G. D. BALK

APPLICATION OF HEURISTIC METHODS TO THE STUDY OF MATHEMATICS AT SCHOOL*

It is well known that heuristic methods, and above all analogy and induction, play an important role in the creative work in any specialized field.

Science, technology, and economics are today presenting mathematics with an increasing number of varied problems that are non-standard in form and content, and it is therefore important to educate the pupil so that he or she can apply general methods in the search for solutions of problems, including "atypical" and non-standard problems.

Although pupils do use such methods of solution and proof, they usually employ them in an elementary and non-systematic fashion. Conscious application of these methods would be much more effective. Unfortunately, the application of heuristic methods to the study of mathematics under secondary-school conditions has not been adequately developed.

The aim of the present communication is to consider a few examples which, if presented to more advanced pupils, will familiarize them with the main principles of the heuristic approach.

ANALOGY

It is frequently found in mathematics that analogous – similar – conditions lead to similar conclusions. To enable the pupils to use these ideas in the study of mathematics, they must be taught how to formulate mathematical propositions by analogy. This can be done through a series of exercises. It is important to emphasize right at the outset that comparison is not proof, and that propositions formulated by analogy may turn out to be incorrect. For example, consider the following two true propositions:

> If the sum of the digits of a number is divisible by 3 then the number itself is divisible by 3.

and

> If the sum of the digits of a number is divisible by 9 then the number itself is divisible by 9.

One can then use analogy to formulate the proposition:

> If the sum of the digits of a number is divisible by 27 then the number itself is divisible by 27.

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However, this proposition is not true. To verify this, consider the number 2799. The sum of the digits is divisible by 27, but the number itself is not.

Here are two exercises for students:

(1) Consider the following valid proposition: "If all the angles in a triangle are equal then its sides are also equal". Formulate the analogous proposition for a hexagon and check its validity.

(2) Consider the following valid proposition: "The sum of the distances between any point within a triangle (or on its contour) and the sides of the triangle is a constant." Formulate the analogous proposition for a polygon and check its validity.

Although propositions formulated by analogy may turn out to be incorrect, it often happens that they are in fact valid. This justifies the use of analogy in the derivation of new mathematical results.

Analogy is, of course, useful not only in the formulation of new (to the pupil) mathematical facts, but even more so in searches for methods of solution of difficult problems or methods of proving complicated theorems. The pupils should be educated into the habit of conscious application of this approach.

Let us consider an instructive experiment at one of the Smolensk schools. The following problem was put to grade IX pupils:

Problem No. 1. Given the sides a, b, c of a triangle ABC, calculate the radius r_1 of a circle touching the side BC inside and the AB and AC outside.

None of the pupils was able to find the solution to this problem during the first 20 minutes. We then suggested to the students to formulate an analogous but simpler or more familiar problem. They proposed the following one:

Problem No. 2. Given the sides a, b, c of a triangle ABC calculate the radius r of a circle inscribed into it.

The solution of this problem (see below) was divided into several "steps". The pupils then readily found that the solution of the original problem (No. 1) could be obtained by analogy with the solution of the auxiliary problem (No. 2). It was sufficient to introduce the analogy at each "step".

The final solutions were as follows:

Auxiliary problem (No. 2)

1. Join the centre O of the inscribed circle to the corners of $\triangle ABC$ (Figure 1)

2. $S_{\triangle ABC}=S_{\triangle AOB}+S_{\triangle BOC}.$ (1)

Original problem (No. 1)

1. Join the centre O_1 of the externally inscribed circle to the corners of $\triangle ABC$ (Figure 2)

2. $S_{\triangle ABC}=S_{\triangle AO_1B}+S_{\triangle AO_1C}-S_{\triangle BO_1C}.$ (1)

Auxiliary problem (No. 2)

3. The area S of $\triangle ABC$ is given by Heron's formula
$$S=\sqrt{s(s-a)(s-b)(s-c)}.$$

4. $S_{\triangle AOB}=\frac{1}{2}cr,$
$S_{\triangle AOC}=\frac{1}{2}br,$
$S_{\triangle BOC}=\frac{1}{2}ar.$

5. From (1) we have
$$S=\tfrac{1}{2}(c+b+a)r=sr,$$
Hence $r=S/s$
and
$$r=\sqrt{\frac{(s-a)(s-b)(s-c)}{s}}.$$

Original problem (No. 1)

3. Ditto

4. $S_{\triangle AO_1B}=\frac{1}{2}cr_1,$
$S_{\triangle AO_1C}=\frac{1}{2}br_1,$
$S_{\triangle BO_1C}=\frac{1}{2}ar_1.$

5. From (1) we have
$$S=\tfrac{1}{2}(c+b-a)r_1=(s-a)r_1.$$
Hence $r_1=S/(s-a)$
and
$$r_1=\sqrt{\frac{s(s-b)(s-c)}{s-a}}.$$
This solves the problem.

This example was used to illustrate a very useful method of finding the solutions of problems. As we have seen, it reduces to the following:

(a) Selection of a problem analogous to the original problem i.e. having similar conditions and conclusions. The auxiliary problem should be simpler than the original, and its solution should be known.

(b) Having solved the auxiliary problem, we carry out an analogous analysis for the original problem.

Analogy is a natural method for problems in solid geometry (here the pupil independently formulates and solves an analogous problem in plane geometry), especially in the solution of problems which involve finding the loci of points in space.

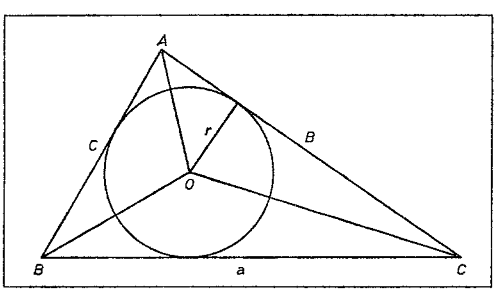

Fig. 1.

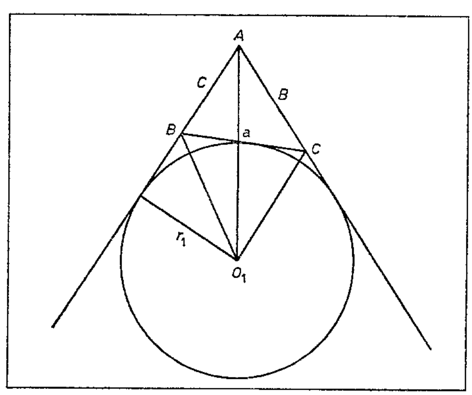

Fig. 2.

It is convenient to use analogy in proving theorems. For example, the theorem on the bisector of the external angle of a triangle can be established by considering step by step the proof of the theorem on the bisector of an internal angle of a triangle.

Here are some problems that can be used as exercises in class:

(3) Construct a point P which externally divides a segment AB in a specified ratio $m:n$, where m and n are specified segments and $m>n$.

In other words, the point P should lie on the extension of AB with $AP/PB = m:n$.

Hint: The first step is to formulate an analogous but simpler problem (internal division of a segment in the ratio $m:n$) and recall its solution.

(4) Find the locus of points for which the difference between the squares of the distances from the ends of a given segment is equal to a constant c^2.

Hint: The first step is to formulate and solve the analogous problem on a plane; the second step is to use it to solve the original problem.

(5) Use analogy to find methods of proving the following theorems:

(a) in a trihedral angle the sum of two plane angles is greater than the third angle;

(b) in a trihedral angle the difference between two plane angles is less than the third angle.

(6) Consider a tetrahedron whose faces are equal triangles. Prove that the sum of distances between any point lying inside the tetrahedron and its faces is a constant.

(7) The lateral edges of a tetrahedron are mutually perpendicular and equal to a, b, c. Find the height of the tetrahedron.

(8) Express the radius of a sphere inscribed into a tetrahedron in terms of the heights of the tetrahedron.

The use of analogy for the definition of new concepts is also a useful method, and systematic exercises in this field would seem to us very desirable. The pupils can be asked to formulate (by analogy) the definitions of: the center of a sphere, diameter of a sphere, plane tangential to a sphere, lines perpendicular and inclined to a plane, parallel planes, right polyhedron, area of the lateral surface of a cone, equivalent systems of equations, midline of a tetragon, diagonal of a tetragon, and so on.

INDUCTION

The importance of the inductive method in the search for mathematical laws has frequently been emphasized in our methodological literature. Such exercises are also used in the teaching of mathematics in school (recall, for example, how one finds, or more precisely guesses, the general term of an arithmetical or geometric progression). An increase in the relative number of school exercises in this field is undoubtedly desirable. A large number of such exercises can be found, for example in [4] and [6].

Here we merely wish to draw the readers' attention to the use of induction in searches for methods of solution of already formulated problems or in proofs of already formulated theorems. The essential point is that preliminary analysis of special cases of the problem may suggest a method of solving the general case. Having encountered a difficult problem we begin our search with the question: "for which special case can we solve the problem?" Having found this special case we ask: "Can we use this solution (or the experience drawn from it) to solve a more general case (though perhaps also a special case)?" This can be repeated several times until the solution of the original problem is found.

We shall confine our attention to two examples.

Example 1

Pupils at a junior mathematics school, preparing for the mathematics olympiad, were given the following problem:

Consider two boxes, one containing m balls and the other n balls ($m>n$). Two players in turn remove balls from the boxes. Each time a player can take any number of balls, but only from one box. The player who takes the last ball is the winner. How many balls should be removed by the first player (i.e. the player who starts the game) to enable him to win?

Since the pupils could not cope with this problem, it was suggested that they should formulate and solve special cases of increasing difficulty, and this led rapidly to the final answer. The pupils considered the following special cases: $n=0$, $m=1$; $n=0$, $m=2$; $n=0$, m arbitrary; $n=1$, m arbitrary; $n=2$, m arbitrary; n, m both arbitrary ($m>n$). As a result, the students arrived at the correct answer: each move of the first player must be such as to equalize the number of balls in the two boxes.

Example 2

The members of the mathematics club formed by grade VII pupils were given the following problem to solve:

Prove that the sum of distances between any point M on the contour of the right triangle ABC, or inside this triangle, and the sides of the triangle is a constant independent of the position of M.

This problem can, of course, be easily solved using the formula for the area of a triangle, but the grade VII pupils were not familiar with this formula. The problem led to serious difficulties. However, it was easily solved after we reminded the class that the inductive approach might be useful. The pupils suggested considering the following cases:

(a) M is a corner of the triangle,

(b) M lies on a side of the triangle (say, on BC); this case reduces to the previous case if we draw a line through M such that $MN \parallel AC$ and let $A' \equiv MN \times AB$ so that M is a corner of $\triangle A'BM$, and

(c) M is any point inside $\triangle ABC$ (general case).

The last case may be reduced to (b) if we draw the straight line $MN \parallel AC$, and consider the points $A' \equiv MN \times AB$, $C' \equiv MN \times BC$ and the triangle $A'BC'$.

It should be noted that when this problem was being solved we first (in cases (a) and (b)) took M on the contour of the triangle. This was not accidental. 'Extreme' cases are particularly important in the inductive approach.

A number of useful exercises may be found in [1]–[4].

LIMIT CASES

The search for solutions is frequently considerably simplified if we first consider an auxiliary problem that has a condition in common with the given problem, but in which this condition or some data are obtained from the condition or the data of the initial problem as a result of transition to the limit. Suppose that some of the figures in the initial problem are placed in limit positions. For example, if the original problem is concerned with a line cutting a circle, we replace it in the auxiliary problem by a tangent line (the limit position is obtained by allowing the distance between the line and the center to approach the radius). Again, if the original problem is concerned with a tetragon then in the auxiliary problem we may consider a triangle (one side of the tetragon tending to zero). Different limit cases can of course be chosen for the same problem. Limiting cases are also useful in establishing the likelihood of a given result (answer to problem or particular formula) and in constructing refutations.

Unfortunately, this valuable device which is close to induction is practically never used at school.

Consider some typical examples.

Example 1

The two sides AD and BC of the tetragon $ABCD$ (Figure 3) are not parallel. Which is greater, the half-sum of these sides or the segment MN joining the midpoints of the two other sides?

Search for solution. The first question that we must answer is, what happens in the limiting case when one of the sides of the tetragon contracts to a

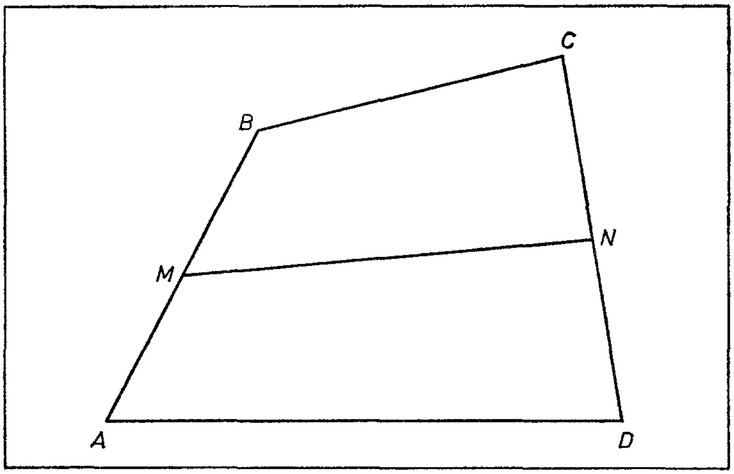

Fig. 3.

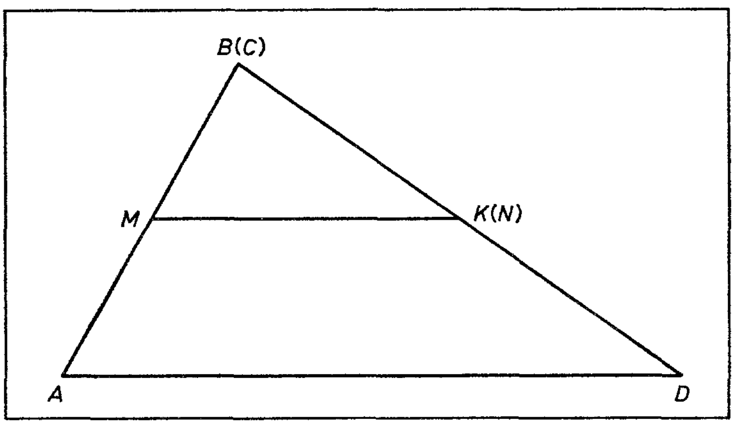

Fig. 4.

point. This construction can be applied either to BC (or AD) or to AB (or CD). Let us take the first alternative and suppose that BC contracts to the point B (Figure 4). In the limit position the point N coincides with the midpoint K of the segment BD, and MN becomes the midline MK of the triangle ABD. In the limit case we have the following auxiliary problem: which is greater, one-half of the side AD of the triangle ABD or the segment MK joining the midpoints of the other two sides. The answer is well-known: $MK=\frac{1}{2}AD$.

We now ask, can the solution of the general problem be reduced to this limiting case. It is readily seen that this is in fact so.

Solution. Let K (Figure 5) be the midpoint of the diagonal BD of the tetragon $ABCD$. In the triangle ABD we have $MK=\frac{1}{2}AD$ and $MK \parallel AD$. In

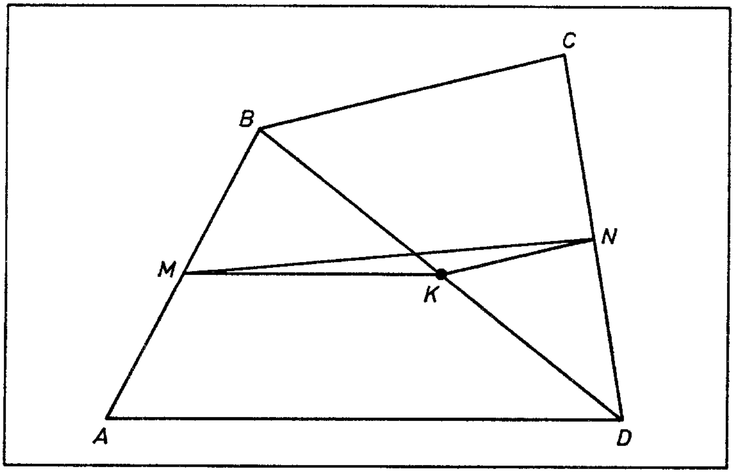

Fig. 5.

the triangle BCD we have $KN = \frac{1}{2}BC$ and $KN \parallel BC$. Since by hypothesis AD is not parallel to BC, it follows that the points M, K, and N do not lie on the straight line. From the triangle MKN it is clear that $MN < MK + KN = \frac{1}{2}(AD + BC)$.

Remark. If we proceed in a different way, and contract the side AB, we obtain a different limiting case (MN is now the median of the triangle CMD), which will suggest to us another method of solution (parallel transport of the segments AD and BC such that A and B are transferred to M).

Example 2

Consider a circle of radius R(Figure 6). Draw a line which cuts the circle from some point A lying outside the circle and located at a distance a from its centre O. The points of intersection B and C are joined to the centre O. Let the angles BOA and COA be denoted by β and γ respectively. Find $\tan \frac{1}{2}\beta \cdot \tan \frac{1}{2}\gamma$.

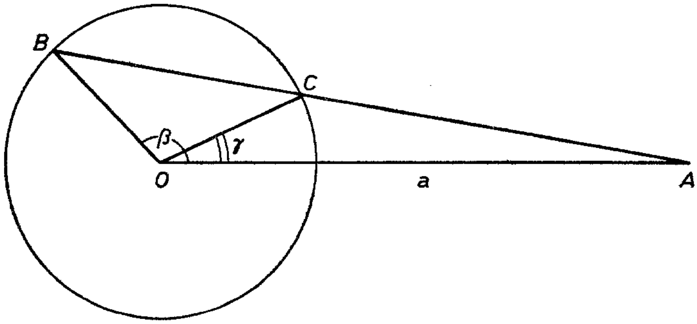

Fig. 6.

Search for the solution. We are to find $\tan\frac{1}{2}\beta\cdot\tan\frac{1}{2}\gamma$ in terms of the data given, i.e. R and a, and the answer should be independent of the choice of the line AB. It is plausible that the same solution can be obtained by considering the limit case in which the line cutting the circle becomes a tangent to it. In this case $\beta=\gamma$ (Figure 7).

$$\tan\frac{\beta}{2}\cdot\tan\frac{\gamma}{2}=\left(\tan\frac{\gamma}{2}\right)^2=\frac{1-\cos\gamma}{1+\cos\gamma}=\frac{1-R/a}{1+R/a}=\frac{a-R}{a+R}.$$

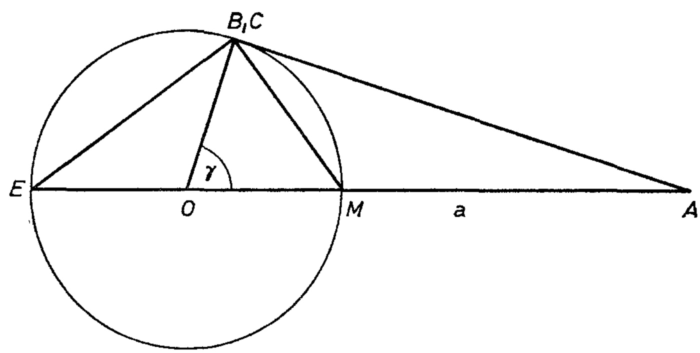

Fig. 7.

Thus, in the limiting case

$$\tan\frac{\beta}{2}\cdot\tan\frac{\gamma}{2}=\frac{a-R}{a+R}.$$

We shall now try to prove that the same relation is valid in the general case. The presence of $(a+R)$ and $(a-R)$ in the formula suggests that these segments should be introduced into the drawing (Figure 8), i.e. we must consider the segments AE and AM where E and M are the points of intersection of the

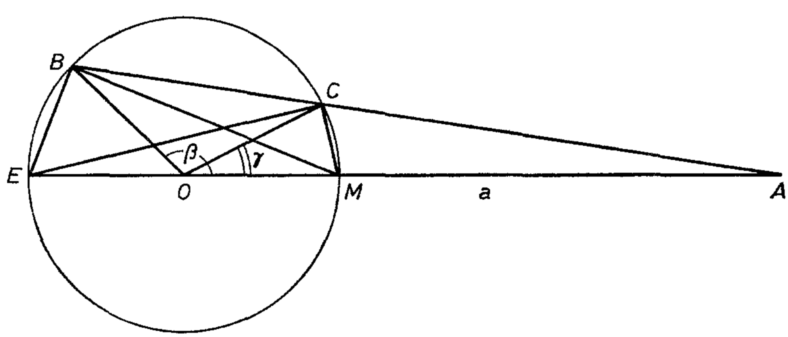

Fig. 8.

line AO with the circle. These preliminary ideas can now be readily used to solve the problem.

Solution. It is clear that $\angle CEM = \frac{1}{2}\gamma$ and $\angle BEM = \frac{1}{2}\beta$ (Figure 8). Since the triangles BEM and CEM are right-angle triangles it follows that $\tan\frac{1}{2}\beta = BM/BE$ and $\tan\frac{1}{2}\gamma = CM/CE$.

$$\tan\frac{\beta}{2}\cdot\tan\frac{\gamma}{2} = \frac{BM}{BE}\cdot\frac{CM}{CE}.$$

The four segments BM, BE, CM and CE or their ratios will now be expressed in terms of $(a+R)$ and $(a-R)$. The segments CM and $(a-R)$ are involved in the triangle AMC which is similar to the triangle ABE ($\angle A$ common, $\angle AMC = \angle ABE$). Consequently,

$$\frac{CM}{BE} = \frac{AM}{AB} = \frac{a-R}{AB}.$$

The segments CE and $(a+R)$ are involved in the triangle ACE:

$$\Delta ACE \sim \Delta AMB$$

($\angle A$ common, $\angle ABM = \angle AEC$).

Therefore

$$\frac{MB}{CE} = \frac{AB}{a+R}.$$

It is now clear that

$$\tan\frac{\beta}{2}\cdot\tan\frac{\gamma}{2} = \frac{a-R}{AB}\cdot\frac{AB}{a+R} = \frac{a-R}{a+R}.$$

Remark. 'Once more and better' (G. Polya). After the above solution is found a shorter one (not connected with the use of limit cases) may be shown to the students. It is based on the formula

$$\tan\frac{\beta}{2}\cdot\tan\frac{\gamma}{2} = \left[\cos\frac{\beta-\gamma}{2} - \cos\frac{\beta+\gamma}{2}\right] : \left[\cos\frac{\beta-\gamma}{2} + \cos\frac{\beta+\gamma}{2}\right].$$

Example 3

It is known that the length of each median in a triangle is less than the sum of the other two medians. Is this proposition also true for the bisectors of a triangle? Is it true for the heights of a triangle?

Hint. Consider an equilateral triangle ACB. Without varying its base AB we shall increase its bisector CC_1 without limit, keeping the other two bisectors bounded (their sum does not exceed $2AB\sqrt{2}$). Hence it follows that for sufficiently large CC_1 we obtain a triangle ACB in which one of the bisectors (CC_1) is greater than the sum of the other two. An analogous solution can be given for the height problem.

CONTINUITY

Mathematicians frequently conclude that a particular mathematical proposition is true or plausible, basing their conclusion on continuity considerations. It is unfortunate that this approach is not used in school teaching, at least not in an explicit form. The reason for this is that a formal rigorous definition of continuity is very difficult to grasp for pupils, and is not usually introduced at school, so that it cannot be employed in proofs. However, in the search for plausible answers there is no loss of rigour in using continuity considerations and in fact this is very desirable.

The idea of continuous variation is intuitively clear to the more advanced pupil, especially if, for the sake of simplicity, the argument of a function is interpreted as the time. Anyone can give examples of quantities which vary continuously with time, e.g. the path traversed by a moving point; the angle swept out by a ray, the area swept out by a moving segment, and so on.

Using these intuitive ideas as a starting point, we can give a mathematical definition of what is meant by saying that a quantity U is a continuous function of time. This means that for any time T_0 within the short interval of time $(t_0-h,\ t_0+h)$ the values of this quantity differ from its value at the time t_0 by an amount that is less than a given deviation d. It must be remembered that d is prescribed in advance and can be as small as we please. We then say that for any such chosen d we can always find a sufficiently small interval of time $(t_0-h,\ t_0+h)$ such that the values of U at any time within this interval differ from its value at t_0 by less than d. In practice, it is usually quite clear whether a given quantity is a continuous function of time.

The following property of continuously varying quantities is particularly useful in the solution of problems and is intuitively obvious (to be given without proof in school); if any particular quantity, e.g. length, angle sum of angles, area, etc., varies continuously over for an interval of time and was less than some given constant a at the initial time, whereas at the end it is greater than a, then it must be equal to a, at some intermediate time.

Let us illustrate this property by an example. The students of grade X in school No. 6 at Smolensk encountered the following problem:

Prove that a tetragonal pyramid whose base sides are proportional to 5, 4, 3 and 6 respectively, cannot be inscribed into a cone.

The students readily established that this problem reduces to the following: can a tetragon whose sides are proportional to 5, 4, 3, and 6 be inscribed into a circle?

To obtain the correct answer we can use the ideas of continuity.

Consider the tetragon $ABCD$ (Figure 9) with sides equal to 5, 4, 3, and 6.

Since $AB+BC=CD+DA$ we can deform the tetragon so that all its corners lie on a single straight line (Figure 10). In this initial position

$$\angle A + \angle C = 0° < 180°.$$

Let us now compress the tetragon of Figure 10 along AC until C lies on BD (Figure 11). In this final position

$$\angle A + \angle C > 180°.$$

Since we have reached a final position by continuously varying the sum $\angle A + \angle C$, it follows that this sum was equal to 180° at some intermediate time. Consequently, there is a position of the tetragon $ABCD$ for which it can be inscribed into a circle (it follows from this that the above pyramid problem contains an incorrect statement).

Here are some further examples of this kind.

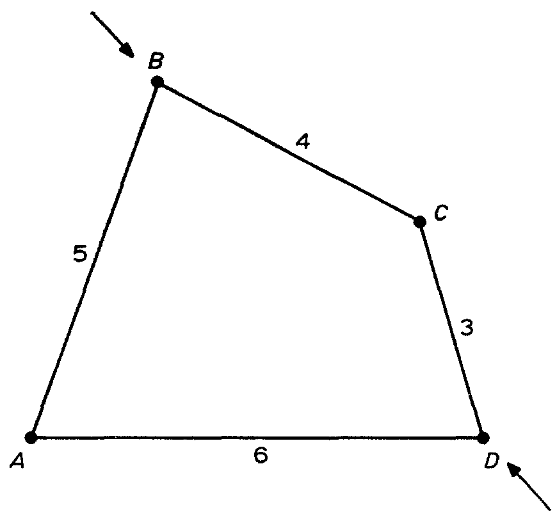

Fig. 9.

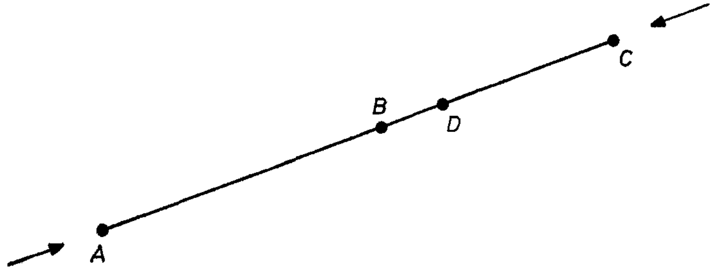

Fig. 10.

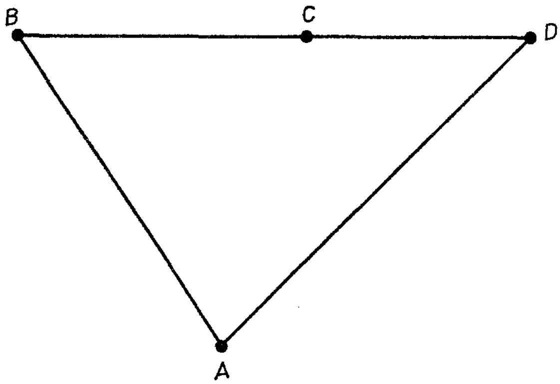

Fig. 11.

(1) It is readily demonstrated that the equation $2^x = 4x$, is satisfied by $x = 4$. Is there another root?

When $x=0$ we have $2^x - 4x > 0$ and for $x=1$ we have $2^x - 4x < 0$. There must therefore be an intermediate value of x between 0 and 1 such that $2^x - 4x = 0$.

(2) Kepler's equation $x + e \sin x = M$ is important to astronomy. Given that $e=0.3$ and $M=6.7$, determine whether this equation has a root in the interval $(2\pi, 3\pi)$.

The quantity $x + e \sin x$ is less than 6.7 for $x = 2\pi$ and greater than 6.7 for $x = 3\pi$. Therefore at some intermediate value x_0 we have

$$x_0 + 0.3 \cdot \sin x_0 = 6.7 .$$

(3) A square and a triangle not overlapping with it are drawn on a plane. Is there a line that would divide both these figures into equal parts?

Draw an arbitrary line l through the centre O of the square, which does not cut the triangle. Uniformly rotate this line about O. Let $U = U(t)$ be the part of the area of the triangle swept out by this line at time t. Initially $U = 0 < \frac{1}{2}S$. (S is the area of the triangle.) Eventually we shall have $U = S > \frac{1}{2}S$. Therefore, at some intermediate time t_0 the quantity U must be equal to $\frac{1}{2}S$. At this time the line l divides both the triangle and the square into two equal parts.

(4) The diameters of two circles are respectively equal to 1 and 0.5 dm. Is there a straight line parallel to the line passing through the centres which forms two chords of these circles with a total length of $\sqrt[1970]{1970}$ dm?

Consider an arbitrary straight line l, parallel to the line of centres $O_1 O_2$, which does not cut the given circles. Displace it parallel to itself towards the line of centres until it coincides with $O_1 O_2$, and suppose that $U = U(t)$ is the total length of chords cut on the line l by the two circles at time t. We have initially

$$U = 0 < \sqrt[1970]{1970},$$

and finally

$$U = 1.5 > \sqrt[1970]{1970}$$

(note that $\sqrt[1970]{1970} < \sqrt[1970]{2^{985}} = \sqrt{2} < 1.5$). Therefore, at some intermediate time t_0, we have

$$U = \sqrt[1970]{1970}\ \mathrm{dm}$$

i.e. at this time l is the required line.

Problems involving the properties of continuous functions in the case of convex figures, which will be suitable for high-school students, can be found in Section III of [5].

It seems to us that the above ideas and examples show sufficiently clearly the extent to which familiarity with heuristic methods should be useful to secondary-school pupils.

Smolensk, U.S.S.R.

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