CELIA HOYLES AND RICHARD NOSS

A PEDAGOGY FOR MATHEMATICAL MICROWORLDS

ABSTRACT. In this paper we attempt to map out some relationships between pedagogy and student behaviour in a mathematical microworld. We illustrate our analysis by a series of episodes which occurred in a Logo-based microworld constructed around the notions of ratio and proportion. We explore the patterns of pedagogy associated with on and off-computer activities and suggest how the teacher has significant roles in both settings; particularly in helping pupils to bridge the discursive disjuncture between the practices of Logo- and school-mathematics.

Our aim in this paper is to try to map the relationship between pedagogy and student behaviour in a mathematical microworld. The problem derives from our appreciation of the inescapable and perhaps unpalatable fact that simply by interacting in an environment, children are unlikely to come to appreciate the mathematics which lies behind its pedagogical intent. The key question is thus to try to outline the relationship between the intended and actual mathematics learning. In doing so, we will have inevitably to confront the nature of pedagogy, the kind of mathematics we intend, and the relationship between the two.

We begin with our perception of a microworld. The *software* at the core of a microworld models mathematical fragments but does not itself embody pedagogic intentions. We have suggested elsewhere that a microworld is best thought of as considerably *more* than software (see for example, Hoyles and Noss, 1987a, 1991). Pieces of knowledge are appropriated (or not) depending upon pupils' own agendas, how they feel about their participation, teacher intervention, and above all, the setting in which the activities are undertaken. Thus it is misguided to argue that simply by interacting with the computer, children are in general likely to 'acquire' specified mathematical ideas (although they might, of course, acquire others).

While pupils structure their activities by their own actions, it is also clear that their actions are structured by the constraints and design of the activities in which they are engaged. That is to say, pupils are constrained (and, perhaps, empowered) by the tools they have available, by the syntax and semantics of the expressive medium they have to hand. Thus the question of pedagogy is of paramount importance: while it would be naive to suggest that pupils normally learn what it is intended they learn, we need to consider carefully the different strands of pedagogy built into microworlds in order to try to disentangle what *might* be learned and *how*.

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This problem is not, of course, unique to computational environments. However there is a level of complexity which enters into attempts to monitor and guide pupil interactions with computers, simply by virtue of the fact that rather little is known about the macro-level of such interactions: i.e. how the computer enters into and interacts with the other elements of the pedagogical setting. Within computational environments which genuinely offer the opportunity for the expression of mathematical ideas, pupils can use ideas *before* they have fully discriminated the relationships involved, and moreover, use them in ways which inevitably push in the direction of mathematical formalisation – even typing the name of a program and running it involves an abstraction at some level (for further discussion see Hoyles, 1987; Hoyles and Noss, 1987b, 1989).

The key question we want to address is what does a viable pedagogy look like in a microworld? This question cannot be investigated in the abstract. From our work over many years (see, for example, Hoyles and Sutherland, 1989[1]), we have evolved a style of pedagogy in computational environments in which we have tended to intervene *concurrently* with pupils' activities on the computer and to do so in relation to *explicit* mathematical agendas – this style is certainly not unique, and we do not offer it as a canonical methodology for intervention. One way of operating has been to build upon pupils' intuitive interactions with the computer by the promotion of cognitive conflict through computer feedback. This kind of intervention involves us in a critical tension: that is, to tread the line carefully between the pupils' room for manoeuvre and exploration on the one hand, and our own intentions and structuring on the other. We are convinced that without the former, much of the interesting potentiality of the computer is put at risk. Yet without the latter, our experience suggests that pupils' mathematical learning is at best haphazard.

We will discuss the specificities of this tension later. But it is worth stating in global terms how we have attempted a resolution in our own research. Broadly, we want pupils to have enough time and space to mathematise their own ideas with the computer. We want, in fact, pupils to *play* with the software.[2] At the same time, we want them to reflect upon the mathematical features with which they are playing, and in particular, to consider precisely the mathematical ideas which we intend. In order to encourage this metacognitive stance to actions on the computer and the feedback received, we adopt a variety of pedagogic approaches: set goals for the pupils (without specifying a solution strategy), intervene in the computer work to shift attention to the mathematical ideas on our agenda and suggest further lines of enquiry. The literature is surprisingly sparse in outlining the specificities and structures of pedagogies which attempt this balance, and it is our intention to explore these issues below. Here we merely outline our central claim, which is that the tension can be eased by recognising the particular ways in which the computer urges pupils towards *formalisation*, and in which pedagogies can be designed which scaffold pupils' sense-making in ways which approach the discourse of school mathematics.

AN EXAMPLE OF A MICROWORLD

Our approach in this paper is to try to point to the general in the context of the particular: that is, we report a series of *episodes* which took place within the context of a particular microworld and which are chosen to highlight specific aspects of the pedagogical relationships we are hoping to elucidate.

We reiterate our conception of the microworld – a set of activities in a computational setting, designed to be "rich" and "dense" as far as the predetermined mathematical domain is concerned. As we shall see, this does *not* imply that all the activities necessarily take place at the computer (for related descriptions of a microworld which place somewhat less emphasis on off-computer activities, see for example Thompson 1987, and Edwards 1991). In the study we report here, we chose as our mathematical domain the notions of ratio and proportion, and we employed Logo as a medium in which to operate. The research was part of a long-term investigation[3] carried out over three years: this is reported in detail elsewhere (Holyles, Noss and Sutherland, 1991).

A Computational Object for Ratio and Proportion

We focus our attention on a single component of the technical part of our microworld, a computational object we call (for obvious reasons!) HOUSE, a fixed Logo procedure (i.e. it has no *inputs* – see Fig. 1). There are three points we would like to make about the structure of HOUSE which was designed to focus the children's attention in specific ways. First, as is evident in Fig. 1, the lengths of the sides of HOUSE are *not* simply related – there is no obvious common unit or multiplying factor connecting them. This was a design feature of our simple software tool arising from our desire that pupils should enlarge HOUSE by using a scale factor rather than unit ratios within the shape. Second, since HOUSE is a closed shape, we expected that attention would be drawn to the necessity of using a

```
TO HOUSE
   HT
   FD 50
   RT 60
   FD 70
   RT 60
   FD 70
   RT 60
   FD 50
   RT 90
   FD 121
   RT 90
END
```

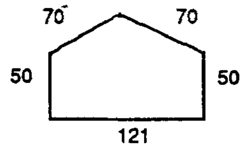

Fig. 1. HOUSE, a fixed procedure.

multiplicative scalar operator since failure to do so would result in unclosed or overlapping shapes (as illustrated in Fig. 2): as we shall see below, the *mathematical* necessity did not always translate into a similar necessity on the part of the pupils. Third, we recognised that turtle-orientation and turtle-turn are sometimes sources of confusion for children with limited Logo experience,[4] and we attempted to avoid these obstacles by providing procedures JUMP and STEP which respectively moved the turtle (without drawing) up and across the screen to the right, when used with positive inputs. Here we wanted – as Sylvia Weir (1987) has put it – the things that

```
TO BIGHOUSE
   HT
   FD 125
   RT 60
   FD 145
   RT 60
   FD 145
   RT 60
   FD 125
   RT 90
   FD 196
   RT 90
END
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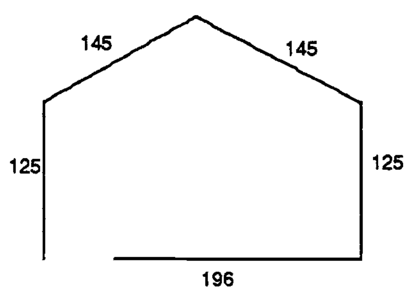

Fig. 2. The result of using an additive strategy to enlarge HOUSE.

matter to be precisely the things that the pupil has commands to change. So in *using* HOUSE as a module to create designs of pupils' own making, we did not particularly want them to concern themselves with turtle-orientation (of course, we did not prohibit them from turning LEFT or RIGHT if they so desired!).

In short we attempted to encapsulate into the mathematical structure of HOUSE precisely the mathematical ideas of ratio and proportion that we wanted to emphasise. We should stress that this is a pedagogical rather than an epistemological process. That is, we do not suggest that the object somehow 'embodies' the mathematics; rather that we structured the objects, *together with the accompanying pedagogy* (how it was presented; what activities we hoped to generate), in order that it could be used in particular ways within particular activity structures, and that within these activities there was a reasonable chance (certainly better than in unstructured Logo activity) that the mathematics *we* had in mind would surface. We are not interested in the figurative features of the objects created, but the structures and relationships which organise them and the actions on them which we define as allowable: for us the visual outcomes should be clues to tap into the mathematical ideas – they are not the ideas themselves. We also note in passing that the mathematical notions of ratio and proportion are not conceptually distinct from a range of other mathematical concepts (for example, decimals and fractions) and that these echoes and interrelationships unavoidably increase the complexity of any pedagogical intent. We will address this issue below.

Methodology of the Microworld Study

For the main study of our project, the ratio and proportion microworld was taught by the researchers to a whole class over a period of 6 weeks, one-and-a-half hours per week during the spring term of 1989. Twelve computers were available for every microworld session. The school in which the research took place was a Secondary Comprehensive school near London. the experimental class consisted of 28, 13-year-old pupils in the third set of six (organised into 3 bands with two classes in each band). Set 1 was classified as the top set according to school tests: thus the experimental group could be regarded as of 'average' mathematical attainment. Calculators formed a part of the normal classroom activities. Prior to the experimental work the pupils had experienced about 25 lessons in Logo of 35 minutes each and had undertaken a Logo project on tessellations over a period of about 20 hours. Thus, most of the pupils were relatively familiar with using the computers and with simple Logo syntax. Prior to the microworld experience, none of the pupils in the experimental class had been formally taught ratio and proportion in their mathematics classes.

The researchers were completely responsible for the teaching of the experimental class, for setting and marking homework assignments and giving feedback to the class teacher. Reciprocally, the class teacher was present at each session and gave us on-going feedback. The rationale for this methodology was that the researchers wished to gain a feel for the pedagogical constraints of teaching a whole class in school as well as be in a position to spontaneously respond to questions from pupils and make immediate assessments of their reactions to the work. The influence on pedagogy of teaching whole classes within the school system – much of our previous work has involved working with small groups – cannot be underestimated. Issues of management (of pupils and computers) become more dominant and monitoring of on-computer work less subtle. Additionally small-group work and class-discussions to compare and synthesise responses have to be *formally* organised rather than relying on them to emerge spontaneously.

The microworld was evaluated summatively in terms of learning outcomes in a number of ways described in detail in Hoyles, Noss and Sutherland (1991). Suffice to say that we constructed and administered a series of written pre- post- and delayed-post-tests to probe change in individual pupils' understandings of ratio and proportion within work problems posed in different settings. These findings showed significant overall improvement for the experimental group, but we do not report them in detail here. Rather our concern is with the interactions among pedagogy, pupils and activities and we therefore concentrate on process data on these interactions which was gained at two levels: first, by data on all the children in the class, and second, by close observation of a small number of pupils who were chosen so as to obtain a spread of mathematics attainment and gender distribution – these we refer to as the 'case-study pupils'. The progress of the case-study pupils during the miroworld experience was monitored by the researchers through observations and clinical probes during the activities (asking 'why' they had adopted a certain strategy for example). These probes were seen by us as part of our intervention strategy.

Thus our sources of data for analysing the teaching and learning within the micrworld are –

for the whole class:
• written notes of our planned aims and intended pedagogic interventions
• marked homework assignments

- print-outs of all procedures
- observation notes

and for the case-study pupils:
- dribble files[5] of all pupil-computer interactions
- observation notes.

Overall, our direct interventions – as much as the pedagogy underlying the activities – stressed the need for consistency, for a precise language and a formalised representation of working strategies. We can schematically present an overview of our pedagogic 'sequence' as follows:

- introducing the mathematical agenda
- a progressive sequence of tasks to be carried out by pupils in pairs on the computer with interventions from us to encourage the mathematisation of the activities and to promote prediction and reflection (inevitably rather haphazard given the whole-class setting)
- paper-and-pencil work arising from and directly related to the work with computers and class discussion of pupil-responses to the computer-based work
- small group activities attempting to bring together on- and off-computer work

Since we wished to endeavour *not* to simply impose a way of working or a set of rules to be learned by rote, we deliberately avoided the following pedagogic strategies:

- the design of specific links to "transfer" the meanings and understandings developed in the microworld to other contexts – in particular to the word-problems of the written tests;
- teach rules for working out proportion questions in any explicit or formal sense (such as the 'rule of three')

EPISODES FROM THE MICROWORLD

We do not intend to give a chronological description of the whole microworld: rather we describe five episodes selected to act as pegs on which we can hang more general pedagogical issues.

Episode 1: Introducing the Mathematical Agenda

Working in groups, the pupils were asked to sort in to classes, pictures which they thought were similar in some way, work out a justification for their sorting, write this down and report their classification to the rest of the class. The aim of this activity was to reveal and make explicit in natural language pupils' intuitions of the meaning of similarity, to confront them with different meanings and to highlight the differences between the use of 'similar' in everyday speech and its more specialised mathematical sense. We learned from our pilot data that a recognition of the mathematical meaning of 'in proportion' was a necessary (but not a sufficient) condition for the exploitation of visual feedback and the appreciation of conflicting evidence during computer-based activity. We hoped that pupils would move from a rough-and-ready categorisation (for example big or small) to more precise definitions provoked by the need to describe and justify their classifications to the class. Our pedagogic agenda was to encourage discussion, emphasise the need for precision and consistency in language, introduce the mathematical meaning of "in proportion", and act as devil's advocate provoking conflict when inconsistencies arose.

Pupils were then given the 'I Task' (see Fig. 3) to complete individually at home. As we have pointed out, pupils' views of their activities determine the way they read and attach meanings to them. In this task, our pedagogic strategy was to make explicit a variety of pupil conceptions[6] of the idea of proportionality which would provide the basis for discussion. The task was designed so that the larger I could be obtained either by simply adding 2 to corresponding lengths or by multiplying horizontal lines by 3 and the vertical line by 2. The prescribed responses range from the 'everyday' focussing on one attribute only (the shape) to a mathematically correct response focussing on relationships *within* or *between* the sides of the figure. We discussed the different answers in class while making a specific effort to emphasise that when constructing figures in proportion, it is important to maintain the relationships *within* a figure and to operate in a similar manner on each of its constituent parts.

Discussion. We begin with the most obvious point. Children came to our microworld – as to every other activity – with their own ways of making sense of what they see and do, and their own ways of expressing it. Specifically, it is clear that *in proportion* has little everyday currency in natural language, and in so far as it does, it often assumes the meaning of 'roughly equivalent'. Thus for us, in trying to generate *mathematical* understandings, it was important to *challenge* existing meanings of 'in proportion' as well as to introduce pupils to the specifically mathematical use of the term. This was the purpose behind the initial classification task where it was very

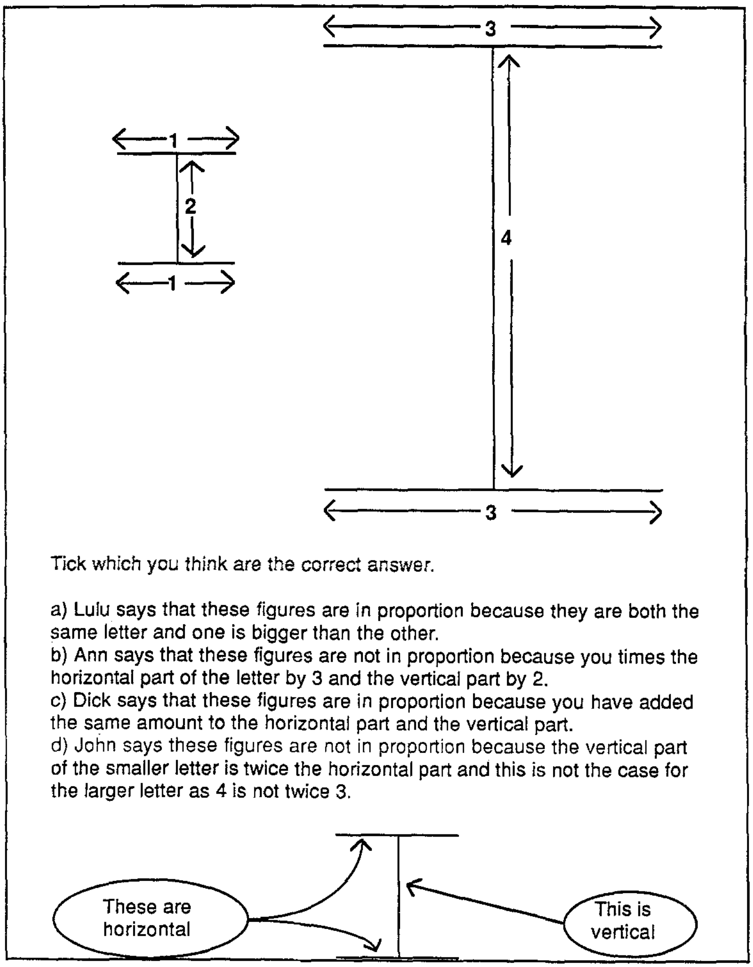

Fig. 3. The I task.

apparent that pupils tended to sort by one or perhaps two attributes rather than by the less tangible relationship *between* attributes.

The pedagogy behind the I task and its implementation had a deeper, more fundamental rationale: namely to try to link *local* mathematical understandings with a more *global* appreciation of mathematics as a coherent system distinct from the discourse of everyday language and activity (in Hoyles and Noss, 1991, we refer to *global* mathematical understandings (*GMU*), as opposed to *local* (*LMU*)). In the UK at least, there is an extreme coyness towards acknowledging that mathematics has its own register, its own rules of discourse and its own semantics. On the contrary, there is a marked tendency for teachers to draw pupils' attention to a (preferably concrete) referent for any abstraction to which they are introduced. This has a number of effects, not least that it emphasises mathematical objects at the expense of relationships: and it is precisely the relationship between objects which gives rise to the power behind the notion of proportionality.

Overall the classes' individual responses to the I task were divided equally between responses (b) and (c) (see Fig. 3). As an example, we consider the case of Toby. He began very uncertainly, and was unable in the classification task to choose between different strategies he had employed to sort similar objects – any strategy was acceptable provided 'it looked alright'. For the I task he picked out as correct answers (a) and (c) – that is, he thought the Is were in proportion as they were the same letter and the same amount was added to each part of the letter to obtain the larger I. The essential insight which Toby seemed to have gained after the activities within Episode 1 was one of a global understanding that mathematical relationships are different from everyday judgements – although, as we see later, he had by no means grasped the notion of ratio and proportion in its entirety! Being the same letter (I) is of incomparably more importance in everyday discourse than the internal or external relationships between the parts of the letter. Mathematically (at least in the scenario we set) the reverse is true, and it is this which we think Toby came to appreciate.

Episode 2: Developing Related Mathematical Intuitions and Exploiting Alternative Strategies

Ratio and proportion calculations frequently involve the use of fractions or decimals. Not unnaturally, a lack of facility with and even fear of decimal numbers (as well as inability to manipulate them) can act as barriers to successfully negotiating an activity which involves their use. Thus we carefully structured an activity to help overcome this barrier.

The pupils were asked to play the *Target Game*, as described in Fig. 4, in order to develop their sense of decimal numbers and the consequences of multiplying by decimal numbers. Before the Target Game we introduced the pupils to Logo's arithmetic operations which can be used to perform calculations – a facet of Logo not widely appreciated by the pupils. Our intention was that by playing the game the pupils would come to recognise the existence of numbers between 0 and 1 and differentiate them from numbers less than 0, to understand the outcome of multiplication by a

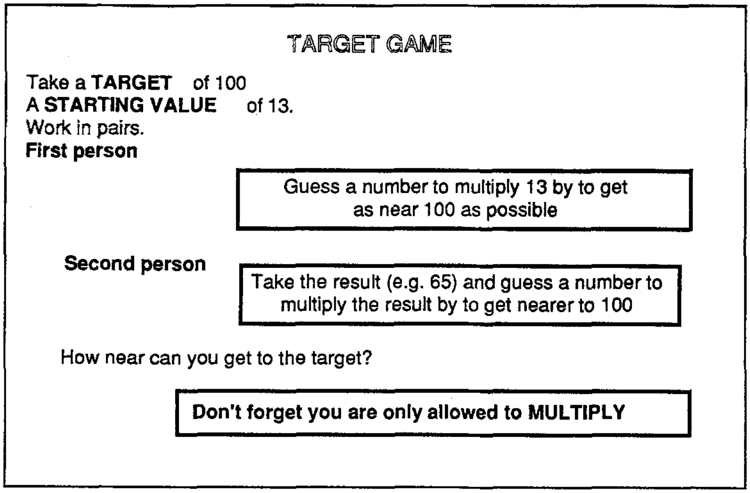

Fig. 4. The target game.

number between 0 and 1 and to begin to recognise that division by x is equivalent to multiplication by $1/x$.[7] In particular, we intended to challenge the pupils to see that multiplication could produce a result by the use of decimals that was smaller than the original number (for example, Bell et al. (1989) have identified this as a formidable conceptual obstacle).

Discussion. On one level, we might interpret this activity as simply a response on our part to a difficulty we anticipated would be encountered by the pupils. But we want to look more closely at the specificities of the activity and the way in which the computer entered into the setting. Primarily, our objective was to extend the range of pupils' *situated abstractions* – the ways in which people make mathematical sense of the results of their actions. These are sense-making devices which are *situated* in that they are derived from experiences within specific mathematisable situations. They are *abstractions* in two senses. First they operate *beyond* the specific experiences in which they arise (compare with Vergnaud's (1982) delimitation of 'theorems' to the realm of *action* or Lave's (1988) focus on 'practice'). Situated abstractions can be mathematically correct or incorrect depending on their domain of operation and how the generalisations are constructed. Second we imply that there is some *conscious* appreciation of the *generalised relationships* among the concepts involved (an extension of Vergnaud's notion of 'theorems in action' where children are seen to operate with relational invariants for specific values of the variables). One example of a situated abstraction might be the 'fisherman's mathematics' reported in Carraher, Schliemann and Carraher (1988) who suggest that fisherman developed an understanding of proportional relations which transcended the acquisition of simple procedural knowledge within their everyday practices. Another example and one which is germane to the discussion in this paper, is that 'multiplication makes bigger' – and it was this abstraction that we wanted to challenge by building *new* experiences based upon action and immediate explicit feedback where multiplication did *not* make bigger.

The Target Game is a useful example of how we see the computer entering onto the scene. Here we have a game, one which apparently caught the imagination of most pupils (a particular motivation was the very large number of decimal places of accuracy which it was possible to obtain) and which – because of the *immediacy* of the feedback generated by the computer – provided experiences of the effects of multiplication within a context different from that which formed the experience of most pupils. In this situation, the effect of multiplication by numbers less than one (and of multiplication by a negative number – apparently a surprise for many pupils) became a part of an activity with meaning, sufficiently integrated into pupils' experiences that for some at least, it enlarged the scope of their abstraction: the other side of the coin is, of course, that the boundaries of the situations over which their abstraction worked were more clearly and explicitly defined.

There is more to say about the role of the computer, but before we do so, we briefly give an example of one pupil's interaction with the Target Game. Jenny had answered correctly on the I-task (see Fig. 3) but we note that this did not require her to *construct* a proportional response. In her Logo work, Jenny was confused between negative and decimal inputs to primitives and procedures. However, in the Target Game, a fragment of which is shown below, she appeared to be developing new understandings. Jenny and her partner, Sue, were trying to reach a target of 100, with a starting value of 13. Sue started by multiplying 13 * 8 giving 104. Jenny's first response was to multiply (she knew the rules of the game!) but by −4. Amidst laughter at the feedback, the pair decided to start again. Jenny again tried 13 * 8, and Sue with a new value of 104 tried 104 * 0.25. The sequence continued with 26 * 3.75 (Jenny) then 97.5 * 1.2 (Sue). Jenny was then faced with 117. She tried 0.2 – the first time she had chosen to use multiplication by a decimal between 0 and 1 to reduce a number. The sequence of the pairs' attempts was then as follows:

23.4 * 4.25
99.45 * 1.01
100.4445 * 0.99
99.440055 * 1.0001
99.4499990055 * 1.001
99.549449004505 * 1.002
99.748547902514 * 1.002
99.948044998319 * 1.0000001
99.948054993123 * 1.000001
99.948154941178 * 1.0004
99.988134203154 * 1.00045
100.03312886354 * 1.00025
100.05813714575 * 0.9999995
100.05808711668 * 0.999995
100.05758682624 * 0.99995
100.05258394689 * 0.9995
100.00255765491 * 0.9995

This sequence appears to show a growing confidence and awareness of the effects of decimal multiplication. Our only interventions were at the beginning of the game to clarify the rules – that is division was *not* allowed and the pupils *must* take turns in their attempts to reach the target. Here at least we might be bold enough to point to the 'effect' of a particular activity: after these experiences, Jenny was 100% correct in her homework questions, an example of which was 'What do you multiply 80 by to give 30?' (a calculator was available).

Let us return to the role of the computer. We contend that the computer opens a range of alternatives, *strategic apertures* through which children can gain access to approaches and solutions which are simply unavailable with pencil-and-paper. In this case, the iterative 'solution' was open to Jenny – indeed it was the only one we offered, since it addressed the conceptual issue of decimal multiplication. This cuts to the heart of the matter: we argue that these kinds of apertures can be used to manipulate the situation pedagogically; that is, the opening of these strategic apertures can be exploited in order to increase the possibility of specific learning outcomes – in this case, enlarging the scope of pupils' situated abstractions to enhance the potential for mathematical learning. Our metaphor of apertures is intended to conjure an image of the pedagogic intentions of a microworld permeating through the openings made available by the computer.

Episode 3: The Computer as a Medium for Exploration

Pupils in pairs at the computer used HOUSE (together with STEP and JUMP as defined above) as they wished – for example to make rows of houses and patterns. The idea behind this activity was that in *using* HOUSE, pupils would begin to discriminate its constituent elements: as we will show in the discussion, such discriminations largely depended on our interventions. The computer activities then progressed to a more specific focus on ratio and proportion. We took the pupils' projects and asked them to build bigger and smaller HOUSEs but *in proportion* – a term introduced earlier. Our pedagogic emphasis was to *require* the pupils to make their strategies clear and to record on paper the operations they had used and the numerical results (in terms of the lengths of the sides of the houses). We attempted to build on pupils' facility with doubling by explicitly 'converting' doubling to ' * 2', asking them to generalise to other specific numbers and then finally to any number. Help had to be given with editing and syntax and also to introduce pupils to procedures with inputs. Figure 5 shows a selection of computer tasks and paper-and-pencil homework sheets given to the pupils following their work.

Discussion. When describing a sequence of activities of this kind, it is all too easy to ignore the need for playfulness and pupil-control. Although we did indeed intervene in the pupils' work, since at the simplest level we needed to encourage them to play – after all, playful activities are not exactly the norm in most mathematical classrooms. Our interventions were mainly motivational and centred around suggesting a range of "interesting" projects: they varied among different pupil pairs since they were supposed to arise from pupils' own projects – management was crucial here! But we also tried to help pupils to discriminate the relationship between the program which generated HOUSE and its visual image: for example, which numbers were the turns? which were the lengths? which lines had to be equal? which turns? Our strategy for achieving this was not by direct questioning but rather by suggesting an activity during which we anticipated the issues which would *have* to be confronted; for example, we suggested to some pupils that they construct a line of houses with no gaps in-between in order that they would have to discriminate, by reference to the HOUSE procedure, the necessary input to the STEP procedure. In

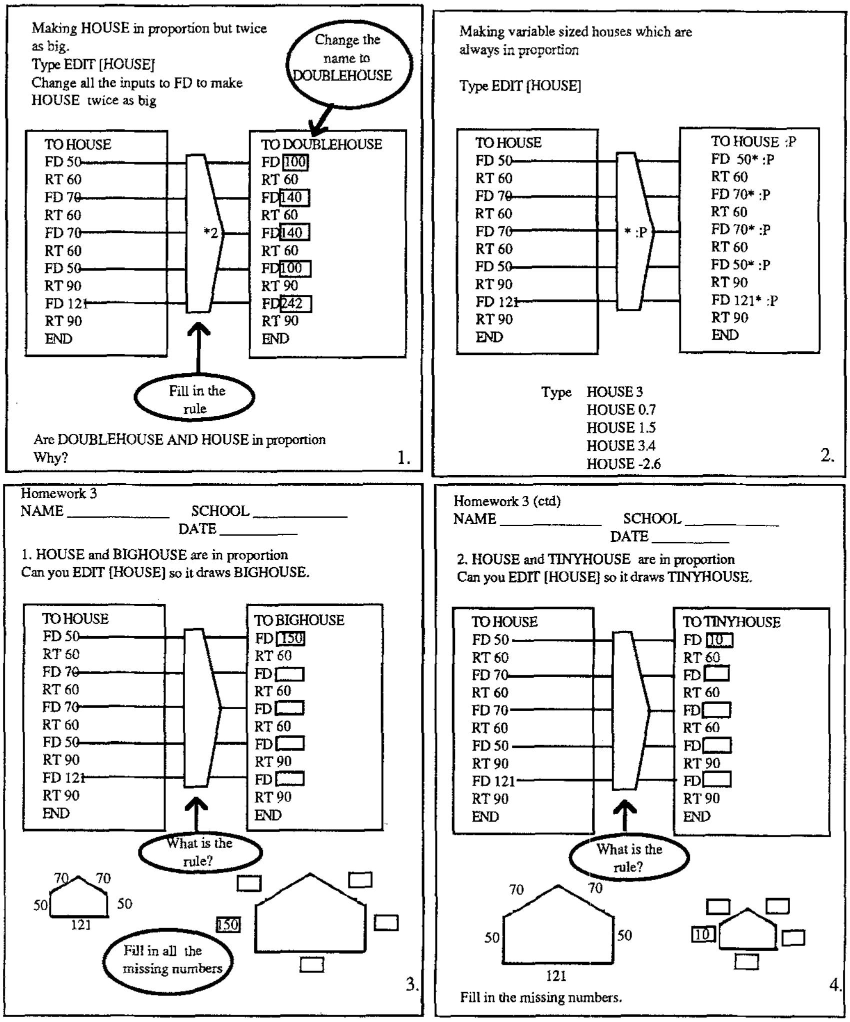

Fig. 5. Sequence of computer and paper and pencil homework tasks.

addition we had a policy to remind pupils to actually *use* the HOUSE procedure and not to "revert" to direct drive to draw their HOUSEs – a frequently-observed practice (see Noss and Hoyles, 1991, for a discussion of the ways in which pupils often *by-pass* Logo tools).

The issue we want to pin-point here is what we have termed the *play paradox*. This derives from the following contradictory situation: on the one hand, the computer offers pupils rich and diverse ways of exploring and solving a problem and building upon rather fragile intuitions of the mathematical domain. Yet in exploiting this diversity, pupils are as likely to avoid encountering the mathematical nuggets so carefully planted by their teachers (although of course they may encounter other – unintended – ones). Our resolution of this paradox, which is of course only provisional, is that our pedagogical interventions were centred upon encouraging pupils to *reflect* on the task at hand. Of course, this is often quite heavy-handed in comparison to the laissez-faire position which some (including, at other times, ourselves) have advocated.

Turning to the construction of porportional HOUSEs on the computer, the first pedagogical point is that we built the activity on doubling. This is at first sight paradoxical, as there is plenty of evidence which shows that students do not perceive doubling as a form of multiplication. In fact, in our earlier versions of the activity, we started by explicitly ruling-out doubling as a 'legal' strategy, and based it around a task which effectively outlawed doubling altogether. However we subsequently reversed this decision, in line with our belief that we could construct an activity from the basis of a strategy which pupils adopted intuitively (doubling), towards a mathematical perspective (a generalised multiplicative strategy) *given the presence of the formal language of the computer environment*. Our contention is that the constraint we imposed upon pupils first to change 2 to 3, then to other numbers and finally to :x, was given meaning through the computer interaction – such meanings are difficult to construct in non-computational learning environments. Pupils could *use* multiplication in ways which allowed them to discriminate and generalise its significance.

It is worth reiterating that at the point when a pupil formalises the relationship between the sides of the houses in a program, his or her strategy becomes explicit (both to the pupil *and* to the teacher/researcher). There is a formal *imperative* involved in the kind of computational environment we designed, and it is this – perhaps more than anything else – which offers the opportunity to sharpen and extend pupils intuitions. Nevertheless, as we hinted earlier, the process by which pupils encounter precisely the mathematics we intended was far from unproblematic as we attempt to show in the following two examples.

From his dribble file we noticed Russell's use of a variable procedure for HOUSE. He was happy simply to take the inputs to FD of the HOUSE and multiply by any integer – or even by a decimal in simple cases. He built HOUSE :P (with help) where :P was the scale factor, and then used this to creat a "reflection" pattern with a range of inputs for HOUSE (see Fig. 6). However, analysis of Russell's paper-and-pencil responses *after* this computer work reveals inconsistencies: he stated in answer to Question 3 in

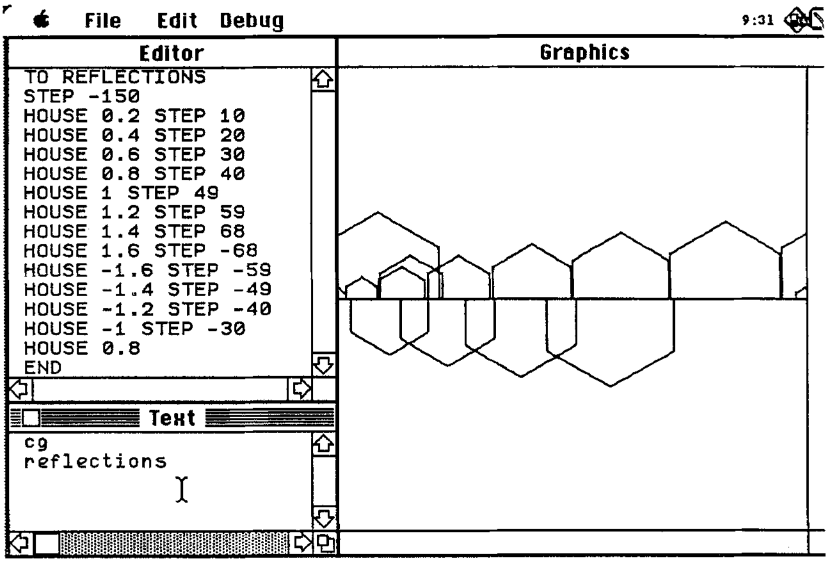

Fig. 6. Russell's paradox.

Fig. 5 that the rule was 'times by 3', but for Question 4 was takeway 40'. These responses can be interpreted in two ways. First, we can view it as an example of the play paradox. On the computer, Russell had used decimal inputs between 0 and 1 to his variable HOUSE procedure to obtain his required visual outcome (smaller HOUSEs) but he had *not yet* discriminated the mathematical 'necessity' of the multiplicative operation. This discrimination was *our* agenda, not Russell's: thanks to the ad-hoc nature of our interventions – inevitable in a class situation – we had simply failed to make an appropriate intervention during Russell's playful activity at the computer. A second interpretation[8] attributes Russell's behaviour to a difficulty of transferring the multiplicative strategy to a non-computer setting: the homeworks could only be done formally – without a computer – so that the visual outcomes of running the Logo procedure were unavailable to assist him in problem solution.

Our second example exhibits a nice interplay between visual and symbolic representations, although as we shall see, this relationship was not necessarily seen as straightforward. Some pupils explicitly – and legitimately (from their perspective) – *avoided* mathematical analysis. A classic example is provided by those who, when adopting a strategy other than multiplication for the construction of larger HOUSEs (see Fig. 2), rather than reassessing their strategy simply closed the gap by 'homing-in' on the correct length: that is, by edging the turtle slowly but surely towards its target by an iterative process.

Our explanation for this kind of perceptual rather than analytic behaviour is that these pupils centrated on the figure as a product rather than thinking of it as a general tool. They were ignoring the *mathematical* requirements for the construction of a proportional product: for these pupils the activity was simply to *draw a larger house*. From our perspective, the response of some pupils to close-up the gap was all the more interesting given that we had stressed in our teaching – to the point, we thought, of boredom – that objects were in proportion *only* if they were 'the same shape but different sizes'. Of course, it may well be that pupils who closed the gap believed that in so doing they were indeed creating a figure which was 'the same shape'; and it was partly as an attempt to resolve this difficulty that our next activity (Episode 4) was designed.

Episode 4: Sharing Meanings in Class-Discussion

The Hole-in-the-House task (Fig. 7) was given to the pupils individually for homework, with the aim that their responses would be compared and contrasted in subsequent class discussion. Dick's response (in the activity) represents an 'everyday' interpretation or proportion, in that HOUSEs A and C 'look alike': Ros's response represents a typical computer-based response arising from an additive approach and a centration on the visual outcome; Ann's response represents a mathematically correct argument. Pupils had to decide which response they agreed with and justify their choice in the subsequent class activity. It should be noted that this activity focussed on the visual product (HOUSE C), its dimensions and how it was constructed from the original house. It did not address the issue of program construction.

Discussion. The first point to note is that this activity was off-computer. This is in itself significant – we see this kind of explicit class discussion as well within the scope of a microworld. On the other hand, we felt it was equally important that the discussion was about activities which had been undertaken by each pupil with the computer. Thus we had a language in which to talk about the activity, and a shared medium for communicating what it was we wanted pupils to focus upon.

To illuminate the relationship between on- and off-computer activities, it is worth considering what sense pupils might have made of such an activity without computer experience. There are two issues. In the first place,

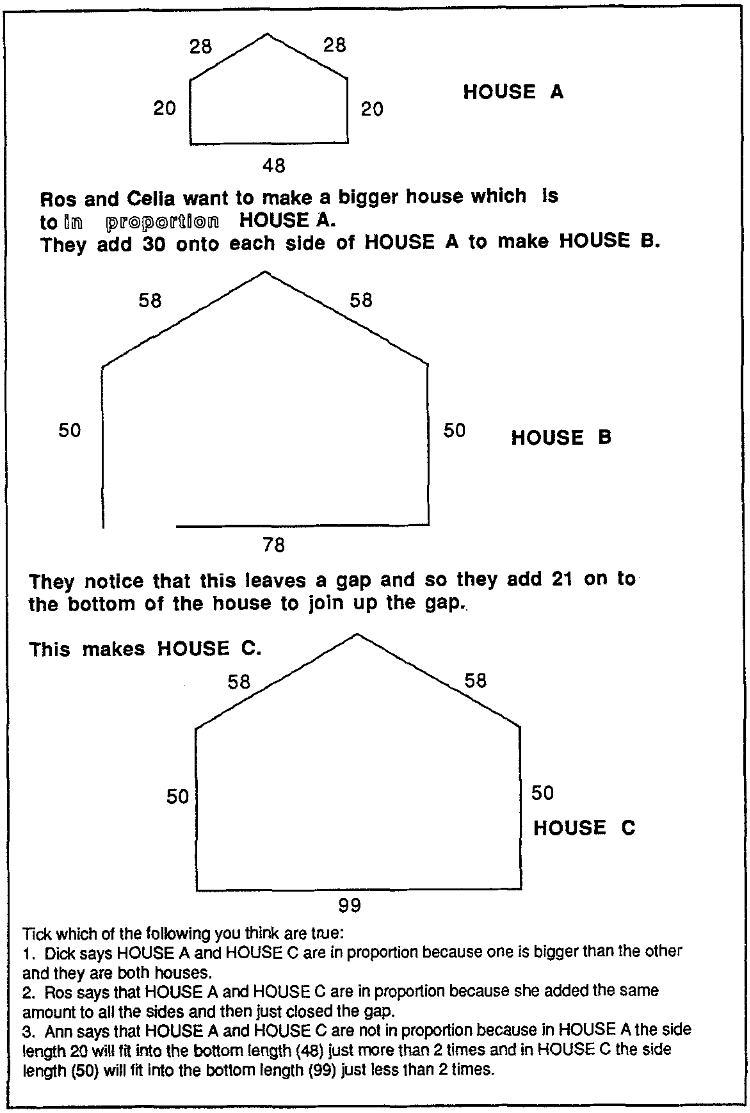

Fig. 7. The Hole-in-the-House task.

without the computer one strategy is very much as good as any other. That is, the pupil has little to assist in judging between strategies – other than by acceding to the whim of the teacher; the computer feedback provides an alternative medium of explanation and validation. Second, what would there be to talk *about*? Each pupil had used HOUSE, and had been encouraged to discriminate the relationships between its constituent components. Thus there was a basis for shared meaning between pupils and teacher, and between the pupils themselves. Interestingly enough, almost all the pupils answered that response number 3 was correct and the discussion was surprisingly listless. A consensus on the 'correct' response had quickly emerged (possibly because of the class's prior experience with the I task?) despite what had happened on the computer. It seems that the discussion was too far 'removed' from the computer work and the emerging classroom culture apparently inhibited pupils from 'admitting' to what they had previously done quite happily.

Episode 5: Pupils' Negotiation of Mathematical Meaning

The format of this activity was a game for a group of four pupils split into pairs (each initially designated as Pair A as in Fig. 8). These pairs were asked to produce 'enormous' houses in proportion to the original HOUSE and record their strategy and the lengths of their new figures. They then challenged the other pair within the group (now designated as pair B) to find how their enormous house had been generated *giving them the value of only one length.* The first stage of the game was designed so that pupils (as Pair A) would construct a similar shape to HOUSE and, if they used a scale factor, would not choose one which was obvious (i.e. not integral) so that it would present a challenge for Pair B. An important aim at this stage of the activity was that the rules of a the game would force pupils to negotiate their methods within their pair and make these explicit in their program.

In the second (off-computer) stage, each pair (now all acting as Pair B's) had to find the 'rule' which the opposite pair had used given the original HOUSE *and one length in the enlargement,* and then use this rule to calculate all the lengths of the enlargement.[9] At this point, if pupils wanted to adopt a multiplicative approach our intervention was to encourage them to work out the scaling factor iteratively using the computer: for example, in order to find the number to multiply the original side of say, 30 by to give 50 (the size of the equivalent enlarged side), we encouraged them to guess a number, try it and improve the guess – effectively to use the strategic aperture opened in the Target Game. In the final stage of the game which was off-computer, the group of four children who had constituted 'opposing' pairs (and who had each played the double roles of Pairs A and B) com- pared the numerical size of their enlarged lengths. If these were different, the pairs were forced to defend their decisions, and explain and justify the strategies adopted – strategies clearly captured in their Logo programs.

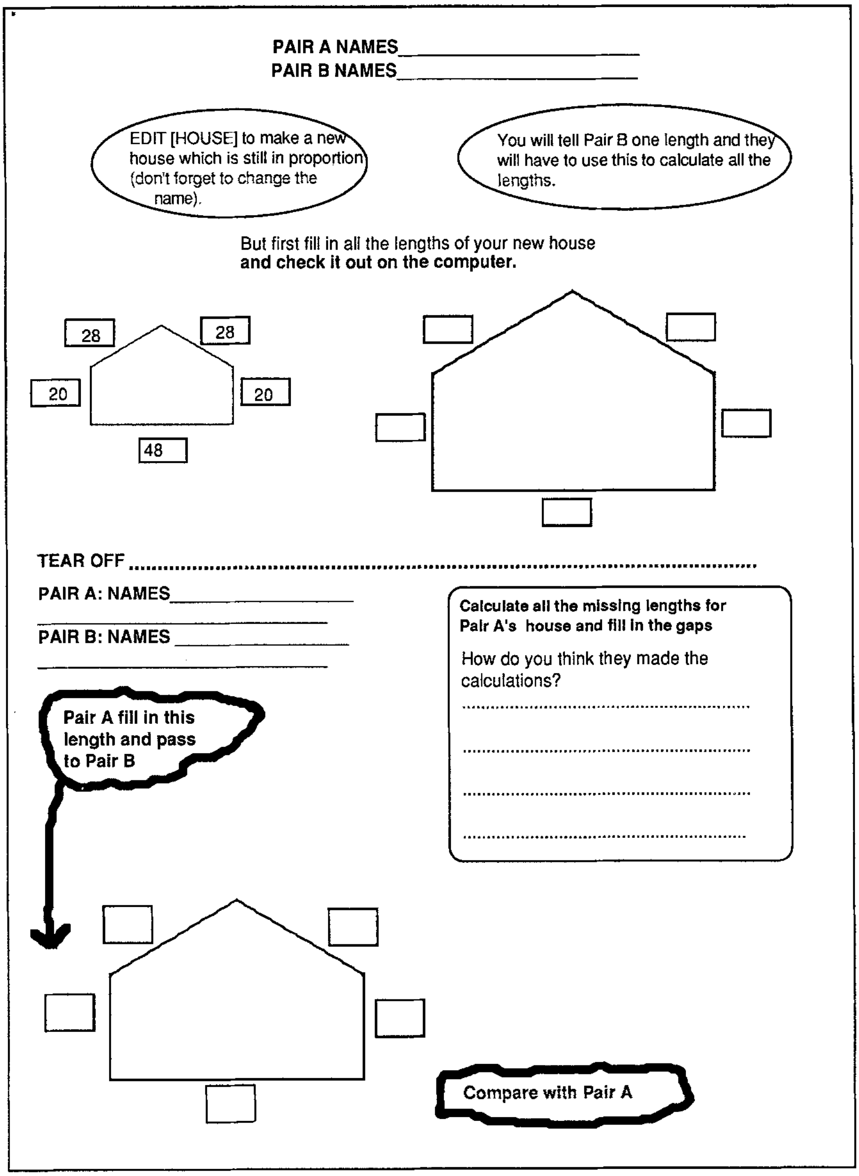

Fig. 8. A group activity: find the rule.

Discussion. We begin by observing that all pairs of pupils in the first stage of the game built enormous houses with a multiplicative scale factor. They were not explicitly told to do so, but their behaviour was not altogether surprising given that their previous experiences on the computer had been moving in this direction. However one result of movement to and from the discourses of Logo and mathematics is the illumination of the fragility of pupils' understandings (see also Balacheff, 1991, who has noted that group work does not necessarily destabilise mathematical conceptions). We found that the *construction* of a HOUSE can call up a different strategy for some pupils compared to *interpretation* (of the opponents' HOUSE). Consider the example of Toby and David. They constructed a larger HOUSE by multiplying each of the sides by 2.379. However their opposing team (Paula and Jane) presented them with a larger house whose vertical side was 53.95 (the original length was 20). Toby and David edited their HOUSE procedure changing the first FD 20 to FD 53.95. At this point, Toby suggested that they added 8 to the next input to FD (the smaller HOUSE had a 'roof length' of 28, 8 more than 20), and they fixed on this additive strategy for the remaining lengths. Of course this left them with a problem for the base of the HOUSE, but despite their apparent understanding of proportional strategies in the construction part of the activity, they happily found the length (106) by a 'homing-in' strategy – in fact, perhaps it is significant that they *wrote down* the length as 106.95 to give spurious accuracy to their result. They justified their result as follows: 'The side was 8 less than the vertical lines we gest [guessed] the bottom line'.

When the group of four came together they were therefore confronted with two different strategies. The girls had constructed their house by multiplying by 2.6975 to give the result shown in Fig. 9. Faced with the boys' solution, the girls immediately thought they had made a mistake and checked all their multiplications (is this a typical response of girls – to assume *they* are wrong!?). Then, convinced that they were *not* in error, the

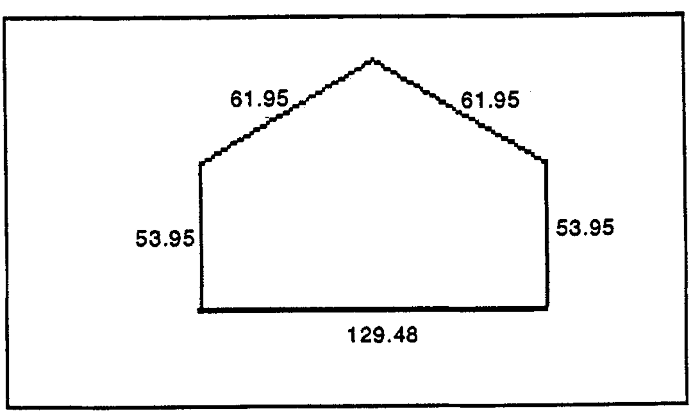

Fig. 9. Paula and Jane's new HOUSE construction.

girls discussed the two strategies and explained to the boys what they had done. David then changed sides and supported the girls, agreeing that he and Toby had previously been in error: so the four ultimately came to a consensus. Toby did make progress – he became more consistent and analytic in the face of proportional situations and was prepared to explicitly state that 'the same had to be done to all the parts': in fact at one point, he was quite vociferous about this in class. We are unable to attribute changes such as these to specific incidents: in fact it might be futile to attempt to do so. But it is at least clear that global understandings of this kind can *only* come about through explicit pedagogy of the kind we have described. The group task was important for Toby, but may *not* have helped Russell. In the work we presented earlier, it is clear that he had no sense that he could construct smaller HOUSEs from larger ones by multiplication. In the group task, he and his partner both correctly constructed a larger HOUSE and found that a scale factor of 1.12 had been used by their opposing team. So, as it turned out, because the 'outcome' HOUSE was *larger* than the original one, Russell's particular 'problem' had *not* been confronted by our pedagogic intervention.

REFLECTIONS AND CONCLUSIONS

In so far as pedagogy has been addressed in the literature on computational environments, it has tended to stress teacher-intervention – that is, the ways in which pupils' activities are 'interrupted' by teachers. While we do not underestimate the importance of such intervention, this paper has provided examples of a pedagogy built into the structure of the activities themselves. We have tried to describe the ways in which carefully constructed computational activities reveal and develop pupils' intuitions and construct a language for representing them – a language which can be exploited by both teachers and pupils.

This kind of pedagogical input is hardly unique to computational environments. But, as we have argued above, the computer does have a rather specific role to play. Playing with HOUSE within the associated pedagogy was crucial in helping pupils make sense of the medium, and the computational objects within it. On the other hand, an important pedagogical facet of the microworld was the way in which we attempted to move pupils between the practices of Logo-mathematics and school-mathematics, each with its own distinct criteria for correctness and justification. It seems likely that without off-computer tasks, our goals (and the pupils') could have been achieved by methods which may have bypassed formal mathematical discourse (e.g. closing the gap in the HOUSE). Our suggestion is that *both* on-computer activities and off-computer negotiation and discussion are critical in bridging the discursive disjuncture between the two practices – helping pupils to link intuitions derived from their Logo/computational interactions and their formal mathematics. We think that the computer plays a specific, even necessary role in this process, but it cannot be considered sufficient.

As far as the off-computer activities are concerned, we suspect that the initial class-discussion was important in helping children at least to begin to appreciate the mathematical rules of the game. They also served to make different computer-based pupil-strategies explicit, and catalysed discussion of 'misconceptions' which might not otherwise have been confronted: such discussions primarily validated the idea of *mathematical* 'correctness', We suggest that the structure and intentions of the discussions were crucial in determining their outcomes – a key facet being that they *started* from the perspective of pupils' responses. Thus the discussions had a two-fold purpose: first to help children *explicitly* to confront mathematical notions and distinguish them from everyday ones; and second, to act as a window into pupils' thinking to enable us to develop sensitive activities within the framework of our microworld.

However, we do not suggest that discussion of itself is a panacea in helping pupils form global mathematical understandings: on the contrary, we have several examples of discussion tacitly providing pupils with a correct 'line' which they could regurgitate when pushed, but which masked (rather than replaced) their underlying intuitions. This points to the pivotal importance of small-group work in serving as a bridge between pupil's own meanings and mathematical meaning – although even here we have illustrated how the best-laid pedagogic plans are sometimes unable to challenge the fine-grained difficulties and conceptions held by individual pupils. The essential elements of these group-activities were that they exploited the formalism of the computational setting by linking graphical and numerical descriptions; they represented pupils' strategies explicitly so that they could be discussed and justified away from the computer; and they made available scaffolding material for the teacher/researcher to develop pupil-understandings of similarity in a non-computational setting.

Turning to the relationship between on- and off-computer activities, it is helpful to recall our distinction between *local* and *global* mathematical understanding: that is, between the understanding of specific and relatively isolated pieces of mathematical content and the insertion of that understanding into a more global appreciation of mathematical structures. In our episodes, we have suggested three mechanisms through which teacher/pupil/computer interaction may potentially develop LMU by extending the range of *situated abstractions* available to pupils; making it *necessary* to communicate in a *formalised language*; and by offering pupils *strategic apertures* through which they can solve problems and explore situations.

As far as the generation of GMU is concerned, we think we can reasonably assert that these kinds of mathematical appreciation necessitate teacher-initiated intervention – yet as we have stressed, this does not relegate the computer to the role of an optional extra. We therefore have a complex situation, in which global mathematical understandings are developed and generated by the teacher, but at the same time mediated by the computer and its associated pedagogy.

We are therefore left with a variety of fundamental questions. How can a microworld be evaluated – specifically, did our evaluation of the Ratio and Proportion Microworld adequately capture the interaction between tasks and interventions? How do the mathematical elements of the microworld fit within the discourse of school mathematics or is it more helpful to conceptualise it as a separate discourse? To what extent are any mathematical outcomes of the microworld dependent upon the setting within which it takes place, and the ways in which the 'computer' is constructed (by both teacher and pupil) as a valid vehicle for learning? What are the social and cultural elements which contribute to or detract from the educational 'outcomes' of the computer – specifically, how does the way in which the computer is mobilised interact with the personal and professional priorities of the teacher?

We have begun to address some of these issues (see for example Noss, Sutherland and Hoyles 1991; Noss, 1991). But from the point of view of pupil-outcomes, we think that it is *impossible* to predict accurately the effect of teaching. Particularly with the introduction of the computer into the equation, there are simply too many variables involved – psychological, social and affective – and slight changes in emphasis can result in major perturbations of the teaching-learning situation. Yet the pedagogy associated with a microworld *does* illustrate a pattern – a pattern influenced by the computer as well as by the implicit and social nature of mathematical knowledge. In our example, pupils undoubtedly developed robust intuitions of multiplicative situations which did 'transfer' (perhaps surprisingly given this was not formally built into our pedagogic sequence) to other contexts in our paper-and-pencil tests. Certainly our microworld maintained a playful atmosphere.

We conclude by returning to the central tension inherent in the situations we have reported – a tension we have characterised as the play paradox. In all our work, we take as axiomatic the need to encourage pupil-autonomy and the requirement to develop pupils' means of mathematical expression. Yet as we have seen, we have provided a rather tight framework within which such autonomy and expression can take place – although pupils could and did break out of the straightjacket we had imposed. In fact, we do not think there is a contradiction, although there *is* certainly a tension – a tension arising from the nature of mathematics itself which deals, as Einstein puts it, '. . . exclusively with the relationships of concepts among themselves, without considering their relationships with experience'. Mathematics is not found in the street, it is not discovered by accident.

It is worth reiterating that our objective is to develop ways of thinking which are specifically mathematical, and it is perhaps too optimistic to ground a pedagogy on a series of accidental encounters – the least we can do is to encourage such meetings. We might think of our pedagogy as a contribution to the computer 'culture' proposed by Papert (1980), and to the subsequent generation of 'conceptual frameworks' that he suggested would follow in such situations – frameworks which can be exploited and developed by teachers to generate *mathematical* understandings.

ACKNOWLEDGMENT

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NOTES

1 In particular pages 140–158.

2 By play we mean to explore, experiment, wonder about and, dare we say it, enjoy.

3 Hoyles, C., Noss, R., and Sutherland, R., The Microworlds Project, Economic & Social Research Council Grant No. C00232364.

4 For elaboration of research into this and other aspects relating to Logo and mathematics, see Hoyles, Noss and Sutherland, 1986; Hillel and Kieran, 1987.

5 A dribble file records all keyboard interactions automatically on disk.

6 We identified common alternative conceptions from our pilotting work and from the literature.

7 For example, if the number received was 200 and the target was 100, pupils had to convert 'divide by 2' to 'multiply by 0.5' in order to conform to the rules of the game.

8 Suggested by Colette Laborde.

9 Note of the guest editor: This kind of situation is what Brousseau (1986, *Recherches en didactique des mathématiques*, 7.2) calls a "situation adidactique", in which the use of a specific piece of knowledge is triggered by the internal logic of the situation, and not by reasons external to the situation, like for example what the student can guess in terms of the teacher's expectations. The asymmetry of the two pairs A and B with respect to information is an essential part of the internal logic of the situation.

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Department of Mathematics, Statistics & Computing
Institute of Education
University of London
20 Bedford Way
London WC1H 0AL