G. PAPY

A FIRST INTRODUCTION TO THE NOTION OF TOPOLOGICAL SPACE

1. The linearization of theories, one of the most characteristic traits of the attractive face of the mathematics of today, appears best when the fundamental notions of differential and integral calculus are put in place – along with the rest of the calculus. Infinitesimal analysis reigns supreme today in nearly all the fields of applied mathematics. Linear algebra and infinitesimal calculus (including the first elements of differential and integral calculus) remain the two essential goals of all worthwhile teaching of mathematics at the secondary level.

2. In certain countries the teaching of infinitesimal analysis at the secondary level began more than half a century ago. In others, one is always involved in the discussion of the opportuneness of introducing such a course three centuries after the discovery of differential calculus by Newton and Leibniz.

Certain university professors regard infinitesimal analysis as a privileged pursuit to be jealously guarded against profanation at the secondary level. They imagine that they alone can teach the delicate notions of infinitesimals without ruining them. No doubt they believe, more or less conscientiously, that such notions should remain under the protection of the elite, and that throwing them to the whims of the uncultured would put in peril a whole precious heritage of intellectual and spiritual values. This typical caste reaction illustrates, by contrast, one of the golden rules of progress: all important discoveries are first reserved for a small elite and finish by appearing sufficiently simple and clear to be understood and used by humans who are not visited by any inspired ideas nor invested with special talents and virtuousity.

3. Analysis at the secondary level has not always been an uncontested success in convincing in a definitive manner a bewildered population. Such courses often remind one of cook-book recipes, designed to be applied with blind confidence and with no light on the inner function of this mysterious machine for solving problems. Such teaching contains nothing formative, nothing to provoke any profound reactions, and scarcely leaves any durable traces. The fundamental notion of limit is left smothered in metaphysical ideas too delicate to handle and too nebulous to be put in context with other mathematical notions that have been already clearly established.

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4. Infinitesimal analysis – comprising differential and integral calculus – is one of the most important discoveries in the history of mathematics, of science, and of thought. Its gestation period was difficult, its liberation from foggy metaphysical ideas was a long and painful work. This progress did not consist in the solution of a single, isolated problem. It marked in fact the entrance into a new universe where a multitude of notions and concepts – often isolated for the first time – began converging, meeting, intersecting, and mutually influencing one another in subtle interaction. Mathematical analysis is at the crossroads. The simultaneous introduction of a large number of new notions has created a superposition of difficulties partly responsible for the failures often incurred in the teaching of analysis both at the secondary level and in the universities.

5. Teaching analysis is unthinkable without the explicit notion of function. It is possible to teach elementary geometry without making explicit use of this concept. This is what Euclid did and what traditional teaching continues to do. If it were possible today to teach elementary mathematics bypassing entirely the notion of function, then it would perhaps be justified to ignore it in geometry. Since the notion of function is inevitable, there is no reason to miss the eminent services it can render in geometry. All plane transformations – translations, central and axial reflections, homotheties, rotations, similitudes – are functions. The groups of geometry are groups of functions. The group law is composition, which classical analysis hides in the theory – so badly named – of functions of functions.

6. Ten or fifteen years ago, during a mathematics teachers convention, two excellent senior high school teachers crossed swords, with great courtesy, on the optimum age for the introduction of the notion of function, of real valued functions of a real variable, as a matter of fact.

– I tried in vain with pupils of sixteen. In spite of all precautions and patience, the notion does not enter. The brain is not developed enough. But at seventeen, no problem; the time is ready. What was impossible to inculcate – no matter how hard the effort – at sixteen, penetrates without difficulty at seventeen.
– I wonder, dear colleague, if it is possible to be as positive as you. Permit me to tell you that I also tried to introduce functions at sixteen. Of course, I don't claim that all the pupils completely assimilated the notion; but they didn't all die of it, and the answers to the exercises showed that they correctly manipulated the proposed functions....
– Allow me, dear colleague, to be somewhat skeptical about the import of the conclusion you seem to draw.... But perhaps our apparent opposition

comes from a misunderstanding about the meaning of the terms we are using. We must not confuse the assimilation of the notion of function with some exercises of manipulation which can be performed without the notion itself being assimilated. Let us not forget that for the pupils the notion of function is difficult, subtle, and delicate. In order to avoid an empty parrot-like verbalism, one must not skip over certain stages.

– I believe we almost agree. Exercises with some simple numerical functions may well serve to introduce progressively the notion at sixteen. But, in any case, before that age nothing can be done along that line for lack of necessary maturity.

At that moment, from the rear of the room came a loud question:

– From what age do you believe children understand the phrase "...has as mother..."?

The question sounded very strange in a discussion on the teaching of mathematics; it provoked some grumbling, from whence finally came out, badly articulated, the strange answer:

– From always!

A moving, naive, and magnificent answer from expert teachers of the 12-to-18 group, for whom – very clearly – the student begins at 12 (and, without doubt, ends at 18)!

Modern reform of the teaching of mathematics has abolished all those groupings.

Today, *Frédérique* teaches the notion of function to 8-year-olds who have been taught modern mathematic since the age of six. To do so she uses multicolored graphs and appropriate pedagogical techniques. [EE], [MM1], [EG], [EM1], [EM2], [EM3]. This initiation to modern mathematics at the age of six [EG], [EM1], seven [EM2], eight [EM3], twelve [MM1], fifteen [EE]... should interest any teacher who has charge of introducing the notions of modern mathematics at whatever age his pupils might be. The experience of modern reform repeatedly confirms that we are all equal before the unknown. The situations best able to initiate a new notion are almost independent of the age of the learner

7. At the secondary level

RELATION = set of couples*

FUNCTION = relation not comprising two different couples with same first component.

* Or ordered pairs.

The notion of relation – natural and without restriction – is simpler than that of function.

Functions are "relations with restriction".

The frame of relations imposes itself if one wishes to present functions in a context large enough to cover counterexamples which are close enough to the notion of function and which, at the secondary level, are nothing but "non-functional relations". Parallels, perpendiculars, divisibility, modular congruence, etc., attest to the unavoidable character of non-functional relations at the secondary level.

The set-theoretic presentation puts in evidence the conceptual link between functions and relations; the utilization of graphs – drawn or imagined – brings it out in a startling manner. A deep comprehension of that link facilitates intelligibility and helps the mastering of the concepts.

The proposed point of view does not represent an extremist set-theoretic tendency. It does not define the couple (x, y) as the set $\{\{x\}, \{x, y\}\}$ (after Wiener-Kuratowski). All one ought to know about couples at the elementary level is condensed in the *definition*

$$(x, y) = (a, b) \Leftrightarrow (x = a \wedge y = b).$$

This moderate set-theoretic-point of view, common language of the great majority of professional mathematicians all over the world, has great pedagogical virtues. It brings to the foreground, and presents as sets – that is, as objects of the theory – some notions which were formerly kept in the background. It simplifies the theory by suppressing the coexistence of notions which are separated only by subtle distinctions and which are often mutually definable. Thus, any relation is its own extension (unhappily named by Bourbaki, its graph).

That presentation – happily – restricts the abstract notion of relation. Thus membership is not a set. ($\{(x, y) \mid x \in y\}$ is a strict class and not a set.)

Later, much later, the children who will continue their mathematical studies could consider the membership from set to set as the class $E = \{(x, y) \mid x \in y\}$ – after Gödel.

Classes of couples – or *correspondences* – generalize relations without sacrificing anything of significance.

The membership correspondence E does not cover the membership from set to strict class, which leads to the notion of "binary predicate". At that level it is true that the "extension of the binary predicate", viz., "the class of all couples satisfying that predicate" does not always define the predicate itself. This distinction between "binary predicate" and "extension of that predicate" may be entirely avoided at the secondary level.

It suffices to think of the graphic representations of functions to realize that the notion of "relations = sets of couples" is unavoidable at the secondary level. Therefore it is that notion of relation, and that one only, that one must adopt.

A parallel presentation of other notions of relation is useless and harmful, as long as one cannot settle the questions that would meaningfully suggest and motivate such an effort.

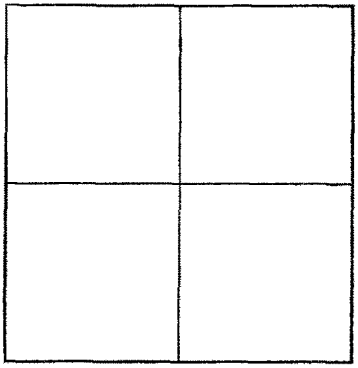
All by-products of relations will be defined as attributes of "relations = sets of couples". For more details see [F3].

8. It is with respect to problems of analysis that set theory started with Cantor. It is via courses in differential and integral calculus that set theory began to penetrate into the universities. It seems that the study of topological properties of sets of points in the plane cannot be avoided in analysis. The very nature of the subject imposes the adoption of a set-theoretic point of view. Since analysis is one of the main goals of any worthwhile teaching of mathematics at the elementary level, it is necessary to prepare the child to adopt that point of view; above all, one must avoid any teaching which would continue to reject it.

9. A strange small school hidden in an old castle surrounded by woods gathered together the children that the occupying armies of that country at war intended to exclude from the human community. For obvious reasons, that underground institution did not have at its disposal specialized personnel for the different subjects nor for the various ages of the children. One of its mathematics teachers, freshly graduated from the university, found himself – without knowing it – in an exceptional pedagogical situation. Sometimes, in the same week, he had to teach students ranging from four to nineteen years of age; a lesson to children of eight or nine might be followed by a course to a student of university age.

The situation was exceptional indeed, for it is not at all usual that the same person teaches simultaneously at the primary level and at the university. Generally, the primary teacher does not have the opportunity to test at the university, ten years later, the good or bad results of his teaching. Our young teacher was in that situation, and for him time was shrunk, as it were, because he saw one hour later the results of his teaching in grade 4 upon students of age 19, because he taught the 9-year-olds in the same traditional way he himself was taught, and in the way his 19-year-old students had also been taught.

One day that young teacher was speaking about the square to a class of 9-year-olds. He asked the children to draw a square and put to them the classical question, "Partition that square into four equal squares". In that story the choice of the word "partition" was perhaps to have important consequences. He was evidently expecting the answer:



Expecting children to answer in accordance with the teacher's wishes gives rise to many surprises. To impose the teacher's point of view may be a grave pedagogical error.

While most of the children quickly drew the desired answer, one of the best pupils did not draw anything at all, and seemed lost in deep meditation. "How is it possible that Isaac has not yet got the right answer?", the teacher asked himself. He could not imagine that it was because of tiredness, lack of attention, or because he hadn't understood the question. The teacher did not jump to conclusions. He wanted to learn why the child did not answer and refused to decide for himself. He accepted the fact that the children could teach him how to teach them to learn.

– So, Isaac, you are not the first to give us the right answer today!

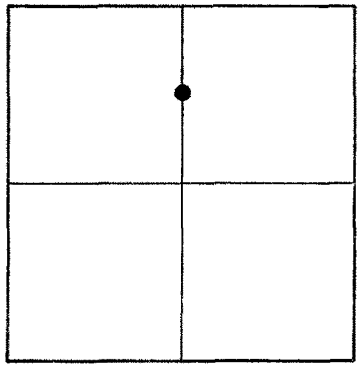
The questioned child gave, from under his beautiful black eyelashes, one of those sidelong looks whose secret is shared by beautiful women and by children. When his glance had rested on the copybooks of his classmates he imperceptibly shrugged his shoulders and said:

– You know very well that they don't have the right answer!

This was the second surprise for the teacher on that memorable day.... But his reaction was good. Without raising his voice, without condemning answer or child, he simply asked:

– Will you come to the blackboard and explain to us why?

After having drawn the "wrong answer", Isaac fattened out one point on a median

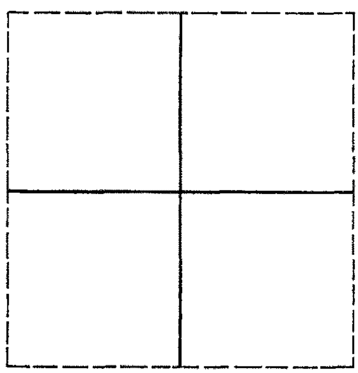


and added

– If in the partition this point goes with the small square on the left, then this small square has one point more and that is no longer a partition into equal squares. If it goes with the small square to the right, etc.

The young teacher was taken aback by the third surprise. Realizing that he was in a very bad position and badly prepared for the answer, he panicked and, in a very loud voice – as loud as his bad conscience – he shouted

– THE POINTS ON THOSE LINES



HAVE NO IMPORTANCE!

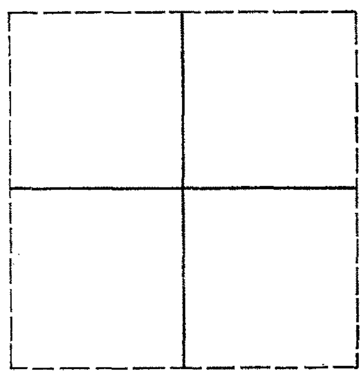
Confronted with that dictatorial, arbitrary assertion, Isaac answered teasingly:

– IF THOSE POINTS HAVE NO IMPORTANCE, then OK.

A quick, logical look at that answer reveals that Isaac agrees without approving. A pedagogically bad situation!

Perhaps in order to ease himself out of it, the teacher had tried what seemed to be a very good means to achieve a very good end. Here, alas, the goal was bad. He tried to give a motivation, an intuitive support, and an apparently plausible justification of his dogmatic assertion, "Those points have no importance".

"When one cuts with scissors, one doesn't ask oneself on what side the points along the cut are going". He took a sheet of paper, folded it in two, and then in two again, unfolded it, showed the cross,



cut in into two pieces, and then in four pieces, emphasizing in a very peremptory manner that "the points on those lines have no importance". Thus, he forced the children to adopt the tailor's point of view, which unfortunately does not fit the mathematics created by Newton and Leibniz.

Some minutes later a new lesson would bring our teacher out of his dream and illusions. He would realize that he had just committed a pedagogical crime.

In the next class our hero had to teach a more or less modern course in analysis. With good sense, he considered as known some non-infinitesimal notions and took them as a basis of departure. He did not intend to redefine the square and the circle but to limit himself to refining those concepts, while making more precise the notions of open square, closed square, and perhaps some intermediate cases neither open nor closed. An enthusiastic lover of exact references and of technical labels, the teacher clearly announced:

– DEFINITION 7.12.32:

One calls *open square* the set of points interior to the square with the exclusion of the boundary points.

– DEFINITION 7.12.33:

One calls *closed square* the set of all the points of the square including those of the boundary.

Marvellous! Make judicious use of known notions to give precise definitions by means of a language which lacks precision. Marvellous, *in abstracto*! In that class there is only one such student, but he exists, and manifests himself with humor, and causes a new surprise in this memorable day in the life of our young teacher by stating:

- PROPOSITION 7.12.34: An open square is closed.

- PROPOSITION 7.12.35: Any closed square is open.

- Certainly my student has not heard well, or he was not attentive. Let us repeat the definition.
  Definition 7.12.32.... Definition 7.12.33....

That pedagogy of repetition is as common as it is inefficient. One can ask oneself by what miracle a person who could not understand a definition, a proposition, or a proof, could suddenly perceive its intelligibility thirty seconds later during a word-for-word repetition of the previous formulation. It is true that the procedure appears foolproof the second time around! In order to get peace, the questioned pupil shortens his torture by stating he has now understood. And there is nothing more like a pupil who has understood than one who has not, but gives the impression that he has. Full of mischief, our "unique" student caused surprise number 5 by repeating again, word-for-word, ...

- Proposition 7.12.34.... Proposition 7.12.35....

This time the teacher reacted properly:

- Very well, you enunciate two propositions... Prove them!

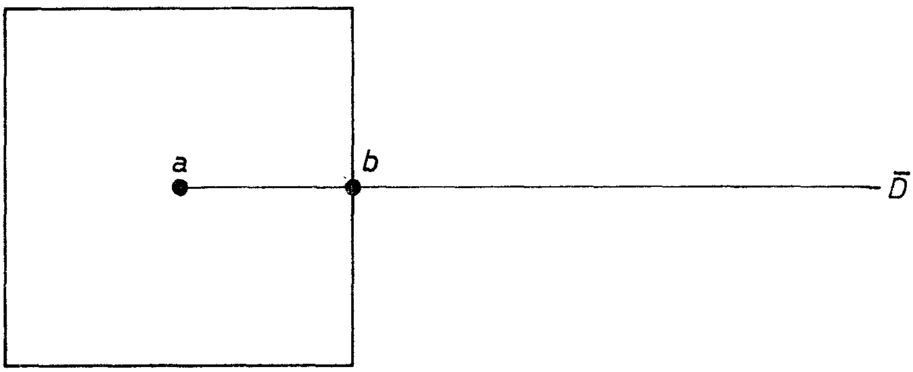
Samuel reflected for some moments, organized his plan of attack, and before the final thrust decided to establish clearly his ground:

- You agree that every closed square comes from an open one and that every open square comes from a closed one?
- Of course, answered the teacher who was beginning to be interested in that attempt at a proof of a proposition he knew to be false. Every open square can be obtained by removing the boundary of a closed one.
- OK, then. Here is a closed square. But as the folks on the boundary are removed, *those that were just behind them become the first* and constitute the new boundary. The open square one gets is thus closed, and the case is won.

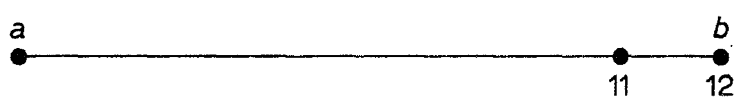
Sixth surprise for the young teacher!

This student was very intelligent. He evidently still is, because he has become a university professor, in chemistry, as a matter of fact. At that time he was extremely brilliant: he solved in a very astute way the riddles of plane geometry. Yet it became suddenly evident that his microscopic vision of the Euclidean plane was unbelievably poor and strewn with errors. The teacher had the good taste not to develop a frontal attack on the proof. It is true that its logical organization was satisfactory, but the fallacy rested upon the erroneous sentence underlined in the above answer.

The young teacher drew a square on the blackboard



- Distance *ab*?
- 12 cm.
- *b* is at 12 cm. from a on the half-line $\bar{D}$.
- Draw a point on the half-line at 11 cm from *a*.
- ...



- Is that the last point strictly before b?
- No, there is one at 11.1.
- Is that the last one?
- No, there is one at 11.2.
- Is that the last one?

11.3, 11.4, ..., 11.9 were produced.

And then 11.91, 11.99, 11.999, ...

And one arrives at the conclusion that between a and b there exists no last point strictly before b. There does not exist a point just behind the point on the boundary! Samuel was crestfallen. He had nothing to say, but he was NOT CONVINCED!

The seventh salutary surprise for the teacher was to learn that

To be convinced by a proof	$\neq$	Not to be able to object to that proof

Nothing is more humiliating than to have to capitulate before a logical argument when one believes one has the truth. Only masochists could prefer punishment by logic to flagellation. Still a bachelor, our young teacher did not know that it is useless – and extremely lacking in gallantry – to try to convince a woman by logical reasoning. One must never beat a woman "not even with a flower" says the proverb. It could add also: "Above all, not with a syllogism".

– Your premises and your reasoning are surely impeccable, because you are so intelligent – said a handsome woman.

But she added immediately,

– It doesn't prevent your conclusion from being stupid!

Samuel was not at all convinced. He considered his teacher a very naughty sophist. He admitted he did not have the necessary perspicacity to object to him, but HE WAS NOT CONVINCED! Happily, he suddenly noticed on the desk the concrete evidence of the crime committed during the previous hour.

Taking between his fingers one of the cut squares, he asked candidly this embarrassing question: (Surprise #8!)

– Is this paper square OPEN or CLOSED?

A very enlightening surprise answer!

There is no answer to that Machiavellian question. Papers and cards constitute a material too coarse to illustrate the set-theoretic point of view. The pedagogical error committed during the first lesson appeared now in a glaring manner.

Little Isaac had spontaneously adopted the set-theoretic point of view. The treasures of pedagogy used by our young colleague had led him into the opposite direction and had prevented him from taking the same point of view.

RESULT: 10 years later – and for our young teacher, one hour later – it was necessary to use new treasures of pedagogy to counteract in the pupils the effects of the slow poison to which they had been exposed.

Naturally, it was necessary to invent a new intuitive support finer than the one of the postcards. The green-red traffic light convention plays that role.

With such means it is possible to introduce topological notions in a natural way, using the children's creativity.

I told elsewhere [F3] the story of the discovery of open sets and of continuous functions.

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WORKS OF FRÉDÉRIQUE AND PAPY

House of Mathematics for Children

Books for children

[GG] Frédérique and Papy, *Graphs Games*, Thomas y Crowell, New York.

Books for teachers

[EG] Frédérique and Papy, *L'Enfant et les Graphes*, Didier, Brussels.
English translation, Algonquin, Montreal.
Futch translation (in preparation), Didier, Brussels.
Meulenhoff, Amsterdam.
Frédérique, *Les Enfants et la Mathématique*, Didier, Brussels.
[EM1] Vol. 1 (6 years-grade 1)
[EM2] Vol. 2 (7 years-grade 2), being printed
[EM3] Vol. 3 (8 years-grade 3), in preparation
English translation, Algonquin, Montréal.
[M] Papy, *Minicomputer*, Ivac, Brussels.
Dutch translation, Ivac, Brussels.
English translation (in preparation), Ivac, Brussels.

House of Mathematics for Teen-agers

Books for teen-agers

Papy, *Mathématique Moderne*, Didier, Brussels, Vol. 1: *Ensembles–Relations, Les débuts de la géométrie. Le groupe des entiers rationnels.*
English translation, Collier-Macmillan, London.
Spanish translation, Eudeba, Buenos-Aires.
Japanese translation, Nippon Hyoron-Sha, Tokyo.
Dutch translation, Didier, Brussels.
Rumanian translation, Tineretului, Bucuresti.
[MM2] Vol 2: *Nombres réels et vectoriel plan.*
English translation, Collier-Macmillan, London.
Dutch translation, Didier, Brussels.
Rumanian translation, Tineretului, Bucuresti.
[MM3] Vol. 3: *Voici Euclide.*
Dutch translation, Didier, Brussels; Meulenhoff, Amsterdam.
[MM5] Vol. 5: *Arithmétique.*
[MM6] Vol. 6: *Géométrie plane.*
[EE] Papy, *Erste Elemente der Modernen Mathematik*, O. Salle, Frankfurt.
[EMG] Papy, *Elemente der Modernen Geometrie*, 1. Heft., Klett, Stuttgart.

Books for teachers

[G] Papy, *Groupes*, Presses Universitaires de Bruxelles.

English translation, Macmillan, London.
Italian translation, Feltrinelli, Milano.
Dutch translation, Plantyn, Antwerpen.

[F1] Papy, *Géométrie affine et nombres réels*, P.U.B.
German translation, Vandenhoeck & Ruprecht, Göttingen.

[F2] Papy, *Initiation aux espaces vectoriels*, P.U.B.
German translation, Vandenhoeck & Ruprecht, Göttingen.
Dutch translation, Plantyn, Antwerpen.

[F3] PAPY, *Le premier enseignement de l'analyse*, P.U.B.
English translation, Algonquin, Montreal.

[G] Papy, *Groupoïdes*, Labor, Bruxelles.
German translation, Vandenhoeck & Ruprecht, Göttingen.