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# VISUALISATION AND MATHEMATICAL GIFTEDNESS

ABSTRACT. Recent research findings are described which uncover the fact that visualisers are seriously under-represented amongst high mathematical achievers at senior high school level. Possible reasons for this phenomenon are discussed, and kinds of imagery which overcome some of the limitations of visual processing in mathematics are described.

## INTRODUCTION

In recent research which investigated the effect on senior high school pupils who were visualisers, of the preferred cognitive modes, attitudes and actions of their mathematics teachers, the strengths and limitations of visual processing in mathematics were examined in a classroom context and in task-based interviews (Presmeg, 1985). In the course of this investigation, the thirteen selected teachers in the study were asked to name any pupils in their final (twelfth) year mathematics classes whom they considered to be "stars". It was found that pupils whose achievements were singled out as being outstanding (7 pupils, out of 277 pupils) were not merely "often", but almost always, nonvisualisers. Even using a less stringent criterion, less than one fifth of pupils singled out for high mathematical achievement were visualisers. Of 27 pupils singled out as "very good" (i.e. about 10% of the sample), only five were members of the study group of visualisers, while eighteen were in the corresponding nonvisual group. A brief description follows of the selection of visualisers for this investigation, and of definitions used (for a full account see Presmeg, 1985).

## DEFINITIONS AND SELECTION OF VISUALISERS

A survey of the psychological literature on visual imagery revealed that many different forms of visual imagery exist. Since one purpose of the research was to identify ways in which individuals use imagery in their solution of problems from the high school mathematics syllabus, a visual image was defined as *a mental scheme depicting visual or spatial information.* This definition is deliberately broad enough to include kinds of imagery which depict shape, pattern or form without conforming to the "picture in the mind" notion of imagery (Clements, 1982), although imagery which attains the vividness and clarity of a picture is also included in this definition. The definition also allows for the possibility that verbal, numerical or mathematical symbols may be arranged spatially to form the kind of numerical or algebraic imagery sometimes designated "number

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forms" (Arnheim, 1969; Paivio, 1971; Skemp, 1971). The position was also taken that imagery is involved in the drawing of a diagram (Piaget and Inhelder, 1971).

Moses (1977; 1980) and Suwarsono (1982) recognized that individuals could be placed in a continuum with regard to their preference for using visual imagery when tackling mathematical problems which could be solved with or without such imagery. Since the instruments used by Suwarsono and by Moses were designed for use with younger children, it was found necessary to construct a new research instrument to measure this preference in teachers and sixteen or seventeen year old pupils. *Visual and nonvisual methods of solution* of mathematical problems were defined as follows:

*A visual method of solution is one which involves visual imagery, with or without a diagram, as an essential part of the method of solution, even if reasoning or algebraic methods are also employed.*
*A nonvisual method of solution is one which involves no visual imagery as an essential part of the method of solution.*

Following Krutetskii's (1976) model, the position was taken that all mathematical problems involve reasoning or logic for their solution. Beyond this requirement, the presence or absence of visual imagery as an essential part of the working determined whether the method was visual or nonvisual. As in Moses' (1977) study, visual methods, then, included solutions involving constructions, drawings, diagrams, tables, charts or graphs, whether written down or in the person's mind.

In addition, another construct *mathematical visuality* (MV) was defined as follows:

*A person's mathematical visuality is the extent to which that person prefers to use visual methods when attempting mathematical problems which may be solved by both visual and nonvisual methods.*

This definition follows that of Suwarsono (1982). Finally, *visualisers and nonvisualisers* were defined as follows:

*Visualisers are individuals who prefer to use visual methods when attempting mathematical problems which may be solved by both visual and nonvisual methods.*
*Nonvisualisers are individuals who prefer not to use visual methods when attempting such problems.*

In this study visualisers were taken to be those individuals whose mathematical visuality (MV) scores on the research instrument exceeded the median scores of the sample from which they were drawn. Possible MV scores ranged from 0 (nonvisual) to 36 (visual), and the actual MV scores of visualisers selected for this investigation (54 pupils, of 277 pupils) ranged from 20 to 33. The median MV score for all pupils was 18.

Of the seven pupils singled out by their mathematics teachers as being outstanding (as discussed in the previous section), six had MV scores of 12 or less; the seventh pupil scored 18, the median score. Four of these pupils scored 4 or less. Thus, in this sample, "stars" were almost always nonvisualisers. This finding is consonant with Krutetskii's (1976) analysis of individual cases of mathematical giftedness; of nine gifted children whom he described, only one, Borya G, was classified as "geometric" in his typology (five were "analytic", one was "abstract-harmonic", one "pictorial-harmonic" and one, Lenya K, was not classified). Clements (1984) described the case of Terence Tao, of whom he wrote, "It is likely that Terence is one of the most mathematically precocious children in the world" (p. 234). Terence is apparently a nonvisualiser, for Clements (ibid, p. 235) wrote that "While he has well developed spatial ability, when attempting to solve mathematical problems he has a distinct, though not conscious, preference for using verbal-logical, as opposed to visual thinking".

If it is the case that mathematical "stars" at high school level are predominantly nonvisualises, the question then arises of why it is so. This question is addressed after a brief discussion of the relevant aspects of Krutetskii's (1969; 1976) research.

### *Krutetskii's Analysis of the Types and Structure of Mathematical Abilities in Gifted Pupils*

The division of schoolchildren into three types with respect to mathematical abilities, antedates Krutetskii's work. In fact, Moses (1977) quoted a study by Haeker and Ziehen (1931) in which they identified the following groups:

(1) subjects for whom the visual element is dominant: abstract problems are solved visually;
(2) subjects for whom the abstract element is dominant: geometry problems are solved logically, abstractly;
(3) subjects for whom the two elements are in equilibrium.

The importance of Krutetskii's research, however, lies in its distinction between *level* of mathematical abilities, determined largely by a verbal-logical component of thinking, and *type* of mathematical giftedness, determined largely by a visual-pictorial component. In the case of the visual-pictorial component, it is not only the ability to use it, but preference for its use, which determines the type of mathematical giftedness of an individual. The name Krutetskii assigned to his geometric type reflects his method of assigning pupils to the two extreme categories: he compared algebra and geometry marks of 1512 pupils in Moscow high schools, and his criterion was a difference of two or more points (out of a possible 5) between algebra and geometry marks. It was a stringent criterion; of 1512 pupils only ten were classed as algebraic and thirteen as geometric.

Components which Krutetskii (1976) did not include in his outline of the structure of mathematical giftedness, although he felt that their presence is useful, are as follows:

(a) Swiftness of mental processing
(b) Computational abilities
(c) Memory for symbols, numbers and formulas
(d) An ability for spatial concepts
(e) An ability to visualise abstract mathematical relationships and dependencies.

That he had not always considered an ability for spatial concepts to be inessential is indicated in earlier writings. Krutetskii had written (1969, p. 99) that "well developed spatial notions ... are essential in geometry", and that logical reasoning and spatial imagination are both necessary in mathematics (1969, p. 117). Krutetskii apparently thought highly of Werdelin's research, to judge by the space he gave to Werdelin's work, and by the comment that Werdelin's study was "the most significant study ... in recent years" (1976, p. 32). Werdelin concluded from his factor-analytic study that "mathematical abilities proper are not related to computational, verbal or visual-spatial ability" (Krutetskii, 1976, p. 34). Following his identification of types of mathematical giftedness, Krutetskii (1976) studied the mathematical processing of 34 capable pupils using his problems in series XXIII, XXIV and XXV, and concluded, like Werdelin, that the ability to visualise abstract mathematical relationships and the ability for spatial geometric concepts are not necessary components in the structure of mathematical abilities, although their strength or weakness determines the type of giftedness.

## WHY ARE THE "STARS" IN SENIOR HIGH SCHOOL MATHEMATICS CLASSES ALMOST ALWAYS NONVISUALISERS?

Possible reasons for this phenomenon may reside in external factors such as the nature of mathematics, or in internal factors such as cognitive preferences, confidence or mathematical abilities.

*Internal Factors*

Although internal factors do not exist in isolation from possible external causes, they probably also play a part in the predominance of nonvisualisers amongst very high achievers. Firstly, the components relating to curtailment, generalisation and logical economy in the structure of mathematical abilities are relevant and may interact with the time factor in the school situation. Analysis of the task-based interview transcripts revealed that each of the nine visualisers in this study who in their final examinations attained an *A*, a *B* or a *C* symbol on higher grade, or an *A* on standard grade, had a way of recalling formulae and procedures rapidly and of using these with ease. Seven of the nine visualisers made use of memory images of formulae, and three (Caron W, Crispin T, and Mark J) used pattern imagery at some stage of their problem solving. Only Laura D and Enrico B made use of neither of these types of imagery in the task-based interviews: their recall of formulae and procedures was facile and largely nonvisual. Memory images of formulae, and pattern images, are two types of imagery which provide a quick means of recall of abstract general principles and procedures, the former in a concrete image which encapsulates a procedure, the latter in a more schematic image which stresses regularities. It emerged strongly in the task-based interviews that for success in school mathematics the one-case concreteness of an image or a diagram must be transcended, and that many if not most visualisers are not aware of how to accomplish this task. For nonvisualisers this problem does not arise.

A second internal factor of relevance, therefore, is preference for particular ways of thinking: those who prefer to use nonvisual methods, especially if this preference is reinforced by the school curriculum, examinations and teaching emphasis, would have a head start in that they would have no need to overcome the stumbling-block of concrete visualisation. Curtailment and generalisation are inherent in symbolic logic, and inessential visual methods may simply be dispensed with entirely. Whether this pattern of thinking is adequate at higher levels in which creative mathematical thought is required, is an open question, but within the constraints of the school system the one-sidedness of nonvisual thinking does not appear to be a disadvantage.

As illustrated in the following extracts from transcripts of interviews, it emerged that increasing facility may move visualisers in the direction of nonvisual methods of solution:

Enrico B: Yes, oh yes, the gradient $y$ is equal to the $\tan\theta$. So the gradient of vector $OA + OB$...

Interviewer (I): Sorry to interrupt, but was that visual? When you thought of your tan $\theta$?

Enrico B: No, I'm just used to working with it. At first it was, when I ... when it was first explained there; when [the teacher] showed us an actual diagram. But since then, I just got used to it.

And from an interview with Mark J:

I: Now I'm just interested in how much visual thinking you needed here, say for your gradient; did you need a visual picture of it?

Mark J: No, not really, no. At one stage I used, I needed it but, you know, gradually I've just got used to it.

Apparently when a topic is first taught, a visual presentation often aids visualisers' understanding, but practice of the procedure or formula may lead to habituation, when an image is no longer necessary. In other words, facility led visualisers away from visual methods. Successful mathematics pupils who still used visual methods had to work hard to achieve the necessary curtailment and generalisation through such means as pattern images and memory images of formulae. Some of the ways in which visualisers may overcome some of the difficulties associated with visual processing are described in the following section.

*How is Visual Processing Effective in High School Mathematics?*

In learning mathematics, the "peculiar excellences" of imagery, of which its limitations were the price (Bartlett, 1977), were found in this study to reside in the following aspects:

(1) vivid images of all types were found to have mnemonic advantages;
(2) an effective combination was found to be concrete imagery alternating with abstract nonvisual modes such as analysis, logic or a facile nonvisual use of formulae;
(3) although used only infrequently in this study, dynamic imagery was found to be potentially effective; and
(4) imagery which served an abstract function was found to be effective.

The mnemonic function of imagery, which has been amply documented in the psychological literature (Paivio, 1969, 1970, 1971 and 1978; Richardson, 1969), was confirmed in the mathematical processing of the visualisers in this study, e.g., in their memory images of standard diagrams of theorems, or of special triangles in trigonometry. Imagery which was vivid was particularly conducive to clear recall. One form of memory imagery which was found to be particularly effective was imagery of formulae; these images were "seen" by a majority of the visualisers in this study, typically on a page of notebook or written on the blackboard.

Visualisers who were able to combine use of concrete imagery and use of abstract nonvisual modes avoided many of the pitfalls associated with use of concrete imagery, e.g., Enrico B's efficient processing alternated between use of concrete imagery and a facile nonvisual use of formulae. Richardson quoted a finding by Mawardi which is relevant in this context:

> She studied a group of professional problem solvers and found that their thinking showed constant alternation between abstract numerical and verbal modes of thought on the one hand and concrete imaging on the other. One of the advantages of this alternation appears to be in providing a basis for breaking an unproductive set towards the solution of a problem.
>
> (Richardson, 1969, pp. 81–82)

Imagery used in this way would need to be controllable. Krutetskii (1976) included flexibility of thought as a necessary aspect of mathematical giftedness, and the present study confirmed that it is an advantage to have visual and nonvisual modes available when tackling problems where one or the other might be more suitable. McKim (1972, p. 62) called a balance between visual and verbal modes, "ambidextrous thinking". This study confirmed the efficacy of ambidextrous thinking in visualisers, a result which points to the desirability of teaching which includes visual and nonvisual aspects.

The qualitative data in this study largely corroborate Krutetskii's (1976) suggestion that pupils of the "geometric" type manifest a certain imbalance in their thinking. Krutetskii considered that, to a certain extent, it is a hindrance to be "riveted" to visual-pictorial schemes. However, he wrote

> But only to a certain extent! The fact is that the graphic schemes used by these pupils [i.e., visualisers of high ability] are a unique synthesis of concrete and abstract. ... Botsmanova has shown convincingly that in a drawing – a graphic scheme – a schematization and a certain generalization of a visual image occur. Supporting thought by, and even 'binding' it to, such a generalized visual image cannot prevent generalized thinking. In such a case this image is in a certain sense the 'bearer' of the sense and content of an abstract concept. The 'geometer' pupils feel the need to interpret a problem on a general plane, but for them this general plane is still supported by such images. In this they differ from pupils of little ability – for whom visual images really bind thinking, push it onto a concrete plane, and hinder the interpretation of a problem in general form.
>
> (Krutetskii, 1976, pp. 325–326)

As Krutetskii recognised, some of the limitations associated with the one-case concreteness of imagery are overcome, therefore, if this imagery can be made to serve an abstract function in mathematics. In the task-based interviews it was found that this objective could be achieved in two ways, as follows. Firstly, this purpose was served by concretising the referent (Werner and Kaplan, in Paivio, 1971), i.e., by making a concrete image the bearer of abstract information. A good example of this aspect is Alison M's use of her "water level" image, as follows.

Alison M: [after drawing $\frac{S\,|\,A}{T\,|\,C}$ to help her to see the positive trigonometrical ratios]: All students ... take chemistry.

I: Is that an image that you keep?

Alison M: Yup! ... Then it would be sine ... second quadrant. So ... 180; you take it from 180, cause that's your water level.

I: Oh is that how you think of it [laughing]? And how does the water level help you to get it?

Alison M: Oh, um ... you've got ... It's like a ship sailing: can't sail that way really [up and down, i.e., *y* axis]. There, that, sort of ... it can. You work it from there. You take most of your ... cause that's your 360 ... it's there as well.

I: Did you think of that for yourself or did someone tell you about the water level?

Alison M: No, I just thought of it.

Alison's image was concrete and idiosyncratic, consonant with Werner and Kaplan's description of images which serve an abstract function. To some extent, memory images of formulae also served the purpose of depicting abstract procedures or processes in a concrete image.

A second way in which an abstract purpose may be served by imagery is in the use of pattern imagery, which embodies the essence of structure without detail. "The regularity, which just helps" was captured in the pattern imagery of Crispin T, who was the only visualiser in this study to achieve the symbol A on higher grade in his final examinations. An example of Crispin's pattern imagery is given in the following extract:

Crispin T: [finding the *x* component of a vector]: The next one is four; four times cos 120. You've got your four quadrants, so cos is, that's going to be in the second quadrant so cos is negative. I'd better just show you how I see that: sine, cos, tan; sine goes $+ + - -$, cos goes $+ - - +$, tan goes $+ - + -$.

I: So you don't actually need a picture of the quadrants: you just need that pattern?

Crispin T: Again, that's another pattern; actually I've got quite a few of them.

I: [summing up later]: The important thing about it was the pattern of it.

Crispin T: Yeah, the regularity, which just helps.

Krutetskii (1969) wrote that "the capable pupil realises that he needs the typical, generalised schemes and that specific data are of no interest in this respect" (p. 98). He recognised that these generalised schemes could be visual images, for he subsequently described graphic schemes of direct relationships in terms of visualisation of abstracted, generalised mathematical dependencies (Krutetskii, 1976). Pattern imagery, in which concrete details are disregarded and pure relationships are depicted in a visual-spatial scheme, is illustrated in the memory images of chess masters as described by the Dutch psychologist, de Groot (Harris, 1980), and by Binet (Paivio, 1971). De Groot showed an unfamiliar chess situation to chess players of various strengths for five seconds. In reconstructing the chessboard from memory, average and even expert players made many errors, while players such as ex-world-champion Max Euwe recalled each situation perfectly. However, when the same pieces were placed randomly on the board, chess masters showed no superiority in recall. Binet found that unlike less experienced players who visualise the board in concrete detail, expert players in blindfolded chess games retained only the geometry of the situation, in images involving patterns of configurations. McKim (1972) wrote that "Pillsbury reported a sort of formless vision of the positions; Alekhine said he visualised the pieces as lines of force" (p. 105). Such imagery may be vague or vivid but its essential feature is that it is pattern-like and stripped of concrete details.

Thus imagery may be used to serve the function of generalisation in mathematics in at least these two ways, i.e., by concretising the referent (as in Alison M's "water level" image) or through the use of pattern imagery. Richardson (1969) wrote, "in the struggle to give form to an idea, imagery provides one of the easiest means of representing it and this can be developed in adults. It is not merely a physical object that can be represented in visual imagery but almost any subtlety of logical relationships" (p. 124). It was felt in this study that too few teachers were aware of the potential effectiveness of imagery in this representation.

Internal factors have been suggested as possible reasons for the preponderance of nonvisualisers amongst mathematical high achievers. These internal factors are likely to interact with external factors, which are considered next.

*External Factors*

In connection with external factors, three possibilities arise, and each of these will be discussed briefly. Firstly, it is possible that mathematics, by its nature, favours the nonvisual thinker. The verbal-logical component of thought is the *sine qua non* of mathematical abilities, whereas the visual-spatial component is not obligatory (Krutetskii, 1976). However, amongst mathematicians there is no dearth of thinkers of the geometric type; indeed, 16 of 34 mathematicians were classified as being of this type in two studies, as follows.

Olson (1975) quoted Ruth Day as writing that "people with different patterns of ability not only excel at different problems (something we know from the old I.Q. studies) but they solve the same problems in quite different ways" (p. 69). This is consonant with Krutetskii's (1976) observation that, although there is a correlation between analytic type and success in learning algebra, and between geometric type and success in learning geometry (implicit, after all, in the way he chose pupils for these categories) the types represent thinking modes which are independent of discipline: "It is impossible to believe that the analytic type is manifested only in algebra, and the geometric in geometry. An analytic cast of mind can be shown in geometry too, and a geometric one in algebra" (p. 316). This fact was confirmed in the writings of Hadamard (1945). Kuyk (1980) quoted from a study of the prodigies of mathematical physics in the first three decades of the nineteenth century, in which Poisson, Laplace, Ampere, Carnot and Lagrange were classified as algebraic, and Euler, Fresnel and Fourier were classed as geometric, and he commented, "The list shows that 'preferred modes' have nothing to do with the choice of discipline of mathematicians. One may be an algebraist with a geometric way of thinking, etc." (p. 21) Kuyk speculatively added a few more names to the lists, as follows:

*Algebraic*: de Fermat, Newton, Weierstrass, Mobius, Kronecker, Dedekind, Polya;

*Geometric*: Descartes, Leibniz, Dirichlet, Galois, Klein, Hamilton, Hermite, Hadamard, E. Noether, Weyl, Einstein, Brouwer, Thom.

Those who show no clear preferred mode ("bimodals") include Gauss, Cauchy, Jacobi, Riemann, Poincare and Hilbert.

Thus it is not simply the nature of mathematics that prevents visualisers from attaining the highest successes at school.

A second possibility is that the school mathematics curriculum, in which achievement is measured by performance in tests and examinations, favours the nonvisual thinker. It is possible that the time constraints of testing procedures militate against extensive use of time-consuming visual methods.

In an analysis of difficulties experienced by the 54 visualisers in this study in solving problems from the senior high school mathematics syllabus as reflected in past examination papers, in task-based interviews, it was found that those difficulties which involved imagery bore out the limitations of imagery suggested in the literature, viz.,

(1) the one-case concreteness of an image or diagram may tie thought to irrelevant details, or may even introduce false data;
(2) an image of a standard figure may induce inflexible thinking which prevents the recognition of a concept in a non-standard diagram;
(3) an uncontrollable image may persist, thereby preventing the opening up of more fruitful avenues of thought, a difficulty which is particularly acute, on the one hand, if the image is vivid;
(4) on the other hand, imagery which is vague needs to be coupled with rigorous analytical thought processes if it is to be helpful.

Difficulties were also experienced in the first three of Krutetskii's five non-obligatory categories, viz., swiftness – since visual methods apparently do require more time – difficulty with computation, and difficulty in remembering formulae. Twenty-one of the 54 visualisers experienced difficulty in remembering formulae as a major stumbling block. The time constraints of examinations in school mathematics might result in a dependency on swiftness and facility with formulae assuming undue importance for successful achievement: examinations may measure abilities such as this non-obligatory ability to remember formulae rapidly to a greater extent than they measure true mathematical abilities as listed in Krutetskii's outline.

A third possibility is that in most school classrooms the teaching emphasis is on nonvisual methods (Moses, 1977) and that even when visual methods occur, these methods of solution are not valued by teachers: textbooks may also stress an orderly logical progression of mathematical material and formal, nonvisual proofs and examples at the expense of visual methods. Classroom and textbook emphasis on nonvisual methods may derive from the need to communicate mathematical ideas. As Skemp (1971) pointed out, more often than not a proof or solution which involves a diagram also requires a nonvisual symbolic explanation for mathematical completeness. The tendency for a nonvisual method of solution to be precipitated by the need to communicate the process was illustrated in an interview with Mark J. In preparing to sketch a parabola Mark had calculated the intercepts on the $x$ and $y$ axes, and he had an image of the shape of the curve.

Mark J: I'd have to ... yeah I'd have to work out the axis of symmetry.
I: Could you find the axis of symmetry from the $x$ intercepts?
Mark J: Well, it would be halfway in between. It has to be.

Mark proceeded to calculate the $x$ co-ordinate of the turning point arithmetically by finding the mean of the two $x$ intercepts. He explained later in the interview that he wanted to find the midpoint by measurement on the $x$ axis but because he was required to communicate his method he used calculation.

In addition to the pre-emptive effect of the need to communicate, the foregoing extract also illustrates that a nonvisual method is usually considered to be more acceptable to a teacher and therefore to a researcher.

It was found that the thirteen teachers in this study fell neatly into three groups with respect to the visuality of their teaching, i.e., a non-visual group ($N=4$), an intermediate group ($N=4$) and a visual group ($N=5$). Teachers in the nonvisual group dispensed with visual presentations whenever possible; teachers in the middle group used visual presentations but devalued them, while teachers in the visual group used and encouraged visual methods.

In the classes of teachers in the nonvisual group, it was found that nonvisual teaching had the effect of leading visualisers to believe that success in mathematics depended on rote memorisation of rules and formulae. In fact, throughout the groups of teachers, there appeared to be an inverse relationship between the teacher's use of pattern-seeking methods and delayed use of symbolism – which were found to be associated with visual teaching – and visualisers' need for a rule or formula. With a nonvisual teacher, very few of the visualisers in this study fulfilled their potential.

Visualisers in classes of teachers in the middle group appeared to benefit from a teaching stress on abstraction and generalisation which was associated with this group of teachers. Pattern imagery and rapid use of curtailed methods were encouraged in the thinking of visualisers with these teachers. The two visualisers who attained *A* or *B* symbols on higher grade in their final examinations were in the classes of middle group teachers. Indeed of nine relatively high achievers (i.e. who attained *A*, *B* or *C* symbols on higher grade or *A* on standard grade) amongst the visualisers, five were with middle group teachers, one was with a nonvisual teacher and three were with visual teachers. The importance, in Krutetskii's (1976) structure, of generalisation, curtailment and logical economy for high mathematical achievement, was confirmed in this study. Teaching by teachers in the middle group was thus found to be optimal for these visualisers. However, no pupil in this study decided to continue with mathematics as a career choice, and the highest achievers with teachers in the middle group manifested cognitive dependence on their teachers, i.e., they tended to feel more secure in their problem solving if they were thinking the way they felt their teacher would have thought. If visualisers who are more successful tend to be cognitively dependent on their teachers, then nonvisualisers might have the edge in attaining the heights of independent excellence which would cause them to be classified as "stars". Most of the teachers in this study were unaware of their pupils' use of imagery and of the associated limitations which they had to overcome. As one teacher commented in an interview, teachers simply do not have the time to probe their pupils' thinking for images. It follows, therefore, that few teachers could lead even those visualisers who were cognitively dependent on them to see and to use "peculiar excellences" of imagery (Bartlett, 1977).

The foregoing consideration led to the writer's hypothesis that visual teachers might be more effective than middle group teachers in teaching visualisers, if these pupils could be educated to overcome some of the difficulties inherent in visual cognition. It is clear that visual presentations are not a universal panacea. Teachers in the visual group were unanimously positive in their attitudes towards visual methods, but they were not always able to lead visualisers to overcome the difficulties, and to make optimal use of the strengths, of visual processing.

In summary, three external factors were discussed in connection with the dearth of high-achieving visualisers in high schools. The nature of mathematics was rejected as a contributing factor because research mathematicians of "geometric" type are well represented, irrespective of the area of mathematics in which they are engaged. Secondly, the time constraints of school testing procedures were seen to militate against extensive use of time-consuming visual methods. Thirdly, it was suggested that textbooks and current teaching methods favour the nonvisualiser insofar as most teachers are unaware of difficulties inherent in visual cognition and of ways of overcoming these difficulties.

In view of the fact that most teachers are unaware of the difficulties associated with visual processing in mathematics, and of the fact that *these difficulties may be overcome*, it seems likely that increased teacher awareness of these issues could aid visualisers in entering the category of "stars". The writer suggests that at the present stage of knowledge about the role of visual imagery in high school mathematics it would be possible to work out a course suitable for teacher-training. Continuing research would, however, be necessary to monitor the effects of such a course. Since it is possible that visualisation may be useful in the creative processing required by research mathematicians, the following quotation from Galton in 1880 is particularly appropriate:

> An over-readiness to perceive clear mental pictures is antagonistic to the acquirement of habits of highly generalised and abstract thought and if the faculty of producing them was ever possessed by men who think hard, it is very apt to be lost by disuse. The highest minds are probably those in which it is not lost, but subordinated, and is ready for use on suitable occasions.
>
> (Bishop, 1980, p. 1)

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