Note on a Class of Exact Solutions in Magneto-hydrodynamics

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1. Introduction

The purpose of this note is to describe a class of exact solutions of the magneto-hydrodynamic equations which are applicable to various flow configurations. Some examples of these solutions have been worked out in some detail by PETER GOTTLIEB; his solutions and some others will be reported in future communications.

The class of solutions considered here is of the "layer-type", including many classical exact solutions of the Navier-Stokes equations as special cases; *e.g.*, the flow near a stagnation point in the two-dimensional and the axially symmetrical cases, and the flow of a viscous fluid over a rotating disc. It represents a further generalization of a class of solutions discussed by the present writer for the steady or unsteady motion of an ordinary viscous fluid.

2. Basic equations

For a uniform electrically conducting incompressible fluid, the equations of magneto-hydrodynamics are as follows*:

$$\frac{\partial v_i}{\partial t} + v_j \frac{\partial v_i}{\partial x_j} - \frac{\mu}{4\pi\varrho} H_j \frac{\partial H_i}{\partial x_j} = - \frac{\partial \Pi}{\partial x_i} + \nu \Delta v_i, \qquad \frac{\partial v_k}{\partial x_k} = 0; \tag{2.1}$$

$$\frac{\partial H_i}{\partial t} + v_j \frac{\partial H_i}{\partial x_j} - H_j \frac{\partial v_i}{\partial x_j} = \eta \Delta H_i, \qquad \frac{\partial H_k}{\partial x_k} = 0. \tag{2.2}$$

In these equations, x_i $(i = 0, 1, 2)$ are the space coordinates, t is the time, v_i are the components of velocity, H_i are the components of the magnetic field. The properties of the fluid are characterized by the density ϱ, the kinematic viscosity ν, the magnetic permeability μ and the electric conductivity σ, in terms of which we define the magnetic diffusivity

$$\eta = (4\pi\mu\sigma)^{-1}. \tag{2.3}$$

The quantity Π is essentially the total pressure and is defined

$$\varrho \Pi = p + \mu H^2/8\pi, \tag{2.4}$$

where p is the ordinary pressure of the fluid. The summation convention is used throughout this paper.

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* *Cf.* COWLING (1957), p. 93

We seek a class of solutions in which the x_0-axis occupies a special position. We allow the velocity field, the magnetic field, and the pressure gradient to depend on the coordinate x_0 and the time t in a fairly general manner, but only *linearly* on the two other coordinates x_α ($\alpha = 1, 2$). To be more specific, we assume

$$\begin{aligned} v_0 &= u_0(x_0, t) \\ v_\alpha &= u_\alpha(x_0, t) + u_{\alpha\beta}(x_0, t)\, x_\beta, \end{aligned} \tag{2.5}$$

$$\begin{aligned} H_0 &= h_0(x_0, t) \\ H_\alpha &= h_\alpha(x_0, t) + h_{\alpha\beta}(x_0, t)\, x_\beta, \end{aligned} \tag{2.6}$$

and

$$\begin{aligned} \frac{\partial \Pi}{\partial x_0} &= \varpi_0(x_0\, t) \\ \frac{\partial \Pi}{\partial x_\alpha} &= \varpi_\alpha(x_0, t) + \varpi_{\alpha\beta}(x_0, t)\, x_\beta. \end{aligned} \tag{2.7}$$

Since we must have

$$\frac{\partial}{\partial x_\alpha}\left(\frac{\partial \Pi}{\partial x_0}\right) = \frac{\partial}{\partial x_0}\left(\frac{\partial \Pi}{\partial x_\alpha}\right), \tag{2.8}$$

the functions $\varpi_\alpha(x_0, t)$ and $\varpi_{\alpha\beta}(x_0, t)$ must actually be independent of x_0. The function Π then takes on the form

$$\begin{aligned} \Pi &= \tfrac{1}{2}\,\varpi_{\alpha\beta}(t)\, x_\alpha\, x_\beta + \varpi_\beta(t)\, x_\beta + \varpi(x_0, t), \\ \varpi_0 &= \frac{\partial \varpi}{\partial x_0}. \end{aligned} \tag{2.9}$$

3. Equations in two independent variables

The particular choice of the form of the solutions is made so that the other terms in the basic equations (2.1) and (2.2) are also *linear* in the variables x_α ($\alpha = 1, 2$). Consider, for example, $v_j \dfrac{\partial H_i}{\partial x_j}$. By using (2.5) and (2.6), it is easily seen that

$$v_j \frac{\partial H_i}{\partial x_i} = v_0 \frac{\partial h_0}{\partial x_0} \tag{3.1}$$

and

$$v_j \frac{\partial H_\alpha}{\partial x_j} = \left(v_0 \frac{\partial h_\alpha}{\partial x_0} + v_\gamma\, h_{\alpha\gamma}\right) + \left(v_0 \frac{\partial h_{\alpha\beta}}{\partial x_0} + v_{\gamma\beta}\, h_{\alpha\gamma}\right) x_\beta. \tag{3.2}$$

Thus, these basic equations can be satisfied by comparing coefficients of x_1 and x_2 and terms independent of them.

For convenience of reference, we list below the formulae useful for such reductions. Each of the vectors v_i and H_i is of the form

$$\{A_i\} = \{a_0,\, a_\alpha + a_{\alpha\beta}\, x_\beta\}, \tag{3.3}$$

where a_0, a_α, $a_{\alpha\beta}$ are functions of x_0 and t. The vector $\partial A_i/\partial t$ is then of the same form. In fact

$$\left\{\frac{\partial A_i}{\partial t}\right\} = \left\{\frac{\partial a_0}{\partial t},\, \frac{\partial a_\alpha}{\partial t} + \frac{\partial a_{\alpha\beta}}{\partial t}\, x_\beta\right\}. \tag{3.4}$$

Also, the Laplacian of A_i is of the form

$$\{\Delta A_i\} = \left\{\frac{\partial^2 a_0}{\partial x_0^2},\ \frac{\partial^2 a_\alpha}{\partial x_0^2} + \frac{\partial^2 a_{\alpha\beta}}{\partial x_0^2} x_\beta\right\}. \tag{3.5}$$

Furthermore, if B_i is another vector of the same form as A_i, then

$$C_i = B_j \frac{\partial A_i}{\partial x_j} \tag{3.6}$$

is also of the same form; indeed,

$$\begin{gathered} c_0 = \frac{\partial a_0}{\partial x_0}, \qquad c_\alpha = b_0 \frac{\partial a_\alpha}{\partial x_0} + a_{\alpha\gamma} b_\gamma, \\ c_{\alpha\beta} = b_0 \frac{\partial a_{\alpha\beta}}{\partial x_0} + a_{\alpha\gamma} b_{\gamma\beta}. \end{gathered} \tag{3.7}$$

With the help of these formulae, it is not difficult to write down the equations for the functions of x_0 and t defined in (2.5) and (2,6).

The equation of continuity in (2.1) is satisfied if

$$\frac{\partial u_0}{\partial x_0} + u_{\beta\beta} = 0. \tag{I}$$

Similarly, the condition for the absence of free magnetic charges is satisfied if

$$\frac{\partial h_0}{\partial x_0} + h_{\beta\beta} = 0. \tag{II}$$

The equations of motion (2.1) lead to

$$\frac{\partial u_0}{\partial t} + u_0 \frac{\partial u_0}{\partial x_0} - \nu \frac{\partial^2 u_0}{\partial x_0^2} = -\varpi_0 + \frac{\mu}{4\pi\varrho}\left(h_0 \frac{\partial h_0}{\partial x_0}\right), \tag{IIIa}$$

$$\frac{\partial u_\alpha}{\partial t} + u_0 \frac{\partial u_\alpha}{\partial x_0} + u_{\alpha\gamma} u_\gamma - \nu \frac{\partial^2 u_\alpha}{\partial x_0^2} = -\varpi_\alpha + \frac{\mu}{4\pi\varrho}\left(h_0 \frac{\partial h_\alpha}{\partial x_0} + h_{\alpha\gamma} h_\gamma\right), \tag{IIIb}$$

$$\frac{\partial u_{\alpha\beta}}{\partial t} + u_0 \frac{\partial u_{\alpha\beta}}{\partial x_0} + u_{\alpha\gamma} u_{\gamma\beta} - \nu \frac{\partial^2 u_{\alpha\beta}}{\partial x_0^2} = -\varpi_{\alpha\beta} + \frac{\mu}{4\pi\varrho}\left(h_0 \frac{\partial h_{\alpha\beta}}{\partial x_0} + h_{\alpha\gamma} h_{\gamma\beta}\right). \tag{IIIc}$$

Similarly, the equations (2.2) for the change of magnetic field lead to

$$\frac{\partial h_0}{\partial t} + u_0 \frac{\partial h_0}{\partial x_0} - \eta \frac{\partial^2 h_0}{\partial x_0^2} = h_0 \frac{\partial u_0}{\partial x_0}, \tag{IVa}$$

$$\frac{\partial h_\alpha}{\partial t} + u_0 \frac{\partial h_\alpha}{\partial x_0} + h_{\alpha\gamma} u_\gamma - \eta \frac{\partial^2 h_\alpha}{\partial x_0^2} = h_0 \frac{\partial u_\alpha}{\partial x_0} + u_{\alpha\gamma} h_\gamma, \tag{IVb}$$

$$\frac{\partial h_{\alpha\beta}}{\partial t} + u_0 \frac{\partial h_{\alpha\beta}}{\partial x_0} + h_{\alpha\gamma} u_{\gamma\beta} - \eta \frac{\partial^2 h_{\alpha\beta}}{\partial x_0^2} = h_0 \frac{\partial u_{\alpha\beta}}{\partial x_0} + u_{\alpha\gamma} h_{\gamma\beta}. \tag{IVc}$$

4. Consistency of the system

It is necessary to examine the above system of equations to see that there is indeed the right number of equations for the unknowns specified in (2.5), (2.6) and (2.7). Clearly, there are seven unknowns in each of (2.5) and (2.6) and only one unknown, *i.e.*, $\varpi_0(x_0, t)$, in (2.7), since ϖ_α and $\varpi_{\alpha\beta}$ are functions of t alone and are therefore expected to be specified by boundary conditions at specified values of x_0 (*e.g.* at infinity). Thus, there are *fifteen* unknowns altogether.

On the other hand, we have one equation for each of the types (I) and (II) and seven equations for each of the types (III) and (IV), making up a total of *sixteen* equations. Thus, one of them must be redundant if we are not going to have the difficulty of too many equations. Indeed, a closer examination shows that the equations (I), (II), (IVa) and (IVc) are not all independent.

If we contract (IVc), we obtain

$$\frac{\partial h_{\beta\beta}}{\partial t} + u_0 \frac{\partial h_{\beta\beta}}{\partial x_0} + h_{\beta\gamma} u_{\gamma\beta} - \eta \frac{\partial^2 h_{\beta\beta}}{\partial x_0^2} = h_0 \frac{\partial u_{\beta\beta}}{\partial x_0} + u_{\beta\gamma} h_{\gamma\beta},$$

or

$$\text{(IVc}') \qquad \frac{\partial h_{\beta\beta}}{\partial t} + u_0 \frac{\partial h_{\beta\beta}}{\partial x_0} - \eta \frac{\partial^2 h_{\beta\beta}}{\partial x_0^2} = h_0 \frac{\partial u_{\beta\beta}}{\partial x_0}.$$

In this form, it is clear that, by using (I) and (II), $u_{\beta\beta}$ and $h_{\beta\beta}$ can be eliminated to give an equation in h_0 and u_0. The resultant equation must then be consistent with (IVa). This fact can be easily verified by noting that the x_0-derivative of (IVa), when added onto (IVc′), leads to an identity by virtue of (I) and (II).

5. Concluding Remarks

We have thus found a large class of exact solutions of the hydromagnetic equations, involving fifteen unspecified functions of the variables x_0 and t. If one restricts oneself to the steady case, the equations above reduce to ordinary differential equations which can be integrated (numerically, if necessary) under appropriate boundary conditions. Symmetry conditions also help to simplify the equations in various cases.

These simplifications are all present in the examples worked out by PETER GOTTLIEB. He considered the flow near a stagnation point both in the two-dimensional case and in the case of axial symmetry with a magnetic field parallel to a solid boundary. These examples serve to illustrate the behavior of the magnetic field as it is carried along a solid surface due to convection currents in the core of the earth. It is clear, from a consideration of the path of the fluid particles, that there can be a considerable stretching of the magnetic lines. In the ideal case of zero viscosity and zero magnetic diffusivity, the field may be expected to become infinite as the boundary is approached. It is to be expected that this tendency to increase will be counteracted by both the hydrodynamic and the magnetic diffusive effects. The resultant picture in the case of steady motion is worked out in a paper to follow. Other interesting problems may be developed within the present framework. For example, one might consider an infinite disc oscillating about its axis (taken to be the x_0-axis) in the presence of a magnetic field in that direction. Some hydromagnetic damping is certainly to be expected. The steady rotation of a disc would be another possible problem, extending that worked out by VON KÁRMÁN & COCHRAN years ago* in the purely hydrodynamic case. Indeed, this may be combined with the problems studied by GOTTLIEB to describe the combined effect of earth rotation and convection near a stagnation point in the problem of geomagnetism. Finally, one might consider the problem of a uniform magnetic field perpendicular to the x_0-axis

* See GOLDSTEIN (1938), p. 110.

and rotating about it (perhaps in an unsteady manner), and attempt to find out whether this would indeed provide an effective "magnetic wall" to hold an approaching fluid. It is clear from elementary considerations of Eq. (2.1) and (2.2) that a magnetic field can act in the manner of a static pressure to hold a fluid in balance, but the added effect of rotation may have the advantage of making the equilibrium more stable. These and other problems are subjects for future studies.

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