H. O. POLLAK

HOW CAN WE TEACH APPLICATIONS OF MATHEMATICS?

I. INTRODUCTION

The purpose of this presentation is to discuss something of the relationship between applications of mathematics and the teaching of mathematics. While we shall not attempt a precise definition of applications of mathematics, it is clear that the phrase connotes a connection of mathematics with something else. More than that, there is the implication that this is a useful connection, that something of 'practical' value might be expected to come of it.

Now how does the student become involved in applications of mathematics? Throughout most of his education, mainly through problems in his textbooks, problems that have come to be known as 'word' problems. We therefore wish to analyze such problems – to see what is right and what is wrong with the problems we use.

II. IMMEDIATE USE OF MATHEMATICS IN EVERYDAY LIFE

We begin with the most obvious, and least controversial, applications of mathematics, namely, immediate uses in everyday living. It is important to realize that not just arithmetic but also algebra, geometry, probability, statistics and in fact most of elementary and secondary mathematics, are likely to come up. Such everyday applications may be either exact or approximate in character. When we check the computation of the sales tax, when we try to figure out how much paint it will take for the living room, when we refigure a recipe for a different number of people, when we try to build or to move a bookcase, or buy a rug of the right size, or win a little money at poker, or plant tomatoes, we are forever using mathematics in everyday life. Problems of this kind quite naturally will also appear in our textbooks. Here are some examples of perfectly sensible everyday problems taken from random texts that happen to be in my office.[1]

Mr. Twiggs changed the price of potatoes in his store from $4\frac{1}{2}$ c a pound to 3 pounds for 14 c.
(a) Did he raise or lower the price?
(b) How much was the increase or decrease per pound?

A large sandbox with a base 10 ft. long and 9 ft. wide is built in a park. A dump truck carrying five cubic yards of sand is emptied into the box. If the sand is leveled off, what is its depth? Give the answer both in feet and in inches.

Educational Studies in Mathematics 2 (1969) 393–404; © *D. Reidel, Dordrecht-Holland*

A boy has 24 ft. of wire fence to make a rectangular pen for his pet rabbit. He plans to use all the fence in making the pen. Could he make a pen 12 ft. long and 12 ft. wide? Why or why not? Could he make a pen 8 ft. long and 3 ft. wide? How about 8 ft. long and 4 ft. wide? Give five examples of lengths and widths he could use for his pen.

In all the discussions about applied mathematics no one has ever questioned the value of such problems. They are realistic and perfectly reasonable. We only wish that there were more, and a much greater variety of them.

III. PROBLEMS THAT USE WORDS FROM EVERYDAY LIFE AND PRETEND, IN VARYING DEGREES TO BE APPLICATIONS

The next class of problems in our textbooks, perhaps the most abundant of all, are those that use words from everyday life outside of mathematics to make the problem sound good. The key feature of all these problems is that a certain amount of translation from English to mathematics is required before you start, and the point presumably is to practice such translation along with practicing the subsequent mathematical technique. The statement of such problems rarely questions the honesty and genuineness of the connection to the real world, but the connection is often false in one or more ways. Some examples:

An electric fan is advertised as moving 3375 cubic feet of air per minute. How long will it take the fan to change the air in a room 27 ft. by 25 ft. by 10 ft.?

The trouble with this is that it pretends that all the old air in a room is removed before any new air comes in. Air simply does not behave in this way. There is, in fact, considerable intermixing and dilution of the old air by the new. Have you ever noticed how long it takes even a powerful fan to get the smell of a burnt pot of beans out of the kitchen? The answer to the problem is at best a lower bound.

The next examples are taken from Dr. J. M. Hammersley, *Bulletin of the Institute of Mathematics and its Applications*, October 1968. They are examination questions.

Question A16. The mean survival period of daisies after being sprayed with a certain make of weed killer is 24 days. If the probability of survival after 27 days is $\frac{1}{4}$, estimate the standard deviation of the survival period.

Professor Hammersley comments on this as follows:

Now, is the candidate expected to apply some cookbook method which assumes that survival times are normally distributed; or is he to try to make his mathematical model realistic? If the latter, then he will have to remember that the distribution is a mixture of two components – the lifetimes of daisies which escape all effects of the spray and of those which do not – and neither of these two distributions are likely to be normal; and he is up against some pretty awkward mathematics. The question hardly suggests that the examiner has extensive experience of constructing mathematical models.

Professor Hammersley's next example and associated comments, are the following:

In the S.M.P. Ordinary Level paper we have some multiple choice questions, and the candidate has to encircle the letter or letters corresponding to any correct answer. Question 16 runs as follows:

"Passengers are allowed 40 lb. of luggage free of charge; any amount in excess of 40 lb. is charged at 3d. per lb." If W lb. is the weight of the luggage (W is an integer) and C shillings is the cost, the regulation quoted above is equivalent to

(a) $C = 3(W + 40)$; (b) $C = \frac{1}{4}(W-40)$;
(c) $C = 40 + 3W$; (d) $C = \frac{1}{4}W-40$.

Since all four choices are false, how does the examiner distinguish between a candidate who answers the question correctly and one who does not attempt it at all? The notable point about this question is that there is no need to scrutinize the individual coefficients in these formulae; all the formulae are linear, and therefore obviously false since the situation is non-linear.

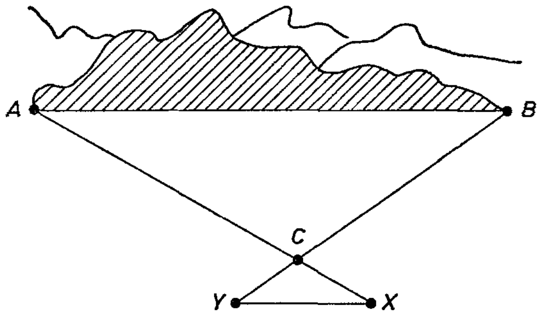
The same spirit of objection applies to all the pipes of various diameters that empty and fill tanks and swimming pools, to railroad schedules, accurate to the nearest second, for constant-velocity infinite-acceleration trains, to all the factories with linear production costs and known customer response to price, and many other stereotypes. Some of these problems perhaps contain a kernel of truth, and provide answers of some qualitative validity. If an attempt were made to discuss their relation to reality, and to be honest with the student generally, they might be acceptable. However, one of the first points about which to be honest with the student is the precision of the data. Rough and approximate calculation is not only excellent mathematical practice, but may, in fact, be the only justifiable response to the approximations made in obtaining the mathematical model of reality which the problem represents.

IV. PROBLEMS THAT USE WORDS FROM OTHER DISCIPLINES

The second main group of applications of mathematics, besides those to everyday life, are those to scientific or engineering disciplines, and, perhaps less commonly, to other scholarly fields. The same troubles that beset applications to everyday life recur in these other situations.

'Word' problems that pretend to come from other scholarly or engineering disciplines tend, once again, to be exercises in translation and in the subsequent mathematical techniques, and the reality of the application is often neglected.

To find out how far to tunnel through a hill, a surveyor lays out $AX = 100$ yards, $BY = 80$ yards, $CX = 20$ yards, and $CY = 16$ yards. He finds, by measurement, that $YX = 30$ yards. How long is the tunnel, AB?



The assumption that A, B, C, X and Y are all in a plane should perhaps be stated; the problem requires it, but it is probably reasonable. The only difficulty is that if the land on the near side of the hill is flat, no one would dream of building a tunnel!

A more suspect example is the following alleged application of combinatorial reasoning to linguistics:

It is a rule in Gaelic that no consonant or group of consonants can stand immediately between a strong and a weak vowel; the strong vowels being a, o, u; and the weak vowels e and i. Show that the whole number of Gaelic words of $n+3$ letters each, which can be formed of n consonants and the vowels aeo is $(2(n+3)!)/(n+2)$ where no letter is repeated in the same word.

The context of Gaelic orthography here is really questionable; doesn't it matter whether the resulting word is pronounceable, and is there really an arbitrary number of consonants? How large is the Gaelic vocabulary of such words?

Another example, which was specifically intended to show the application of mathematics to science, is the following:

The specific weight of water (s) at a temperature $t°c$ is given by the equation:

$$s = 1 + at + bt^2 + ct^3 \quad (\text{for } 0° < t < 100°\text{C})$$

where t = temperature in degrees centigrade, $a = 5.3 \times 10^{-5}$, $b = -6.53 \times 10^{-6}$, $c = 1.4 \times 10^{-8}$.

At which temperature will the water have the maximum specific weight?

Solution: $(ds)/(dt) = a + 2bt + 3ct^2$. When $(ds)/(dt) = 0$, $t = 4.09$°C.

Since the second derivative, $(d^2s)/(dt^2) = 2b + 6ct$, is negative in the above-mentioned range ($0 < t < 100$ – all temperatures at which water is liquid – under pressure conditions) s is a maximum when $t = 4.09$.

Now it is perfectly possible that for future purposes, one might want a cubic approximation to the experimental curve plotting the specific weight of water against temperature. It would, of course, be much more exciting if we had some physical intuition leading us to the reasonableness of a cubic, and if a, b, and c, had an interesting interpretation. However, it is unlikely that the cubic approximation to the real data, or any other analytic fit, would be used to find the maximum specific weight – you would certainly refer to the original data for that. Eyeballing the experimental points will give a far better guess for the location of the extremum than fitting a cubic over a very big range and then differentiating it. Of all the purposes for which you *might* use a cubic approximation to the specific weight of water, locating the point near 4°C of maximum density is perhaps the least likely.

It is probably not immediately relevant, but I should like to mention at this point a saying due to R. W. Hamming: "If the design of this airplane depends on the difference between the Riemann and the Lebesgue Integral, I don't want to fly it."

V. PROBLEMS OF WHIMSY

At the far end of typical textbook problems that are made to look like applications are problems which can best be described as pure whimsy. These will use words from daily existence or from other disciplines, but it will be quite clear to everyone that no real application is intended. In my experience, mathematicians love these kinds of problems better than any other kind. The function of such problems, and there are many of them in our textbooks, is not quite clear. Perhaps they serve to bring a tolerant smile from a weary student, or to distract him momentarily from an otherwise dreary lesson by diverting the imagination to some more pleasant scene. They provide comic relief in the Shakespearean sense, and probably do a lot of good – although not as applied mathematics. Here are some examples:

Two bees working together can gather nectar from 100 hollyhock blossoms in 30 minutes. Assuming that each bee works the standard eight-hour day, five days a week, how many blossoms do these bees gather nectar from in a summer season of fifteen weeks?

In working on a batch of 100 blossoms, one of the bees stops after 18 minutes (just to smell the flowers), and it takes the other bee 20 minutes to finish the batch. How long would it take the diligent bee to gather nectar from 100 blossoms if she worked all by herself?

Sometimes such problems can have considerable mathematical interest. For example, at his summer school for secondary school students in 1966, Professor Kolmogorov gave the following problem (I do not have his exact phrasing):

A bee and a lump of sugar are located at different points inside a triangle. The bee wishes to reach the lump of sugar, while traveling a minimum distance, under the requirement that it must touch all three sides of the triangle before coming to the sugar. What is the shortest path?

In this series of apiarian problems no actual relation to the real world is implied. The stories serve to introduce some simple algebra problems on the one hand, or a highly ingenious and educational geometry problem on the other.

VI. GENUINE APPLICATIONS IN REAL LIFE

The applications of mathematics that we have examined so far, whether genuine or false or whimsical, have all been simple specific problems whose solution required only the direct translation of the story into mathematical terms and the application of standard mathematical technique. Actual applications of mathematics, of course, are often not as simple as that. Rather than beginning from a specific problem, we will probably be given a messy fuzzy situation which we are trying to understand. It may often be more difficult to find the right problem to solve than to solve the problem after you have found it. For example, consider the simple question, 'What is the best way to get from here to the airport?' The difficulty begins with trying to understand what you mean by 'best'. If you have a rented car and are in no particular hurry, then probably you mean minimum distance. Under other conditions, you might well mean minimum expected time. Depending on the time of day you might be satisfied with a reasonably short drive, but require the minimum number of intersections at which you do not have the right of way. You may also wish to take into account the annoyance or the danger or the police patrol on any particular route. Have you ever been in the area before, and what is your probability of getting lost on a non-standard path?

If you are late because of bad weather, is the airplane also likely to be late? After talking to a number of people I personally have reached the conclusion that they mean, by the 'best' route to an airport, the route not with minimum expected time but with minimum variance in time. People are willing to put up with a longer average time if the spread is small.

Supermarkets are a wonderful source of real situations which are subject to useful mathematization, and I recommend that mathematicians frequently do the family shopping so that they may see these problems. When a product comes in several sizes, which is the cheapest? The immediate level of this question can be answered by simple arithmetic. There may, however, be other angles. If you buy too large an amount is it likely to become spoiled, stale or moldy before you finish it? If you get a free bath towel with one size, and a free dish towel with another, which is really cheaper? And the extra large roll of paper towels is no good to you if it is too fat to fit the kitchen rack.

An excellent exercise in applications of mathematics may be achieved by forming a group of students into a firm of junior consultants for an hour. Here is an example of how this might be used. In most supermarkets you typically find one check-out line labeled 'express lane', for n packages or less. When I experimented with this pedagogic technique, the students, in their experience, had found n to vary between 5 to 15. Obviously, if a number varies this much, people do not understand what it should be. The problem, therefore, is to find out how many packages you ought to allow in an express lane. The group of students had a wonderful time with this. First of all, what is the express line for? Obviously you want to make the most money for the stockholders, but how do you do that? Do you want to minimize the average waiting time? If so, do you mean average per package or average per customer? Do you want to minimize the expected maximum waiting time? Do you wish to minimize the probability that the wait exceeds 10 minutes, or any other number you care to choose? Next, if you have agreed on what you are trying to accomplish, how do you make a mathematical model? What is the relationship between check-out time and the number of packages? Is it good enough to assume a linear relation? Obviously, such a straight line would not go through the origin. However, if the number of packages is sufficiently large, a second sack will be needed. Perhaps a discontinuous, piece-wise linear function should be used! Is it sufficient to assume a deterministic model for check-out time, or is a probabilistic description necessary? How should the model differ between supermarkets that weigh produce at the check-out position and those that weigh and price at the produce counter? Next, what do you say about the arrival behavior of the customers? How far apart are they likely to be? There are, of course, many other questions that might come up. These will be hotly debated by the students. The hour will go very fast.

There are innumerable situations in everyday life that can lead to similar mathematical questions. Can you build a perfect cubical box out of six identical pieces of lumber? What is the best strategy for raking leaves on a lawn? If you wash a pile of socks and hang them up on a line in the basement, what are the chances of finding a pair adjacent to each other? What is the best distance for spacing cars in a tunnel? Given the pattern of the traffic lights in New York City, what is the quickest way to walk between two locations? It is quite clear that not every situation of this sort will be formulated successfully into a precise mathematical problem, not to mention one that the students can already solve. Furthermore, you will not be sure at the beginning what kind of mathematics will result. Nevertheless, it is terribly important for students to have practice in seeing situations in which mathematics might be helpful, and in trying their hand at formulating useful problems. In fact, one of the most valuable lessons which comes from trying real applications of mathematics is that *finding* a problem that is 'right' for a particular fuzzy situation is itself a real mathematical achievement. This is important in the classroom not only because it is the honest truth, but also because it helps to de-emphasize the 'answer' as the sole goal of mathematics, and helps to shift emphasis toward mathematical structure and process.

In the couse of practicing mathematization from real life, students will, incidentally, discover that for some situations mathematics is quite irrelevant. This too is very valuable.

VII. GENUINE APPLICATIONS TO OTHER DISCIPLINES

Applications of mathematics to other scientific, scholarly, and engineering disciplines follow very much the same pattern. Once again, there are situations whose understanding needs to be improved. Once again, formulating the right mathematical problem is likely to be half the battle. It is important to realize that situations to be mathematized will arise from many different disciplines. Besides physics, which has long been recognized as a major field of application of mathematics, all branches of engineering, all the other physical sciences, as well as the social and biological sciences, are nowadays leading to interesting mathematics. For example, biologists make models of the prey-predator cycle and attempt to analyze the spread of epidemics, as well as continuing to develop the more familiar theory of mathematical genetics. Engineers apply topology to printed circuits, probability to random vibrations, and modern optimization techniques to production and inventory control, as well as using lots of differential equations.

Even lawyers sometimes get into serious mathematics. The following is an extract from Section 217, on the taxability of reimbursed moving expenses, of the current (1969) Internal Revenue Code:

(a) Deduction Allowed – There shall be allowed as a deduction moving expenses paid or incurred during the taxable year in connection with the commencement of work by the taxpayer as an employee at a new principal place of work....
(c) Conditions for Allowance – No deduction shall be allowed under this section unless – (1) the taxpayer's new principal place of work – (A) is at least 20 miles farther from his former residence than was his former principal place of work, or (B) if he had no former principal place of work, is at least 20 miles from his former residence.

Where are all former residences satisfying these conditions? How is the problem different between air miles and highway miles?

It is crucial that mathematics problems which claim to be applications of mathematics to other disciplines be *honest* applied mathematics. This means that the relationship between the mathematical model and the situation in the outside world that is being mathematized must be clearly understood. It is rather ridiculous to just say some words from another discipline, and then exhibit some equation which you claim is relevant. How did you find it? What approximations did you make in obtaining this equation? How will you tell whether the mathematical conclusions are meaningful in the original real-world situation?

Here is an example in which the process of creating the mathematical model for the physical situation is perhaps insufficiently explored:

The reaction of the body to a dose of a drug can be represented by the following function:

$$R(D) = D^2\left(\frac{C}{2} - \frac{D}{3}\right),$$

where C = a positive constant, R = the strength of the reaction (for example – if R is the change in blood pressure, it is measured in mm mercury, if R is the change in temperature, it is measured in degrees, and so on), D = the amount of the drug.

We will assume that whenever the drug is administered, the concentration of the drug already in the body is insignificant; for if there is already a certain concentration of the drug in the body, the reaction will depend upon this initial concentration (Weber-Fechner Law).

D is defined in the range of 0 to C. (In other words, C is the maximum amount that may be given.)

Find the range of dose for which the medicine has maximum sensitivity, in other words, where there is the greatest change in R for a small change in D, or equivalently where $R'(D)$ is at a maximum.

The equation in this problem is certainly suspect, if only because unfamiliar. What led you to believe that this relation is true? How does the functional form arise from first principles? What exactly do you measure, and why? How can the units possibly work out right? What kinds of useful understanding can be derived from such a model? A lot needs to be said to make this example into good applied mathematics.

Even applications of mathematics in classical physics frequently suffer from being badly presented as applied mathematics. It is an unfortunate fact that in much of our physics teaching the relation between the phenomena that we are trying to understand and the corresponding approximate mathematization has been lost in the shuffle. People tend to start a derivation in physics by writing down the mathematical equation and going on from there. The student is typically not given the opportunity to participate in making the abstraction from the physical reality to the mathematical model. Why not? It is of course possible to argue that the particular mathematization has been familiar for many years, that it is well known that it works very well and that it is a waste of time to rederive it. Similarly, it is sometimes argued that it is a waste of time to worry about existence and uniqueness for solutions of the resulting differential equations since this is after all obvious physically. The trouble with these arguments is that they are not good physics but just bad applied mathematics. The student has as much right to participate in the derivation of the mathematical model and in checking the degree of its validity as he has to repeat any experiment in order to satisfy himself of *its* validity.

Take for example, the old and familiar example of the motion of a pendulum. We assume frictionless suspension, weightless inextensible string, no air resistance, and point concentration of mass. Perhaps we should make an attempt to estimate the influence of these approximations, but at least we have mentioned them to the student, and appealed to his personal experience to make the neglect of these factors appear plausible. Part of the way through the derivation, however, we make the further approximate assumption that we may replace $\sin\theta$ by θ because 'θ is small'. The student has no 'physical feeling' for this approximation, and we owe him an opportunity to see how much of a mistake we make in the period – especially since it can easily be done within the student's knowledge of analysis at this stage. If we define T_1 as the period obtained with the replacement of $\sin\theta$ by θ, and T_2 as the period without this approximation, then, if α is the maximum angle,

$$T_1 = 2\pi\sqrt{l/g} = 4\sqrt{l/g}\int_0^\alpha \frac{d\theta}{\sqrt{\alpha^2-\theta^2}}$$

$$T_2 = 4\sqrt{l/g}\int_0^\alpha \frac{d\theta}{\sqrt{2\cos\theta - 2\cos\alpha}}.$$

On the next page, we shall show that

$$(\alpha^2-\theta^2)\frac{\sin\alpha}{\alpha} \leqslant 2\cos\theta - 2\cos\alpha \leqslant (\alpha^2-\theta^2)\left(1-\tfrac{1}{3}\sin^2\frac{\alpha}{2}\right),$$

so that *both* upper and lower bounds on the error are easy.

Therefore

$$\frac{1-\cos\alpha}{12} \leqslant \frac{T_2-T_1}{T_1} \leqslant \left(\frac{\alpha}{\sin\alpha}\right)^{1/2} - 1.$$

In particular, for example, if $\alpha = 6°$

$$0.00045 \leqslant \frac{T_2-T_1}{T_1} \leqslant 0.0009.$$

A grandfather's clock 5 feet long, with a maximum swing of 6 inches of the tip, is to be designed. Is the approximate period T_1 good enough? The above formulas show that the error is *between* 0.6 and 1.2 min in a day, which is *not* good enough. A more precise formula is used for designing clocks.

The above inequalities are both proved from the fact that $(\sin x)/x$ has a local maximum at $x=0$.

$$\frac{\sin x}{x} \geqslant \frac{\sin \alpha}{\alpha}, \ |x| \leqslant \alpha \leqslant \frac{\pi}{2}$$

$$\int_{\theta}^{\alpha} \sin x \, dx \geqslant \frac{\sin \alpha}{\alpha} \int_{\theta}^{\alpha} x \, dx$$

$$2(\cos\theta - \cos\alpha) \geqslant \frac{\sin\alpha}{\alpha}(\alpha^2 - \theta^2) \tag{1}$$

$$\frac{\sin y}{y} \geqslant \frac{\sin \alpha/2}{\alpha/2}, \quad |y| \leqslant \frac{\alpha}{2}.$$

$$1 - \cos x = 2\sin^2\frac{x}{2} \geqslant 2\frac{x^2}{\alpha^2}\sin^2\alpha/2, \quad |x| \leqslant \alpha. \tag{2}$$

Integrate from 0 to x:

$$2\sin x \leqslant 2x - \frac{4x^3}{3}\frac{\sin^2\alpha/2}{\alpha^2}.$$

Integrate from θ to α:

$$\begin{aligned} 2\cos\theta - 2\cos\alpha &\leqslant (\alpha^2 - \theta^2)\left(1 - \frac{\sin^2\alpha/2}{3\alpha^2}(\alpha^2 + \theta^2)\right) \\ &\leqslant (\alpha^2 - \theta^2)\left(1 - \frac{\sin^2\alpha/2}{3}\right). \end{aligned}$$

VIII. CONCLUSION

We have made an attempt to classify problems that are intended to give the student the feeling that he is applying mathematics, and we have seen some of the difficulties in making these problems realistic. Much work remains to be done to make real applications of mathematics an integral part of the classroom experience. It is important to realize however, that formulating the 'right' problem in a situation outside of mathematics is creative activity much like discovering mathematics itself. Thus our continuing efforts to bring the discovery method to the classroom naturally go hand-in-hand with attempts to bring genuine applications to the classroom. The two efforts reinforce each other, and both are essential for a complete and honest presentation of mathematics in our schools.

Bell Telephone Laboratories, Inc.
Murray Hill, N.J.

REFERENCE

[1] References are, in most cases, simply not available, because they were chosen essentially at random; the books, in turn, were sent to me by publishers who were kind enough to overlook my lack of a regular teaching appointment. My apologies in advance to any authors, editors, and publishers whose favorite problems may be treated a bit unkindly in the sequel.