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GEOMETRICAL ACTIVITIES FOR THE UPPER ELEMENTARY SCHOOL

INTRODUCTION

In Germany, high school starts with grade 5. In grades 5 and 6 about one hour per week is devoted to intuitive geometry. The aim is to develop the *geometric intuition* of the child and to familiarize him with many important *geometrical concepts*. The course appears somewhat unsystematic, since it consists of a great number of geometrical activities which are supposed to develop various *intellectual* and *manual* skills. In grade 7 there is a gradual transition to a more systematic development of geometry.

In this paper I shall present a selection of activities which I have used in grades 5 to 7 for the past 16 years. Well known topics will be mentioned only in passing; less familiar ones will be considered in greater detail. At this level I prefer topics which are not treated later, but which are still interesting, important and challenging. In fact, they are more interesting and instructive than many specialized curiosities which are proved in higher grades. These examples will show that, even at an early age, one can reach rather deep results in a short time and starting from scratch. It is in geometry that children are for the first time confronted with nontrivial mathematics.

COMBINATORIAL ACTIVITIES

For young children, activities with a combinatorial touch are best suited. They are easy to comprehend, and they often have a recreational twist which appeals to the natural curiosity of the child. Here are some examples:

(a) A square can be cut into 4, 6, 7, 8, 9, 10, 11,... squares. Show how!

Figure 1 shows the solution. It needs no explanation.

Problem 1. Solve the same problem for the equilateral triangle.

(b) Two rectangles are drawn in the plane. Into how many parts can they subdivide the plane? Think of all the possibilities!

Figure 2 shows subdivisions into 2, 3, 4, 5, 6, 7, 8, 9, 10 parts.

Problem 2. Study in a similar way subdivisions of the plane by

(a) a rectangle and a circle
(b) a rectangle and a triangle
(c) a triangle and a circle
(d) two triangles.

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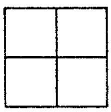
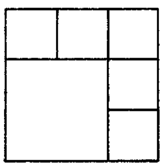
Problem 3. Study subdivisions of the plane by 1, 2, 3, 4, ... circles. Do you see a pattern? Can you explain the pattern? What happens if one circle is added?

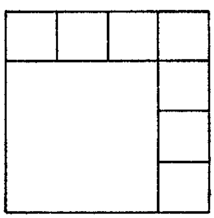
Problem 4. Cut a cake by 1, 2, 3, 4, ... straight cuts. Do you see a pattern? Explain the pattern. What happens if an additional line is drawn?



$4,\ 7, 10, 13, \dots, 3n+1$ for all $n \geqslant 1$



$6,\ 9, 12, 15, \dots, 3n$ for all $n > 1$



$8, 11, 14, 17, \dots, 3n+2$ for all $n > 1$.

Fig. 1.

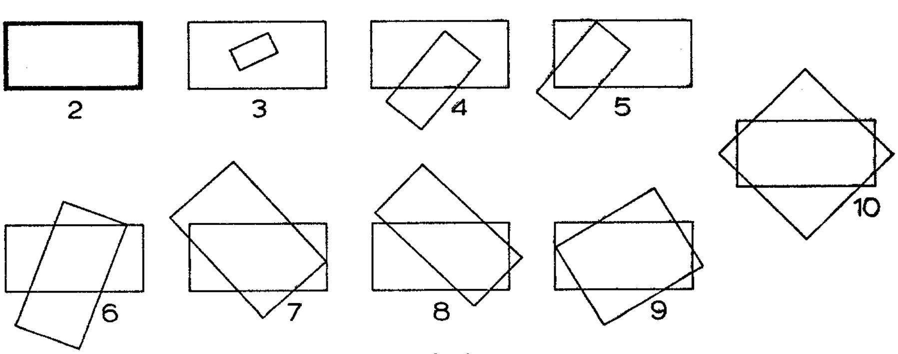


Fig. 2.

Problem 5. (This is a challenging problem in space visualization.) Into how many parts can a cube and a sphere subdivide space? If a challenging problem is posed to children without any help, they fumble around aimlessly and loose interest. They must be taken on a guided tour.

You are in a big cubical room; there is a small sphere in the room, its center coinciding with the center of the cube. We have a subdivision of space into 3 parts. Now the sphere starts expanding. When it touches the 6 walls there are still 3 parts only. If the sphere increases slightly, the number of parts jumps to 9. Now the sphere cuts each face in a circle. This circle expands and it becomes the incircle of the face. Now one part disappears and 8 parts appear. That is, we have 16 parts, the maximum possible number.

Problem 6. It is beyond the power of visualization of a child to see the subdivisions of space by two cubes. What is the maximum number of parts?

Covering problems are the source of many instructive activities. Here are some examples. Later we will meet more.

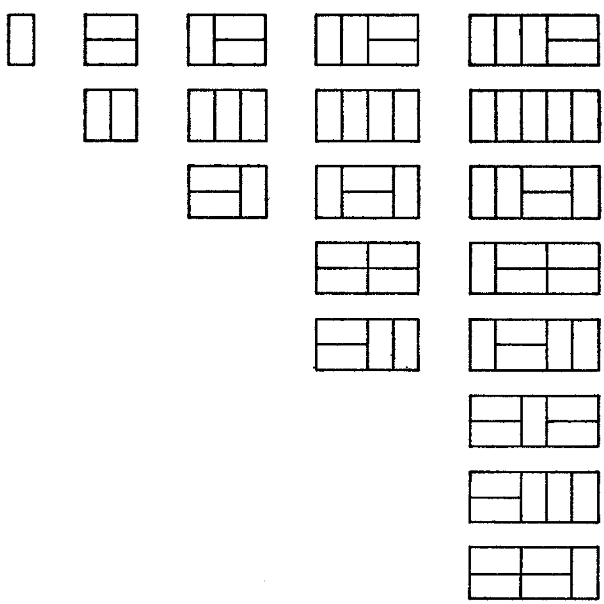
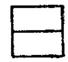


Fig. 3.

(a) Children are given a supply of 2×1 rectangles. They are challenged to pave a $2\times n$ road and find the number p_n of distinct patterns. By trial and error they find p_1, p_2, p_3, p_4, p_5.

By now a pattern emerges. Suppose you have a 2×6 road. You can start in two ways:

either  or 

In the first case, a 2×5 road remains which can be paved in p_5 ways. In the second case, a 2×4 road remains which can be paved in p_4 ways. Hence

$$p_6 = p_5 + p_4.$$

This activity does not fill one hour. When it is completed, one must have other activities in store which use the 2×1 rectangles.

Here are two possibilities: The rectangles are now bricks, and the children are challenged to build earthquake-proof walls. Figure 4 shows two unsuccessfull and two successful attempts. The first wall can slide vertically and the second can slide horizontally.

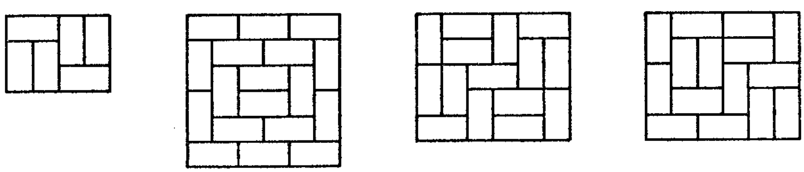


Fig. 4.

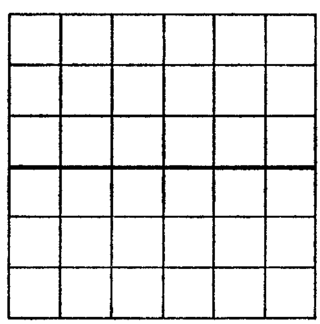


Fig. 5.

Children have more fun with two-person covering games. Two persons place alternately 2×1 rectangles on the board (Figure 5). Each rectangle must cover two squares on the same side of the heavy line in the middle. The player who makes the last move wins. There are two obvious winning strategies for the player who makes the second move. Whenever his opponent puts down a rectangle, he places a rectangle symmetrically either with respect to the heavy line or with respect to the center of the board. Children quickly recognize the strategy used by the teacher, especially if they are familiar with symmetry.

Problem 7. You have an unlimited supply of black and white 1×1 square tiles. You are to pave a $1 \times n$ road with the restriction that no two black tiles are next to each other. Let p_n be the number of distinct patterns. Find $p_1, p_2, p_3, p_4, \ldots$.

Problem 8. Figure 6 shows a tile consisting of 5 rookwise-connected unit

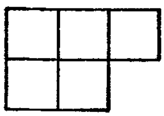


Fig. 6.

squares.* These tiles can be used to tile enlarged scale models of themselves, as is shown by Figure 7. Suppose you have an unlimited supply of congruent tiles consisting of n rookwise-connected unit squares. Prove that if they tile

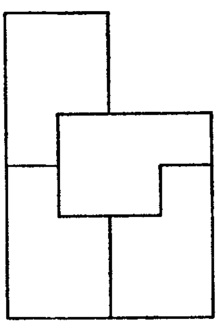


Fig. 7.

a rectangle then they also tile enlarged scale models of themselves. Suppose you can tile an $a \times b$ rectangle. This rectangle can be used to tile an $ab \times ab$ square, and the square can be used n times to cover an enlarged copy of the original tile.

Problem 9. The plane can be covered by congruent triangles, quadrangles, and centrally symmetric hexagons.

Solution: Two congruent triangles pave a parallelogram, which paves a

* Any two squares are said to be rookwise-connected if, in the game of chess, a rook can move from one to the other.

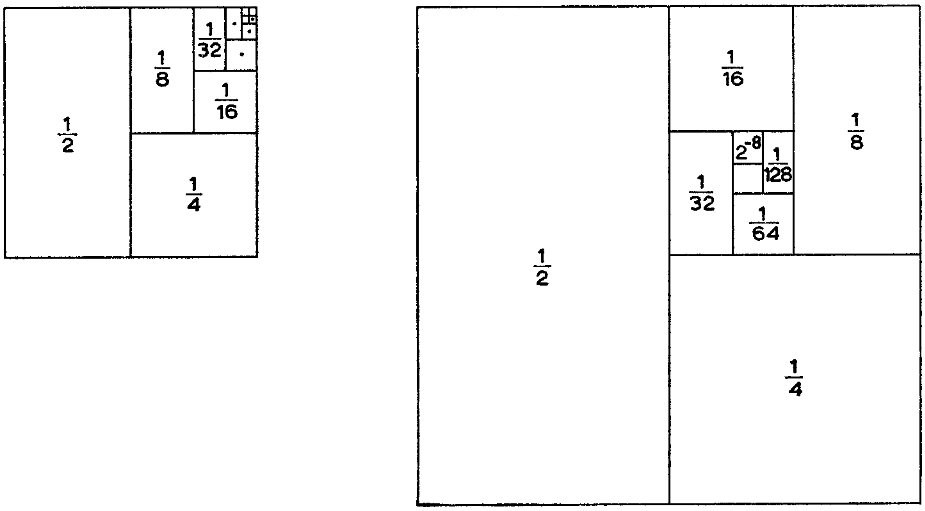


Fig. 8.

strip, which in turn paves the plane. Two congruent quadrangles pave a centrally symmetric hexagon, which paves a staircase strip, which in turn paves the plane.

Infinite tiling problems are fascinating and instructive. Take the tiling problem corresponding to

$$\tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \tfrac{1}{16} + \tfrac{1}{32} + \cdots$$

Figure 8 shows two of infinitely many possibilities.

COMBINATIONAL GEOMETRY AND SYMMETRY

Classification is an important activity and it is accessible at an early age. The discrete case of a finite group acting on a finite set is an unlimited source of fascinating and important activities; it is also one of the best ways to develop space intuition and the idea of equivalence. The following examples will show its rich mathematical content. I will not go into applications.

(a) You have an unlimited supply of congruent sticks in two colors, black and red. How many distinguishable tetrahedra can you build?

There are 12 distinguishable tetrahedra. Figure 9 shows six. To get the remaining 6 switch colors.

Problem 10. In how many distinguishable ways can you color the vertices of a tetrahedron with (a) two (b) three colors?

Problem 11. In how many distinguishable ways can you color the faces of a cube if two colors, black and white, are available?

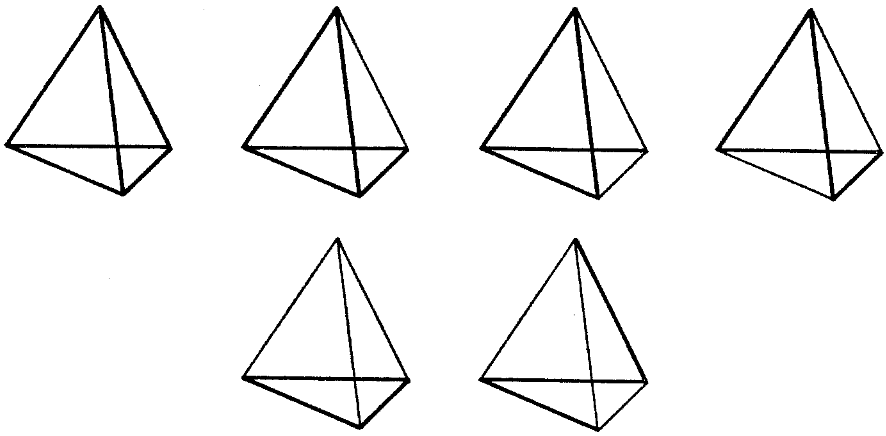


Fig. 9.

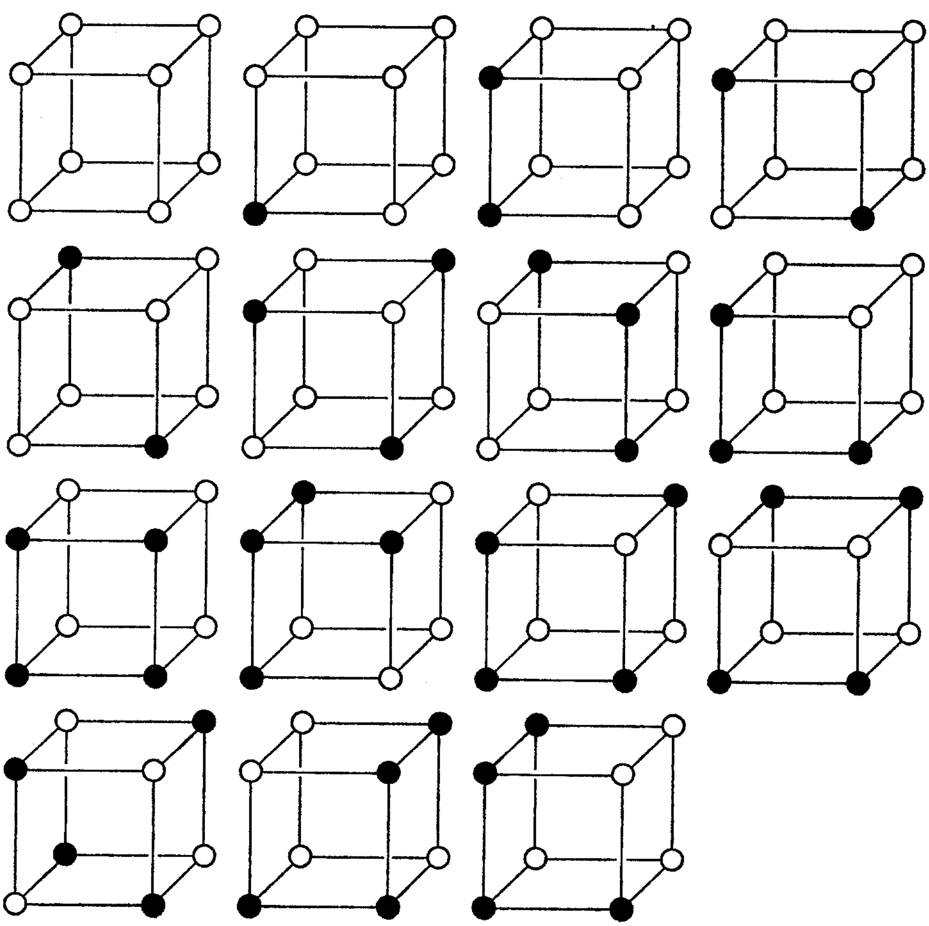


Fig. 10.

(b) There are two colors available, black and white. In how many distinguishable ways can you color the 8 vertices of a cube?

Under rotation there are 23, under rotation and reflection there are 22 distinct colorings. This is also the number of essentially different Boolean functions with 3 variables.

Problem 12. Find the only pair of colorings which is equivalent under reflection but not under rotation. Solve the same problems for the tetrahedron in (a).

Figure 10 shows some of the colorings. The remaining cases can be obtained by switching colors.

(c) Color the 2×2 chessboard with 2 colors. All distinguishable colorings (Figure 11) are bilaterally symmetric. So it does not matter which group operates on the board, the cyclic group C_4 or the dihedral group D_4.

(d) With 3 colors the situation is different. The number of distinct colorings is 24 under C_4 and 21 under D_4. There are 3 pairs of asymmetric colorings. It is a waste of time to ask children to find all 24 colorings. But to find the 6 asymmetric colorings is a perfectly good activity (Figure 12).

(e) *Two-Color Strips (Ties)*. The two strips in Figure 13 are equivalent under a half-turn or reflection. How many distinct black-white strips of length 5 (6) are there?

Here is the solution, which is easy to generalize:

strip length	5	6
all colorings	2^5	2^6
symmetric colorings	2^3	2^3
asymmetric colorings	$2^5 - 2^3$	$2^6 - 2^3$
inequivalent colorings	$2^3 + \frac{(2^5 - 2^3)}{2} = 20$	$2^3 + \frac{(2^6 - 2^3)}{2} = 36$

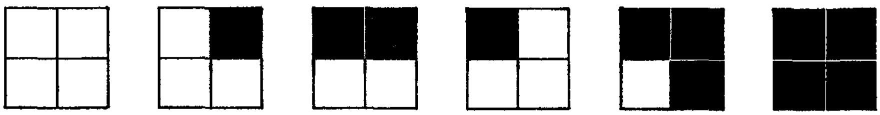


Fig. 11.

Fig. 12.

Problem 13. Show generally: The number of distinct 2-color strips of length n is $\frac{1}{2}(2^n+2^{n/2})$ for n even, and $\frac{1}{2}(2^n+2^{(n+1)/2})$ for n odd.

(f) Bend a strip of prime length p into a ring. In how many distinct ways can the ring be colored by two colors? (No flipping over of the ring!)

Fig. 13.

All colorings: 2^p.

Colorings symmetric under rotation: 2.

Colorings asymmetric under rotation: 2^p-2.

Nonequivalent colorings under rotation: $C_p=2+(2^p-2)/p$.

Problem 14. Show that with a colors there are $C_p=a+(a^p-a)/p$ nonequivalent colorings under rotation. Deduce Fermat's theorem from this result.

For non-primes the formula is more complicated.

Obviously, we are dealing here with the famous *necklace* problem: You have an unlimited supply of pearls in two (a) colors. How many distinct necklaces of n pearls can you make, if a necklace may not be flipped over?

(g) *Distinguishable friezes of 0's and 1's.* Consider all periodic doubly infinite sequences of 0's and 1's. Let S_n be the set of distinct (under shift) sequences of period n. Find S_1, S_2, S_3, S_4, S_5.

$$S_1 = \{\ldots 0 \ldots, \ldots 1 \ldots\}$$
$$S_2 = \{\ldots 00 \ldots, \ldots 11 \ldots, \ldots 10 \ldots\}$$
$$S_3 = \{\ldots 000 \ldots, \ldots 111 \ldots, \ldots 010 \ldots, \ldots 101 \ldots\}$$
$$S_4 = \{\ldots 0000 \ldots, \ldots 1111 \ldots, \ldots 1010 \ldots, \ldots 1100 \ldots, \ldots 1110 \ldots, \ldots 0001 \ldots\}$$
$$S_5 = \{\ldots 00000 \ldots, \ldots 11111 \ldots, \ldots 10000 \ldots, \ldots 11000 \ldots, \ldots 11100 \ldots, \ldots 11110 \ldots, \ldots 10101 \ldots, \ldots 01010 \ldots\}$$

What has this problem to do with necklaces under rotation? The general formula is

$$\# S_n = \frac{1}{n} \sum_{d/n} \varphi(d)\, 2^{n/d}$$

(h) Children enjoy solving problems like this: Find all distinct necklaces with 7 white and 3 black pearls. A necklace may be rotated and flipped over. Identify the asymmetric necklaces.

This is an outstanding activity. The child gets a sheet of paper with pictures

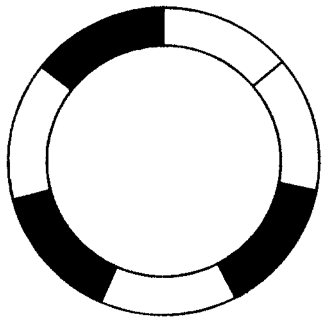


Fig. 14.

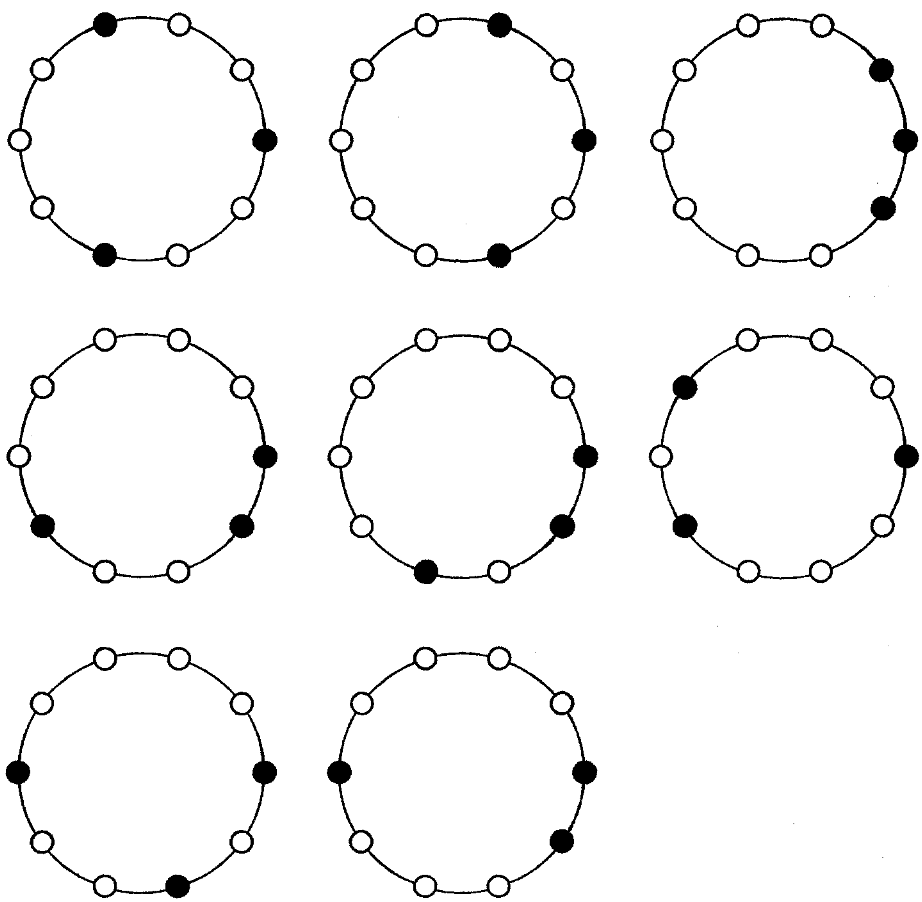


Fig. 15.

of necklaces with 10 white pearls. He colors 3 of the pearls with a black pencil. It requires a considerable effort on his part to see if two colorings are equivalent or not.

Figure 15 shows the 8 necklaces distinct under the dihedral group D_{10}.

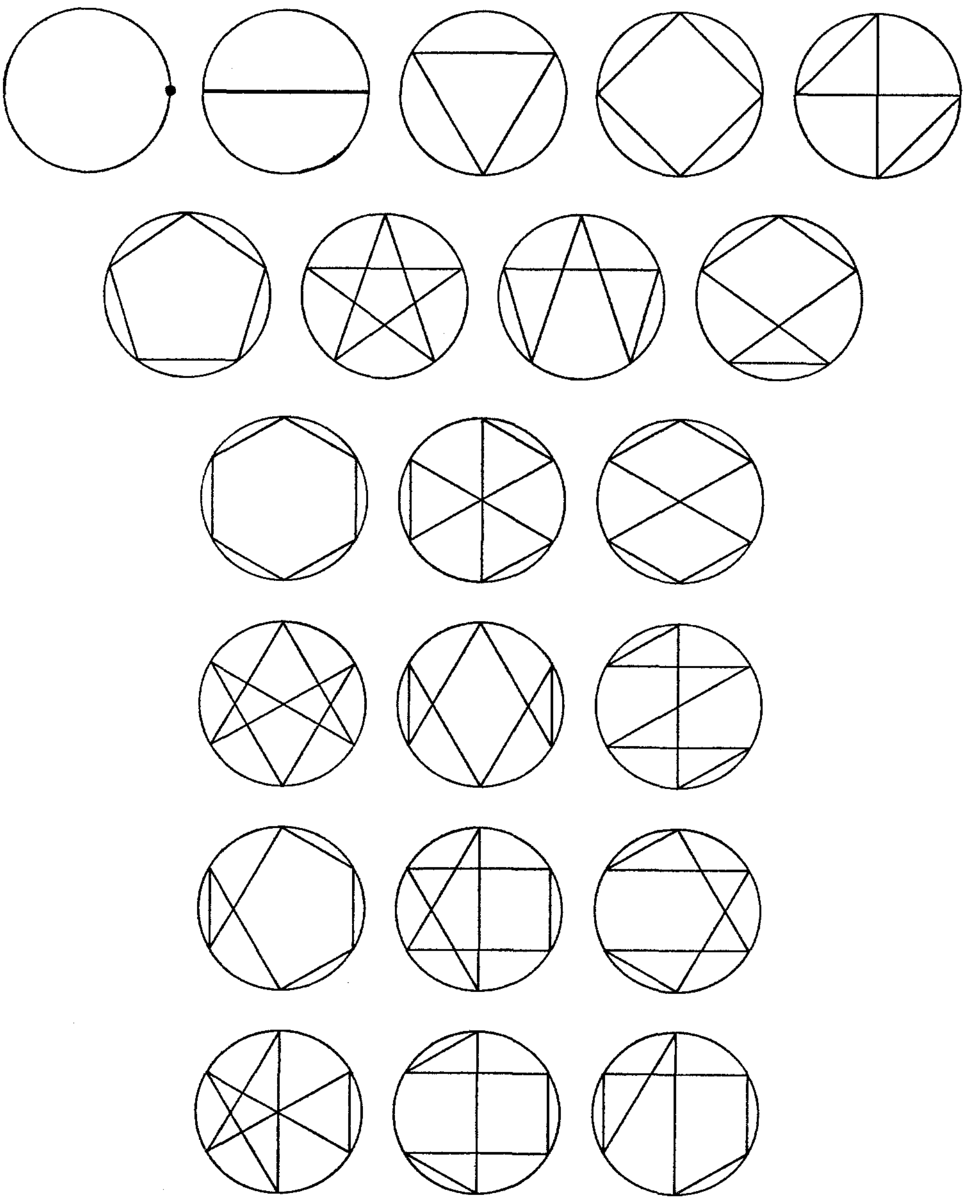


Fig. 16.

Since 4 of the necklaces are asymmetric, there are 12 distinct necklaces under the cyclic group C_{10}.

(i) Here is another activity with a beautiful result: Take n equally spaced points on a circle and draw all distinguishable n-gons. Children find by trial and error the solution for $n=1, 2, 3, 4, 5, 6$. For $n=6$ the number of distinct polygons should be told or else they will overlook some cases. Figure 16 shows the result. For $n=6$ there are two asymmetric 6-gons.

(j) You have a cube. In how many ways can you make a die? On a die the points on opposite faces add up to 7. Place a cube in front of you. Put one dot on the top face and 6 dots on the bottom face. Next put two dots on the front face and 5 dots on the back face. Now the cube is rigid and the face for the three dots can be chosen in two ways: left or right face. Hence there are two distinct labelings of a die shown in Figure 17. Look at some dice to see if both labelings are used.

Suppose you drop the restriction that the sum of the points on opposite

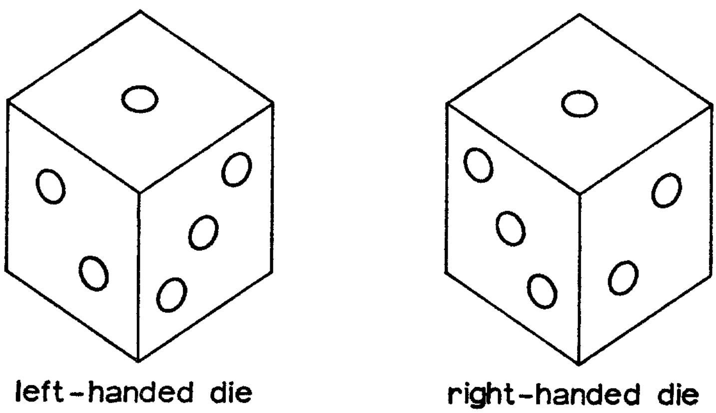


Fig. 17.

faces be 7. In how many distinguishable ways can you label (color) the 6 faces of the cube?

Rotate the cube so that 1 is in front. Face 2 is either adjacent or opposite. If adjacent, rotate face 2 to the top. The cube is now rigid. Then all $4!=24$ different ways of labeling the remaining faces are distinguishable.

If face 2 is opposite face 1, rotate face 3 to the top. The cube is now rigid, and there are $3!=6$ ways to label the remaining 3 faces.

Hence, the cube can be labeled in $4!+3!=30$ distinct ways.

Problem 15. Two faces of a cube are to be colored black, two white, and two red. What is the number of distinct colorings?

GEOMETRICALLY DEFINED POINT SETS

Here is another class of activities. A particle moves in the plane or in space. But its movements are restricted by various strings or rigid rods. Find the set of points the particle can reach.

(a) Figure 18 shows a fence *BACDA*. A goat *G* is tied to the fence at *A* by a string of length 3. Shade and describe the area it can graze.

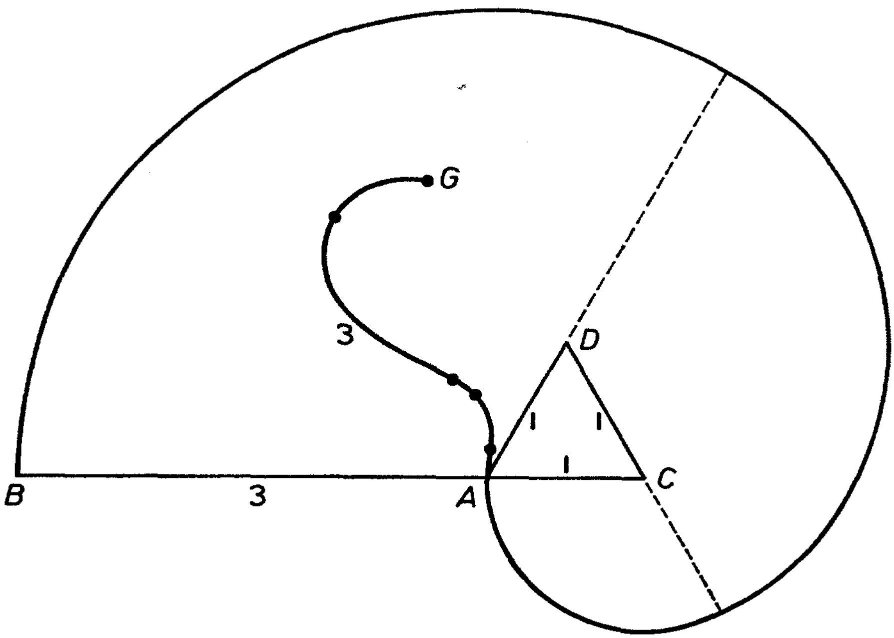


Fig. 18.

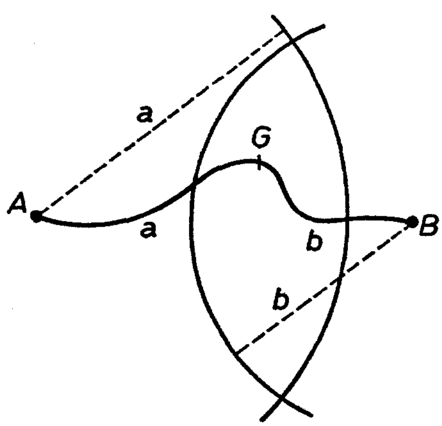


Fig. 19.

(b) A goat is tied to points A and B by ropes of lengths a and b. Shade its grazing area (Figure 19).

(c) A goat is tied to the vertices of a unit square by strings of unit length. Find the grazing area. Suppose you want it to reach $A \cup B$. Which string must be cut? How about $A \cup B \cup C$? (Figure 20).

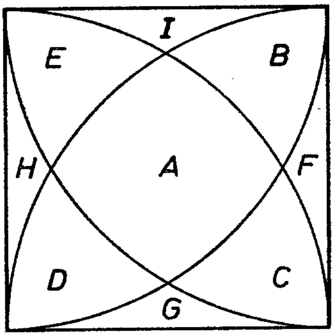


Fig. 20.

(d) A man has a semicircular lawn in front of his house. He wants to tie a cow in such a way that she can graze the entire inside of the lawn without being able to overstep its boundary. Can he do it with a string, three short pegs, and a ring? (Figure 21).

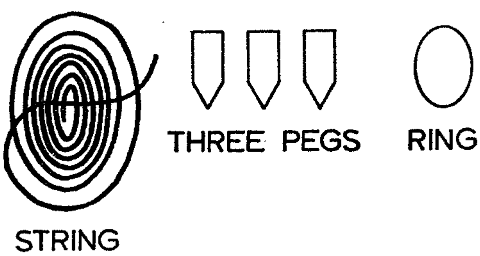


Fig. 21.

Figure 22 shows a beautiful and theoretically perfect solution which does not work in practice. Why? There is a simple, practical solution which is almost perfect. Find it!

(e) A dog is tied to a fence with a leash of length c. He can crawl under the fence and run around it both on the outside and on the inside. Which points of the plane are unsafe for a cat?

Surprisingly, there is a great difference between the outside and the inside. Suppose c is increased until the inside safety zone shrinks to a point. Which

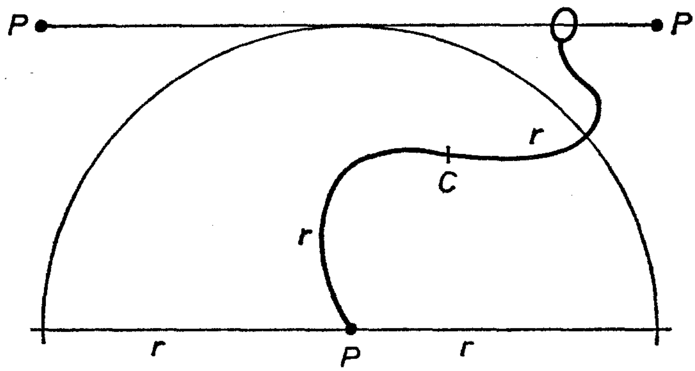


Fig. 22.

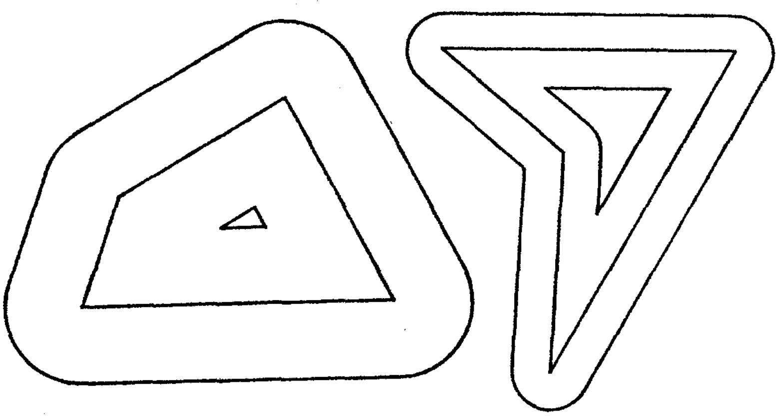


Fig. 23.

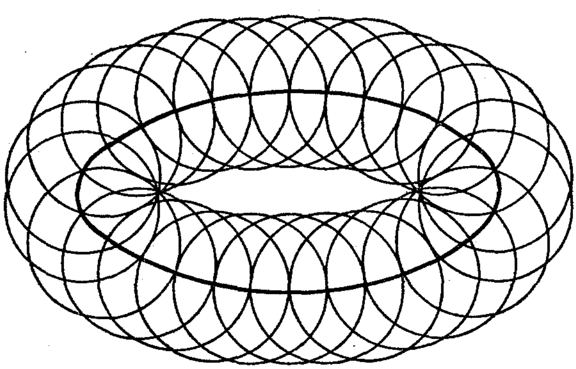


Fig. 24.

point is it? How do you construct this point? Figure 23 shows the unsafe points for a convex and a nonconvex quadrilateral.

(f) A dog is tied to an elliptical fence by a leash of length c (Figure 24). Find the set of unsafe points for a cat.

(g) A strong flying insect is leashed to the surface of a cube. The end of the leash on the cube can move freely along the surface by magnetic rollers. Describe the set of points in space it can reach (Figure 25). This is a very challenging problem in space visualization.

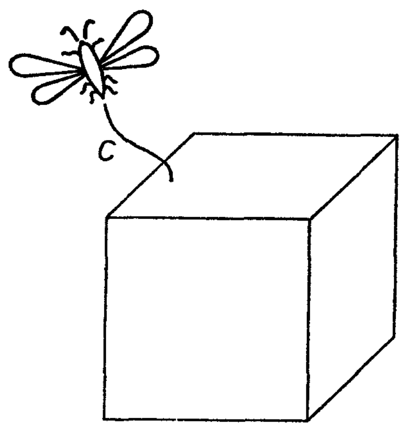


Fig. 25.

(h) Given two fixed points A and B in the plane. Find the set of all points C such that the triangle ABC has acute angles.

C must lie in the black area in Figure 26.

(i) A rider leaves point O at noon. He can move along the roads AB and OC 20 km/h and in the lower half-plane (outside the roads) at 10 km/h. Find the set of all points he can reach after 1 hour (Figure 27). The region of possible locations is bounded by AB and the envelopes of 3 families of circles.

(j) Given two point sets S_1 and S_2. Find the set of midpoints of all line segments XY with $X \in S_1$ and $Y \in S_2$.

By appropriate choices of S_1 and S_2 one gets many interesting and instructive problems. Two examples are shown in Figures 28 and 29. In Figure 28, S_1 and S_2 are two skew face diagonals of a cube. The set of midpoints is the shaded square.

In Figure 29, S_1 and S_2 are two circles. The set of midpoints is the shaded ring.

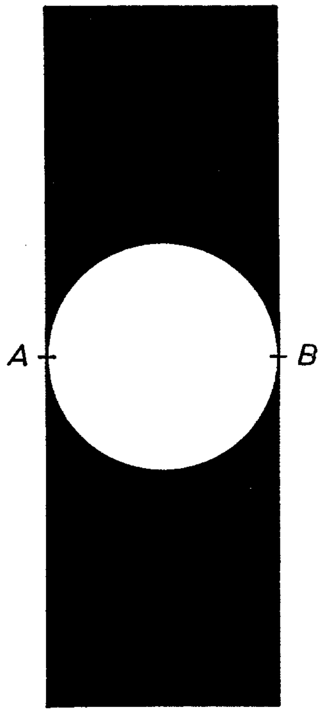


Fig. 26.

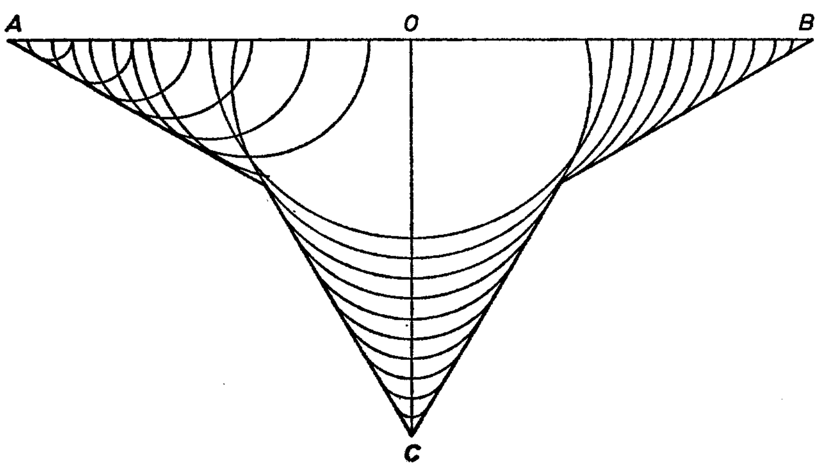


Fig. 27.

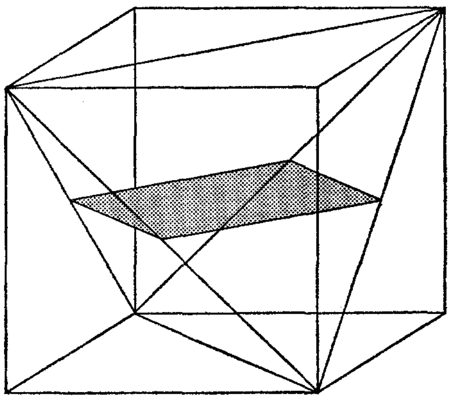


Fig. 28.

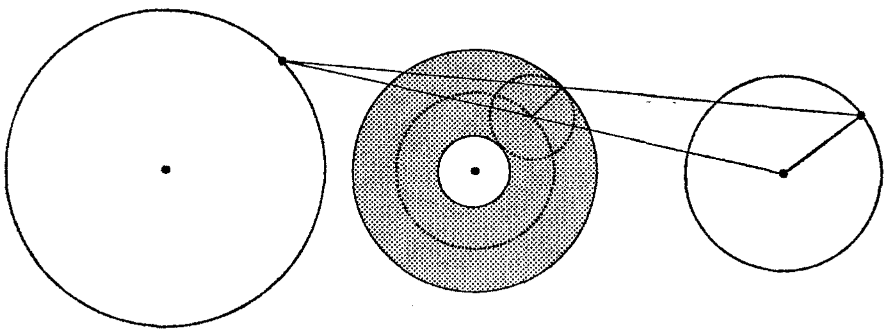


Fig. 29.

FOUR STICKS

What can you do with four sticks? Here are some activities which require no prerequisities:

You have four sticks of length *a*, *b*, *c*, *d*. Can you always make a quadrangle? What conditions must be satisfied? Is the quadrangle rigid? When can you deform it into a triangle?

Problem 16. A polygon with more than four vertices can always be deformed into a triangle. (Tough problem, not for young children).

How many distinct quadrangles can you make (a) with turning over (b) without turning over?

The fact that a quadrangle is not rigid is of utmost technological importance. The *four-bar linkage* is the simplest and most basic mechanism. It has literally thousands of applications from steam shovels to ploughs. If none of the lines is fixed we have a kinematic chain. We are free to fix any of the four links.

In Figure 30, the longest link a is fixed. The result is a *crank-and-rocker mechanism.* When the shortest link d makes a complete revolution the rocker b only oscillates between two extreme positions indicated in the figure. How do you find these extreme positions?

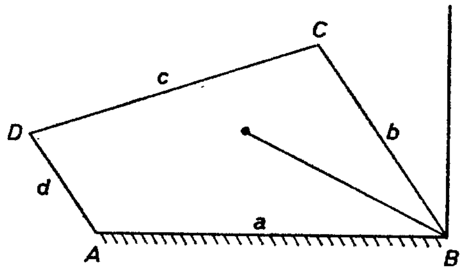


Fig. 30.

When can the shortest link d make a complete revolution? Show that the inequality

$$a + d \leqslant b + c$$

(shortest link + longest link $\leqslant$ the other two links) must be satisfied.

Fixing link c, instead of a, gives a similar mechanism. By fixing link b opposite the shortest one you get a *double-rocker mechanism.*

Figure 31 shows two positions of the mechanism. Link d makes a complete

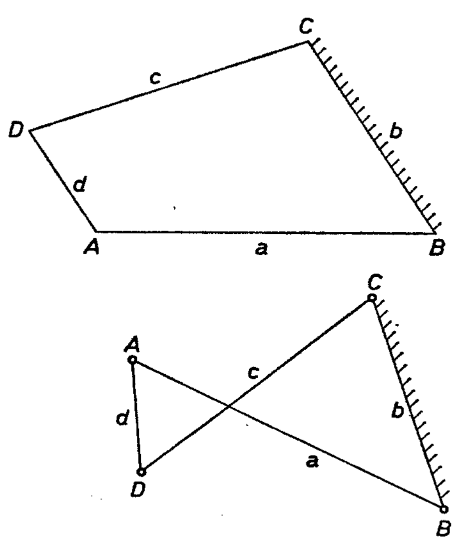


Fig. 31.

revolution during the operation of the mechanism, whereas a and c are the oscillating rockers.

Finally, by fixing the shortest link d we get *a double-crank linkage* (Figure 32). Links a and c make complete revolutions. What about b?

OPTIMIZATION

You have three sticks of lengths a, b, c. Can you always make a triangle? What condition must be satisfied? Here the *triangular inequality* appears for the first time, and it is used immediately to solve nontrivial problems.

(a) Six consumption centers A, B, C, D, E, F are located at the vertices of a regular hexagon. They are to be supplied from one supply depot P. Where should you place P so as to make the sum of the distances from P to all consumers a minimum? (Figure 33). A straightforward application of the triangular inequality gives

$$(PA + PD) + (PB + PE) + (PC + PF) > 2r + 2r + 2r = 6r$$

for each $P \neq 0$. Hence O is the optimal point for P.

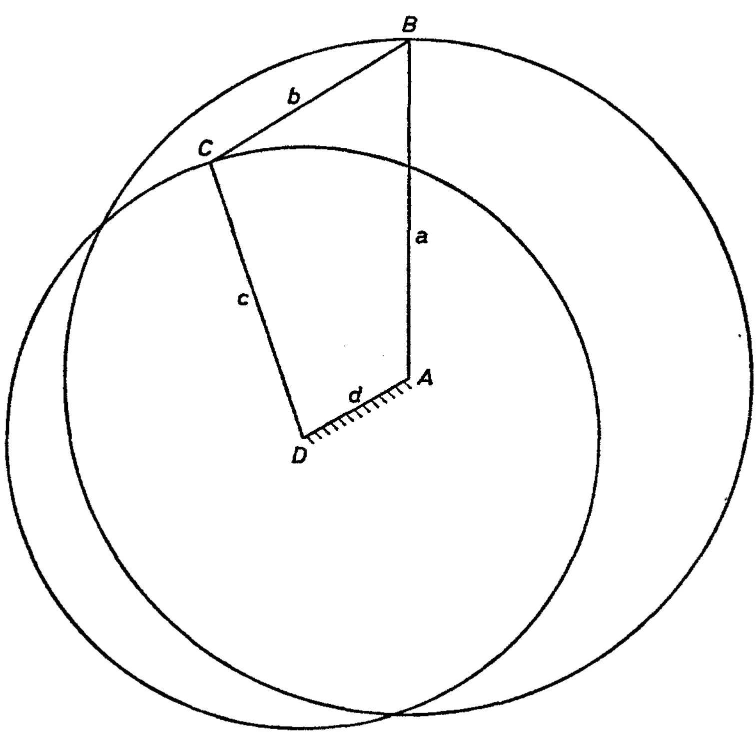


Fig. 32.

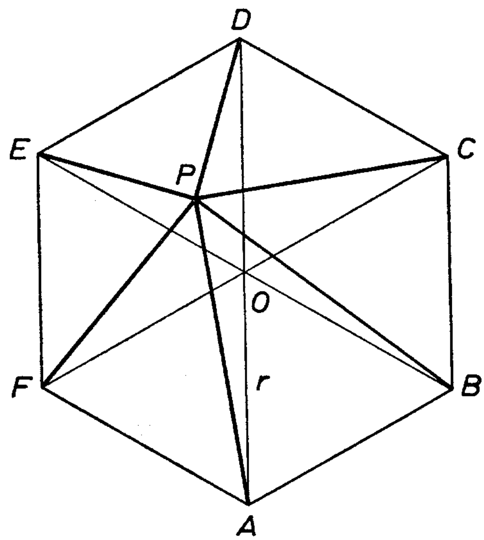


Fig. 33.

Problem 17. (Tough). Solve the same problem for a regular pentagon.

(b) Four consumption centers are the vertices of a convex quadrangle *ABCD*. Find the optimal location for a supply depot. Two applications of the triangular inequality show immediately that the point *O* of intersection of the diagonals is the optimal location (Figure 34).

But suppose *A*, *B*, *C*, *D* are not the vertices of a convex polygon. Suppose *D* lies inside the triangle *ABC*. First show, by two applications of the triangular inequality to Figure 35, that

$$a + b > c + d\,.$$

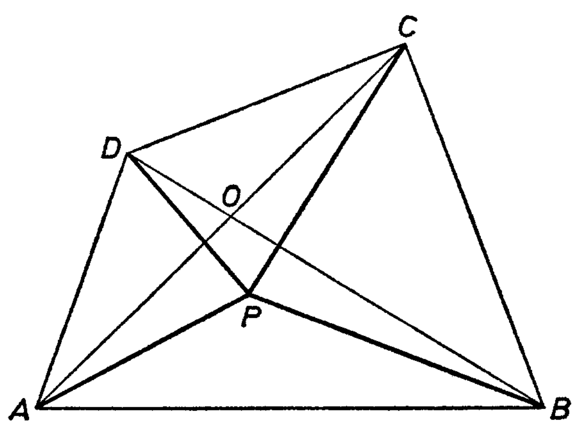


Fig. 34.

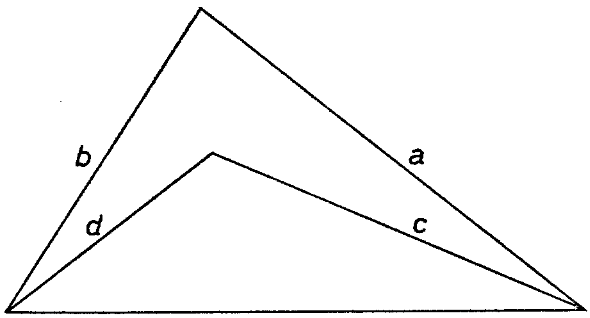


Fig. 35.

Now look at Figure 36. By the previous result, $BP+CP>BD+CD$, and by the triangular inequality

$$AP + DP > AD.$$

By adding the inequalities we get

$$PA+PB+PC+PD>DA+DB+DC$$

Hence, D is the optimal location.

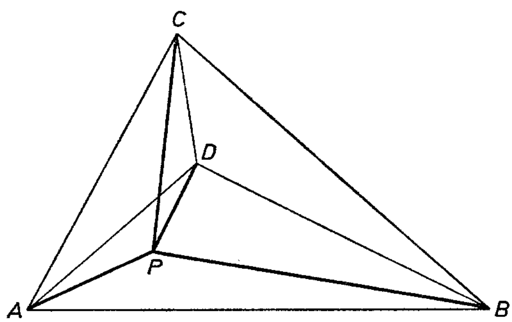


Fig. 36.

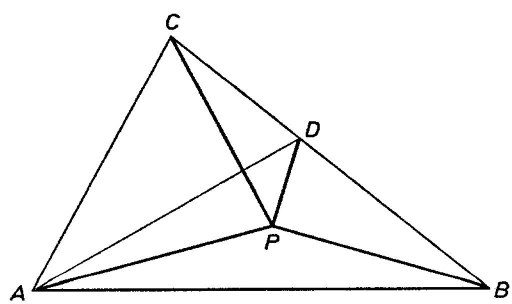


Fig. 37.

Suppose D lies on a side of the triangle ABC (Figure 37). We have immediately

$$\frac{\begin{array}{l} PA + PD > DA \\ \qquad\qquad PB + PC > DB + DC \end{array}}{PA + PB + PC + PD > DA + DB + DC}.$$

This means D is again the optimal location.

(c) But what if the points A, B, C, D all lie on a straight line? This leads to a highly interesting problem which can be solved completely for any number of points: n friends live at $x_1 < x_2 < x_3 < \ldots < x_n$ on the same street. Find a meeting place so that the total distance travelled is minimal.

For $n=2$ any point $x \in [x_1, x_2]$ will give minimum distance $x_2 - x_1$. Let $n=3$. For x_1 and x_3 any point in $[x_1, x_3]$ will do. Of these points x_2 is optimal for x_2 itself. Hence x_2 is the optimal point.

Let $n=4$. For x_1, x_4 any point in $[x_1, x_4]$ will do. For x_2, x_3 any point in $[x_2, x_3]$ will do. Hence, any point x in $[x_1, x_4] \cap [x_2, x_3] = [x_2, x_3]$ is an optimal meeting place (Figure 38).

Generally: for n even, any point in the innermost interval $[x_{n/2}, x_{n/2+1}]$ is optimal. For n odd, the innermost point $x_{(n+1)/2}$ is the optimal point.

Let us generalize further: At x_1, x_2, x_3, x_4, x_5 live 20, 50, 70, 80, 100 people, respectively (Figure 39). Find the optimal meeting place.

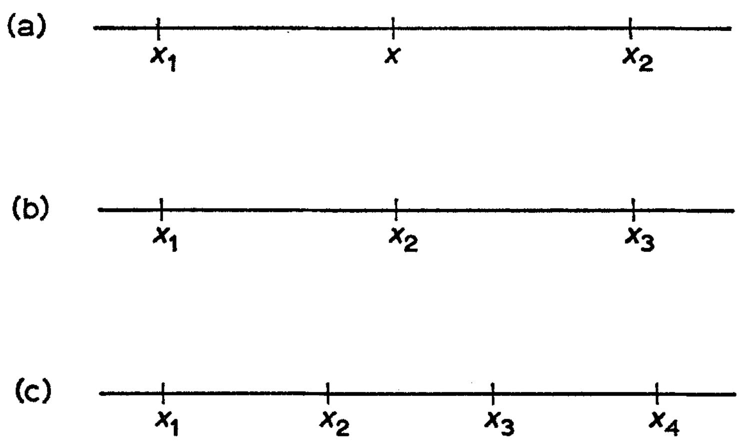


Fig. 38.

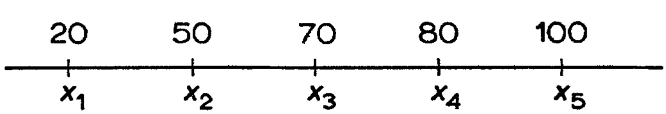


Fig. 39.

We start with a trial point on the line and decide by majority voting whether this point should be moved to the right or to the left. The optimal point is x_4, since a 180:140 vote is against moving it to the left and a 220:100 vote is against moving it to the right.

Let us generalize further to two dimensions. Figure 40 shows 7 con-

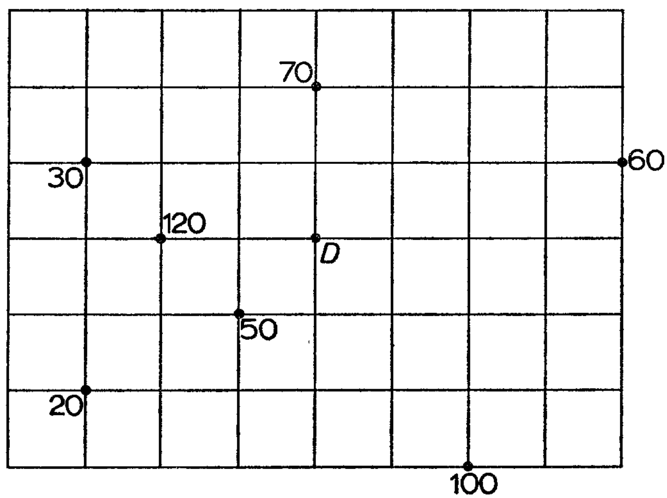


Fig. 40.

sumption centers with daily consumptions indicated. Find the optimal location D for a supply depot. Again, by majority voting, the best location D is found.

The last problem shows that in the taxicab metric many optimization problems become trivial. In the Euclidean metric the corresponding problem is extremely difficult.

Problem 18. (Easy). Figure 41 shows the street system of a small town

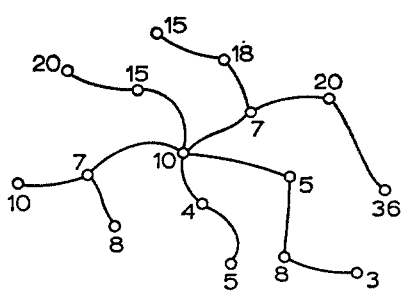


Fig. 41.

together with the consumption centers and their daily demand. Find the optimal location for a supply depot.

(d) Now suppose the cost of travel is the square of the distance travelled. Find the optimal location of a meeting place. For two points one sees by geometric insight that the mid-point is best (Figure 42).

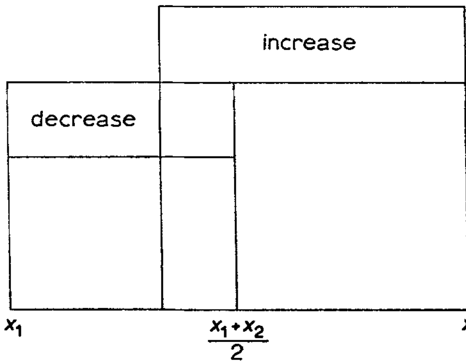


Fig. 42.

In general it is the centroid. A proof is not possible at this stage. But one can check by numerical cases. For instance, take $x_1=2$, $x_2=3$, $x_3=7$.

Location of meeting place	2	3	4	5	6
Cost of travel	26	17	14	17	26

At the mean $\bar{x}=(2+3+7)/3=4$ the cost is a minimum.

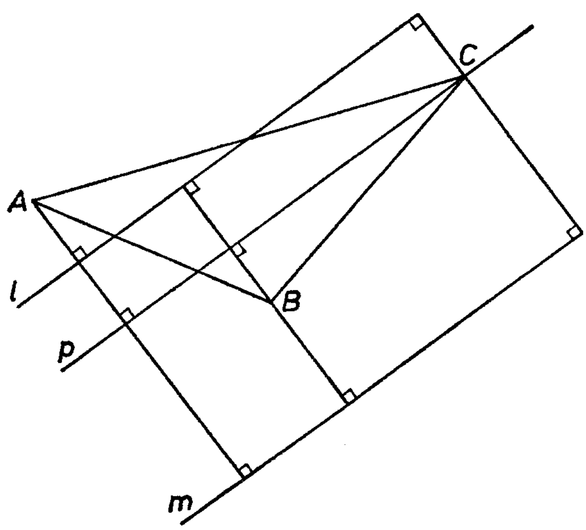


Fig. 43.

(e) A, B, C are three oil wells. A pipeline p is to be laid with prescribed direction parallel to line m (Figure 43). The sum of the distances from A, B, C to p should be as small as possible.

The optimal location p of the pipeline can be found again by a majority vote of the oil well proprietors.

Problem 18. For what directions of line m does the best pipeline pass through A, B, C?

Problem 19. Suppose now that the direction of p is not fixed. What is the best location for p?

MINIMAX AND MAXIMIN

These activities are inspired by game theory and linear programming.

(a) Consider the points of the plane with positive coordinates. Choose a closed subset S of these points and tell the children: You may choose any point $(x, y) \in S$. The smaller of the two numbers x, y is your gain. Which points of S are optimal for you? That is, the student is challenged to find a point of S with the smaller coordinate as large as possible:

$$\max_{(x, y) \in S} \min(x, y).$$

The game is played with different choices of S (Figures 44a–f) until the

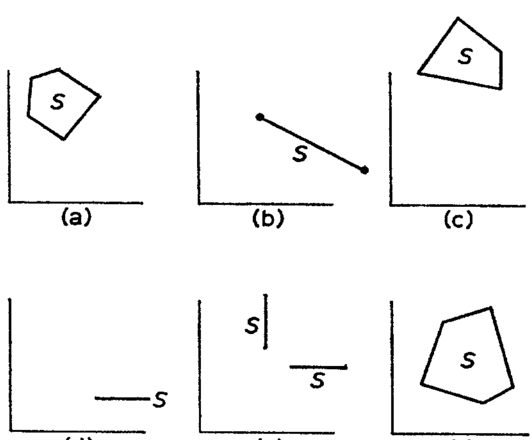


Fig. 44.

children come up with a simple algorithm for finding the optimal points: Take a right angle with vertex on the first diagonal $y=x$, like in Figure 45. Translate it in the direction of the origin parallel to $y=x$ until it hits S. The points hit first are the optimal points.

(b) Again, the child chooses a point $(x, y) \in S$. But this time the larger

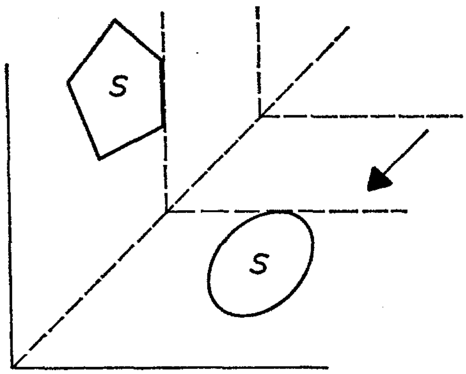


Fig. 45.

of the two number x, y is his loss. So he tries to make max (x, y) as small as possible. He is looking for $\min_{(x,y)\in S} \max(x, y)$.

Problem 20. Give a simple algorithm for finding the optimal points.

(c) Children choose a point $(x, y) \in S$. Their gain (loss) is the difference $|x-y|$. Find the optimal points.

(d) Children choose a point $(x, y) \in S$. Their gain (loss) is the sum $x+y$. Find the optimal points.

(e) Two children play. The first player chooses a point $P \in S$, his opponent knowing the choice. The second player chooses a point $Q \in S$. The first player forfeits to the second the distance PQ. What are the best choices for the first player? How does the second player find his optimal choices?

(f) Suppose S is a circular disc with radius 1. Each of two children chooses a point of S. But this time each child is ignorant of the other's choice. The first player pays to the second the distance of the two chosen points.

The first player can avoid losing more than 1 per game by always choosing the center of the disc. And he will lose more, on the average, if he chooses any other point P. The second player needs only choose a diameter HT at random. Then he tosses a coin. If head comes up his choice is H and if tail comes up his choice is T. His expected gain is $\frac{1}{2}(HP+PT)$. But $HP+PT \geq$ $\geq HT=2$.

Hence, the expected gain is $\geqslant 1$. The equality sign is valid only if P happens to lie on HT. The probability for this is 1 if P coincides with the center of the disc and is 0 otherwise (Figure 46).

IMPOSSIBILITY PROOFS

There are many crackpots who try all their life to do something which can be proved to be impossible. Angle trisectors will never die out. For this reason a child should meet convincing impossibility proofs early in his life. Best suited for this level are parity proofs, and coloring proofs from combinatorial geometry. These proofs require no prerequisites; they are short, elegant, and totally convincing.

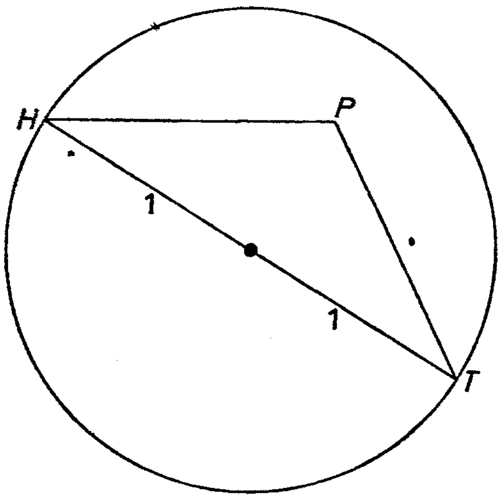


Fig. 46.

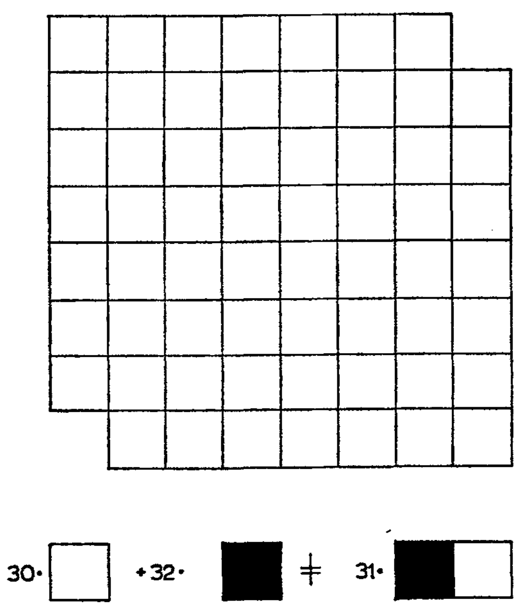


Fig. 47.

Since this is a well known subject, I only mention three examples of increasing difficulty. It is easy to devise any number of similar examples.

(a) The classical example is the following: Can you cover the 62 squares of the mutilated chessboard in Figure 47 by 31 dominoes, each covering 2 adjacent squares of the board? This is impossible and the concise proof is summarized in one line below the board.

(b) It is easy to cover an 8×8 chessboard (Figure 49) with 16 *T*-tetrominoes (Figure 48b). But can you cover the chessboard with one square tetromino (Figure 48a) and 15 *T*-tetrominoes?

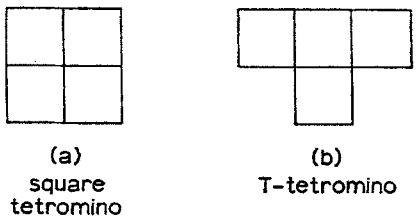


Fig. 48.

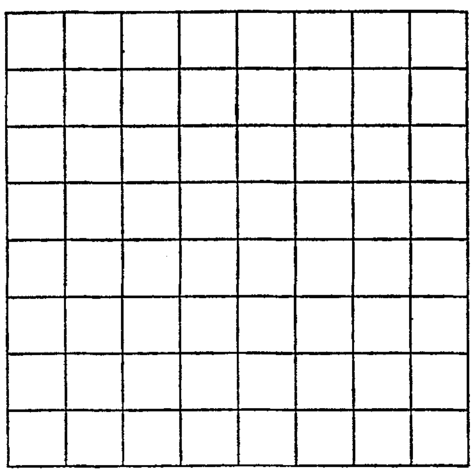


Fig. 49.

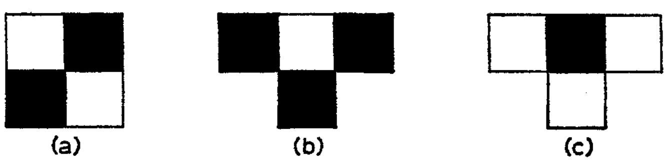


Fig. 50.

This is not possible. The standard chessboard has 32 black squares. The square tetromino covers two of these (Figure 50a). The 30 remaining black squares cannot be covered by 15 T-tetrominoes. For each tetromino covers either 3 black squares or 1 black square (Figures 50b, c). But a sum of 15 odd numbers (3's or 1's) is an odd number, and cannot be equal to the even number 30.

(c) Look at the funny face in Figure 52. Can you cover its 72 white squares with 24 tetrominoes of the type shown in Figure 51?

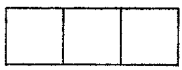


Fig. 51.

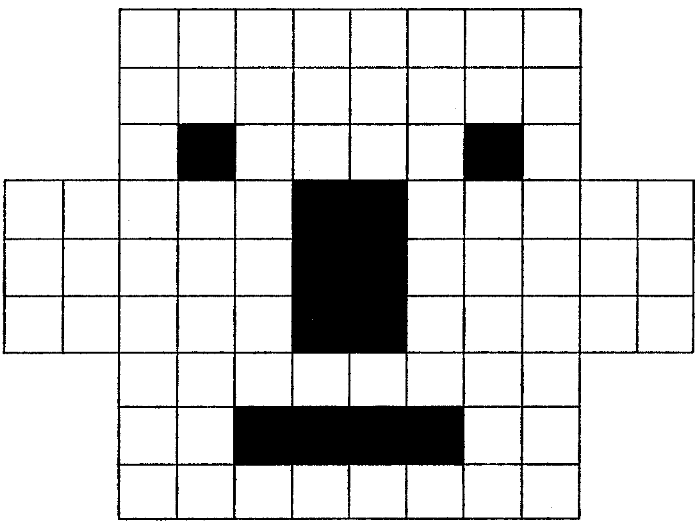


Fig. 52.

It is not possible, but to prove it you have to use 3 colors. Let us label the 3 colors by 1, 2, 3. Now the concise impossibility proof is summarized in one line below Figure 53.

Here is another classic impossibility proof. Figure 54 shows the Schlegel diagram of a rhombic dodecahedron. We interpret it as a road map for 14 cities. Is there a path passing through each city exactly once? No! There are 6 black cities and 8 white cities. Each path passes alternately through black and white cities. But a sequence of the type

bwbwbw... or wbwbwb...

cannot contain 6 b's and 8 w's.

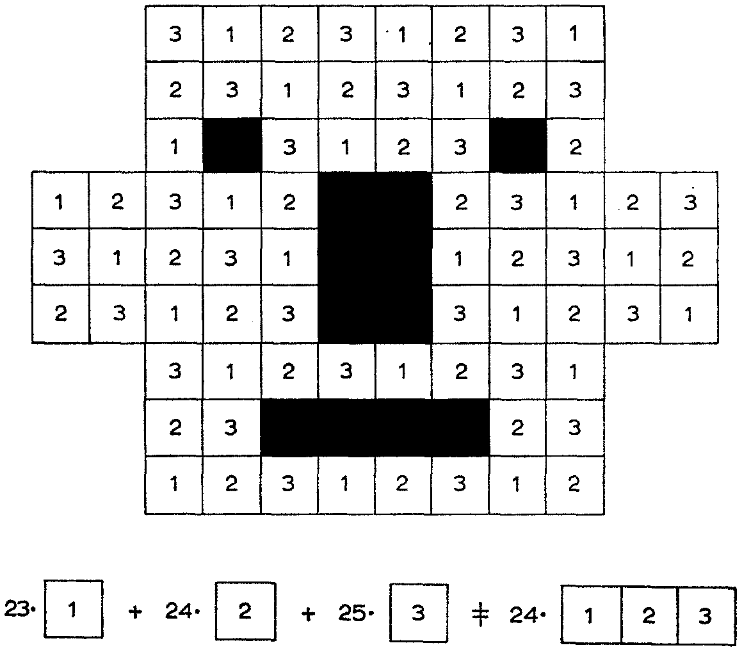

3 1 2 3 1 2 3 1
2 3 1 2 3 1 2 3
1 3 1 2 3 2
1 2 3 1 2 2 3 1 2 3
3 1 2 3 1 1 2 3 1 2
2 3 1 2 3 3 1 2 3 1
3 1 2 3 1 2 3 1
2 3 2 3
1 2 3 1 2 3 1 2
23· 1 + 24· 2 + 25· 3 ≠ 24· 1 2 3

Fig. 53.

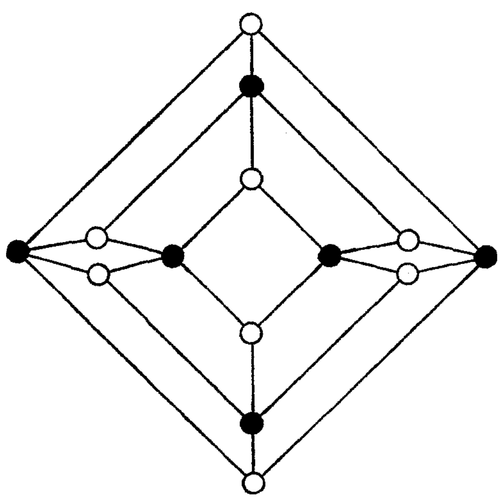


Fig. 54.

Problem 21. Our last impossibility proof is of an entirely different kind. Figure 55 shows a space polygon. Show that such a space polygon cannot exist.

PROBLEMS IN A STORY SETTING

A teacher should dramatize important geometric ideas by some story which the children will never forget. Experienced teachers make extensive use of this effective device. Here is a good example:

The Tragic Mistake of the Poor Tailor of Sikinia

In Sikinia people are very poor, but everyone owns a ferocious dog. These dogs tear triangular holes into the clothes of passers-by. One expensive

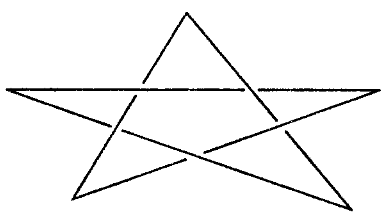


Fig. 55.

breed of dogs tears holes in the shape of perfect squares. The victims don't throw their clothes into trash cans. They go to a tailor to mend them. There

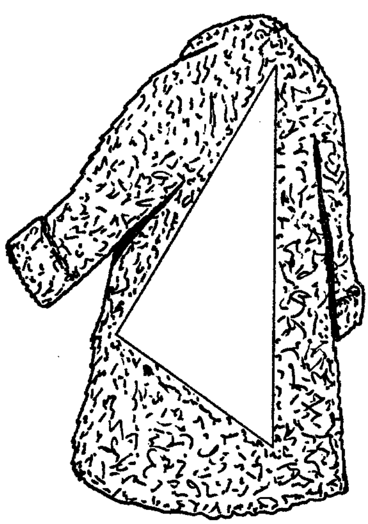


Fig. 56. Mink coat.

was a poor tailor who was making a living patching up holes. When the hole was a square he cut a square piece of cloth. To test it he would fold it along each diagonal. Is this a good test? Is there a folding test for a square?

The secret dream of our poor tailor was to become rich by mending mink coats. One day he had his big chance. A lady came with a mink coat which had a huge triangular hole on the back (Figure 56). Our poor tailor had never mended furs before, but only regular cloth. And he made a tragic mistake. On mink, hair grows on one side only. The other side is clean shaven. You cannot turn it over like cloth which looks the same on both sides. But our poor tailor had to learn this the hard way. He cut a patch to fit the hole, but it fit only on the wrong side (Figure 57). What to do now? How can we help our poor tailor?

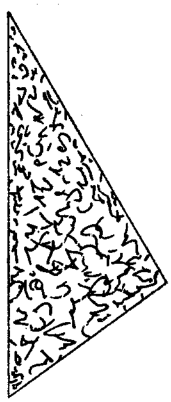


Fig. 57. A part to mend the coat. It fits in the wrong way.

Symmetric pieces can be turned over. So we must cut up the patch into symmetric pieces. But you should do it in as few cuts as possible. Two cuts are always sufficient, as is shown by Figure 58.

One cut is sufficient for the two shapes in Figures 59 and 60. Can you find the cut?

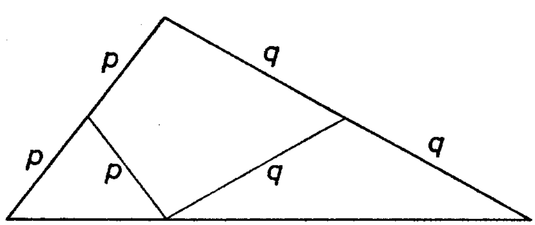


Fig. 58.

Sharing a Pie

Cain

There was once a couple, Adam and Eve. They had a son, Cain. To develop Cain's intellect Adam liked to pose problems. One day he showed Cain a triangular pie Π, and he said: You may choose any point O in the plane of Π and reflect through O to Π'. The intersection $\Pi \cap \Pi'$ is yours.

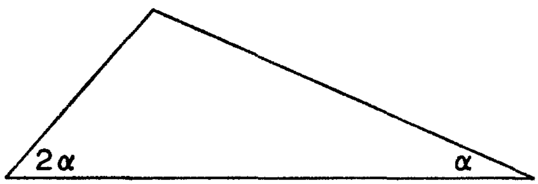


Fig. 59.

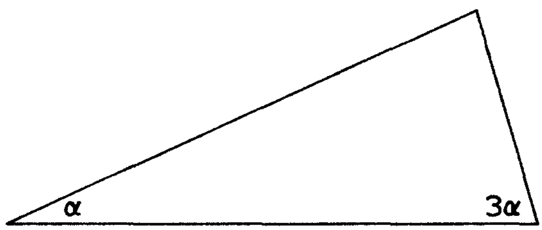


Fig. 60.

This is a rich geometrical situation: the children are confronted with a battery of about 30 geometrical problems arising from it. Here are some:

(a) Cain got nothing. Where did he choose O?

(b) Cain got a piece in the shape of a parallelogram. Where did he choose O?

(c) Cain got a hexagon. Where did he choose O? What shape is the hexagon?

(d) Can he get a triangular piece of pie?

(e) What else can he get?

(f) Where should he choose O to get as much as possible?

(g) For what shapes of the pie can Cain get all of it?

(h) The pie was a polygon and he managed to get all of it. Deduce as many properties of the polygon as you can.

These problems, with the exception of (f), require no prerequisites for their solution.

Solutions

(a) He chose O outside Π (Figure 61a).

(b) He chose O inside Π, but outside or on the triangle of midpoints (Figures 61b, d).

(c) He chose O inside the triangle of midpoints (Figure 61c).

(d) No! He cannot get a triangle. $\Pi \cap \Pi'$ has O as its center of symmetry. But a triangle cannot have a center of symmetry. Why not?

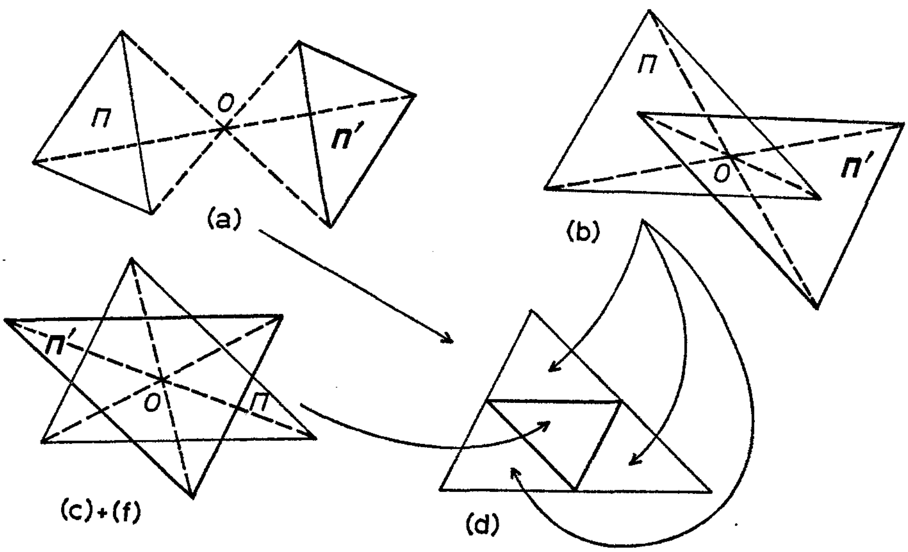


Fig. 61.

(e) $\Pi \cap \Pi'$ can be a hexagon, a parallelogram, a line segment, or a point.

(f) To get as much as possible ($\frac{2}{3}$ of the whole pie), O must coincide with the centroid. This cannot be proved at this stage. It is a question for empirical exploration.

(g) Obviously Π must have a center of symmetry. Cain should choose this center as his point O.

(h) The polygon has a center of symmetry. Hence, it has an even number of sides. Opposite sides are equal and parallel.

Cain and Abel

After some time Abel arrived, a very clever boy. Adam liked him very much and gave him preferential treatment. This behavior is dangerous and it will eventually end in a tragedy.

One day Adam showed the two brothers a pie. And he said to Abel: You may cut off a piece as big as you can. The remainder is for Cain. But your cut must be straight and it must go through a point O which is chosen by Cain.

Around this geometrical situation a whole battery of geometrical problems arises. Here are some with solutions:

(a) Name a class of fair pies. Pies with a center of symmetry are fair if Cain is smart enough to place O in the center.

(b) Has every fair pie a center of symmetry? Yes! But the proof is not so simple and can be postponed for higher grades.

(c) Does there exist a shape which is favorable to Cain? This is a trivial question for us, but children often have to think about it.

(d) Suppose Abel does not want to take advantage of his brother's blunders. Can he always be fair for every shape of the pie, and for every choice of point O? Yes, the existence of a fair cut through every point of the plane is obvious, but its construction is not obvious. A continuity argument is involved here.

(e) The pie is *rectangular* and Cain failed to choose the center. Which cut is most favorable to Abel?

Reflect the pie Π at O to Π' (Figure 62). Then it becomes completely

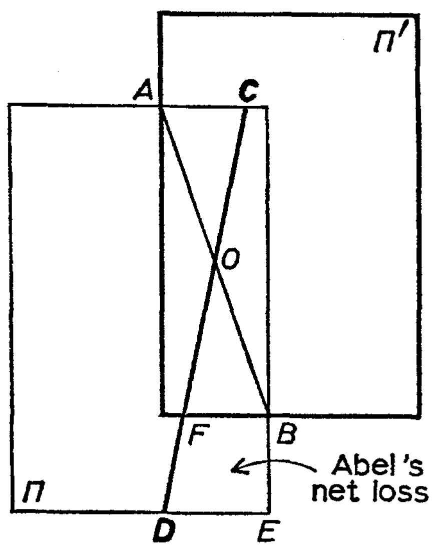


Fig. 62.

obvious that the cut AB, which is bisected at O, is best for Abel. Indeed, take another cut, say CD. This cut results in a net loss to Abel of the piece $DEBF$.

(f) The pie is *circular* and Cain failed to choose the center. Find the optimal cut for Abel.

Reflect Π at O to Π'. Again the cut AB, which is bisected at O, is optimal for Abel. Another cut *CD* leads to a net loss for Abel equal to the shaded area. (Figure 63).

(g) The method used in (e) and (f) works for any *convex pie*. The best cut has its center at *O*, and it is easy to construct by reflecting *Π* at *O*.

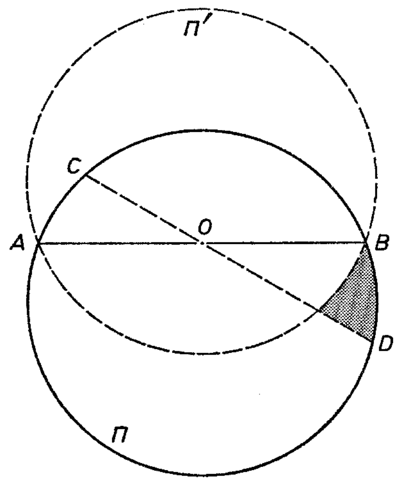


Fig. 63.

(h) For *nonconvex pies* complications arise. In Figure 64 there are 3 cuts bisected at *O*. Which one is the best for Abel?

Turn the cut around *O* and make a balance of gains and losses.

(i) One *square pie* is fair. Cain chooses the center. Two congruent square pies, as in Figure 65, are also fair, since Cain can choose the point *O* which

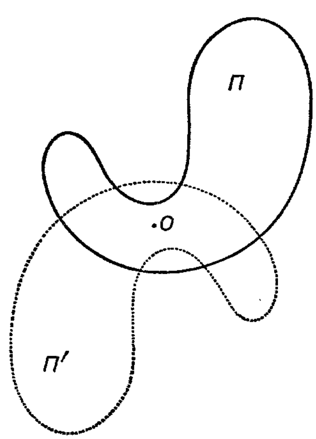


Fig. 64.

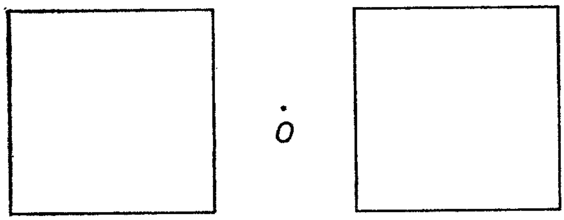


Fig. 65.

guarantees him one half of the pies. But suppose Abel may translate one of the pies before Cain chooses the point D. Can he increase his share? What if he may rotate a pie? Investigate the case where the second pie in Figure 65 shrinks to a smaller size. Study the case of two unequal circular pies.

(k) *Pies in the sky*. Three different pies are floating in the sky. They have the shape of boxes (Figure 66). Abel wants to bisect all three simultaneously

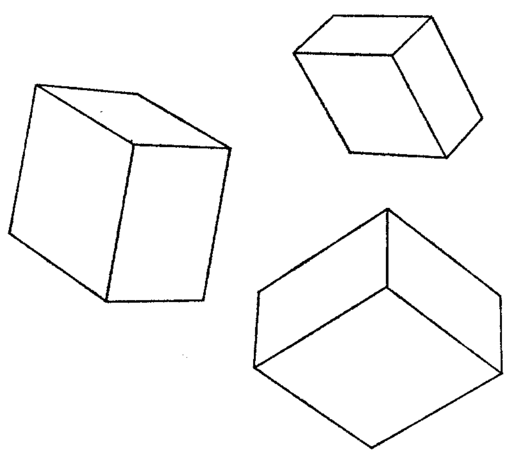


Fig. 66.

by a plane cut. Can he always do it, and if yes, how? For what locations of the pies are there infinitely many solutions? If the problem causes trouble, one should solve its plane analog first (see Figure 67). The solution is extremely simple and elegant. Each box has a center of symmetry. Each plane through its center bisects a box. The plane through the three centers bisects all boxes simultaneously. There are infinitely many solutions if the centers are collinear.

Abel runs for his life

In Figure 68 Abel is at A and Cain is 100 yards due North at C. Cain chases Abel. Both are running at the same speed. Abel is running East. Cain's path is more complicated. Every 15 yards he locates Abel's current position and runs in the direction of that position. In the long run he will

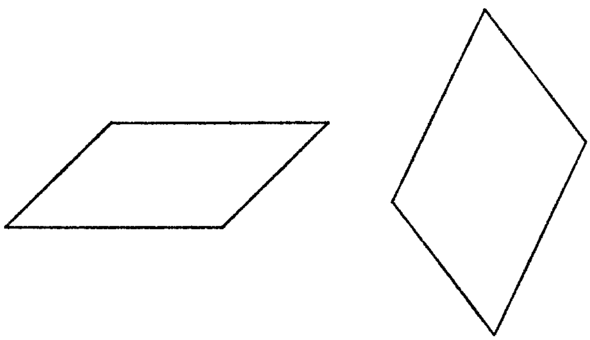


Fig. 67.

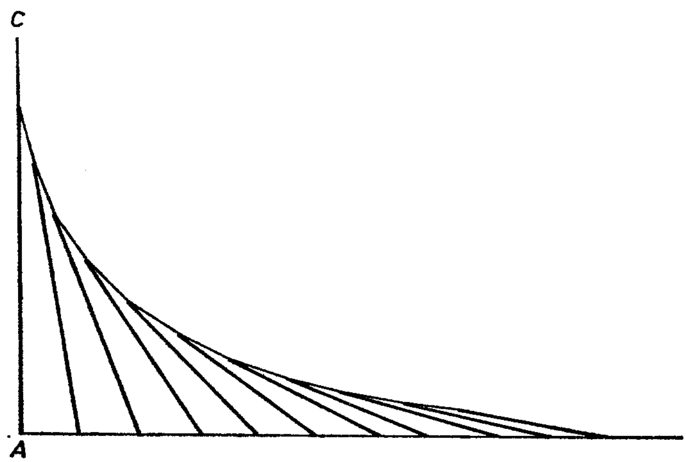


Fig. 68.

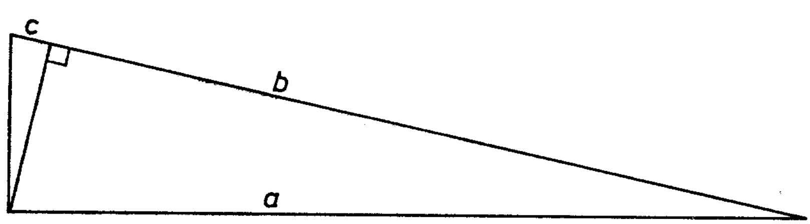


Fig. 69. $b < a < b + c$.

cut the original distance by one half. By using a different strategy Cain could reduce the distance to less than c for every $c>0$. Look at Figure 69 and prove this.

RECTANGULAR PROJECTORS

In some activities students become familiar with important concepts, in others they find empirically an important result. But in some activities only geometric intuition is developed. Here is one such example: Figure 70

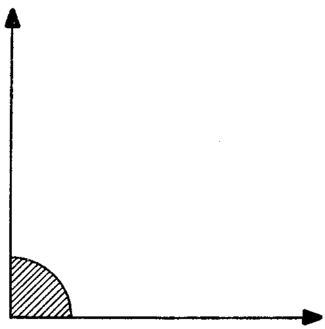


Fig. 70.

shows a rectangular projector. It illuminates a quarter of the plane. Four of these projectors are placed in the plane. Can you always illuminate the whole plane? Or can you find some positions for which this is not possible?

Children assume different positions and try to cover the plane by projectors placed at these positions. They usually succeed, sometimes after many fruitless attempts.

At the end of the activity comes the proof that it is always possible. Let A, B, C, D be the four positions of the projectors. Draw a line g separating A, B from C, D. The projectors A and B can illuminate the half-plane of C, D. The projectors C and D can illuminate the half-plane of A, B.

Problem 22. Take 8 space projectors, each illuminating one octant of space. Can you always illuminate the whole space?

ASSUMPTIONS AND CONSEQUENCES

An important class of problems consists in making assumptions and drawing consequences. A teacher should have hundreds in store. In one hour one can use up 5 to 10 of these problems. It takes a teacher many years until he has a store of problems which can be solved with a bare minimum of prerequisities. Here are three examples:

(a) The opposite edges of a tetrahedron are equal. Show that all faces are congruent. (See Figure 71.)

(b) All faces of a hexahedron are congruent parallelograms. Show that the faces are rhombi, i.e., the hexahedron is a rhombohedron. (Figure 72). Is

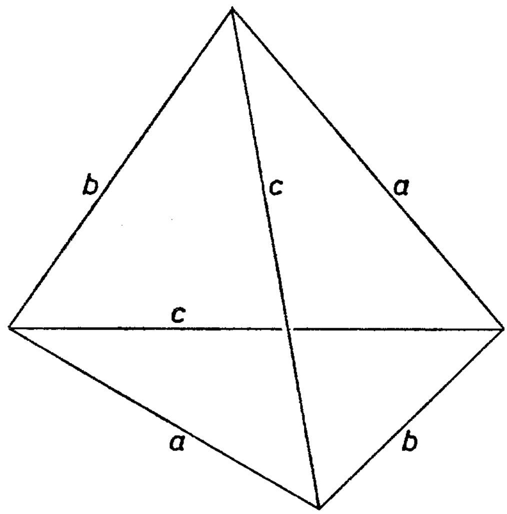


Fig. 71.

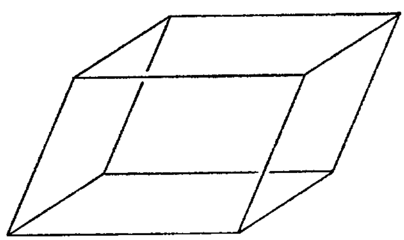


Fig. 72.

the rhombohedron symmetric? Yes, it has three planes of symmetry, each passing through a pair of opposite edges.

(c) Definition of a regular space polygon: It is a closed sequence of congruent segments which do not lie in the same plane. The angles between two successive segments are equal. Show that regular space polygons with 4, 6, 7, 8, 9, 10, 11, 12, ... vertices exist.

The case of an even number of edges is especially easy. Here are two solutions:

(1) Take a strip of congruent squares and fold it at right angles to form a staircase (Figure 73).

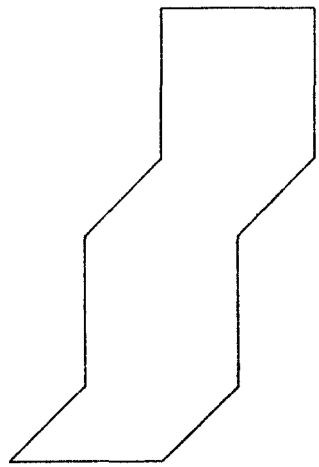


Fig. 73.

(2) Take a regular plane polygon and raise every second vertex by the same amount.

Problem 23. It is somewhat more difficult to find regular space polygons with 7, 9, 11, 13, ... vertices. Find such polygons!

Problem 24. A regular space polygon with 5 vertices does not exist. This is a famous and tough problem.

NOTE

I would like to thank Father Larry N. Lorenzoni, SDB, CSMP Senior Editor for helping me eliminate some crimes against the English language.

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