the proof of an implication statement through the assumption of its antecedent. Two precautions can be taken to try to avoid this misconception. First, it is essential that the student should be taught both the meaning and the methods of proof of implication statements. Second, as many authorities including the Mathematical Association (1957) suggest, the apparent circularity of the Principle of Mathematical Induction is reduced by expressing it in a two-variable form. For example:

If $P(1)$, and if all for all $k$ $P(k)$ implies $P(k + 1)$, then for all $n$ $P(n)$.

In this form the variable $k$ occurring in the inductive hypothesis is localised to the induction step. This makes it easier for the student to understand that the inductive hypothesis is only assumed for the induction step and that the assumption is discharged during this step. A further advantage in the two-variable form is that "one is able to make sensible statements such as 'if the proposition is true when $n = k$, then it is true when $n = k + 1$'" [(Wheeler (1981, p. 274)].

(iii) The previous misconception considered, that of the apparent circularity of the PMI, is a particular difficulty arising from the logical form of the principle. Other conceptual difficulties can also be ascribed to the logical form of the principle, for example, its use of quantifiers, or at least of universally quantified variables. An author who feels strongly on this point writes: "I am satisfied that the major source of confusion concerning induction is that students don't understand the role of the quantifier . . ." [Brumfiel (1974, p. 617)]. Although rather forcefully put, this claim undoubtedly has some substance behind it. The manipulation of free variables in deductions, the use of quantifiers and the relationship between these two areas are matters both subtle and abstract. It is all too easy to assume that the skills and understanding involved will be acquired 'by osmosis'. Such skills need to be explicitly taught and practised.

The combination of universal quantification and implication, and indeed multiple occurrences of both, in the statement of the PMI create a further difficulty: the sheer logical complexity of its statement. This observation is not new. It was noted over 25 years ago that the form of the principle ". . . is logically complicated and difficult to grasp . . ." [Mathematical Association (1957, p. 13)]. Perhaps one should not be surprised if students who have not been given practice in interpreting logically complex statements find the form of the PMI difficult to comprehend. The remedies to the above difficulties are evident: explicit teaching of the use of free and quantified variable statements and practice with sentences of this type and more complex types.

(iv) A misconception that sometimes occurs is the view that one of the components of an inductive proof is not really essential. This is most common in the case of the basis of the induction. "Getting the induction started, i.e. verifying $P_1$ (in most instances) – or in general $P_{n_0}$ – is often treated as a formality. That it is essential is best shown by example . . ." [Baxandall (1978, p. 85)]. As suggested, examples of fallacious induction arguments are perhaps the best way of dispelling this misconception. There are many examples which can be almost proved except for the basis, e.g., $2n + 1$ is even. This remedial tactic of criticising faulty inductive arguments has a wider application than merely to show that all the components of an induction are essential. Setting as an exercise the task of finding the errors in arguments is analogous to the provision of non-examples to a given concept. Such a move serves to sharpen understanding by focusing attention on the boundaries of what is admissible.

(v) A further misconception concerning mathematical induction arises from its exclusive use in summing finite series. It needs to be stressed "that it is not merely a device for proving very special results about finite series." [Mathematical Association (1956, p. 10).] One author finds this misconception so prevalent that "to most students arriving at university, induction means: (A) take an equation involving $n$ and add something to both sides so as to produce a similar equation with $n + 1$ in place of $n$" [Woodall (1981, p. 100)]. The remedy for this problem is obvious. Students should apply the method of induction to a much wider range of problems than those involving only the sums of finite series. An indication of the range of possible problems is given in Wheeler (1981), although he neglects problems drawn from geometry. For a monograph devoted entirely to geometrical applications of induction see Golovina and Yaglom (1963). A further range of exercises in induction may be found by consulting the appropriate sections of Avital and Hansen (1976), Douglas (1970), Sominskii (1961) and last, but not least, Polya (1954).

(vi) The final conceptual difficulty to be considered is not a misconception as such, but a lack of understanding. Many students encountering the method of proof by induction wonder why this rather complex and seemingly arbitrary principle is adopted. Unlike many axioms and principles, mathematical induction is neither self evident nor a generalisation of previous more elementary experience. Thus the introduction of the principle raises the questions; what is the basis of the principle? how can it be justified? Any answers to these questions must refer to the well ordering of the natural numbers. A typical explanation runs as follows.

The natural numbers have the unique property that they can all be generated from a single initial number (one) by the iterated formation of successors. Thus the set of all natural numbers forms a (well) ordered sequence. To show that all natural numbers have a certain property this ordering can be exploited. For if the initial number (one) has the property and if it is passed along the ordered sequence from any natural number to its successor, then the property will hold for all natural numbers, since they all occur in the sequence. To justify this inference the Principle of Mathematical Induction is adopted as a basic principle or axiom of mathematics. The relationship between the method of induction and the ordering of the natural numbers is well illustrated by means of several analogies. These analogies include:

(i) The ascent of a ladder, step by step.
(ii) The transmission of a message down a line of soldiers.
(iii) The departure of a railway train, car by car.
(iv) The entry of a princess into all the locked rooms of a palace, given that she has the key to the first room and that each room contains the key to the next room.
(v) The knocking down of a long line of dominos.

This last, and perhaps best known, example contains the following analogical pairs.

| *Features of example* | *Corresponding feature of induction* |
|---|---|
| A domino | A natural number |
| Linear arrangement of dominos | Ordering of natural numbers |
| Falling down of domino | Property of number |
| 'Knock on' effect | Induction step |
| Knocking over of first domino | Basis of the induction |

The analogy can be strikingly illustrated by means of a picture, as in Figure 3, or better still by a demonstration.

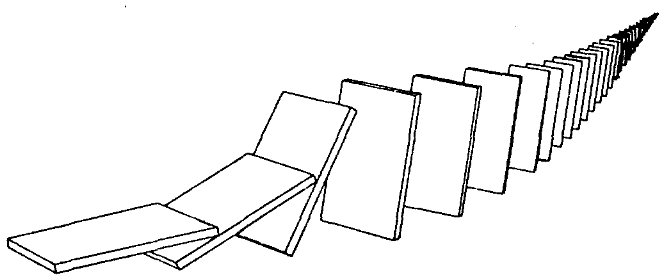

Fig. 3. The domino analogy for mathematical induction

## 6. ANALYSING TEXTBOOK TREATMENTS OF INDUCTION

The previous three sections contain pedagogical discussions of mathematical induction from differing viewpoints. In particular, the last section contains explicit suggestions for the teaching of mathematical induction with a view to remedying or anticipating conceptual difficulties. It is intended that these discussions should provide a foundation on which instructional sequences for teaching induction can be based. In addition to informing the development of curriculum materials these discussions can be used as a basis for the analysis of, and ultimately the evaluation of, existing materials. In particular, to provide a set of criteria for the analysis of written materials, especially text books. This is the subject matter of the present section, as well as the application of these criteria to a selection of published texts. The subsequent evaluation of the texts is left largely to the reader.

### *Criteria for the Analysis of Texts*

*Number of pages.* The number of pages devoted to induction. This is a crude indicator, since a ten-page treatment may be more extensive than one of 20 pages, depending on page size, print density and other factors. Nevertheless, it gives some indication of the extent of the treatment. Included in the pages counted are initial discussions of generalisation and the need for induction, examples of proofs, and student exercises on induction. Subsequent revision exercises in induction are included, provided they are grouped together.

*Number of examples*. The number of worked examples of proofs by the method of mathematical induction.

*Number of exercises*. The number of exercises provided for the student is a useful (though crude) indicator of the extent of the treatment.

*On Series*. The number of exercises on the summation of finite series.

*On algebra*. The number of student exercises which are basically algebraic. Applications of induction to calculus are counted under this heading.

*On number theory*. The number of exercises on number theory, most commonly of divisibility results.

*On geometry*. The number of exercises on geometry. These either include diagrams or, more commonly, refer to geometric figures.

The remaining criteria relate to qualitative features and therefore are not assigned a numerical value.

*Explicit form*. The principle is explicitly formulated, as opposed to simply being used in an example, and named.

*Two-variable form*. The form of induction contains two variables, one confined to the induction step and the second occurring in the conclusion.

*Faulty proof shown.* One or more examples of deliberately faulty proofs by induction are presented, either as part of the exposition or as a 'debugging' exercise for the student.

The next two criteria concern the explanatory framework used in the presentation of induction.

*Analogy given*. An analogy, like the domino analogy, is used to give an intuitive understanding of the method.

*Explanation*. An explanation for the working of the method is presented, typically in terms of the ordering of the natural numbers. The minimum that is acceptable as an explanation is that the basis of an induction establishes P(1), and by repeated use of the induction step, P(2), P(3), P(4), . . . are successively established.

The final subset of proposed criteria concern the treatment of the conceptual background erected prior to, or concurrent with, the introduction of the method.

*Proof concept*. There is a general discussion of the nature and purpose of of proof based on elementary examples.

*Implication*. There is a discussion of the meaning of this connective, and in particular of the ways in which an implication statement can be proved.

*Induction vs MI*. A distinction is made between the heuristic method of induction and the deductive method of proof by mathematical induction. In particular, the use of the latter method as a possible way to sanction the fruits of the former is indicated.

These, then, constitute the proposed criteria for the analysis of text book treatments of induction. Further criteria have been considered and rejected. Some possible criteria, for example the explicit treatment of recurrence within the background conceptual framework, have been rejected as they relate more to the syllabus underlying the text than to the treatment of induction. Other possible criteria, including the distinctions between formal and verbal explicit statements of the principle, and between explanations and justifications of the method, have been rejected as too vague and subjective.

The proposed criteria for analysing treatments of induction are applied to a selection of 17 texts, and the resulting information is displayed in Table I. The books chosen are mainly British sixth form (high school) texts. Among the exceptions to this are the three oldest books, all published prior to the Second World War, which are probably university texts.

Table I reveals, first of all, the results of applying the analytic criteria to each of the texts considered. Thus the table permits the evaluation of each text according to its fulfilment of the criteria. Beyond this, the table permits a direct comparison of different texts according to the criteria used. Finally, the table can be used as a basis for speculation, for an examination of the table reveals some interesting tendencies. In the sample of texts surveyed, there is a loose correlation between the thoroughness of the treatment of induction and the modernity of the book. From the mid-nineteen-sixties onwards the

TABLE I

Analysis of some textbook treatments of induction

| Criterion | Text | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | a | b | c | d | e | f | g | h | i | j | k | l | m | n | o | p | q |
| No. of pages | 2 | 2 | 1 | | | 2 | 2 | 7 | 6 | 5 | 6 | 7 | 9 | 10 | 4 | 7 | 6 |
| No. of examples | 2 | 1 | 1 | 1 | 1 | 1 | 2 | 6 | 4 | 2 | 1 | 5 | 4 | 5 | 4 | 5 | 1 |
| No. of exercises | 14 | 20 | 12 | 5 | | 16 | 16 | 23 | 59 | 18 | 9 | 36 | 19 | 43 | 10 | 39 | 27 |
| On series | 8 | 15 | 8 | 5 | | 14 | 16 | 7 | 20 | 5 | 9 | 14 | 15 | 16 | 6 | 17 | 26 |
| On algebra | 2 | 3 | 1 | | | 2 | | 6 | 21 | 7 | | 9 | 3 | 13 | | 11 | |
| On no. theory | 4 | 2 | 3 | | | | | 7 | 18 | 4 | | 10 | 1 | 8 | 2 | 9 | 1 |
| On geometry | | | | | | | | 3 | | 2 | | 3 | | 6 | 2 | 2 | |
| Explicit form | | | | | | | X | X | | X | X | X | X | X | X | X | X |
| Two-variable form | | | | | | | X | | X | X | X | X | X | X | X | X | X |
| Faulty proof shown | | | | | | | | X | | X | X | X | | X | | X | X |
| Analogy given | | | | | | | | | | | | | | X | | | X |
| Explanation | | | | | | | X | X | | X | X | X | X | X | X | X | |
| Proof concept | | | | | | | | | | E | | X | X | E | | X | |
| Implication | | | | | | | | | | E | | X | X | E | | X | |
| Induction vs MI | | | X | | | | | X | | X | X | X | | X | X | X | X |

Key:
[a] Baker and Bourne (1904), [b] Durrell (1932), [c] Brewster (1939), [d] Porter (1951), [e] Tranter (1953), [f] Backhouse and Houldsworth (1957), [g] Dakin and Porter (1964), [h] Snell and Morgen (1965), [i] Clarke (1967), [j] School Mathematics Project (1967), [k] Brand *et al.* (1969), [l] Parsonson (1970), [m] Hunter and Monk (1971), [n] School Mathematics Project (1973), [o] Turner (1976), [p] Heard and Martin (1978), [q] Sherlock, Roebuck, Heneage and Beck (1979).
X = Treated.
E = Treated earlier in the SMP series.

books devote more pages to induction and often use more examples and give more exercises. The principle is explicitly stated, there is a greater variety in the types of exercises and more consideration is given to the mode of presentation of the method, to the explanatory framework used and to the treatment of the conceptual background, according to the criteria employed. To some extent, this increasing thoroughness of treatment continued to increase with the transition to the nineteen seventies.

Further speculation is possible if the sample of books is taken as representative of comparable books published during the same period. Given this assumption it seems that the treatment of induction over the passage of time becomes increasingly explicit, and is accompanied by an increasing awareness of the underlying skills, concepts and explanatory background. It may be hoped that this results in the method becoming increasingly accessible to a broader range of students.

There is an interesting analogy between this speculated tendency and the emergence of the Principle of Mathematical Induction during the development of mathematics. The speculated tendency consists of increasingly explicit presentations of the method of mathematical induction reflecting an increasing awareness of the underlying pedagogical issues over the passage of time. This parallels in form, but follows in time, a similar increase in the explicitness of statements of the Principle of Mathematical Induction itself, from the time of Euclid to that of Peano, as documented in Ernest (1982).

Finally, it should be remarked that the proposed criteria for the analysis and evaluation of textbook treatments of mathematical induction are entirely *a priori*. Needless to say, more reliable grounds for preferring one text's approach to another can be found by empirically testing the approaches with students. However, such a procedure is expensive and time consuming and should be preceded by an analysis similar to that above. For these reasons no apology is offered for a wholly theoretical study.

This concludes the pedagogical discussion of mathematical induction. It is hoped it will help to inform the teaching of the topic or, at the very least, stimulate further debate.

## NOTE

[1] I am grateful to an anonymous reviewer for suggesting a seventh source of difficulty, namely "the conceptual/technical difficulty of handling the vital step $P(n) \rightarrow P(n+1)$. Most encounters of students with things like $P(n+1)$ have been straight substitutions – substitutions of $n+1$ for some variable in a known expression (function). But in the induction proof the student has to handle almost the inverse of this. For example he takes $P(n)$ and *adds* something to it, then has to arrange the new expression to show that it is in fact $P(n+1)$. This requires getting it into the form $P(x)$ while simultaneously 'thinking' of $n+1$ and not $x$ as the variable. This is *really* hard, in many cases, because one is not using the algebra to simplify but to force a correspondence to a certain model. Where else are students required to do this? No wonder they find it baffling."

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