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HIERARCHIES IN MATHEMATICS EDUCATION

ABSTRACT. In mathematics education literature the term 'hierarchy' is used in a number of ways. It is important that the mathematics educator consider the usefulness of the hierarchies presented by various researchers and theorists, in the light of their application to teaching. Current works on mathematical learning hierarchies are illustrated and in particular the work of the mathematics team of the research project 'Concepts in Secondary Mathematics and Science' is examined.

INTRODUCTION

The word 'hierarchy' when applied to how and in what order children learn mathematics, is used in a number of ways. It can be used to describe:

(i) a learning sequence or sequence of understanding, which is essentially in the learner

(ii) a teaching sequence which the teacher uses

(iii) a logic sequence which is in the topic.

These are not necessarily the same but can be considered as interdependent. For successful learning in school, the three aspects must be closely matched or failure is the result. In each case "hierarchy" implies a string of skills/levels/stages/concepts which are ordered from simple to complex. A classification of mathematical content into subordinate grades according to any of the three criteria mentioned, does not necessarily imply that each of these grades is subsumed into higher ones.

HIERARCHIES OF STAGES AND CONCEPTS

Initially let us consider work on stages or concepts rather than skills. The word 'stage' is usually applied when some generalised form of behaviour is being described whereas 'skill' refers to specific identifiable prerequisites for some needed performance.

It is useful at this point to consider Wohlwill's ideas of stages which embody the requirement that one grade subsumes another. For Wohlwill (1973), 'stage' is taken as a construct within a structurally defined system, having the property of unifying a set behaviour. Wohlwill describes it in these terms and further exemplifies his requirements with reference to the works of Shirley (1931) Erikson (1959) and Piaget and Inhelder (1958) as shown in figure one. 'Stage' in the work of all three writers is concerned with developmental

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levels of the child, which of course influence a child's learning or understanding. The stages are not however specific to mathematics and the work of Shirley and Erikson are mentioned solely as a comparison with that of Piaget on whose idea of stages much sequencing of mathematics experiences has been based.

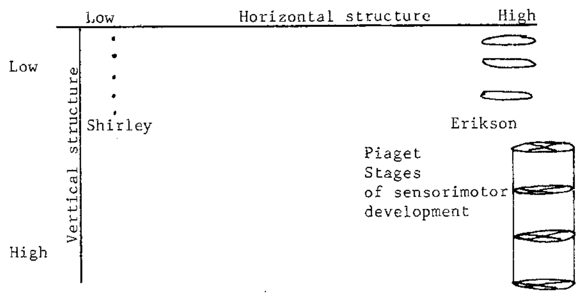

Fig. 1.

Shirley's work with babies provides the description of a sequence of behaviours characteristic of the development of locomotion in infancy. They form a set of highly concrete, specific motor response patterns. Each is a qualitatively different motoric activity which appears in a predictable order during the course of the development of the infant. Erikson's work on the other hand refers to a constellation of emotions, feeling and dispositions which do not have any direct reference to overt behaviour patterns. Each phase is useful in describing commonly encountered sources and types of emotional conflict in the growing individual's personal and interpersonal life.

As we can see in the work of Shirley, the order of emergence of each of the activities is based on evidence but the activities are in no way generalisable nor do they extend to cover a multitude of behaviours; in the diagram this is shown by low horizontal structure. The work of Erikson on the other hand has wider criteria for each stage, i.e. high horizontal structure, but there is no question of one phase being integrated into another.

Piaget's theory of cognitive development as we see from Wohlwill's diagram goes rather further. His stage theory involves integration of lower levels into higher levels, gradual consolidation, unifying characteristics shown by inter-related behaviours, concepts and skills and equilibrium. It is not a model of an additive nature, stage A need not be obvious within stage B and the integration of the two would constitute a transformation and entail structuring and coordination. Consolidation always involves an aspect of the recently acquired behaviour and an aspect of preparation for the behaviour of the following level. The actions or operations of a given level are not simply juxtaposed in an additive fashion but are organically interconnected and form a total "structure d'ensemble". There should also be a succession of levels of equilibrium at the heart of the evolutionary process. In addition the theory of Piaget is concerned with not only the mental development of the child but also his physical development and is thus closely tied to age.

A theory of stages of cognitive development is also present in the work of the Van Hieles whose investigations in Geometry have led to considerable research in the USSR (from Wirszup 1976) and currently in the United States. The Van Hieles' ideas on levels of development are described in terms of pedagogical needs and are concerned with the teaching needed to match the level of the child rather than the needs of the child in vacuo. At each level there appears in an extrinsic manner what was intrisic on the preceding level and each level has its own language. Thus a teacher who is talking at a level higher than that of the pupil has no meaningful form of communication with him. The Van Hieles see the learning process as discontinuous; the jumps reveal the presence of the levels. Learning stops from time to time and then starts again. In Geometry they suggest that at level one the child judges geometric figures according to their appearance. The child recognises a square, a rhombus etc. but does not see that a square is a rhombus, the figures are distinct. At level two however he sees relationships among components of the figures and between figures. At level three the pupil establishes the properties of figures and can logically order them. It is not until level four that he grasps the significance of deduction in formulating geometric theory and level five is recognised by the child's ability to abstract and deal with geometry without any concrete representations. The Russian research results indicate that children taught in conformity with the Van Hiele theory, that is being given experiences to match each level, were performing after eight years in a manner comparable to their compatriots in the traditional eleven year school.

The theory of Piaget has been subject to validation type research in a number of ways. The age of acquisition of certain stages, the sequence of abilities to solve certain tasks and the matching of Piagetian stages against attainment in certain areas, have all been investigated very often in the hope that if one could find the cognitive level of the child then one could match instruction to it. Many studies e.g. Renner and Paske (1977), Lovell (1961), show that only a minority of adolescents seem capable of, or find it useful to use, formal operational thought. Renner investigated College students, very often using "the use of proportion" as an indicator of the presence of formal operational thought. He also introduces the notion of assigning errors to levels of understanding as indeed has Piaget.

One of Renner's examples (1977) was that the shadows of a building and a post were measured at the same time of day. The building's shadow was

50 metres, the height of the post 3 metres and its shadow 2 metres, the problem being to find the height of the building. The responses were analysed carefully and ordered as follows:

Categories 1 and 2. Naïve answers involving irrelevant or manufactured numbers.

3. A relationship was recognised, resulting in answers such as 51 m.
4. The ratio 3 : 2 was recognised and stated but not used.
5. The ratio was used incorrectly i.e. 2/3 of 50 instead of 3/2 of 50.
6. Proper use of the ratio.
7. Proper use of the ratio in addition to mentioning other problem variables such as "at the same time of day".

Karplus also endeavoured to identify levels of thinking by looking at the errors made, although he kept clear of stating which was formal operational thought. His early version of the Mr. Short, Mr. Tall problem (Karplus and Peterson, 1970) involved measurement with paper clips of two sizes, 'biggies' and 'smallies'. Mr. Short being 4 biggies and Mr. Tall 6 biggies. The children in the study were asked to measure Mr. Short using a smaller paperclip and predict the comparable height of Mr. Tall. The replies were categorised in seven ways:

N	no explanation
I	intuition or guessing
IC	use of data in an illogical or haphazard way, some inaccurate reasoning present
A	use of all the data but the difference applied "the little man was four of his and six of mine so I added 2"
S	multiplication but not by the correct factor (90 per cent of these subjects doubled)
AS	addition and multiplication "I think 2 smallies are as big as 1 biggie, so I added 4 smallies for the 2 extra biggies"
P	setting up and using a ratio.

Karplus' ordering of the responses was based not on the original experiment but on the responses given two years later by the same children to the same problem. He stated levels P and AS were more advanced than the other categories because 28 per cent of those in other categories moved into these two over the two years. Categories I and IC were regarded as the most naive since 65 per cent of subjects in these two categories moved into others over the two years. The study did not provide evidence for the ordering of categories A and S since the same number of children moved from A to S as moved from S to A. It is interesting to note that strategy A, although stated by Piaget as being indicative of late concrete operational thinking is considered by Karplus to be not indicative of a level of thinking since its incidence was very low for some of the populations in the seven countries investigated in a later study (Karplus et al., 1975).

Piaget tended to investigate one problem with certain children and then another problem with a different group of children, although the general descriptions given to the responses may have been made according to the same set of levels, early concrete – late formal. Certain researchers have taken what is stated to be a sequence of acquisition of concepts and with the same children attempted to validate the order suggested. Elim Kofsky (1966) is one such researcher. She took a sequence of classificatory skills with children aged 4 to 9 years. She ranked the tasks by frequency of success and then split them into six groups the order of which correlated significantly with the hypothesised Piagetian order at $p < 0.01$ level. Then she looked at the requirement that the mastery of a particular task showed that all previous tasks had been mastered and used a Loevinger coefficient to quantify the connection (which showed that this hypothesis was not supported). In addition she looked at the success pattern for each child; there should have been twelve patterns if success at harder meant success at each easier (in order) task. Thus arranging the tasks in order of difficulty we should have the picture in figure two. Note that a score of four obtained by the pattern 000110011 would display attainment on harder items without all easier tasks being solved. Only 27 per cent of the sample showed perfect patterns, 63 patterns in all occurred. Thus the integrated nature of the stages was open to doubt.

Score	Easy				Hard							
1	1	0	0	0	0	0	0	0	0	-	-	-
2	1	1	0	0	0	0	0		-	-	-	-
3	1	1	1	0	0	0	-	-	-	-	-	-

Fig. 2.

The same type of experiment was carried out by Wohlwill (1960) who investigated the development of number concepts with young children aged 4 to 7 years. The child had to match numbers in seven tasks which had an hypothesised order of acquisition.

A pass on a set of trials was taken as 5/6 or 10/12; subjects who attained this mark were given three additional tasks but since they almost invariably succeeded on these items the tougher pass mark of 8/9 was abandoned in favour of 5/6. Only 45 of the 72 subjects had perfect scale type responses and when those with everything correct or everything incorrect were deleted, half of the remaining group had perfect responses. Wohlwill concluded that the data did not warrant the acceptance of each test representing a discrete, well determined point on the scale of conceptual development and he was particularly concerned that there was no significant separation between pairs of adjacent tasks. He looked too at the performance within each task to see whether the children tended to get all or nothing correct.

Some recent research by Noelting (1978) is also in the Piagetian mode. He has used one task throughout the experiment and increased its complexity by varying the numbers involved. The task involved mixtures of water and orange juice; the child being required to state the relative taste of the mixture. The subjects were aged 6–16 and came from mathematically advanced classes in the upper middle class socio-economic group in Quebec City. The items were ranked according to difficulty, and a Guttman scalogram analysis used to test for scalability. Noelting then assigned Piagetian levels to sub-groups on the basis of his knowledge of Piagetian theory and the ages of the children tested. This leads one to question whether one needs a cross check that the children who are labelled 'early formal' because of their performance on this task should also be given a further task or tasks which have been described by Piaget. Is a task at the formal operational level because (a) it is successfully completed by children capable of/consistently using formal operational thought; (b) it demands abstract thought for completion (the abstraction judged by the experimenter); (c) it is successfully completed mainly by children in late adolescence; (d) it is hard and so completed by few children?

To assign a child to an operational level is difficult if he is presented with different tasks; on one task one may be able to distinguish between the quality of responses but on a number of tasks the child performance is often task-related. If one gives a task which requires formal thought for successful completion then the lower level responses are of necessity incorrect in that they contain erroneous reasoning. To try to assign a child to a cognitive level based entirely on precise errors on particular tasks is bound to be difficult. Very often the error has not appeared in the descriptions given by Piaget and one must interpret its significance in the light of his general theory, but besides this the level of response (or type of error) might be very task specific.

HIERARCHIES OF INSTRUCTIONAL CONTENT

An often used description of hierarchies is that of "a sequence of instructional content", that is, an ordering of some subject matter one might wish to impart. All teachers order the material they present, in some form, if their teaching is to be effective. Whether they change the order when the teaching proves not to be effective is open to question. The order of presentation is usually assumed to follow 'the logic' of what is being taught; that logic is dictated by people who have already mastered the subject and can see the topic as a whole. At this point it is useful to consider a piece of research by Robert Mager (1961) who attempted to compare a 'logical' sequence with a 'student preference' sequence, the topic being electronics and the students adults. Although this is somewhat removed from a hierarchy of mathematics understanding or a sequence of mathematics topics, it points to aspects of learning we should bear in mind when considering order of presentation. The fact that the subjects are adults makes it possible for them to articulate what they see as the next requirement for them to obtain the desired knowledge.

The experiment was "to determine whether a learner-generated sequence would be similar to an instructor-generated sequence, and whether or not there was any commonality among sequences generated by independent learners". The topic was electronics and each of six adults had stated that they wished to learn about it but initially knew nothing of the topic. They were told that they had complete control over the curriculum and that the instructor would try to behave as a responsive mechanism and offer information only in response to questions. The sessions were of about an hour and the number of sessions dependent on the participant (between one and seven sessions).

The first important pointer from the results was that it was very difficult to instruct in the method required by the experiment i.e. when the student asked a specific question which was foreign to what the instructor expected, the instructor had to reorganise his planned discourse. It was also difficult for the instructor to shorten his explanation if the questioner required only a sentence or two and was then satisfied and moved on to another topic. If the teacher finds it difficult to keep up with the material when the learner dictates the sequence might not the opposite be equally as difficult? The second important finding was that each participant entered the experiment with a large amount of background knowledge, although not the specifics of electronics and there was a strong tendency on his part to associate new information with old. Both the information and the new associations could of course be erroneous. Every student began 'his' course with an entirely different topic than did the instructor. Within the first 40 minutes all the participants asked about a vacuum tube whereas in eight different courses normally taught by industry the first topics in an electronics course were magnetism or electron theory. At the start all the subjects tended to want to know *how* something works. The initial interest was very much in the concrete rather than the abstract and in function rather than structure.

If the same type of experiment was carried out in mathematics education would we obtain any clues as to the order in which a child wishes to learn? Does discovery learning match the aspect of concrete and function rather than structure and abstraction? Edith Biggs' ideas on discovery learning in mathematics were very much concerned with presenting a problem so that the child wanted to learn the mathematics needed to solve it. Do we as teachers, however, err as the instructors in the experiment and embark on excessive explanations? It is possible that the sequencing of material by an expert who knows 'the whole' is not necessarily the best way of matching the order of learning natural to a child.

Gagné (1968) has proposed the idea of a learning hierarchy and has further allied this with ideas of sequence of instruction. To Gagné learning hierarchies are characterised as "an ordered set of intellectual skills such that each entity generates a substantial amount of positive transfer to the learning of a not previously acquired higher-order capability". The assumptions made by Gagné include:

(1) Any human task may be analysed into a set of component tasks which are quite distinct from each other in terms of the operations needed.

(2) The presence of these tasks insures positive transfer to a final performance, their absence reduces such transfer to near zero.

(3) The design of instruction requires (a) the identification of the component tasks (b) ensuring each task is fully achieved (c) arranging the total in a sequence which will produce optimal transfer.

The problems one faces with this type of hierarchy are numerous, to start with, how does one construct the hierarchy and how does one know that it *is* a hierarchy? Gagné suggested that to form the tasks one simply asks "what would the individual already have to know how to do in order to learn this new capability simply by being given verbal instructions". He also suggested that the hierarchy can be checked empirically for positive transfer by taking as a hypothesis that an individual will not be able to learn a particular topic if he has failed to achieve any of the subordinate topics that support it. The study by Gagné and Paradise (1961) tested this hypothesis by administering an examination immediately after the instructional programme and considering pass/fail patterns. The index used was

PPT – Proportion Positive Transfer

$$\text{PPT} = \frac{1+2}{1+2+3} \quad \text{where}$$

1: Number of children who correctly answered questions for higher skills and all relevant lower skills

2: Number of children who were incorrect on a higher skill and at least one lower skill

3: Number of children who were correct on a higher skill and incorrect on at least one lower skill.

White and Clark (1974) pointed out that although PPT values of greater than 0.91 were found for all the ratios, the experiment should be tried on different samples, the criterion for success on a skill was open to debate and that high values of PPT could be obtained even if the subordinate skills were independent.

This raises a number of points about learning hierarchies viewed as a sequence of tasks: (1) one can show that a skill is a prerequisite if it is to a far greater degree less complex than the next in the chain e.g. Figure 3; (2) one can obtain a high PPT value even if the skills are not immediately relevant to each other, i.e. the hierarchy is not valid in a content sense, for example a high PPT value could arise from attainment on three skills as shown in Figure 4; (3) the gaps between composite skills seem to be significant; although one needs each of the lower skills, what else is required?

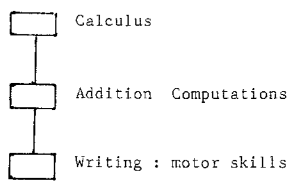

Fig. 3.

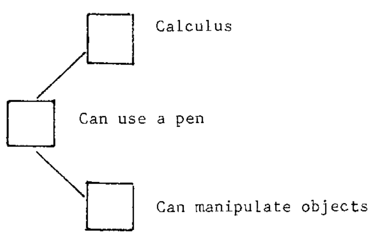

Fig. 4.

Merrill's (1965) work illustrates some of these points. He tested the hypothesis that learning and retention of a hierarchical task are facilicated by mastering each successive component of the hierarchy before continuing in the instructional programme. Merrill chanelled students who failed on a component into a review/correction procedure but his results tended to imply that it was not necessary to master one level before proceeding to the next. Briggs (1968) suggested that Merrill's task analysis might be faulty – he questioned the validity of the hierarchy. Note too that we must also distinguish between what the student knows and what the student can do although we tend to test the former by the latter.

One might consider that instructional programmes written in sequence were always better than ones which had been deliberately scrambled, there is some research on this and the evidence suggests that is not a clear cut case. Roe et al. (1962) carried out a study in which a 71-item program on elementary probability was presented to two groups, one received the program in its normal ordered form and one received a scrambled version of it. A criterion test was administered at the end of the program. There was no significant difference reported on time required for learning, error score during learning, criterion test score or time required for the criterion test. Roe (1962) reported contradictory results when using an extended version of the same probability program, the random sequence group performing significantly worse. The length and complexity of the program seemed to be highly significant. Payne et al. (1967) examined the effect of scrambling in three teaching programs which varied in the judged logical inter-connectedness of the material from high to low. The hypothesis that the effect of scrambling would be greatest for those programs dealing with topics have the most internal logical development was not confirmed. The reasons given were that there was a repetition in each frame so that even when scrambled there was a clue to the meaning of others and that there was an amount of redundant information in each frame.

The analysis of computational skills into a sequence of pre-requisites has become a common practice in the field of diagnosis and remediation. This approach assumes that a child is 'ready', can be taught the first step and with practice can acquire a skill which will enable him to understand the teaching of the next step. Some examples of such computation hierarchies are available in Underhill et al. (1980), the problem of validating the sequences is also mentioned. The crucial issue of whether a sequence intended for original learning is the same as the sequence that is apparent when children are tested after being taught i.e. a retention hierarchy, is also addressed. Underhill stated that "studies examining relationships within hierarchies conducted in the sixties provided evidence to support the hierarchical structure of knowledge, but produced no validated hierarchies". He goes on to mention the apparent effectiveness of validating learning sequences by focussing on two or three concepts or skills in a given hypothesised hierarchy as carried out by Uprichard (1970).

The picture that emerges is unclear; dissecting a topic into component skills is not easy nor necessarily effective. Suppes (1966) says:

> For anyone interested in the psychological foundations of mathematical concept formation it is natural to ask what is the sort of connection that holds between the logical structure of mathematical concepts and the psychological processes of acquisition of the concept. As far as I know, not very much has been written in the psychological literature about this kind of question. My present view, based partly on

our experiments and partly on conjecture, is that the psychological stratification of mathematical concepts will seldom, if ever, do violence to the logical structure of these concepts; but it will markedly deviate from the mathematical analysis of the same concepts with respect to the amount of detail that must be considered.

The mathematics wing of the research project 'Concepts in Secondary Mathematics and Science' based at Chelsea College has been investigating hierarchies in eleven different mathematical topics which commonly appear in the British secondary school curriculum. The hierarchies were based on success on a number of word problems in each topic, the word problems being written to reflect key ideas in the topic as it is taught in schools and to span a wide range of difficulty. The word problems were first used in interviews with children in order to ascertain the methods they would commonly use to solve them. Then written tests containing the word problems were given to some 10,000 children aged 11+ to 15+ years. The sample was chosen to be representative of the normal distribution of I.Q. scores, on each test at each age level. The results of the written examination were then interpreted in the light of the interviews.

The analysis used to form the hierarchies was concerned primarily with success on the items in the written test. The items were firstly grouped according to facility and then a measure of association (the ϕ coefficient) was worked out item/item. Items which fell within the same facility band (roughly spanning less than 20 per cent facility gap) *and* which had a ϕ value of greater than a stated criterion, with each other and with harder and easier items, were said to form a group. Children were deemed to have succeeded on a certain group of items if they had 2/3 (or the nearest possible amount to 2/3) correct. In addition, success on other than the easiest group must entail success on all easier groups; a Guttman scalogram analysis was used to test this. If less than 93 per cent of the sample did not behave in this way, the analysis was repeated until the error rate was reduced. The method described resulted in the rejection of a number of items and further required decisions to be made on the cut-off points between groups, unless there was a noticeable facility gap. These decisions were based on

(a) the mathematical coherence of the items

(b) the consistency of a certain type of error or child method used within the group of items

(c) whether the inclusion of an item reduced or increased the error rate (lack of success on all easier groups)

(d) whether the ϕ links for a particular item were stronger with the easier or harder items.

The groups were then described in terms of mathematical demand and methods available to the child for success on the items they contained. The method used by CSMS did assure the researcher of the subsumption of one level into another; it did however depend on the items used and resulted in very general descriptions of each level. The results of the research and attempts to match levels in individual hierarchies with each other to form 'stages' are described in two books (Hart, 1980, CSMS team 1980). The hierarchies obtained were based not on computational skills (as in Underhill et al. 1980) but on the solution of word problems so that although retention was a major factor it was not retention of specifically taught facts.

CONCLUSIONS

This paper has attempted to describe different types of hierarchies in the learning of mathematics and certain criteria by which they might be judged. An amount of research on different types of hierarchies has been carried out but the need to match the logical development of mathematics to the cognitive level of the child and then to base a teaching sequence on this information is still apparent. Ausubel (1968) has stated that new subject matter being taught to a child should be 'attached' to what he already knows. His idea of an advance organiser which is "... introductory material at a higher level of abstraction, generality and inclusiveness than the learning passage itself" (Ausubel 1978) has been somewhat modified by adherents to his theory, working in Science Education (Novak 1980). Bruner's (1960) early theory was based on that of Piaget but the distinction between the two appears to rest on Bruner's application of theory to classroom practice and his belief that by adaptation of the material to be taught the learning process could be accelerated. Dienes (1967) postulated a number of different learning modes from the playing of games to abstraction, the underlying theme being the *structure* of mathematics. What the teacher needs is well-documented evidence that one particular teaching sequence is preferable to another *and* some means by which he/she can ascertain whether the child is ready (the criteria being either cognitive level or prerequisite knowledge) for that sequence.

In reading the literature on hierarchies in mathematics the following questions might form a useful guide for judging their effectiveness:

(1) Is the sequence obtained from studying the logical development of a topic, decided upon by the expert (who can see the topic as a whole) or based on the development in the child?

(2) Has attention be paid to the large amount of background knowledge (sometimes erroneous) already present?

(3) How does one know that a learning hierarchy (in Gagné's sense) is what

it purports to be? Do the gaps between the component tasks have a significant effect on the transfer aspect? What is success on a component task?

(4) What effect do redundant and repetitive materials have on instructional programmes written in sequence?

(5) Does the age of the learner need to be considered in that success on component tasks or levels should improve with age?

(6) Is the hierarchy and its demands based on errors made by the learner (Renner, Karplus) rather than success? Does the absence of an error in the performance of a particular sample mean that it cannot be indicative of a level?

(7) For a series of tasks to form a hierarchy does the response pattern of each child need to be a perfect scale type e.g. 1100 not 00110; i.e. must success on a level entail success on all previous levels?

(8) Does there need to be a significant separation between adjacent parts of the hierarchy?

(9) Does the sample chosen affect the results?

(10) How is the data fitted to theory and vice versa (Noelting)?

(11) How dependent is the hierarchy on the items used: would a different hierarchy result if different items were used? Would this different hierarchy contradict the first?

(12) Is there a need for a longitudinal study before a hierarchy can be validated?

(13) What is the effect of teaching, cultural background and experience?

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