# *On an Analogue of the Euler-Cauchy Polygon Method for the Numerical Solution of* $u_{xy}=f(x, y, u, u_x, u_y)$

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Contents



**Abstract.** This paper[1] develops, with an eye on the numerical applications, an analogue of the classical Euler-Cauchy polygon method (which is used in the solution of the ordinary differential equation

$$\frac{dy}{dx}=f(x,y), \quad y(x_0)=y_0)$$

for the solution of the following characteristic boundary value problem for a hyperbolic partial differential equation

$$u_{xy}=f(x,y,u,u_x,y_y),$$

$$u(x,y_0)=\sigma(x),$$

$$u(x_0,y)=\tau(y),$$

where $\sigma(x_0)=\tau(y_0)$. The method presented here, which may be roughly described as a process of bilinear interpolation, has the advantage over previously proposed methods that only the tabulated values of the given functions $\sigma(x)$ and $\tau(y)$ are required for its numerical application. Particular attention is devoted to the proof that a certain sequence of approximating functions, constructed in a specified way, actually converges to a solution of the boundary value problem under consideration. Known existence theorems are thus proved by a process which can actually be employed in numerical computation.

[1] This paper was issued on 16 January 1957 as NAVORD Report 4451, U. S. Naval Ordnance Laboratory, White Oak, Maryland, and was presented to the American Mathematical Society in October 1956.

## § 1. Introduction

The classical initial value problem for the ordinary differential equation

$$\frac{dy}{dx} = f(x, y),$$

(where the real valued continuous function $f(x, y)$ is defined for $x_0 \leqq x \leqq x_0 + a$ and $-\infty < y < +\infty$) consists in the determination of a real valued function $y(x)$, defined on $x_0 \leqq x \leqq x_0 + a$, which satisfies the given ordinary differential equation on this interval, and also satisfies the initial condition

$$y(x_0) = y_0,$$

where $y_0$ is a given real number.

Among the many methods which have been employed for proving the existence of a solution $y(x)$ to this problem, mention will be made here only of PICARD's method of successive approximations (see *e.g.*, G. SANSONE [*21*, vol. I, pp. 9–14], E. L. INCE [*12*, pp. 63–65], E. A. CODDINGTON & N. LEVINSON [*28*, p. 11–13], or E. KAMKE [*16*, pp. 51–56]); of L. TONELLI's method (see, *e.g.*, L. TONELLI [*13*], G. SANSONE [*21*, vol. I, pp. 45–48]); and of the Euler-Cauchy polygon method (see, *e.g.*, G. SANSONE [*21*, vol. I, pp. 36–45, vol. II, pp. 208–283], E. L. INCE [*12*, pp. 75–81], E. A. CODDINGTON & N. LEVINSON [*28*, pp. 3–7], E. KAMKE [*16*, pp. 62–64], or G. A. BLISS [*9*, pp. 86–92]).

For the numerical purpose of the actual construction of a solution the Euler-Cauchy polygon method is usually the most advantageous. The construction of the Euler-Cauchy polygons may be described as follows. For each positive integer $m$, let

$$x_0 \equiv x_{0,m} < x_{1,m} < x_{2,m} < \cdots < x_{m-1,m} < x_{m,m} \equiv x_0 + a,$$

be a subdivision of the interval $x_0 \leqq x \leqq x_0 + a$ into $m$ closed subintervals $x_{k,m} \leqq x \leqq x_{k+1,m}$, where $k = 0, 1, \ldots, m-1$. On each such subinterval the ordinary differential equation is, so to speak, replaced by one whose right-hand side is a (suitably chosen) constant, so that the corresponding function approximating a solution turns out to be a linear function on each subinterval. More precisely put, the polygonal function $y_m$, which is an approximation to a solution, is defined recurrently by the equations

$$\begin{array}{lll}
\frac{dy_m}{dx}(x) = f(x_{0,m}; y_0), & y_m(x_{0,m}) = y_0, & \text{on } x_{0,m} \leqq x \leqq x_{1,m}, \\
\frac{dy_m}{dx}(x) = f(x_{1,m}; y_1), & y_m(x_{1,m}) = y_1, & \text{on } x_{1,m} \leqq x \leqq x_{2,m}, \\
\vdots & \vdots & \vdots \\
\frac{dy_m}{dx}(x) = f(x_{k,m}; y_k), & y_m(x_{k,m}) = y_k, & \text{on } x_{k,m} \leqq x \leqq x_{k+1,m}, \\
\vdots & \vdots & \vdots
\end{array}$$

for $k = 0, 1, \ldots, m-1$. Notice that, for simplicity in writing these equations, the symbol $y_k$ is used to denote the value of the function $y_m(x)$ at $x_{k,m}$, a value which is obtained from the definition of $y_m$ as a linear function on the preceding subinterval $x_{k-1,m} \leqq x \leqq x_{k,m}$ and which is used as an initial value for the function

$y_m(x)$ for the "miniature" initial value problem (of the same kind as the original one, but whose differential equation has a *constant* right-hand side):

$$\frac{d y_m}{dx}(x) = f(x_{k,m}; y_k), \qquad y_m(x_{k,m}) = y_k,$$

on the next subinterval $x_{k,m} \leqq x \leqq x_{k+1,m}$. For each positive integer $m$, the function $y_m(x)$ is continuous on the interval $x_0 \leqq x \leqq x_0 + a$, but its derivative will, in general, not exist throughout the interval, since it may jump at the subdivision numbers $x_{k,m}$.

Under the sole additional hypothesis that the function $f(x, y)$ is bounded in absolute value on $x_0 \leqq x \leqq x_0 + a$, $-\infty < y < +\infty$, it follows that the sequence of functions $\{y_m(x)\}$ is equibounded in absolute value and equicontinuous on the interval $x_0 \leqq x \leqq x_0 + a$, and hence, by ASCOLI's theorem [*1*] (see also TONELLI [*11*, p. 76–86]) there is a subsequence of the sequence $\{y_m(x)\}$ which converges uniformly to a continuous limit function on $x_0 \leqq x \leqq x_0 + a$. If, further, it is supposed that the maximum length of the subintervals of the subdivision of $x_0 \leqq x \leqq x_0 + a$ approaches zero, *i.e.*

$$\lim_{m \to \infty} \left[ \max_{k=0,1,\ldots,m-1} (x_{k+1,m} - x_{k,m}) \right] = 0,$$

then every such continuous limit function is a solution of the original initial value problem, whose solution need not be unique. (It should be noticed that the condition on the maximum length of the subintervals is automatically satisfied in the most common case when the $m^{\text{th}}$ subdivision consists of $m$ subintervals of equal length, namely $a/m$.) If, besides this, the function $f(x, y)$ satisfies a Lipschitz condition with respect to $y$, *i.e.* there is a number $L \geqq 0$ such that

$$|f(x, y_1) - f(x, y_2)| \leqq L |y_1 - y_2|,$$

whenever $x_0 \leqq x \leqq x_0 + a$, then the whole sequence $\{y_m(x)\}$ converges uniformly on $x_0 \leqq x \leqq x_0 + a$ to the (known to be unique) solution of the original initial value problem.

The purpose of the present paper is to develop, with an eye on the numerical applications, an analogue of the Euler-Cauchy polygon method for the solution of the characteristic boundary value problem for the hyperbolic partial differential equation

$$u_{xy} = f(x, y, u, u_x, u_y),$$

(where the real-valued continuous function $f(x, y, z, p, q)$ is defined for all $(x, y, z, p, q)$ satisfying

$$x_0 \leqq x \leqq x_0 + a, \quad y_0 \leqq y \leqq y_0 + b, \quad \text{and} \quad -\infty < z, p, q < +\infty).$$

The problem in question consists in the determination of a real-valued function $u(x, y)$ which satisfies the given partial differential equation on the rectangle $x_0 \leqq x \leqq x_0 + a$, $y_0 \leqq y \leqq y_0 + b$, and also satisfies the conditions

$$u(x, y_0) = \sigma(x) \quad \text{for } x_0 \leqq x \leqq x_0 + a,$$

$$u(x_0, y) = \tau(y) \quad \text{for } y_0 \leqq y \leqq y_0 + b,$$

where $\sigma(x_0)=\tau(y_0)$ and $\sigma(x)$ and $\tau(y)$ are given continuously differentiable functions on the characteristics $y=y_0$ and $x=x_0$ of the given hyperbolic equation. (The treatment of this boundary value problem by successive approximations goes back to E. PICARD [5] and has been considered by various other methods by many writers since that time.) For each pair of positive integers $m$ and $n$, consider the following subdivisions of the intervals

$$x_0 \leqq x \leqq x_0 + a \quad \text{and} \quad y_0 \leqq y \leqq y_0 + b,$$

$$x_0 \equiv x_{0,m} < x_{1,m} < x_{2,m} < \cdots < x_{m-1,m} < x_{m,m} \equiv x_0 + a,$$

$$y_0 \equiv y_{0,n} < y_{1,n} < y_{2,n} < \cdots < y_{n-1,n} < y_{n,n} \equiv y_0 + b,$$

which produce a subdivision of the rectangle $x_0 \leqq x \leqq x_0 + a$ and $y_0 \leqq y \leqq y_0 + b$. The miniature problem in the present method (see Section 3 for details) is as follows:

$$\frac{\partial^2 u_{mn}}{\partial x\,\partial y}(x,y) = A_{kl}, \quad \text{for } x_k \leqq x \leqq x_{k+1}, \quad y_l \leqq y \leqq y_{l+1},$$

$$u_{mn}(x,y_l) = D_{kl} + B_{kl}(x - x_k), \quad \text{for } x_k \leqq x \leqq x_{k+1},$$

$$u_{mn}(x_k,y) = D_{kl} + C_{kl}(y - y_l), \quad \text{for } y_l \leqq y \leqq y_{l+1},$$

where $A_{kl}$, $B_{kl}$, $C_{kl}$ and $D_{kl}$ are suitable constants, depending on the subrectangle (for simplicity in writing, $x_k$ has been written for $x_{k,m}$ and $y_l$ for $y_{l,n}$ in the formulation of the boundary value problem for the subrectangle). This means that on each subrectangle, the approximating function $u_{mn}$ is bilinear in $(x, y)$, *i.e.* it is a hyperbolic paraboloid:

$$u_{mn}(x,y) = A_{kl}(x - x_k)(y - y_l) + B_{kl}(x - x_k) + C_{kl}(y - y_l) + D_{kl}.$$

The process just described reduces in the special case of the equation $u_{xy}=f(x,y,u)$ and equal subdivisions of the intervals $x_0 \leqq x \leqq x_0 + a$, $y_0 \leqq y \leqq y_0 + b$ to the process given by G. ZWIRNER [24, pp. 222–223], who did not consider the more general equation treated here. Similar methods, analogous to the one described above, have been employed to prove existence theorems for the same boundary value problem by P. HARTMAN & A. WINTNER [26], R. H. MOORE [29] and R. CONTI [27], but they do not appear to be as convenient for numerical purposes as the one described above, which requires knowledge only of the tabulated values of the given functions $\sigma(x)$ and $\tau(y)$ (from which the difference quotients needed may easily be calculated) and does not require the tabulated values of the first derivatives $\sigma'(x)$ and $\tau'(y)$. Mention is also made of a different, but closely related, method, also analogous to the Euler-Cauchy polygon method, given by H. LEWY [14] (see also H. BECKERT [22]) for the solution of the initial value problem for second order quasilinear partial differential equations in two independent variables, which appears to require more differentiability assumptions than the present method.

The statement of the known main results and their connection with the existing literature is given in Section 2. Section 3 contains the precise description of the analogue of the Euler-Cauchy polygon method and the construction of the double sequence of functions $\{u_{mn}(x, y)\}$ approximating a solution. Each function $u_{mn}$

is continuous, but not necessarily differentiable with respect to $x$ and $y$ on the rectangle $x_0 \leqq x \leqq x_0 + a$, $y_0 \leqq y \leqq y_0 + b$. Section 4 contains an inequality, termed the convergence inequality, which is used, together with a theorem of C. ARZELÀ [7, pp. 119—125] on the convergence of certain not necessarily continuous functions to continuous limit functions, in order to complete the proof of the existence of a solution in Sections 5 and 6.

## § 2. Statement of known results

**Theorem 1.** *If*

(1) *the real-valued function* $f(x, y, z)$ *is defined for all* $(x, y, z)$ *such that*

$$x_0 \leqq x \leqq x_0 + a, \quad y_0 \leqq y \leqq y_0 + b, \quad -\infty < z < +\infty,$$

*where* $x_0, y_0, a, b$ *are real numbers, and* $a \geqq 0$, $b \geqq 0$, *and if* $f(x, y, z)$ *is continuous and bounded in absolute value, so that for a certain non-negative constant* $M$ *one has*

$$|f(x, y, z)| \leqq M$$

*for all these* $(x, y, z)$;

(2) *the real-valued function* $\sigma(x)$ *is defined for all* $x$ *such that* $x_0 \leqq x \leqq x_0 + a$ *and possesses a continuous first derivative* $\sigma'(x)$ *for all these* $x$, *while the real-valued function* $\tau(y)$ *is defined on the set* $y_0 \leqq y \leqq y_0 + b$ *and possesses a continuous first derivative* $\tau'(y)$ *for all these* $y$ *(it being understood, of course, that* $\sigma'(x_0)$, *for example, denotes the right-hand derivative of* $\sigma$ *at* $x_0$, *etc.)*; *then*

(3) *there is at least one real-valued function* $u(x, y)$ *defined on the rectangle*

$$R: \quad x_0 \leqq x \leqq x_0 + a, \quad y_0 \leqq y \leqq y_0 + b,$$

*which is continuous, together with its partial derivatives* $\partial u/\partial x$, $\partial u/\partial y$, $\partial^2 u/\partial x \cdot \partial y$ $(= \partial^2 u/\partial y\, \partial x)$ *on* $R$, *satisfies the partial differential equation*

$$\frac{\partial^2 u}{\partial y\, \partial x}(x, y) = f(x, y, u(x, y)) \quad \text{for } (x, y) \text{ in } R$$

*and the characteristic conditions*

$$u(x, y_0) = \sigma(x) \quad \text{for } x_0 \leqq x \leqq x_0 + a,$$

$$u(x_0, y) = \tau(y) \quad \text{for } y_0 \leqq y \leqq y_0 + b.$$

It is to be noticed that this theorem asserts the existence of at least one solution to the characteristic initial value problem under consideration, but that the uniqueness of the solution is not asserted, and is, in fact, in general not true. (See P. MONTEL [8, pp. 279—283].) One need only consider the following simple example of a characteristic problem (*cf.* P. HARTMAN & A. WINTNER [26, p. 84] and P. LEEHEY [23, p. 23]) consisting of the partial differential equation

$$\frac{\partial^2 u}{\partial x\, \partial y} = |u|^{1/2} \quad \text{for } 0 \leqq x \leqq a, \quad 0 \leqq y \leqq b,$$

and the initial conditions

$$u(x, 0) = 0 \quad \text{for } 0 \leqq x \leqq a,$$

$$u(0, y) = 0 \quad \text{for } 0 \leqq y \leqq b,$$

which has as solutions both

$$u_1(x, y) = 0,$$

and

$$u_2(x, y) = \tfrac{1}{16} x^2 y^2,$$

on the rectangle $0 \leqq x \leqq a$, $0 \leqq y \leqq b$.

**Theorem 2.** *If*

(1) *the real-valued function $f(x, y, z, p, q)$ is defined for all $(x, y, z, p, q)$ such that*

$$x_0 \leqq x \leqq x_0 + a, \quad y_0 \leqq y \leqq y_0 + b, \quad -\infty < z, p, q < +\infty,$$

*and is continuous and bounded in absolute value, so that for a certain non-negative constant $M$ one has*

$$|f(x, y, z, p, q)| \leqq M$$

*for all these $(x, y, z, p, q)$, and if $f$ satisfies a Lipschitz condition in the three arguments $z, p, q$ (that is, there is a constant $L \geqq 0$ such that one has*

$$|f(x, y, z, p, q) - f(x, y, z_1, p_1, q_1)| \leqq L|z - z_1| + L|p - p_1| + L|q - q_1|,$$

*for any $(z, p, q)$ and $(z_1, p_1, q_1)$, whenever $(x, y)$ lies in the rectangle*

$$R: \quad x_0 \leqq x \leqq x_0 + a, \quad y_0 \leqq y \leqq y_0 + b);$$

(2) *the real-valued function $\sigma(x)$ is defined for all $x$ such that $x_0 \leqq x \leqq x_0 + a$ and possesses a continuous first derivative $\sigma'(x)$ for all these $x$, while the real-valued function $\tau(y)$ is defined for all $y$ such that $y_0 \leqq y \leqq y_0 + b$ and possesses a continuous first derivative for all these $y$; then*

(3) *there is one and only one real-valued function $u(x, y)$ defined on the rectangle $R$, which is continuous together with its partial derivatives*

$$\frac{\partial u}{\partial x}, \ \frac{\partial u}{\partial y}, \ \frac{\partial^2 u}{\partial x \, \partial y} \quad \left(= \frac{\partial^2 u}{\partial y \, \partial x}\right) \text{ on } R,$$

*satisfies the partial differential equation*

$$\frac{\partial^2 u}{\partial x \, \partial y}(x, y) = f\left(x, y, u(x, y), \frac{\partial u}{\partial x}(x, y), \frac{\partial u}{\partial y}(x, y)\right) \quad \text{for } (x, y) \text{ in } R,$$

*and the characteristic conditions*

$$u(x, y_0) = \sigma(x) \quad \text{for } x_0 \leqq x \leqq x_0 + a,$$

$$u(x_0, y) = \tau(y) \quad \text{for } y_0 \leqq y \leqq y_0 + b.$$

This second theorem does not contain the first theorem as a special case, since the function $f(x, y, z)$ of Theorem 1 is not assumed to satisfy a Lipschitz condition in the argument $z$. However, if in Theorem 2 the function $f(x, y, z, p, q)$ does not depend on $p$ and $q$, then Theorem 2 yields the additional information that if $f(x, y, z)$ of Theorem 1 does satisfy a Lipschitz condition in the argument $z$, then the solution whose existence is assured by Theorem 1 is indeed unique. Theorem 2 is the classical theorem of PICARD [5] mentioned in the Introduction.

**Theorem 3.** *If*

(1) *the real-valued function* $f(x, y, z, p, q)$ *is defined for all* $(x, y, z, p, q)$ *such that*

$$x_0 \leqq x \leqq x_0 + a, \quad y_0 \leqq y \leqq y_0 + b, \quad -\infty < z, p, q < \infty,$$

*and is continuous and bounded in absolute value, so that for a certain non-negative constant* $M$ *one has*

$$|f(x, y, z, p, q)| \leqq M$$

*for all these* $(x, y, z, p, q)$, *and if* $f$ *satisfies a Lipschitz condition in the two arguments* $p, q$ *(that is, there is a constant* $L \geqq 0$ *such that one has*

$$|f(x, y, z, p, q) - f(x, y, z, p_1, q_1)| \leqq L|p - p_1| + L|q - q_1|$$

*for any* $(p, q)$ *and* $(p_1, q_1)$ *whenever* $(x, y)$ *lies in the rectangle*

$$R: \quad x_0 \leqq x \leqq x_0 + a, \quad y_0 \leqq y \leqq y_0 + b,$$

*and* $z$ *is any real number);*

(2) *the real-valued function* $\sigma(x)$ *is defined for all* $x$ *such that*

$$x_0 \leqq x \leqq x_0 + a,$$

*and possesses a continuous first derivative* $\sigma'(x)$ *for all these* $x$, *while the real-valued function* $\tau(y)$ *is defined for all* $y$ *such that*

$$y_0 \leqq y \leqq y_0 + b$$

*and possesses a continuous first derivative for all these* $y$; *then*

(3) *there is at least one real-valued function* $u(x, y)$ *defined on the rectangle* $R$ *which is continuous together with its partial derivatives* $\partial u/\partial x$, $\partial u/\partial y$, $\partial^2 u/\partial x\, \partial y$ $(= \partial^2 u/\partial y\, \partial x)$ *on* $R$, *satisfies the partial differential equation*

$$\frac{\partial^2 u}{\partial x\, \partial y}(x, y) = f\left(x, y, u(x, y), \frac{\partial u}{\partial x}(x, y), \frac{\partial u}{\partial y}(x, y)\right) \quad \textit{for } (x, y) \textit{ in } R,$$

*and the characteristic conditions*

$$u(x, y_0) = \sigma(x) \quad \textit{for } x_0 \leqq x \leqq x_0 + a,$$

$$u(x_0, y) = \tau(y) \quad \textit{for } y_0 \leqq y \leqq y_0 + b.$$

This third theorem contains the first theorem as a special case (and the same example used there is applicable here). The hypotheses made in the third theorem are such that the part of the second theorem concerning the existence of a solution follows, while the second theorem yields the additional information that if the function $f(x, y, z, p, q)$ satisfies a Lipschitz condition in $(z, p, q)$ together, rather than just in $(p, q)$, the solution $u(x, y)$ whose existence is asserted by the third theorem is indeed unique. Theorem 3 was first proved by P. Leehey [*23*] and P. Hartman & A. Wintner [*26*]. For more general theorems see R. Conti [*27*] and A. Alexiewicz & W. Orlicz [*30*].

## § 3. The double sequence of functions approximating a solution

Let $m$ and $n$ be positive integers and consider the corresponding subdivisions of the intervals $x_0 \leqq x \leqq x_0 + a$ and $y_0 \leqq y \leqq y_0 + b$, as follows:

$$x_0 \equiv x_{0,m} < x_{1,m} < x_{2,m} < \cdots < x_{m-1,m} < x_{m,m} \equiv x_0 + a,$$

$$y_0 \equiv y_{0,n} < y_{1,n} < y_{2,n} < \cdots < y_{n-1,n} < y_{n,n} \equiv y_0 + b.$$

These subdivisions of the intervals $x_0 \leqq x \leqq x_0 + a$ and $y_0 \leqq y \leqq y_0 + b$ produce a subdivision of the closed rectangle $R$ into $m \cdot n$ closed subrectangles $R_{kl}^{mn}$, where $k = 0, 1, \ldots, m-1$ and $l = 0, 1, \ldots, n-1$. The closed subrectangle $R_{kl}^{mn}$ consists in all $(x, y)$ of $R$ which satisfy the inequalities

$$x_{k,m} \leqq x \leqq x_{k+1,m}, \qquad y_{l,n} \leqq y \leqq y_{l+1,n}.$$

Given the functions $\sigma(x)$ and $\tau(y)$, defined on the closed intervals $x_0 \leqq x \leqq x_0 + a$ and $y_0 \leqq y \leqq y_0 + b$ respectively, a continuous function $u_{mn}(x, y)$ will be defined on the rectangle $R$ by a recurrent process, consisting in solving, on each subrectangle $R_{kl}^{mn}$, a boundary value problem of the form $\partial^2 u_{mn}/\partial x\, \partial y =$ constant, with assigned (linear) values for $u_{mn}$ on the two rectilinear closed intervals of the boundary of $R_{kl}^{mn}$ which intersect at its lower left hand vertex $(x_{k,m}, y_{l,n})$. Of course, the constant involved in the partial differential equation, and also the linear boundary values, both depend on $k$ and $l$ (and on $m$ and $n$). The fact that two adjacent rectangles, say $R_{kl}^{mn}$ and $R_{k+1,l}^{mn}$ for instance, have a common boundary interval (since they are both *closed* subrectangles) will create no difficulty concerning the definition of the function $u_{mn}$ for points lying on the common boundary intervals, since the specific process employed in defining $u_{mn}$ will be such that the values assigned to $u_{mn}$ will coincide in this situation.

Suppose, for the moment, that $u_{mn}$ has already been defined on the subrectangle of $R$ with lower left vertex $(x_0, y_0)$ and upper right vertex $(x_{k,m}, y_{l,m})$, *i.e.*, the subrectangle defined by the inequalities

$$x_0 \leqq x \leqq x_{k,m}, \qquad y_0 \leqq y \leqq y_{l,n},$$

where $1 \leqq k < m-1$ and $1 \leqq l < n-1$. Then the definition of the function $u_{mn}$ will be extended to the slightly larger subrectangle defined by the inequalities

$$x_0 \leqq x \leqq x_{k+1,m}, \qquad y_0 \leqq y \leqq y_{l+1,n},$$

by first defining it on the closed subrectangles

$$R_{k,0}^{mn}, R_{k,1}^{mn}, \ldots, R_{k,l-1}^{mn}$$

in numerical succession (*i.e.*, passing from $R_{k,0}^{mn}$ to $R_{k,1}^{mn}$, and so on); then defining it on the closed subrectangles

$$R_{0,l}^{mn}, R_{1,l}^{mn}, \ldots, R_{k-1,l}^{mn}$$

in numerical succession (*i.e.*, passing from $R_{0,l}^{mn}$ to $R_{1,l}^{mn}$, and so on); and finally defining it on the remaining closed subrectangle $R_{kl}^{mn}$ in order to complete the definition of $u_{mn}$ on the rectangle

$$x_0 \leqq x \leqq x_{k+1,m}, \qquad y_0 \leqq y \leqq y_{l+1,m}.$$

(A simply drawn figure will readily make the process intuitive to the reader.) Alternatively, the function $u_{mn}(x, y)$ may first be determined on the $m$ subrectangles in a row:

$$R_{0,l}^{mn},\ R_{1,l}^{mn},\ \ldots,\ R_{m-1,l}^{mn},$$

for the rows $l = 0, 1, 2, \ldots, n-1$ in succession. There remains only to make precise just exactly what boundary value problem, *i.e.*, what partial differential equation and what boundary conditions, is to be solved on each subrectangle $R_{kl}^{mn}$. This will be done by showing how the process is started in the initial subrectangle $R_{00}^{mn}$ and how the step-by-step scheme indicated above can then be carried out, using the given data, the given functions $\sigma(x)$ and $\tau(y)$. The final result will be an explicit formula for $u_{mn}(x, y)$ at any point $(x, y)$ of a typical subrectangle $R_{kl}^{mn}$.

On the rectangle $R_{00}^{mn}$ the function $u_{mn}$ is required to satisfy the partial differential equation (with constant right-hand side)

$$\frac{\partial^2 u_{mn}}{\partial x\,\partial y}(x, y) = f\left(x_0,\ y_0,\ \sigma(x_0),\ \frac{\sigma(x_{1m}) - \sigma(x_{0m})}{x_{1m} - x_{0m}},\ \frac{\tau(y_{1n}) - \tau(y_{0n})}{y_{1n} - y_{0n}}\right) \quad \text{for } (x, y) \text{ in } R_{00}^{mn},$$

subject to the boundary conditions

$$u_{mn}(x, y_0) = \sigma(x_0) + \frac{\sigma(x_{1,m}) - \sigma(x_0)}{x_{1,m} - x_0}(x - x_0) \qquad \text{for } x_0 \leqq x \leqq x_{1,m},$$

$$u_{mn}(x_0, y) = \tau(y_0) + \frac{\tau(y_{1,n}) - \tau(y_0)}{y_{1,n} - y_0}(y - y_0) \qquad \text{for } y_0 \leqq y \leqq y_{1,n}.$$

Roughly speaking, what is done in defining $u_{mn}$ on $R_{00}^{mn}$ is to take as boundary conditions along its left boundary edge and its lower boundary edge certain linear functions derived in a natural manner from the given functions $\tau(y)$ and $\sigma(x)$, and to use the value of $u_{mn}$ at $(x_0, y_0)$ and the slopes of these linear functions in determining the constant value to be assigned to $\partial^2 u_{mn}/\partial x\,\partial y$ on $R_{00}^{mn}$. It is clear that, the boundary value problem for $u_{mn}$ on $R_{00}^{mn}$ being explicitly solvable,

$$\begin{aligned} u_{mn}(x, y) = {} & u_{mn}(x_0, y_0) + \\ & + \frac{u_{mn}(x_{1,m}; y_0) - u_{mn}(x_0, y_0)}{x_{1,m} - x_{0,m}}(x - x_0) + \\ & + \frac{u_{mn}(x_0; y_{1,n}) - u_{mn}(x_0, y_0)}{y_{1,n} - y_{0,n}}(y - y_0) + \\ & + f\left(x_0,\ y_0,\ u_{mn}(x_0, y_0),\ \frac{u_{mn}(x_{1,m}; y_0) - u_{mn}(x_0, y_0)}{x_{1,m} - x_{0,m}},\right. \\ & \left.\frac{u_{mn}(x_0; y_{1,n}) - u_{mn}(x_0, y_0)}{y_{1,n} - y_{0,n}}\right)(x - x_0)(y - y_0) \end{aligned}$$

for $(x, y)$ in $R_{00}^{mn}$, where, for uniformity in the writing of formulas to appear later, $u_{mn}(x_0, y_0)$ has been written instead of $\sigma(x_0)$ or $\tau(y_0)$ *etc.* It is to be noticed that $u_{mn}$ is bilinear in $(x, y)$ on $R_{00}^{mn}$, *i.e.*, it is linear in $x$ for each fixed $y$ and linear in $y$ for each fixed $x$. (From this point of view the process of defining $u_{mn}$ being described may be thought of as a process of bilinear interpolation, so to speak.)

Consider now the definition of $u_{mn}$ on the rectangle $R_{kl}^{mn}$, it being assumed that $u_{mn}$ is *already known* as a linear function on the left boundary edge, where $x = x_{k,m}$, and on the lower boundary edge, where $y = y_{l,n}$, of the closed rectangle $R_{kl}^{mn}$. Then $u_{mn}$, on the rectangle $R_{kl}^{mn}$, is required to satisfy the partial differential equation with known constant right-hand side

$$\frac{\partial^2 u_{mn}}{\partial x\,\partial y}(x,y) = f\Bigl(x_{k,m};\ y_{l,n};\ u_{mn}(x_{k,m};\ y_{l,n});\ \frac{u_{mn}(x_{k+1,m};\ y_{l,n}) - u_{mn}(x_{k,m};\ y_{l,n})}{x_{k+1,m} - x_{k,m}};\ \frac{u_{mn}(x_{k,m};\ y_{l+1,n}) - u_{mn}(x_{k,m};\ y_{l,n})}{y_{l+1,n} - y_{l,n}}\Bigr) \quad \text{for } (x,y) \text{ in } R_{kl}^{mn},$$

and to coincide with an already known linear function of $y$ on the left boundary edge, where $x = x_{k,m}$, and with another already known linear function of $x$ on the lower boundary edge, where $y = y_{l,n}$. It is clear that $u_{mn}$ is bilinear in $(x, y)$ on $R_{kl}^{mn}$ and that

$$\begin{aligned} u_{mn}(x,y) = {} & u_{mn}(x_{k,m};\ y_{l,n}) + \\ & + \frac{u_{mn}(x_{k+1,m};\ y_{l,n}) - u_{mn}(x_{k,m};\ y_{l,n})}{x_{k+1,m} - x_{k,m}}(x - x_{k,m}) + \\ & + \frac{u_{mn}(x_{k,m};\ y_{l+1,n}) - u_{mn}(x_{k,m};\ y_{l,n})}{y_{l+1,n} - y_{l,n}}(y - y_{l,n}) + \\ & + f\Bigl(x_{k,m};\ y_{l,n};\ u_{mn}(x_{k,m};\ y_{l,n});\ \frac{u_{mn}(x_{k+1,m};\ y_{l,n}) - u_{mn}(x_{k,m};\ y_{l,n})}{x_{k+1,m} - x_{k,m}}; \\ & \frac{u_{mn}(x_{k,m};\ y_{l+1,n}) - u_{mn}(x_{k,m};\ y_{l,n})}{y_{l+1,n} - y_{l,n}}\Bigr)(x - x_{k,m})(y - y_{l,n}), \quad \text{for } (x,y) \text{ in } R_{kl}^{mn}. \end{aligned}$$

This last formula does not exhibit the explicit dependence of the function $u_{mn}(x, y)$ on the given functions $\sigma(x)$, $\tau(y)$. In order to obtain a formula which makes evident this explicit dependence on $\sigma$ and $\tau$, which will be essential in the convergence proofs to follow, it is convenient to use an abbreviated notation yielding more manageable formulas. For example, when considering the function $u_{mn}$, with $m$ and $n$ regarded as fixed throughout the discussion, a functional value such as

$$u_{mn}(x_{k,m};\ y_{l,n})$$

will be denoted simply by $u_{kl}$, and a functional value such as

$$f\Bigl(x_{k,m};\ y_{l,n};\ u_{mn}(x_{k,m};\ y_{l,n});\ \frac{u_{mn}(x_{k+1,m};\ y_{l,n}) - u_{mn}(x_{k,n};\ y_{l,n})}{x_{k+1,m} - x_{k,m}};\ \frac{u_{mn}(x_{k,m};\ y_{l+1,n}) - u_{mn}(x_{k,m};\ y_{l,n})}{y_{l+1,n} - y_{l,n}}\Bigr)$$

will be denoted merely by $f_{kl}$. Further, $x_k$ and $y_l$ will replace $x_{k,m}$ and $y_{l,n}$, respectively.

In this notation, the above formula for $u_{mn}(x, y)$, for $(x, y)$ in $R_{kl}^{mn}$, may be rewritten

$$u_{mn}(x,y) = u_{kl} + \frac{u_{k+1,l} - u_{kl}}{x_{k+1} - x_k}(x - x_k) + \frac{u_{k,l+1} - u_{kl}}{y_{l+1} - y_l}(y - y_l) + f_{kl}\cdot(x - x_k)(y - y_l).$$

Using this abbreviated notation, one has the following formulas for $u_{mn}$ on each of the rectangles $R_{00}^{mn}$, $R_{10}^{mn}$, $R_{01}^{mn}$, and $R_{11}^{mn}$, which are all special cases of the last formula just written for $R_{kl}^{mn}$, for $(k, l) = (0, 0)$, $(1, 0)$, $(0, 1)$ and $(1, 1)$, respectively. In the first place

$$u_{mn}(x, y) = u_{00} + \frac{u_{10} - u_{00}}{x_1 - x_0}(x - x_0) + \frac{u_{01} - u_{00}}{y_1 - y_0}(y - y_0) + f_{00}(x - x_0) \cdot (y - y_0)$$

for $(x, y)$ in $R_{00}^{mn}$, that is, when $x_0 \leqq x \leqq x_1$ and $y_0 \leqq y \leqq y_1$. In the second place

$$u_{mn}(x, y) = u_{10} + \frac{u_{20} - u_{10}}{x_2 - x_1}(x - x_1) + \frac{u_{11} - u_{10}}{y_1 - y_0}(y - y_0) + f_{10} \cdot (x - x_1)(y - y_0),$$

for $(x, y)$ in $R_{10}^{mn}$, that is, when $x_1 \leqq x \leqq x_2$ and $y_0 \leqq y \leqq y_1$. In the third place

$$u_{mn}(x, y) = u_{01} + \frac{u_{11} - u_{01}}{x_1 - x_0}(x - x_0) + \frac{u_{02} - u_{01}}{y_2 - y_1}(y - y_1) + f_{01} \cdot (x - x_0)(y - y_1),$$

for $(x, y)$ in $R_{01}^{mn}$, that is, when $x_0 \leqq x \leqq x_1$ and $y_1 \leqq y \leqq y_2$. In the fourth place

$$u_{mn}(x, y) = u_{11} + \frac{u_{21} - u_{11}}{x_2 - x_1}(x - x_1) + \frac{u_{12} - u_{11}}{y_2 - y_1}(y - y_1) + f_{11} \cdot (x - x_1)(y - y_1),$$

for $(x, y)$ in $R_{11}^{mn}$, that is, when $x_1 \leqq x \leqq x_2$ and $y_1 \leqq y \leqq y_2$.

The formulas for $R_{01}^{mn}$, $R_{10}^{mn}$ and $R_{11}^{mn}$ will now be rewritten so as to reveal the exact influence of the given functions $\sigma(x)$ and $\tau(y)$. From the formula for $(x, y)$ in $R_{00}^{mn}$ it follows that

$$u_{11} = u_{10} + u_{01} - u_{00} + f_{00} \cdot (x_1 - x_0)(y_1 - y_0).$$

Substituting this expression for $u_{11}$ into the formulas for $(x, y)$ in $R_{10}^{mn}$ and $R_{01}^{mn}$, one obtains

$$u_{mn}(x, y) = u_{10} + \frac{u_{20} - u_{10}}{x_2 - x_1}(x - x_1) + \frac{u_{01} - u_{00}}{y_1 - y_0}(y - y_0) + \\ + f_{00} \cdot (x_1 - x_0)(y - y_0) + f_{10} \cdot (x - x_1)(y - y_0),$$

when $(x, y)$ is in $R_{10}^{mn}$, and that

$$u_{mn}(x, y) = u_{01} + \frac{u_{10} - u_{00}}{x_1 - x_0}(x - x_0) + \frac{u_{02} - u_{11}}{y_2 - y_1}(y - y_1) + \\ + f_{00} \cdot (x - x_0)(y_1 - y_0) + f_{01}(x - x_0)(y - y_1),$$

when $(x, y)$ is in $R_{01}^{mn}$.

Now from these last two formulas for $(x, y)$ in $R_{10}^{mn}$ and $R_{01}^{mn}$ one obtains

$$u_{21} = u_{01} + u_{20} - u_{00} + f_{00}(x_1 - x_0)(y_1 - y_0) + f_{10}(x_2 - x_1)(y_1 - y_0),$$

and

$$u_{12} = u_{10} + u_{02} - u_{00} + f_{00}(x_1 - x_0)(y_1 - y_0) + f_{01}(x_1 - x_0)(y_2 - y_1);$$

these, together with the already known equation

$$u_{11} = u_{10} + u_{01} - u_{00} + f_{00} \cdot (x_1 - x_0)(y_1 - y_0),$$

can be used to rewrite the formula for $(x, y)$ in $R_{11}^{mn}$ as follows:

$$u_{mn}(x,y) = u_{10} + u_{01} - u_{00} + \frac{u_{20} - u_{10}}{x_2 - x_1}(x - x_1) + \frac{u_{02} - u_{01}}{y_2 - y_1}(y - y_1) + $$
$$+ f_{00} \cdot (x_1 - x_0)(y_1 - y_0) + f_{10} \cdot (x - x_1)(y_1 - y_0) +$$
$$+ f_{01} \cdot (x_1 - x_0)(y - y_1) + f_{11}(x - x_1)(y - y_1),$$

for $(x, y)$ in $R_{11}^{mn}$.

From the preceding considerations, the following general formula may be obtained by a process of mathematical induction:

$$u_{mn}(x,y) = u_{k,0} + u_{0,l} - u_{00} + \frac{u_{k+1,0} - u_{k,0}}{x_{k+1} - x_k}(x - x_k) + \frac{u_{0,l+1} - u_{0,l}}{y_{l+1} - y_l}(y - y_l) +$$
$$+ \sum_{i=1}^{k} \sum_{j=1}^{l} f_{i-1,j-1}(x_i - x_{i-1})(y_j - y_{j-1}) + \sum_{j=1}^{l} f_{k\,j-1}(x - x_k)(y_j - y_{j-1}) +$$
$$+ \sum_{i=1}^{k} f_{i-1,l}(x_i - x_{i-1})(y - y_l) + f_{kl}(x - x_k)(y - y_l)$$

for $(x, y)$ in $R_{kl}^{mn}$, that is, when $x_k \leqq x \leqq x_{k+1}$ and $y_l \leqq y \leqq y_{l+1}$, where $k = 0, 1, \ldots, m-1$ and $l = 0, 1, \ldots, n-1$. It is readily seen that by putting $(k, l)$ equal to $(0, 0)$, $(1, 0)$, $(0, 1)$, $(1, 1)$ in turn one obtains the formulas given above for $R_{00}^{mn}$, $R_{10}^{mn}$, $R_{01}^{mn}$, $R_{11}^{mn}$, respectively, as special cases.

For each pair of positive integers $m$ and $n$, there has been defined a subdivision of the rectangle $R$ into $m \cdot n$ closed subrectangles, and there has also been defined on the rectangle $R$ a real valued continuous function $u_{mn}(x, y)$. This double sequence of continuous functions $\{u_{mn}(x, y)\}$ is equibounded in absolute value on $R$. For let $A$, $B$, $C$, $D$ denote non-negative real constants such that

$$|\sigma(x)| \leqq A, \quad |\sigma(x) - \sigma(x^*)| \leqq C|x - x^*|,$$

whenever $x_0 \leqq x \leqq x_0 + a$, and $x_0 \leqq x^* \leqq x_0 + a$; and

$$|\tau(y)| \leqq B, \quad |\tau(y) - \tau(y^*)| \leqq D|y - y^*|,$$

whenever $y_0 \leqq y \leqq y_0 + b$, and $y_0 \leqq y^* \leqq y_0 + b$. (The existence of these constants $A$, $B$, $C$, $D$ follows from the assumptions made about the functions $\sigma(x)$ and $\tau(y)$ in any of the three theorems of Section 2.) Then, given $(x, y)$ in $R$, one has $x_k \leqq x \leqq x_{k+1}$ and $y_l \leqq y \leqq y_{l+1}$ for some suitable pair of integers $k$ and $l$, with $0 \leqq k \leqq m-1$ and $0 \leqq l \leqq n-1$. Hence

$$|u_{mn}(x,y)| \leqq |u_{k,0}| + |u_{0,l}| + |u_{00}| + \left|\frac{u_{k+1,0} - u_{k,0}}{x_{k+1} - x_k}\right| |x - x_k| + \left|\frac{u_{0,l+1} - u_{0,l}}{y_{l+1} - y_l}\right| |y - y_l| +$$
$$+ \sum_{i=1}^{k} \sum_{j=1}^{l} M(x_i - x_{i-1})(y_j - y_{j-1}) + \sum_{j=1}^{l} M(x - x_k)(y_j - y_{j-1}) +$$
$$+ \sum_{i=1}^{k} M(x_i - x_{i-1})(y - y_l) + M(x - x_k)(y - y_l),$$

where $M \geqq 0$ is an upper bound for the absolute value of the function $f$ (see the hypotheses of Theorems 1 to 3). Thus, by use of the definitions of the constants $A$, $B$, $C$, $D$ just given, it follows that

$$|u_{mn}(x,y)| \leqq 2A + B + Ca + Db + Mab,$$

where the numerical constant on the right-hand side does not depend on the point $(x, y)$ of $R$ or on the pair of positive integers $(m, n)$. This proves that the double sequence of continuous functions $\{u_{mn}(x, y)\}$ is equibounded in absolute value on $R$.

Now, for each pair of positive integers $(m, n)$, let the positive numbers $\alpha_m$ and $\beta_n$ be defined by

$$\alpha_m = \max_{k=0,1,\ldots,m-1} (x_{k+1} - x_k)$$

and

$$\beta_n = \max_{l=0,1,\ldots,n-1} (y_{l+1} - y_l),$$

so that the product $\alpha_m \cdot \beta_n$ is certainly not less than the area of the largest subrectangle of the subdivision of $R$ corresponding to the pair of positive integers $(m, n)$. Under the additional restriction that

$$\lim_{m\to\infty} \alpha_m = 0 \quad \text{and} \quad \lim_{n\to\infty} \beta_n = 0$$

(which implies, but is not implied by, the fact that the maximum area of the largest subrectangle of the $(m, n)^{\text{th}}$ subdivision of $R$ approaches zero) it will be shown that the double sequence of continuous functions $\{u_{mn}(x, y)\}$ is an equicontinuous double sequence of functions on $R$. By this is meant that if $\{u_{m_r n_r}(x, y)\}$ is any *singly* infinite sequence of functions (with $\lim_{r\to\infty} m_r = \lim_{r\to\infty} n_r = \infty$) extracted from the double sequence $\{u_{mn}(x, y)\}$, then the set of all functions $u_{m_r n_r}$, where $r = 1, 2, 3, \ldots$, is an equicontinuous set of functions.

In order to show this, one has to find an upper bound for the absolute value of the difference $u_{mn}(\bar{x}, \bar{y}) - u_{mn}(x, y)$, where $(\bar{x}, \bar{y})$ and $(x, y)$ are points of $R$. There are really four cases to consider, depending on the relative positions of the points $(\bar{x}, \bar{y})$ and $(x, y)$ with respect to each other namely; $x \leqq \bar{x}$ and $y \leqq \bar{y}$; $\bar{x} \leqq x$ and $\bar{y} \leqq y$; $x \leqq \bar{x}$ and $\bar{y} \leqq y$; $\bar{x} \leqq x$ and $y \leqq \bar{y}$. The first two cases are essentially the same by symmetry, *i.e.* by interchanging the roles of $(x, y)$ and $(\bar{x}, \bar{y})$, and a similar remark applies to the last two cases. Only the first case mentioned will be considered here, since the treatment in the third case is exactly analogous to it. In the first case one has $x \leqq \bar{x}$, $y \leqq \bar{y}$ and $x_k \leqq x \leqq x_{k+1}$, $y_l \leqq y \leqq y_{l+1}$, and $x_{\bar{k}} \leqq \bar{x} \leqq x_{\bar{k}+1}$, $y_{\bar{l}} \leqq \bar{y} \leqq y_{\bar{l}+1}$ for suitable pairs of integers $(k, l)$ and $(\bar{k}, \bar{l})$. Further, $x_k \leqq x_{\bar{k}}$, $y_l \leqq y_{\bar{l}}$ and $x_{k+1} \leqq x_{\bar{k}+1}$, $y_{l+1} \leqq y_{\bar{l}+1}$.

From the definition of the function $u_{mn}$ it follows that

$$\begin{aligned} u_{mn}(\bar{x}, \bar{y}) - u_{mn}(x, y) = {} & u_{\bar{k},0} - u_{k,0} + u_{0,\bar{l}} - u_{0,l} + \\ & + \frac{u_{\bar{k}+1,0} - u_{\bar{k},0}}{x_{\bar{k}+1} - x_{\bar{k}}} (\bar{x} - x_{\bar{k}}) + \frac{u_{0,\bar{l}+1} - u_{0,\bar{l}}}{y_{\bar{l}+1} - y_{\bar{l}}} (\bar{y} - y_{\bar{l}}) - \\ & - \frac{u_{k+1,0} - u_{k,0}}{x_{k+1} - x_k} (x - x_k) - \frac{u_{0,l+1} - u_{0,l}}{y_{l+1} - y_l} (y - y_l) + \\ & + \left[ \sum_{i=1}^{\bar{k}} \sum_{j=1}^{\bar{l}} - \sum_{i=1}^{k} \sum_{j=1}^{l} \right] [f_{i-1,j-1}(x_i - x_{i-1}) (y_j - y_{j-1})] + \\ & + \sum_{j=1}^{\bar{l}} f_{\bar{k},j-1}(\bar{x} - x_{\bar{k}}) (y_j - y_{j-1}) + \sum_{i=1}^{\bar{k}} f_{i-1,\bar{l}}(x_i - x_{i-1}) (\bar{y} - y_{\bar{l}}) - \\ & - \sum_{j=1}^{l} f_{k,j-1}(x - x_k) (y_j - y_{j-1}) - \sum_{i=1}^{k} f_{i-1,l}(x_i - x_{i-1}) (y - y_l) + \\ & + f_{\bar{k}\bar{l}}(\bar{x} - x_{\bar{k}}) (\bar{y} - y_{\bar{l}}) - f_{kl}(x - x_k) (y - y_l). \end{aligned}$$

Hence

$$|u_{mn}(\bar{x},\bar{y}) - u_{mn}(x,y)| \leqq C(x_{\bar{k}+1} - x_k) + D(y_{\bar{l}+1} - y_l) + 2C\alpha_m + 2D\beta_n + \\ + M\cdot[(x_{\bar{k}} - x_0)(y_{\bar{l}} - y_0) - (x_k - x_0)(y_l - y_1)] + 2Mb\alpha_m + 2M\alpha\beta_n + 2M\alpha_m\beta_n,$$

in terms of the constants $A$, $B$, $C$, $D$, $M$ which were introduced earlier. However,

$$x_{\bar{k}+1} - x_k \leqq (\bar{x} + \alpha_m) - (x - \alpha_m) = (\bar{x} - x) + 2\alpha_m,$$

and similarly

$$y_{\bar{l}+1} - y_l \leqq (\bar{y} + \beta_n) - (y - \beta_n) = (\bar{y} - y) + 2\beta_n$$

while

$$\begin{aligned}(x_{\bar{k}} - x_0)&(y_{\bar{l}} - y_0) - (x_k - x_0)(y_l - y_0)\\ &\leqq (\bar{x} + \alpha_m - x_0)(\bar{y} + \beta_n - y_0) - (x - \alpha_m - x_0)(y - \beta_n - y_0)\\ &\leqq (\bar{x} - x_0)(\bar{y} - y_0) - (x - x_0)(y - y_0) + \\ &\qquad + \alpha_m[(\bar{y} - y_0) + (y - y_0)] + \beta_n[(\bar{x} - x_0) + (x - x_0)]\\ &\leqq (\bar{x} - x_0)(\bar{y} - y_0) - (x - x_0)(y - y_0) + 2b\alpha_m + 2a\beta_n,\end{aligned}$$

so that finally

$$|u_{mn}(\bar{x},\bar{y}) - u_{mn}(x,y)| \leqq 4(C + Mb)\alpha_m + 4(D + Ma)\beta_n + \\ + C(\bar{x} - x) + D(\bar{y} - y) + 2M\alpha_m\beta_n + M[(\bar{x} - x_0)(\bar{y} - y_0) - (x - x_0)(y - y_0)].$$

Suppose $\varepsilon > 0$ is given. Since

$$\lim_{m\to\infty} \alpha_m = \lim_{n\to\infty} \beta_n = 0,$$

there are positive integers $m_\varepsilon$ and $n_\varepsilon$ such that

$$4(C + Mb)\alpha_m + 4(D + Ma)\beta_n + 2M\alpha_m\beta_n < \tfrac{1}{2}\varepsilon$$

whenever $m > m_\varepsilon$ and $n > n_\varepsilon$. Further, in view of the continuity of the functions involved, there is a number $\delta_\varepsilon > 0$, which does not depend on $m$ and $n$ and is such that

$$C(\bar{x} - x) + D(\bar{y} - y) + M[(\bar{x} - x_0)(\bar{y} - y_0) - (x - x_0)(y - y_0)] < \tfrac{1}{2}\varepsilon$$

whenever $|x - \bar{x}| < \delta_\varepsilon$ and $|y - \bar{y}| < \delta_\varepsilon$. Thus, whenever $m > m_\varepsilon$ and $n > n_\varepsilon$ and

$$|x - \bar{x}| < \delta_\varepsilon, \quad |y - \bar{y}| < \delta_\varepsilon,$$

one has

$$|u_{mn}(\bar{x},\bar{y}) - u_{mn}(x,y)| < \varepsilon.$$

Now, let $\{u_{m_r n_r}(x, y)\}$, where $\lim_{r\to\infty} m_r = \lim_{r\to\infty} n_r = \infty$, be a *singly* infinite subsequence extracted from the *double* sequence $\{u_{mn}(x, y)\}$. Given $\varepsilon > 0$, one certainly has $m_r > m_\varepsilon$ and $n_r > n_\varepsilon$ for all but a finite number of positive integers $r$, and hence

$$|u_{m_r n_r}(\bar{x},\bar{y}) - u_{m_r n_r}(x,y)| < \varepsilon \quad \text{whenever both} \quad |x - \bar{x}| < \delta_\varepsilon \text{ and } |y - \bar{y}| < \delta_\varepsilon.$$

Since only a *finite* number of values of $r$ are excluded and the corresponding *finite* number of excluded functions $u_{m_r n_r}$ are continuous (hence uniformly continuous) on the rectangle $R$, it easily follows that the set of functions $u_{m_r n_r}$, where $r = 1, 2, 3, \ldots$, is an equicontinuous set of functions as desired.

It follows then from ASCOLI's theorem (see ASCOLI [1] or TONELLI [11]) that there is a subsequence of $\{u_{m_r n_r}(x, y)\}$ which converges uniformly on $R$ to a continuous limit function. This information is all that is really needed to complete the proof of Theorem 1 of Section 2 (where $f$ depends only on $(x, y, z)$), as can be easily seen by particularizing the considerations of the following sections, and for this reason the proof will not be carried out in detail here.

The formula for $u_{mn}(x, y)$ given above was derived by carrying out a step-by-step process such as would take place in an actual numerical solution. An alternative derivation of the formula for $u_{mn}(x, y)$ will now be given. This second derivation seems to have the advantage of leading more quickly than the step-by-step method to a formula of the desired kind for other boundary value problems as well as for the present one.

First, it will be recalled that if the function $F(x, y)$ is continuous for $(x, y)$ in $R$, and the functions $G(x)$ and $H(y)$ are continuously differentiable on $x_0 \leqq x \leqq x_0 + a$ and $y_0 \leqq y \leqq y_0 + b$, respectively (and $G(x_0) = H(y_0)$), then there is one and only one function $w(x, y)$ which is continuous in $R$, together with $\partial w/\partial x$, $\partial w/\partial y$, and $\partial^2 w/\partial x\, \partial y$ $(= \partial^2 w/\partial y\, \partial x)$ and satisfies the boundary value problem

$$\frac{\partial^2 w}{\partial x\, \partial y}(x, y) = F(x, y) \qquad \text{for } (x, y) \text{ in } R,$$

$$w(x, y_0) = G(x) \qquad \text{for } x_0 \leqq x \leqq x_0 + a,$$

$$w(x_0, y) = H(y) \qquad \text{for } y_0 \leqq y \leqq y_0 + b.$$

The function $w(x, y)$ is given by the formula

$$w(x, y) = G(x) + H(y) - w(x_0, y_0) + \int_{x_0}^{x} \int_{y_0}^{y} F(\xi, \eta)\, d\xi\, d\eta,$$

where $w(x_0, y_0) = G(x_0) = H(y_0)$.

Consider the subdivisions

$$x_0 < x_1 < x_2 < \cdots < x_m < x_0 + a,$$

$$y_0 < y_1 < y_2 < \cdots < y_n < y_0 + b,$$

which were employed in the step-by-step process leading to the equation for $u_{mn}(x, y)$. By use of this subdivision of the rectangle $R$, the formula for $w(x, y)$ may be rewritten as follows:

$$w(x, y) = G(x) + H(y) - w(x_0, y_0) + \sum_{i=1}^{k} \sum_{j=1}^{l} \int_{x_{i-1}}^{x_i} \int_{y_{j-1}}^{y_j} F(\xi, \eta)\, d\xi\, d\eta +$$

$$+ \sum_{j=1}^{l} \int_{x_k}^{x} \int_{y_{j-1}}^{y_j} F(\xi, \eta)\, d\xi\, d\eta + \sum_{i=1}^{k} \int_{x_{i-1}}^{x_i} \int_{y_l}^{y} F(\xi, \eta)\, d\xi\, d\eta + \int_{x_k}^{x} \int_{y_l}^{y} F(\xi, \eta)\, d\xi\, d\eta.$$

This rewriting of the equation for $w(x, y)$ makes no difference under the assumptions made about the functions $F(x, y)$, $G(x)$, and $H(y)$. But it makes a difference when the differentiability and continuity requirements concerning $F(x, y)$, $G(x)$ and $H(y)$ are relaxed slightly. Specifically, suppose that $F(x, y)$ is bounded in absolute value throughout $R$ and continuous at all *interior* points of each subrectangle $R_{kl}^{mn}$, with possible discontinuities allowed on the boundary of any such subrectangle. Suppose also that $G(x)$ is continuous throughout $x_0 \leqq x \leqq x_0 + a$ and continuously differentiable for each $x$ interior to a subinterval (*i.e.*, such that $x_k < x < x_{k+1}$ for some $k$) but that the derivative of $G(x)$ need not exist for the subdivision numbers $x_k$. Similarly, suppose also that $H(y)$ is continuous throughout $y_0 \leqq y \leqq y_0 + b$ and continuously differentiable for each $y$ interior to a subinterval (*i.e.*, such that $y_l < y < y_{l+1}$ for some $l$) but that the derivative of $H(y)$ need not exist for the subdivision numbers $y_l$. The requirement that $G(x_0) = H(y_0)$ is still retained. Under these relaxed assumptions, the rewritten formula for $w(x, y)$ shows immediately that $w(x, y)$ is continuous on $R$ and satisfies the partial differential equation

$$\frac{\partial^2 w}{\partial x\, \partial y}(x, y) = \frac{\partial^2 w}{\partial y\, \partial x}(x, y) = F(x, y)$$

whenever $(x, y)$ is *interior* to a subrectangle $R_{kl}^{mn}$. Further

$$w(x, y_0) = G(x) \quad \text{for } x_0 \leqq x \leqq x_0 + a,$$
$$w(x_0, y) = H(y) \quad \text{for } y_0 \leqq y \leqq y_0 + b.$$

This last observation and the rewritten formula for $w(x, y)$ furnish immediately the desired formula for $u_{mn}(x, y)$ upon taking $F(x, y)$, $G(x)$, and $H(y)$ to be certain suitably chosen functions. One need only take for $F(x, y)$ the following (piecewise constant) function defined on $x_0 \leqq x \leqq x_0 + a$, $y_0 \leqq y \leqq y_0 + b$, by

$$F(x, y) = f_{kl} \quad \text{for } x_k \leqq x \leqq x_{k+1} \quad \text{and} \quad y_l \leqq y \leqq y_{l+1},$$

while for $x = x_0 + a$ and $y = y_0 + b$

$$F(x_0 + a, y) = f_{m-1,l} \quad \text{for } y_l \leqq y < y_{l+1},$$
$$F(x, y_0 + b) = f_{k,n-1} \quad \text{for } x_k \leqq x < x_{k+1},$$
$$F(x_0 + a, y_0 + b) = f_{m-1,n-1},$$

where $k = 0, 1, \ldots, m-1, \quad l = 0, 1, \ldots, n-1,$

while for $G(x)$ and $H(y)$, respectively, one takes the polygonal functions (compare the description of the Euler-Cauchy polygon method in the introduction):

$$G(x) = \begin{cases} \sigma(x_k) + \dfrac{\sigma(x_{k+1}) - \sigma(x_k)}{x_{k+1} - x_k}(x - x_k) \\ \qquad\qquad \text{for } x_k \leqq x < x_{k+1}, \quad \text{and} \quad k = 0, 1, \ldots, m-1 \\ \sigma(x_0 + a) \qquad \text{for } x = x_0 + a \equiv x_m, \end{cases}$$

and

$$H(y) = \begin{cases} \tau(y_l) + \dfrac{\tau(y_{l+1}) - \tau(y_l)}{y_{l+1} - y_l}(y - y_l) \\ \qquad\qquad \text{for } y_l \leqq y < y_{l+1}, \quad \text{and} \quad l = 0, 1, \ldots, n-1 \\ \tau(y_0 + b) \qquad \text{for } y = y_0 + b \equiv y_n. \end{cases}$$

In verifying this remark, it must be remembered that, in the abbreviated notation, one has for example

$$u_{00} \equiv u(x_0, y_0), \quad \sigma(x_k) \equiv u_{k,0}, \quad \tau(y_l) \equiv u_{0,l}.$$

## § 4. The convergence inequality

In order to complete the proof of the theorems of Section 2, one has to consider the two double sequences of "partial derivatives" with respect to $x$ and $y$ of the double sequence of approximating functions of the last section. The quotation marks enclosing the phrase "partial derivatives" are a reminder that these functions must be precisely defined on $R$, especially along the boundaries of the subrectangles of $R$, where jumps may occur. The exact definition of what is meant by "partial derivatives" will be taken up in Section 5. Since the "partial derivatives" in question are not necessarily continuous functions on $R$, in considering their convergence one cannot make use of the theorem of ASCOLI [*1*] on equibounded, equicontinuous sequences of functions employed in Section 3 above. Instead, appeal will be made to a theorem of ARZELÀ [*7*, pp. 119—125] asserting the convergence of certain sequences of not necessarily continuous functions to continuous limit functions. The lemma of the present section furnishes an inequality concerning finite sums which serves as a basis for the application of ARZELÀ'S theorem in Section 5. The result of the lemma is termed here the "convergence inequality" because of the central role it plays in the convergence proof of Section 5. It is remarked that in the theory of the ordinary differential equation $dy/dx = f(x, y)$, an entirely similar rôle is played by another convergence inequality (see, for example, BLISS [*9*, pp. 88—89]). The proof of the inequality of the lemma below resembles that given by M. BRELOT [*18*, pp. 31—32] for an inequality occurring in the theory of the ordinary differential equation $dy/dx = f(x, y)$. Compare also the inequality employed by H. BECKERT [*22*, p. 13].

**Lemma.** *If*

(1) *$t$ is a positive integer, $f_0, f_1, \ldots, f_t$ is a sequence of $t+1$ non-negative numbers, and $z_0, z_1, \ldots, z_t$ is a non-decreasing sequence of $t+1$ real numbers (so that $z_j - z_{j-1} \geqq 0$ for $j = 1, 2, \ldots, t$);*

(2) *the numbers $L \geqq 0$ and $\varepsilon \geqq 0$ are such that the inequality*

$$f_l \leqq \varepsilon + L \sum_{j=1}^{l} f_{j-1}(z_j - z_{j-1})$$

*is valid for $l = 1, 2, \ldots, t$; then*

(3) *the inequality*

$$f_l \leqq \left\{ \prod_{i=1}^{t} [1 + L(z_i - z_{i-1})] \right\} \{\varepsilon + L f_0(z_i - z_0)\}$$

*holds for $l = 1, 2, \ldots, t$.*

*Proof.* It will be shown by mathematical induction that

$$\varepsilon + L \sum_{j=1}^{l} f_{j-1}(z_j - z_{j-1}) \leqq \left\{ \prod_{i=1}^{l} [1 + L(z_i - z_{i-1})] \right\} \{\varepsilon + L f_0(z_1 - z_0)\}$$

for $l = 1, 2, \ldots, t$, which implies the desired conclusion of the lemma, since $1 + L(z_i - z_{i-1}) \geqq 1$ for $i = 1, 2, \ldots, t$ and $\varepsilon + L f_0(z_1 - z_0) \geqq 0$.

For $l = 1$ the asserted inequality follows from hypothesis (2) and the fact that $1 + L(z_1 - z_0) \geqq 1$, because

$$\varepsilon + L f_0(z_1 - z_0) \leqq \{1 + L(z_1 - z_0)\}\{\varepsilon + L f_0(z_1 - z_0)\}.$$

Now for the inductive step. Suppose that the inequality to be shown holds for a positive integer $l \leqq t-1$, then it will be shown to hold also for the integer $l+1$ in place of $l$. This is readily seen, because then, by the inductive hypothesis and hypothesis (2) of the lemma, one has

$$f_l \leqq \varepsilon + L\sum_{j=1}^{l} f_{j-1}(z_j - z_{j-1}) \leqq \left\{\prod_{i=1}^{l}\left[1 + L(z_i - z_{i-1})\right]\right\}\left\{\varepsilon + L f_0(z_1 - z_0)\right\}$$

which, together with the equality

$$\varepsilon + L\sum_{j=1}^{l+1} f_{j-1}(z_j - z_{j-1}) = \left\{\varepsilon + L\sum_{j=1}^{l} f_{j-1}(z_j - z_{j-1})\right\} + f_l \cdot L \cdot (z_{l+1} - z_l),$$

implies that

$$\varepsilon + L\sum_{j=1}^{l+1} f_{j-1}(z_j - z_{j-1}) \leqq \left\{\prod_{i=1}^{l+1}\left[1 + L(z_i - z_{i-1})\right]\right\}\left\{\varepsilon + L f_0(z_1 - z_0)\right\},$$

and the proof is complete.

It is of some interest, although it is not needed in the considerations that follow, to point out that the inequality contained in the lemma just proved is a finite difference analogue of an inequality due to T. H. GRONWALL [*10*, p. 293], in the continuous case. (See also G. SANSONE [*21*, vol. I, pp. 30—31].) Making suitable changes (in order to conform with the present notation) in the statement of GRONWALL's inequality, as given by SANSONE, one obtains the following result:

If $f(z)$ is a non-negative continuous function defined on the interval $z_0 \leqq z \leqq z_0 + a$ and there exist numbers $\varepsilon \geqq 0$ and $L \geqq 0$ such that

$$0 \leqq f(z) \leqq \varepsilon + \int_{z_0}^{z} L f(t)\, dt$$

for $z_0 \leqq z \leqq z_0 + a$, then

$$0 \leqq f(z) \leqq \varepsilon\, e^{La} \quad \text{for } z_0 \leqq z \leqq z_0 + a.$$

In order to *compare* this last inequality with the inequality of the lemma proved here, for each positive integer $t$ consider the following subdivision of the interval $z_0 \leqq z \leqq z_0 + a$:

$$z_0 \equiv z_{0,t} \leqq z_{1,t} \leqq z_{2,t} \leqq \cdots \leqq z_{t-1,t} \leqq z_{t,t} \equiv z_0 + a,$$

and suppose that the hypothesis (2) of the lemma holds, with $z_j$ and $f_j$ being replaced, respectively, by $z_{j,t}$ and $f(z_{j,t})$. Then the conclusion of the lemma proved reads

$$f(z_{j,t}) \leqq \left\{\prod_{i=1}^{t}\left[1 + L(z_{i,t} - z_{i-1,t})\right]\right\}\left\{\varepsilon + L f_0(z_{1,t} - z_{0,t})\right\}$$

$$\leqq \left\{\prod_{i=1}^{t} e^{L(z_{i,t} - z_{i-1,t})}\right\}\left\{\varepsilon + L f_0(z_{1,t} - z_0)\right\},$$

that is,

$$f(z_{j,t}) \leqq e^{La} \cdot \left\{\varepsilon + L f_0(z_{1,t} - z_0)\right\},$$

whose relationship in the limit to the inequality of GRONWALL cited above,

$$f(z) \leqq \varepsilon\, e^{La},$$

is clear.

## § 5. The double sequence of functions approximating the partial derivatives of a solution

Consider the double sequence of approximating continuous functions $u_{mn}$ defined in Section 3. It has been pointed out at the beginning of Section 4 that the partial derivative with respect to $x$ of $u_{mn}$ exists in the usual sense and is finite on $R$ save possibly when $x$ is equal to one of the finite set of numbers (recall the abbreviated notation introduced at the end of Section 3)

$$x_1 < x_2 < \cdots < x_{m-1},$$

where jumps may occur. (Of course, it is understood that when $x = x_0$ and $x = x_0 + a$, by the "partial derivative with respect to $x$" of the function $u_{mn}$ are meant the one-sided limits

$$\lim_{\substack{\bar{x} \to x_0 \\ \bar{x} > x_0}} \frac{u_{mn}(\bar{x}, y) - u_{mn}(x_0, y)}{\bar{x} - x_0}$$

and

$$\lim_{\substack{\bar{x} \to x_0 + a \\ \bar{x} < x_0 + a}} \frac{u_{mn}(\bar{x}, y) - u_{mn}(x_0 + a, y)}{\bar{x} - (x_0 + a)},$$

respectively.) A similar statement applies to the derivative with respect to $y$ of the function $u_{mn}$, the possible jumps now occurring when $y$ is equal to one of the finite set of numbers

$$y_1 < y_2 < \cdots < y_{n-1},$$

a corresponding agreement being made about the "partial derivatives with respect to $y$" of the function $u_{mn}$ when $y = y_0$ and $y = y_0 + b$. For reasons of symmetry, it is clear that one may restrict attention to the $x$ derivative, similar considerations being applicable in the case of the $y$ derivative. Intuitively speaking, it will now be shown, using the lemma of Section 4, that the magnitude of the jumps in $\partial u_{mn}/\partial x$ can be made arbitrarily small by choosing both $m$ and $n$ sufficiently large.

First, consider the function $u_{mn}$ on the closed subrectangle $R_{kl}^{mn}$, where $k = 0, 1, \ldots, m-1$ and $l = 0, 1, \ldots, n-1$. By its very construction, the function $u_{mn}$ is bilinear in $x$ and $y$ on the subrectangle $R_{kl}^{mn}$. In view of the formula for $u_{mn}$ given in Section 3, when $(x, y)$ is a point of the rectangle $R_{kl}^{mn}$ which is *not* on its closed left and right-hand rectilinear boundary intervals (*i.e.*, when the point $(x, y)$ satisfies the inequalities $x_k < x < x_{k+1}$ and $y_l \leq y \leq y_{l+1}$), then

$$\frac{\partial u_{mn}}{\partial x}(x, y) = \frac{u_{k+1,0} - u_{k,0}}{x_{k+1} - x_k} + \sum_{j=1}^{l} f_{k,j-1}(y - y_{j-1}) + f_{kl}(y - y_l).$$

On the other hand, when the point $(x, y)$ is on the closed left-hand rectilinear boundary interval

$$x = x_k, \quad y_l \leq y \leq y_{l+1},$$

then the right-hand $x$ derivative

$$\frac{\partial^+ u_{mn}}{\partial x}(x_k, y) = \lim_{\bar{x} \to x_k} \frac{u_{mn}(\bar{x}, y) - u_{mn}(x_k, y)}{\bar{x} - x_k},$$

where $x_k < \bar{x} \leq x_{k+1}$ and $y_l \leq y \leq y_{l+1}$, exists and is a linear function of $y$. Similarly, when the point $(x, y)$ is on the closed right-hand rectilinear boundary interval

$$x = x_{k+1}, \quad y_l \leq y \leq y_{l+1},$$

then the left-hand $x$ derivative

$$\frac{\partial^- u_{mn}}{\partial x}(x_{k+1,y}) = \lim_{\bar{x}\to x_{k+1}} \frac{u_{mn}(\bar{x}, y) - u_{mn}(x_{k+1}, y)}{\bar{x} - x_{k+1}},$$

where $x_k \leq \bar{x} < x_{k+1}$ and $y_l \leq y \leq y_{l+1}$ exists and is a linear function of $y$. It is to be noticed that the "partial derivative with respect to $x$" is constant on $R_{kl}^{mn}$ for each fixed $y$; that is, for each $y$ such that $y_l \leq y \leq y_{l+1}$ one has

$$\frac{\partial^+ u_{mn}}{\partial x}(x_k, y) = \frac{\partial u_{mn}}{\partial x}(x, y) = \frac{\partial^- u_{mn}}{\partial x}(x_{k+1,y})$$

for all $x$ satisfying $x_k < x < x_{k+1}$.

The maximum absolute value of the difference between the values of the "partial derivative with respect to $x$" of $u_{mn}$ on two subrectangles $R_{kl}^{mn}$ and $R_{\bar{k}l}^{mn}$ at the same $y$ level will now be estimated by use of the lemma of Section 4. Suppose, for definiteness, that $\bar{k} \geq k$. For the rectangle $R_{\bar{k}l}^{mn}$ there are formulas for $\partial u_{mn}/\partial x$, *etc.*, similar to those just derived for $R_{kl}^{mn}$, which need not be recorded here explicitly. One also has that for each $y$ such that $y_l \leq y \leq y_{l+1}$ the equality

$$\frac{\partial^+ u_{mn}}{\partial x}(x_{\bar{k}}, y) = \frac{\partial u_{mn}}{\partial x}(x, y) = \frac{\partial^- u_{mn}}{\partial x}(x_{\bar{k}+1}, y)$$

holds for all $x$ satisfying $x_{\bar{k}} < x < x_{\bar{k}+1}$. Consequently, the problem of estimating the maximum absolute value of the difference between the values of the "partial derivative with respect to $x$" of $u_{mn}$ on the two subrectangles $R_{kl}^{mn}$ and $R_{\bar{k}l}^{mn}$ reduces simply to the estimation of maximum absolute value of the difference of the two functions of $y$,

$$\frac{\partial^+ u_{mn}}{\partial x}(x_k, y) \quad \text{and} \quad \frac{\partial^+ u_{mn}}{\partial x}(x_{\bar{k}}, y),$$

which are *linear* functions of $y$ on the interval $y_l \leq y \leq y_{l+1}$. In view of the linearity of the two functions involved, the desired maximum absolute value of their difference,

$$\max_{\substack{y_l \leq y \leq y_{l+1} \\ y_l \leq \bar{y} \leq y_{l+1}}} \left| \frac{\partial^+ u_{mn}}{\partial x}(x_k, y) - \frac{\partial^+ u_{mn}}{\partial x}(x_{\bar{k}}, \bar{y}) \right|$$

is just equal to the maximum of the four numbers

$$\left| \frac{\partial^+ u_{mn}}{\partial x}(x_k, y_{l+1}) - \frac{\partial^+ u_{mn}}{\partial x}(x_{\bar{k}}, y_{l+1}) \right|,$$

$$\left| \frac{\partial^+ u_{mn}}{\partial x}(x_k, y_l) - \frac{\partial^+ u_{mn}}{\partial x}(x_{\bar{k}}, y_l) \right|,$$

$$\left| \frac{\partial^+ u_{mn}}{\partial x}(x_k, y_{l+1}) - \frac{\partial^+ u_{mn}}{\partial x}(x_{\bar{k}}, y_l) \right|,$$

$$\left| \frac{\partial^+ u_{mn}}{\partial x}(x_k, y_l) - \frac{\partial^+ u_{mn}}{\partial x}(x_{\bar{k}}, y_{l+1}) \right|.$$

It is to be noticed that only the estimation of the first two of these numbers requires special attention, since it will turn out that the last two can be made arbitrarily small whenever the difference $y_{l+1}-y_l$ is chosen sufficiently small, the reason for this being the continuity of $\partial^+ u_{mn}/\partial x$ with respect to $y$ for each fixed $x$. For example,

$$\left|\frac{\partial^+ u_{mn}}{\partial x}(x_k, y_{l+1}) - \frac{\partial^+ u_{mn}}{\partial x}(x_{\bar{k}}, y_l)\right|$$
$$\leqq \left|\frac{\partial^+ u_{mn}}{\partial x}(x_k, y_{l+1}) - \frac{\partial^+ u_{mn}}{\partial x}(x_{\bar{k}}, y_{l+1})\right| + \left|\frac{\partial^+ u_{mn}}{\partial x}(x_{\bar{k}}, y_{l+1}) - \frac{\partial^+ u_{mn}}{\partial x}(x_{\bar{k}}, y_l)\right|.$$

Further, since the first number is obtainable from the second merely by replacing $l$ by $l+1$, all that remains is to estimate, for each pair of fixed integers $\bar{k}\geqq k$, the $n+1$ numbers

$$\left|\frac{\partial^+ u_{mn}}{\partial x}(x_k, y_l) - \frac{\partial^+ u_{mn}}{\partial x}(x_{\bar{k}}, y_l)\right|,$$

where $l=0, 1, \ldots, n$. For $l=0$ this absolute value can be made arbitrarily small, and the lemma of Section 4 will now be used in showing that the absolute values for $l=1, \ldots, n$ can also be made arbitrarily small.

Now, from the definition of $\frac{\partial^+ u_{mn}}{\partial x}(x_k, y_l)$ and $\frac{\partial^+ u_{mn}}{\partial x}(x_{\bar{k}}, y_l)$ (recall, for example, that $u_{mn}(x, y_l)$, for $x_k\leqq x\leqq x_{k+1}$, is a *linear* function of the single variable $x$) together with the previous formula for $\partial^+ u_{mn}/\partial x$ obtained in this section, it follows that (recall that, for example, $u_{mn}(x_{k+1}, y_l)\equiv u_{k+1,l}$)

$$\frac{\partial^+ u_{mn}}{\partial x}(x_k, y_l) - \frac{\partial^+ u_{mn}}{\partial x}(x_{\bar{k}}, y_l) = \frac{u_{k+1,l}-u_{k,l}}{x_{k+1}-x_k} - \frac{u_{\bar{k}+1,l}-u_{\bar{k},l}}{x_{\bar{k}+1}-x_{\bar{k}}}$$
$$= \frac{u_{k+1,0}-u_{k,0}}{x_{k+1}-x_k} - \frac{u_{\bar{k}+1,0}-u_{\bar{k},0}}{x_{\bar{k}+1}-x_{\bar{k}}} + \sum_{j=1}^{l}(f_{k,j-1}-f_{\bar{k},j-1})(y_j-y_{j-1}) \quad \text{for } l=1, \ldots, n.$$

The proof that the absolute value of this last difference can be made arbitrarily small provided that $m$ and $n$ are chosen sufficiently large will now be completed, at first under the hypothesis required of the function in Theorem 2, *i.e.*, that $f$ satisfies a Lipschitz condition in all three of its last arguments $z, p, q$ (see hypothesis (1) of Theorem 2). The argument will be carried out first in this case because it is somewhat simpler than the corresponding argument when $f$ satisfies a Lipschitz condition only in its last two arguments $p, q$ (see hypothesis (1) of Theorem 3). *It will also be supposed at first,* again for the sake of simplicity in writing, that the function $f(x, y, z, p, q)$ does not depend on $x$ and $y$, that is $f\equiv f(z, p, q)$.

Accordingly, under the hypothesis (1) of Theorem 2, one has that (recall the description of the abbreviated notation $f_{k,j-1}$ introduced in Section 3):

$$|f_{k,j-1}-f_{\bar{k},j-1}| \leqq L\left\{|u_{k,j-1}-u_{\bar{k},j-1}| + \left|\frac{u_{k+1,j-1}-u_{k,j-1}}{x_{k+1}-x_k} - \frac{u_{\bar{k}+1,j-1}-u_{\bar{k},j-1}}{x_{\bar{k}+1}-x_{\bar{k}}}\right| + \right.$$
$$\left. + \left|\frac{u_{k,j}-u_{k,j-1}}{y_j-y_{j-1}} - \frac{u_{\bar{k},j}-u_{\bar{k},j-1}}{y_j-y_{j-1}}\right|\right\},$$

and consequently

$$\left|\frac{\partial^+ u_{mn}}{\partial x}(x_k, y_l) - \frac{\partial^+ u_{mn}}{\partial x}(x_{\bar{k}}, y_l)\right| = \left|\frac{u_{k+1,l} - u_{k,l}}{x_{k+1} - x_k} - \frac{u_{\bar{k}+1,l} - u_{\bar{k},l}}{x_{\bar{k}+1} - x_{\bar{k}}}\right|$$
$$\leq \left|\frac{u_{k+1,0} - u_{k,0}}{x_{k+1} - x_k} - \frac{u_{\bar{k}+1,0} - u_{\bar{k},0}}{x_{\bar{k}+1} - x_{\bar{k}}}\right| +$$
$$+ L \cdot \sum_{j=1}^{l} \left\{ \left| u_{k,j-1} - u_{\bar{k},j-1} \right| + \left| \frac{u_{k+1,j-1} - u_{k,j-1}}{x_{k+1} - x_k} - \frac{u_{\bar{k}+1,j-1} - u_{\bar{k},j-1}}{x_{\bar{k}+1} - x_{\bar{k}}} \right| + \right.$$
$$\left. + \left| \frac{u_{k,j} - u_{k,j-1}}{y_j - y_{j-1}} - \frac{u_{\bar{k},j} - u_{\bar{k},j-1}}{y_j - y_{j-1}} \right| \right\} (y_j - y_{j-1}).$$

The term in the last inequality which involves the difference quotients with respect to $y$ requires special attention. Consider the function $u_{mn}$ on the rectangle $R^{mn}_{k,j-1}$. From the formula for $u_{mn}$ given in Section 3 it follows that

$$u_{k,j-1} \equiv u_{k,0} + u_{0,j-1} - u_{00} + \sum_{i=1}^{k} \sum_{J=1}^{j-1} f_{i-1,J-1}(x_i - x_{i-1})(y_J - y_{J-1}),$$

and

$$u_{k,j} \equiv u_{k,0} + u_{0,j} - u_{00} + \sum_{i=1}^{k} \sum_{J=1}^{j-1} f_{i-1,J-1}(x_i - x_{i-1})(y_J - y_{J-1}) +$$
$$+ \sum_{i=1}^{k} f_{i-1,j-1}(x_i - x_{i-1})(y_j - y_{j-1});$$

hence

$$\frac{u_{k,j} - u_{k,j-1}}{y_j - y_{j-1}} = \frac{u_{0,j} - u_{0,j-1}}{y_j - y_{j-1}} + \sum_{i=1}^{k} f_{i-1,j-1}(x_i - x_{i-1}).$$

Similarly

$$\frac{u_{\bar{k},j} - u_{\bar{k},j-1}}{y_j - y_{j-1}} = \frac{u_{0,j} - u_{0,j-1}}{y_j - y_{j-1}} + \sum_{i=1}^{\bar{k}} f_{i-1,j-1}(x_i - x_{i-1}),$$

and thus

$$\frac{u_{k,j} - u_{k,j-1}}{y_j - y_{j-1}} - \frac{u_{\bar{k},j} - u_{\bar{k},j-1}}{y_j - y_{j-1}} = - \sum_{i=k+1}^{\bar{k}} f_{i-1,j-1}(x_i - x_{i-1}).$$

Further, in view of this

$$\sum_{j=1}^{l} \left| \frac{u_{k,j} - u_{k,j-1}}{y_j - y_{j-1}} - \frac{u_{\bar{k},j} - u_{\bar{k},j-1}}{y_j - y_{j-1}} \right| (y_j - y_{j-1}) = \sum_{j=1}^{l} \left| \sum_{i=k+1}^{\bar{k}} f_{i-1,j-1}(x_i - x_{i-1}) \right| (y_j - y_{j-1})$$
$$\leq M \left[ \sum_{i=k+1}^{\bar{k}} (x_i - x_{i-1}) \right] \left[ \sum_{j=1}^{l} (y_j - y_{j-1}) \right] \leq M \left[ \sum_{i=k+1}^{\bar{k}} (x_i - x_{i-1}) \right] \cdot b = M b (x_{\bar{k}} - x_k).$$

The inequality for

$$\left| \frac{\partial^+ u_{mn}}{\partial x}(x_k, y_l) - \frac{\partial^+ u_{mn}}{\partial x}(x_{\bar{k}}, y_l) \right|$$

may now be rewritten in the form

$$\left| \frac{u_{k+1,l} - u_{k,l}}{x_{k+1} - x_k} - \frac{u_{\bar{k}+1,l} - u_{\bar{k},l}}{x_{\bar{k}+1} - x_{\bar{k}}} \right| \leq \left| \frac{u_{k+1,0} - u_{k,0}}{x_{k+1} - x_k} - \frac{u_{\bar{k}+1,0} - u_{\bar{k},0}}{x_{\bar{k}+1} - x_{\bar{k}}} \right| +$$
$$+ L \sum_{j=1}^{l} \left| u_{k,j-1} - u_{\bar{k},j-1} \right| (y_j - y_{j-1}) + M b (x_{\bar{k}} - x_k) +$$
$$+ L \cdot \sum_{j=1}^{l} \left| \frac{u_{k+1,j-1} - u_{k,j-1}}{x_{k+1} - x_k} - \frac{u_{\bar{k}+1,j-1} - u_{\bar{k},j-1}}{x_{\bar{k}+1} - x_{\bar{k}}} \right| (y_j - y_{j-1}) \quad \text{for } l = 1, 2, \ldots, n.$$

This last inequality is precisely of the same type as that of hypothesis (2) of the lemma in Section 4, upon identifying, in particular, $t$ with $n$, the $f_j$ and $z_j$ occurring there (for $j=0, 1, \dots, n$) with the present

$$\left|\frac{u_{k+1,j}-u_{k,j}}{x_{k+1}-x_k}-\frac{u_{\bar{k}+1,j}-u_{\bar{k},j}}{x_{\bar{k}+1}-x_{\bar{k}}}\right|$$

and $y_j$, respectively, and the $\varepsilon$ of the lemma with

$$\max_{0\leqq k<\bar{k}\leqq m-1}\left\{\left|\frac{u_{k+1,0}-u_{k,0}}{x_{k+1}-x_k}-\frac{u_{\bar{k}+1,0}-u_{\bar{k},0}}{x_{\bar{k}+1}-x_{\bar{k}}}\right|+\right.$$
$$\left.+L\sum_{j=1}^{l}|u_{k,j-1}-u_{\bar{k},j-1}|\,(y_j-y_{j-1})+M\,b\,(x_{\bar{k}}-x_k)\right\},$$

which, as will now be shown, can be made arbitrarily small merely by choosing $m$ and $n$ sufficiently large and $|x_k-x_{\bar{k}}|$ sufficiently small (in view of the assumed continuity of the derivative $\sigma'(x)$ and the equicontinuity of any subsequence $\{u_{m_r n_r}(x,y)\}$, with $\lim_{r\to\infty} m_r=\lim_{r\to\infty} n_r=\infty$, which was shown in Section 3). In verifying this, one can use the mean value theorem of the differential calculus, since for $k=1,\dots,m-1$

$$\frac{u_{k+1,0}-u_{k,0}}{x_{k+1}-x_k}-\frac{u_{\bar{k}+1,0}-u_{\bar{k},0}}{x_{\bar{k}+1}-x_{\bar{k}}}=\sigma'(x_k^*)-\sigma'(x_{\bar{k}}^*),$$

where $x_k<x_k^*<x_{k+1}$ and $x_{\bar{k}}<x_{\bar{k}}^*<x_{\bar{k}+1}$. Let $\varepsilon>0$ be given, then there exist (see Section 3) positive integers $m_\varepsilon$ and $n_\varepsilon$ and a positive number $\delta_\varepsilon$ such that whenever $m_r>m_\varepsilon$, $n_r>n_\varepsilon$ and $|x_k-x_{\bar{k}}|<\delta_\varepsilon$ one has

$$\max_{0\leqq k<\bar{k}\leqq m-1}\left\{L\sum_{j=1}^{l}|u_{k,j-1}-u_{\bar{k},j-1}|\,(y_j-y_{j-1})+M\,b\,(x_{\bar{k}}-x_k)\right\}<\tfrac{1}{2}\,\varepsilon,$$

where $u(x,y)$ is written for $u_{m_r n_r}(x,y)$. Also, in view of the *uniform* continuity of the function $\sigma'(x)$ on the interval $x_0\leqq x\leqq x_0+a$, it follows that

$$\left|\frac{u_{k+1,0}-u_{k,0}}{x_{k+1}-x_k}-\frac{u_{\bar{k}+1,0}-u_{\bar{k},0}}{x_{\bar{k}+1}-x_{\bar{k}}}\right|<\frac{1}{2}\,\varepsilon.$$

Consequently, from the conclusion of the lemma of Section 4 it follows that

$$\left|\frac{u_{k+1,l}-u_{k,l}}{x_{k+1}-x_k}-\frac{u_{\bar{k}+1,l}-u_{\bar{k},l}}{x_{\bar{k}+1}-x_{\bar{k}}}\right|$$
$$\leqq\left\{\prod_{i=1}^{n}[1+L(y_i-y_{i-1})]\right\}\{\varepsilon+L\,\varepsilon\,(y_1-y_0)\}\leqq\left\{\prod_{i=1}^{n}e^{L(y_i-y_{i-1})}\right\}\{\varepsilon+L\,\varepsilon\,(y_1-y_0)\}$$
$$=e^{Lb}\{\varepsilon+L\,\varepsilon\,(y_1-y_0)\}\qquad\text{for } l=1,2,\dots,n.$$

(It should be noticed that the last inequality also holds for $l=0$.)

The last inequality has been obtained under the *two* assumptions that the function $f$ satisfies a Lipschitz condition in all three variables $z, p, q$ (hypothesis (1) of Theorem 2) *and* that $f$ does not depend explicitly on $x$ and $y$; that is, $f\equiv f(z,p,q)$. The derivation of a similar inequality, in the case when $f\equiv f(x,y,z,p,q)$ satisfies *only* a Lipschitz condition in the two variables $p, q$ (hypothesis (1) of Theorem 3) will now be sketched. As in the previous case, the initial step, where the Lipschitz condition is applied, is in estimating the absolute value of the difference $f_{k,j-1} - f_{\bar{k},j-1}$. This can now be done as follows by adding and subtracting the number

$$f\left(x_{\bar{k}};\ y_{j-1};\ u_{\bar{k},j-1};\ \frac{u_{k+1,j-1} - u_{k,j-1}}{x_{k+1} - x_k};\ \frac{u_{k,j} - u_{k,j-1}}{y_j - y_{j-1}}\right)$$

to the difference in question. One obtains

$$\begin{aligned} f_{k,j-1} - f_{\bar{k},j-1} = {} & f\left(x_k;\ y_{j-1};\ u_{k,j-1};\ \frac{u_{k+1,j-1} - u_{k,j-1}}{x_{k+1} - x_k};\ \frac{u_{k,j} - u_{k,j-1}}{y_j - y_{j-1}}\right) - \\ & - f\left(x_{\bar{k}};\ y_{j-1};\ u_{\bar{k},j-1};\ \frac{u_{k+1,j-1} - u_{k,j-1}}{x_{k+1} - x_k};\ \frac{u_{k,j} - u_{k,j-1}}{y_j - y_{1-j}}\right) + \\ & + f\left(x_{\bar{k}};\ y_{j-1};\ u_{\bar{k},j-1};\ \frac{u_{k+1,j-1} - u_{k,j-1}}{x_{k+1} - x_k};\ \frac{u_{k,j} - u_{k,j-1}}{y_j - y_{j-1}}\right) - \\ & - f\left(x_{\bar{k}};\ y_{j-1};\ u_{\bar{k},j-1};\ \frac{u_{\bar{k}+1,j-1} - u_{\bar{k},j-1}}{x_{\bar{k}+1} - x_{\bar{k}}};\ \frac{u_{\bar{k},j} - u_{\bar{k},j-1}}{y_j - y_{j-1}}\right). \end{aligned}$$

Using this and the Lipschitz condition with respect to $p$ and $q$, one has

$$\begin{aligned} & \left|\frac{u_{k+1,l} - u_{k,l}}{x_{k+1} - x_k} - \frac{u_{\bar{k}+1,l} - u_{\bar{k},l}}{x_{\bar{k}+1} - x_{\bar{k}}}\right| \\ & \leqq \left|\frac{u_{k+1,0} - u_{k,0}}{x_{k+1} - x_k} - \frac{u_{\bar{k}+1,0} - u_{\bar{k},0}}{x_{\bar{k}+1} - x_{\bar{k}}}\right| + \sum_{j=1}^{l} |f_{k,j-1} - f_{\bar{k},j-1}|\,(y_j - y_{j-1}) \\ & \leqq \left|\frac{u_{k+1,0} - u_{k,0}}{x_{k+1} - x_k} - \frac{u_{\bar{k}+1,0} - u_{\bar{k},0}}{x_{\bar{k}+1} - x_{\bar{k}}}\right| + \\ & \quad + \sum_{j=1}^{l} \left| f\left(x_k;\ y_{j-1};\ u_{k,j-1};\ \frac{u_{k+1,j-1} - u_{k,j-1}}{x_{k+1} - x_k};\ \frac{u_{k,j} - u_{k,j-1}}{y_j - y_{j-1}}\right) - \right. \\ & \quad \left. - f\left(x_{\bar{k}};\ y_{j-1};\ u_{\bar{k},j-1};\ \frac{u_{k+1,j-1} - u_{k,j-1}}{x_{k+1} - x_k};\ \frac{u_{k,j} - u_{k,j-1}}{y_j - y_{j-1}}\right)\right| (y_j - y_{j-1}) + \\ & \quad + L \sum_{j=1}^{l} \left\{\left|\frac{u_{k+1,j-1} - u_{k,j-1}}{x_{k+1} - x_k} - \frac{u_{\bar{k}+1,j-1} - u_{\bar{k},j-1}}{x_{\bar{k}+1} - x_{\bar{k}}}\right| + \right. \\ & \quad \left. + \left|\frac{u_{k,j} - u_{k,j-1}}{y_j - y_{j-1}} - \frac{u_{\bar{k},j} - u_{\bar{k},j-1}}{y_j - y_{j-1}}\right|\right\} (y_j - y_{j-1}). \end{aligned}$$

The term in the last summation involving the explicit difference quotients with respect to $y$ may be handled exactly as before, yielding the same result:

$$\sum_{j=1}^{l} \left|\frac{u_{k,j} - u_{k,j-1}}{y_j - y_{j-1}} - \frac{u_{\bar{k},j} - u_{\bar{k},j-1}}{y_j - y_{j-1}}\right| (y_j - y_{j-1}) \leqq M\,b\,(x_{\bar{k}} - x_k).$$

Thus, one has finally

$$\begin{aligned} & \left|\frac{u_{k+1,l} - u_{k,l}}{x_{k+1} - x_k} - \frac{u_{\bar{k}+1,l} - u_{\bar{k},l}}{x_{\bar{k}+1} - x_{\bar{k}}}\right| \leqq \left|\frac{u_{k+1,0} - u_{k,0}}{x_{k+1} - x_k} - \frac{u_{\bar{k}+1,0} - u_{\bar{k},0}}{x_{\bar{k}+1} - x_{\bar{k}}}\right| + \\ & \quad + \sum_{j=1}^{l} \left| f\left(x_k;\ y_{j-1};\ u_{k,j-1};\ \frac{u_{k+1,j-1} - u_{k,j-1}}{x_{k+1} - x_k};\ \frac{u_{k,j} - u_{k,j-1}}{y_j - y_{j-1}}\right) - \right. \\ & \quad \left. - f\left(x_{\bar{k}};\ y_{j-1};\ u_{\bar{k},j-1};\ \frac{u_{k+1,j-1} - u_{k,j-1}}{x_{k+1} - x_k};\ \frac{u_{k,j} - u_{k,j-1}}{y_j - y_{j-1}}\right)\right| (y_j - y_{j-1}) + \\ & \quad + M\,b\,(x_{\bar{k}} - x_k) + L \sum_{j=1}^{l} \left|\frac{u_{k+1,j-1} - u_{k,j-1}}{x_{k+1} - x_k} - \frac{u_{\bar{k}+1,j-1} - u_{\bar{k},j-1}}{x_{\bar{k}+1} - x_{\bar{k}}}\right| (y_j - y_{j-1}) \end{aligned}$$

for $l = 1, 2, \ldots, n$.

This inequality is again precisely of the same type as that of hypothesis (2) of the lemma in Section 4, upon identifying, in particular, $t$ with $n$, the $f_j$ and $z_j$ occurring there (for $j=0,1,\ldots,n$) with the present

$$\left|\frac{u_{k+1,j}-u_{k,j}}{x_{k+1}-x_k}-\frac{u_{\bar{k}+1,j}-u_{\bar{k},j}}{x_{\bar{k}+1}-x_{\bar{k}}}\right|$$

and $y_j$, respectively, and the $\varepsilon$ of the lemma with

$$\max_{0\leq k<\bar{k}\leq m-1}\left\{\left|\frac{u_{k+1,0}-u_{k,0}}{x_{k+1}-x_k}-\frac{u_{\bar{k}+1,0}-u_{\bar{k},0}}{x_{\bar{k}+1}-x_{\bar{k}}}\right|+\right.$$
$$+\sum_{j=1}^{l}\left|f\left(x_k;\ y_{j-1};\ u_{k,j-1};\ \frac{u_{k+1,j-1}-u_{k,j-1}}{x_{k+1}-x_k};\ \frac{u_{k,j}-u_{k,j-1}}{y_j-y_{j-1}}\right)-\right.$$
$$\left.\left.-f\left(x_{\bar{k}};\ y_{j-1};\ u_{\bar{k},j-1};\ \frac{u_{k+1,j-1}-u_{k,j-1}}{x_{k+1}-x_k};\ \frac{u_{k,j}-u_{k,j-1}}{y_j-y_{j-1}}\right)\right|(y_j-y_{j-1})+Mb\,(x_{\bar{k}}-x_k)\right\},$$

which (as will now be indicated, without entering into the detailed argument) can be made arbitrarily small (*i.e.*, less than any positive number given in advance) merely by choosing $m$ and $n$ sufficiently large and $|x_k-x_{\bar{k}}|$ sufficiently small. In showing this, use is made of the assumed continuity of the derivative $\sigma'(x)$; of the equicontinuity of any subsequence $\{u_{m_r n_r}(x,y)\}$ with $\lim_{r\to\infty} m_r = \lim_{r\to\infty} n_r=\infty$, which was shown in Section 3; and of the *uniform* continuity of the function $f(x,y,z,p,q)$ on any closed and bounded set of points $(x,y,z,p,q)$ satisfying

$$(x,y)\text{ in }R,\qquad -Z\leqq z\leqq Z,\qquad -P\leqq p\leqq P,\qquad -Q\leqq q\leqq Q,$$

with $Z, P, Q$ positive numbers. Notice that it can readily be seen, from the definition of $u_{mn}$ and of the difference quotients involved, that there exist positive numbers $Z, P, Q$ such that

$$|u_{k,l}|\leqq Z,$$
$$\left|\frac{u_{k+1,l}-u_{k,l}}{x_{k+1}-x_k}\right|\leqq P\quad\text{for } k=0,1,\ldots,m-1\quad\text{and}\quad l=0,1,\ldots,n,$$
$$\left|\frac{u_{k,l+1}-u_{k,l}}{y_{j+1}-y_j}\right|\leqq Q\quad\text{for } k=0,1,\ldots,m\qquad\text{and}\quad l=0,1,\ldots,n-1$$

and for *any* pair of positive integers $m$ and $n$, where one uses the abbreviated notation, $u_{kl}\equiv u_{mn}(x_k,y_l)$, *etc.* In particular, since

$$\frac{u_{k+1,l}-u_{k,l}}{x_{k+1}-x_k}=\frac{u_{k+1,0}-u_{k,0}}{x_{k+1}-x_k}+\sum_{j=1}^{l}f_{k,j-1}(y_j-y_{j-1}),$$

one may choose

$$P=C+Mb,$$

in terms of the constants $C, M$ and $b$ of Sections 2 and 3. This being granted, one obtains exactly as before, by an application of the lemma of Section 4, that if $\varepsilon>0$ is given, then there exist positive integers $m_\varepsilon$ and $n_\varepsilon$ and a positive number $\delta_\varepsilon$ such that whenever $m_r>m_\varepsilon$ and $n_r>n_\varepsilon$, and $|x_k-x_{\bar{k}}|<\delta_\varepsilon$ then

$$\left|\frac{u_{k+1,l}-u_{k,l}}{x_{k+1}-x_k}-\frac{u_{\bar{k}+1,l}-u_{\bar{k},l}}{x_{\bar{k}+1}-x_{\bar{k}}}\right|\leqq e^{Lb}\{\varepsilon+L\,\varepsilon(y_1-y_0)\}\quad\text{for } l=0,1,2,\ldots,n.$$

It is now time to define the double sequence of functions $\{p_{mn}(x, y)\}$ corresponding to the double sequence $\{u_{mn}(x, y)\}$ of Section 3. The double sequence $\{p_{mn}(x, y)\}$ will be, roughly speaking, a sequence of functions approximating the partial derivative with respect to $x$ of a solution. In view of the possibility of jumps in $\partial u_{mn}/\partial x$, the function $p_{mn}(x, y)$ has to be defined carefully in $R$, to make sure it is single-valued. For each pair of positive integers $m$ and $n$, the function $p_{mn}$ is defined as follows, for points $(x, y)$ in the closed rectangle $R$:

$$p_{mn}(x,y) = \begin{cases} \dfrac{\partial u_{mn}}{\partial x}(x,y) & \text{whenever } x_k < x < x_{k+1} \text{ for some } k = 0, 1, \ldots, m-1, \\ \dfrac{\partial^+ u_{mn}}{\partial x}(x,y) & \text{whenever } x = x_k \text{ for some } k = 0, 1, \ldots, m-1, \\ \dfrac{\partial^- u_{mn}}{\partial x}(x,y) & \text{whenever } x = x_m = x_0 + b. \end{cases}$$

The function $p_{mn}$ possibly has jump discontinuities only when $x = x_1, \ldots, x_{m-1}$ and is continuous in the two independent variables $x$ and $y$ at all other points of $R$.

This double sequence of functions $\{p_{mn}(x, y)\}$, as may be readily seen from the formulas given for $\partial u_{mn}/\partial x$, $\partial^+ u_{mn}/\partial x$ and $\partial^- u_{mn}/\partial x$ given earlier in this section, is equibounded in absolute value on $R$. That is to say, there is a positive number $P$, which is independent of $m$, $n$ and of $(x, y)$, such that

$$|p_{mn}(x,y)| \leqq P$$

for any positive integers $m$ and $n$, and any point $(x, y)$ of $R$.

Let $\{p_{m_s n_s}\}$ denote any *singly* infinite subsequence of functions (with $\lim_{s\to\infty} m_s = \lim_{s\to\infty} n_s = \infty$) extracted from the double sequence $\{p_{mn}(x, y)\}$. Let $\varepsilon > 0$. From the preceding considerations it follows that there exist positive integers $m_\varepsilon$ and $n_\varepsilon$ and a number $\delta_\varepsilon > 0$ such that whenever $(\bar{x}, \bar{y})$ and $(x, y)$ are points of $R$ satisfying

$$|x - \bar{x}| < \delta_\varepsilon, \quad |y - \bar{y}| < \delta_\varepsilon,$$

and $m_s > m_\varepsilon$, $n_s > n_\varepsilon$, then

$$|p_{m_s n_s}(\bar{x}, \bar{y}) - p_{m_s n_s}(x, y)| < \varepsilon.$$

(In ARZELÀ's terminology [7, p. 119], the subsequence of $\{p_{m_s n_s}(x, y)\}$ for which $m_s > m_\varepsilon$ and $n_s > n_\varepsilon$ is equioscillating by less than $\varepsilon$. This can be proved by an argument similar to that used in Section 3 in showing that the sequence $\{u_{m_r n_r}(x, y)\}$ is equicontinuous. There are again four cases to consider, depending on the relative positions of the points $(\bar{x}, \bar{y})$ and $(x, y)$ with respect to each other. As in Section 3, only the case when $x \leqq \bar{x}$ and $y \leqq \bar{y}$ need be considered in detail. Here one has $x_k \leqq x \leqq x_{k+1}$; $y_l \leqq y \leqq y_{l+1}$; and $x_{\bar{k}} \leqq \bar{x} \leqq x_{\bar{k}+1}$; $y_{\bar{l}} \leqq \bar{y} \leqq y_{\bar{l}+1}$ for suitable pairs of integers $(k, l)$ and $(\bar{k}, \bar{l})$. Further $x_k \leqq x_{\bar{k}}$; $y_l \leqq y_{\bar{l}}$ and $x_{k+1} \leqq x_{\bar{k}+1}$; $y_{l+1} \leqq y_{\bar{l}+1}$. The inequalities deduced earlier in this section for

$$\left| \frac{u_{k+1,l} - u_{k,l}}{x_{k+1} - x_k} - \frac{u_{\bar{k}+1,l} - u_{\bar{k},l}}{x_{\bar{k}+1} - x_{\bar{k}}} \right|$$

may then readily be employed to obtain the desired result, the details being as follows. Now

$$p_{mn}(\bar{x},\bar{y}) - p_{mn}(x,y) = [p_{mn}(\bar{x},\bar{y}) - p_{mn}(\bar{x},y)] + [p_{mn}(\bar{x},y) - p_{mn}(x,y)],$$

where the point $(\bar{x}, y)$ is in the subrectangle $R_{\bar{k}l}^{mn}$ because $x_{\bar{k}} \leqq \bar{x} \leqq x_{\bar{k}+1}$ and $y_l \leqq y \leqq y_{l+1}$. This, together with the definition of the function $p_{mn}$, implies the inequality

$$\begin{aligned}
|p_{mn}(\bar{x},\bar{y}) - p_{mn}(x,y)| &\leqq |p_{mn}(\bar{x},\bar{y}) - p_{mn}(\bar{x},y)| + |p_{mn}(\bar{x},y) - p_{mn}(x,y)| \\
&= \left|\left\{\frac{u_{\bar{k}+1,\bar{l}} - u_{\bar{k},\bar{l}}}{x_{\bar{k}+1} - x_{\bar{k}}} + (\bar{y} - y_{\bar{l}})\, f_{\bar{k}\bar{l}}\right\} - \left\{\frac{u_{\bar{k}+1,l} - u_{\bar{k},l}}{x_{\bar{k}+1} - x_{\bar{k}}} + (y - y_l)\, f_{\bar{k}l}\right\}\right| + \\
&\quad + \left|\left\{\frac{u_{\bar{k}+1,l} - u_{\bar{k},l}}{x_{\bar{k}+1} - x_{\bar{k}}} + (y - y_l)\, f_{\bar{k}l}\right\} - \left\{\frac{u_{k+1,l} - u_{k,l}}{x_{k+1} - x_k} + (y - y_l)\, f_{kl}\right\}\right| \\
&\leqq \left|\frac{u_{\bar{k}+1,\bar{l}} - u_{\bar{k},\bar{l}}}{x_{\bar{k}+1} - x_{\bar{k}}} - \frac{u_{\bar{k}+1,l} - u_{\bar{k},l}}{x_{\bar{k}+1} - x_{\bar{k}}}\right| + \left|\frac{u_{\bar{k}+1,l} - u_{\bar{k},l}}{x_{\bar{k}+1} - x_{\bar{k}}} - \frac{u_{k+1,l} - u_{k,l}}{x_{k+1} - x_k}\right| + \\
&\quad + |(y - y_{\bar{l}})\, f_{\bar{k}\bar{l}}| + 2|(y - y_l)\, f_{\bar{k}l}| + |(y - y_l)\, f_{kl}| \\
&\leqq \left|\sum_{j=l+1}^{\bar{l}} f_{\bar{k},j-1}(y_j - y_{j-1})\right| + \left|\frac{u_{\bar{k}+1,l} - u_{\bar{k},l}}{x_{\bar{k}+1} - x_{\bar{k}}} - \frac{u_{k+1,l} - u_{k,l}}{x_{k+1} - x_k}\right| + 4M\beta_n,
\end{aligned}$$

in case $x_k \leqq x \leqq x_{k+1}$ for some integer $k = 0, 1, \ldots, m-1$ (this only excludes $x = x_0 + a$, which will be treated separately below) and the integer $\bar{k}$ is chosen (if possible) so that $x_{\bar{k}} \leqq \bar{x} \leqq x_{\bar{k}+1}$, with $\bar{k} = 0, 1, \ldots, m-1$ (if $\bar{x} = x_0 + a$, which is seemingly excluded at first, the inequality just written still continues to hold, but with $\bar{k}$ replaced by $m-1$). If $x = x_0 + a$, a case definitely excluded above, then one must necessarily have $\bar{x} = x_0 + a$ $(= x_m)$ too, and then

$$\begin{aligned}
&|p_{mn}(\bar{x},\bar{y}) - p_{mn}(x,y)| \\
&\quad = \left|\left\{\frac{u_{m,\bar{l}} - u_{m-1,\bar{l}}}{x_m - x_{m-1}} + (\bar{y} - y_{\bar{l}})\, f_{m-1,\bar{l}}\right\} - \left\{\frac{u_{m,l} - u_{m-1,l}}{x_m - x_{m-1}} + (y - y_l)\, f_{m-1,l}\right\}\right. \\
&\quad \leqq \left|\sum_{j=l+1}^{\bar{l}} f_{m-1,j-1}(y_j - y_{j-1})\right| + 2M\beta_n.
\end{aligned}$$

These inequalities now readily furnish the desired "equioscillation" property of the singly infinite sequence $\{p_{m_s n_s}\}$.

Since the sequence of functions $\{p_{m_s n_s}\}$ is equibounded in absolute value, and since for each $\varepsilon > 0$ there are positive integers $m_\varepsilon$ and $n_\varepsilon$ and a number $\delta_\varepsilon > 0$ such that for all points $(\bar{x},\bar{y})$ and $(x, y)$ of $R$ satisfying $|x - \bar{x}| < \delta_\varepsilon$, $|y - \bar{y}| < \delta_\varepsilon$ and for all $m_s$ and $n_s$ satisfying $m_s > m_\varepsilon$, $n_s > n_\varepsilon$ one has

$$|p_{m_s n_s}(\bar{x},\bar{y}) - p_{m_s n_s}(x,y)| < \varepsilon,$$

it follows from a theorem of ARZELÀ [*7*, pp. 119—125] that there is a continuous function $p(x, y)$ defined on $R$ and a subsequence of the sequence $\{p_{m_s n_s}\}$ which converges *uniformly* to the continuous function $p(x, y)$ on $R$. For a proof of this particular result needed here, carried out under the equivalent hypothesis that the given sequence of functions has zero "Grenzschwankung" (see CARATHÉODORY [*17*, p. 3] for the definition of this term), reference is made to H. BECKERT [*22*, pp. 24—27].

For reasons of symmetry, without further discussion it is clear how the double sequence $\{q_{mn}(x, y)\}$, which approximates the $y$ derivative of a solution is defined. It is also clear that there is a positive number $Q$, which is independent of $m$, $n$ and of $(x, y)$ such that

$$|q_{mn}(x, y)| \leqq Q$$

for any positive integers $m$ and $n$ and any point $(x, y)$ of $R$. Let $\{q_{m_t n_t}\}$ denote any *singly* infinite subsequence of functions with $\lim_{t\to\infty} m_t = \lim_{t\to\infty} n_t = \infty$ extracted from the double sequence of functions $\{q_{mn}(x, y)\}$. Again, by ARZELÀ's theorem, one concludes that there is a *continuous* function $q(x, y)$ defined on $R$ and a subsequence of the sequence $\{q_{m_t n_t}(x, y)\}$ which converges *uniformly* to $q(x, y)$ on $R$.

## § 6. The existence of a solution

Consider the double sequences of functions $\{u_{mn}(x, y)\}$, $\{p_{mn}(x, y)\}$, and $\{q_{mn}(x, y)\}$. In Section 5 it was pointed out that there exist positive numbers $Z$, $P$, and $Q$ such that for any positive integers $m$, $n$ and any $(x, y)$ in $R$, one has

$$|u_{mn}(x, y)| \leqq Z, \quad |p_{mn}(x, y)| \leqq P, \quad |q_{mn}(x, y)| \leqq Q.$$

It is remarked, since use will be made of this fact immediately, that the continuous function $f(x, y, z, p, q)$ is *uniformly* continuous in $(x, y, z, p, q)$ on the closed and bounded five dimensional set of points defined by

$$x_0 \leqq x \leqq x_0 + a; \quad y_0 \leqq y \leqq y_0 + b, \quad |z| \leqq Z, \quad |p| \leqq P, \quad |q| \leqq Q.$$

That is, given $\varepsilon > 0$ there is $\delta_\varepsilon > 0$ (which may be chosen to be less than $\varepsilon$, for later convenience) such that whenever $(\bar{x}_1, \bar{y}_1, \bar{z}_1, \bar{p}_1, \bar{q}_1)$ and $(\bar{x}_2, \bar{y}_2, \bar{z}_2, \bar{p}_2, \bar{q}_2)$ satisfy the inequalities

$$x_0 \leqq \bar{x}_i \leqq x_0 + a, \quad y_0 \leqq \bar{y}_i \leqq y_0 + a, \quad |\bar{z}_i| \leqq Z, \quad |\bar{p}_i| \leqq P, \quad |\bar{q}_i| \leqq Q \quad \text{for } i = 1, 2$$

and

$$|\bar{x}_1 - \bar{x}_2| < \delta_\varepsilon, \quad |\bar{y}_1 - \bar{y}_2| < \delta_\varepsilon, \quad |\bar{z}_1 - \bar{z}_2| < \delta_\varepsilon, \quad |\bar{p}_1 - \bar{p}_2| < \delta_\varepsilon, \quad |\bar{q}_1 - \bar{q}_2| < \delta_\varepsilon,$$

then

$$|f(\bar{x}_1, \bar{y}_1, \bar{z}_1, \bar{p}_1, \bar{q}_1) - f(\bar{x}_2, \bar{y}_2, \bar{z}_2, \bar{p}_2, \bar{q}_2)| < \varepsilon.$$

Let $\{u_{m_r n_r}(x, y)\}$, where $\lim_{r\to\infty} m_r = \lim_{r\to\infty} n_r = \infty$, be a singly infinite subsequence of the double sequence $\{u_{mn}(x, y)\}$, and suppose further that (see Section 3)

$$\lim_{r\to\infty} u_{m_r n_r}(x, y) = u(x, y),$$

where the convergence to the continuous function $u(x, y)$ holds over the rectangle $R$. From Section 5, it follows that the corresponding subsequence $\{p_{m_r n_r}(x, y)\}$ itself contains a subsequence which converges uniformly on $R$ to a continuous function $p(x, y)$. For simplicity, suppose the subscripts have been chosen so that the subsequence $\{p_{m_r n_r}\}$ itself converges uniformly on $R$ to $p(x, y)$. Making a similar agreement about subscripts, it may also be supposed that the corresponding subsequence $\{q_{m_r n_r}(x, y)\}$ itself converges uniformly on $R$ to a continuous function $q(x, y)$. Summarizing, one concludes that

$$\lim_{r\to\infty} u_{m_r n_r}(x, y) = u(x, y),$$

$$\lim_{r\to\infty} p_{m_r n_r}(x, y) = p(x, y),$$

$$\lim_{r\to\infty} q_{m_r n_r}(x, y) = q(x, y),$$

the convergence to the continuous functions $u$, $p$, $q$ being uniform on $R$. It will now be shown that the function $u(x, y)$ is a solution of the boundary value problem under study.

In view of the above mentioned uniform continuity of $f$ on a certain closed and bounded five-dimensional set of points, it follows that

$$\lim_{r\to\infty} f\big(x, y, u_{m_r n_r}(x, y), p_{m_r n_r}(x, y), q_{m_r n_r}(x, y)\big) = f\big(x, y, u(x, y), p(x, y), q(x, y)\big),$$

the convergence to the continuous limit function being again uniform on $R$. Furthermore, since the limit function

$$f\big(x, y, u(x, y), p(x, y), q(x, y)\big)$$

is continuous on $R$, the following Riemann integrals exist for all $(x, y)$ in $R$:

$$\int_{x_0}^{x}\int_{y_0}^{y} f\big(\xi, \eta, u(\xi, \eta), p(\xi, \eta), q(\xi, \eta)\big)\, d\xi\, d\eta,$$

$$\int_{x_0}^{x} f\big(\xi, y, u(\xi, y), p(\xi, y), q(\xi, y)\big)\, d\xi,$$

$$\int_{y_0}^{y} f\big(x, \eta, u(x, \eta), p(x, \eta), q(x, \eta)\big)\, d\eta;$$

the order of integration with respect to $\xi$ and $\eta$ may be interchanged in the double integral without altering its value. All this information will now be used in order to show that the function $p$ is precisely the $x$ derivative of the function $u$ and that the function $q$ is precisely the $y$ derivative of the function $u$.

Let $\varepsilon > 0$, and let $\delta_\varepsilon > 0$ be such that $\varepsilon > \delta_\varepsilon > 0$ (the restriction $\varepsilon > \delta_\varepsilon$ is made for later convenience) and that also

$$|f(\bar{x}_1, \bar{y}_1, \bar{z}_1, \bar{p}_1, \bar{q}_1) - f(\bar{x}_2, \bar{y}_2, \bar{z}_2, \bar{p}_2, \bar{q}_2)| < \varepsilon$$

whenever the points $(\bar{x}_i, \bar{y}_i, \bar{z}_i, \bar{p}_i, \bar{q}_i)$ satisfy both

$$x_0 \leqq \bar{x}_i \leqq x_0 + a, \quad y_0 \leqq \bar{y}_i \leqq y_0 + a, \quad |\bar{z}_i| \leqq Z, \quad |\bar{p}_i| \leqq P, \quad |\bar{q}_i| \leqq Q,$$

and

$$|\bar{x}_1 - \bar{x}_2| < \delta_\varepsilon, \quad |\bar{y}_1 - \bar{y}_2| < \delta_\varepsilon, \quad |\bar{z}_1 - \bar{z}_2| < \delta_\varepsilon, \quad |\bar{p}_1 - \bar{p}_2| < \delta_\varepsilon, \quad |\bar{q}_1 - \bar{q}_2| < \delta_\varepsilon.$$

In view of the uniform continuity of the functions $u(x, y)$, $p(x, y)$ and $q(x, y)$ on $R$, there is another number $\delta_\varepsilon^* > 0$ (which for convenience will be chosen such that $\varepsilon > \delta_\varepsilon > \delta_\varepsilon^* > 0$) such that

$$|u(\xi_1, \eta_1) - u(\xi_2, \eta_2)| < \delta_\varepsilon < \varepsilon,$$
$$|p(\xi_1, \eta_1) - p(\xi_2, \eta_2)| < \delta_\varepsilon < \varepsilon,$$
$$|q(\xi_1, \eta_1) - q(\xi_2, \eta_2)| < \delta_\varepsilon < \varepsilon,$$

whenever the points $(\xi_1, \eta_1)$, $(\xi_2, \eta_2)$ of $R$ satisfy the inequalities

$$|\xi_1 - \xi_2| < \delta_\varepsilon^*, \quad |\eta_1 - \eta_2| < \delta_\varepsilon^*.$$

Further, there is a positive integer $N_\varepsilon$ such that

$$|u(\xi, \eta) - u_{m_r n_r}(\xi, \eta)| < \delta_\varepsilon^* < \delta_\varepsilon,$$
$$|p(\xi, \eta) - p_{m_r n_r}(\xi, \eta)| < \delta_\varepsilon^* < \delta_\varepsilon,$$
$$|q(\xi, \eta) - q_{m_r n_r}(\xi, \eta)| < \delta_\varepsilon^* < \delta_\varepsilon,$$

and (*cf.* Section 3 for the definitions of $\alpha_m$ and $\beta_n$) also

$$\alpha_{m_r} < \delta_\varepsilon^* < \delta_\varepsilon, \quad \beta_{n_r} < \delta_\varepsilon^* < \delta_\varepsilon,$$

whenever

$$m_r > N_\varepsilon, \quad n_r > N_\varepsilon,$$

and $(\xi, \eta)$ is any point of the rectangle $R$.

Let $(x, y)$ be a point of $R$ and $m_r$ and $n_r$ be positive integers such that $m_r > N_\varepsilon$ and $n_r > N_\varepsilon$. These positive integers $m_r$ and $n_r$ and the numbers $\varepsilon$, $\delta_\varepsilon$, $\delta_\varepsilon^*$ will be supposed fixed during the immediate discussion. The notation of Section 2 (for example, writing $x_k$ instead of $x_{m_r, k}$) will be used in the next computation for simplicity in writing. There are integers $k$ and $l$, with $0 \leqq k \leqq m_r - 1$ and $0 \leqq l \leqq n_r - 1$, such that $x_k \leqq x \leqq x_{k+1}$ and $y_l \leqq y \leqq y_{l+1}$, *i.e.*, such that the point $(x, y)$ being examined lies in the closed subrectangle $R_{kl}^{m_r n_r}$. Recall that $\sigma(0) = \tau(0)$ and consider the difference

$$u_{m_r n_r}(x, y) - \left[\sigma(x) + \tau(y) - \sigma(0) + \int_{x_0}^{x} \int_{y_0}^{y} f(\xi, \eta, u(\xi, \eta), p(\xi, \eta), q(\xi, \eta))\, d\xi\, d\eta\right],$$

which may be written

$$\begin{aligned} u_{m_r n_r}(x, y) - \Big[\sigma(x) + \tau(y) - \sigma(0) &+ \\ &+ \sum_{i=1}^{k} \sum_{j=1}^{l} \int_{x_{i-1}}^{x_i} \int_{y_{j-1}}^{y_j} f(\xi, \eta, u(\xi, \eta), p(\xi, \eta), q(\xi, \eta))\, d\xi\, d\eta + \\ &+ \sum_{j=1}^{l} \int_{x_k}^{x} \int_{y_{j-1}}^{y_j} f(\xi, \eta, u(\xi, \eta), p(\xi, \eta), q(\xi, \eta))\, d\xi\, d\eta + \\ &+ \sum_{i=1}^{k} \int_{x_{i-1}}^{x_i} \int_{y_l}^{y} f(\xi, \eta, u(\xi, \eta), p(\xi, \eta), q(\xi, \eta))\, d\xi\, d\eta + \\ &+ \int_{x_k}^{x} \int_{y_l}^{y} f(\xi, \eta, u(\xi, \eta), p(\xi, \eta), q(\xi, \eta))\, d\xi\, d\eta\Big]. \end{aligned}$$

Recalling the definition of $u_{m_r u_r}(x, y)$ from Section 2, and the fact that

$$f_{i-1,j-1} = f(x_{i-1}, y_{j-1}, u_{m_r n_r}(x_{i-1}, y_{j-1}), p_{m_r n_r}(x_{i-1} y_{j-1}), q_{m_r n_r}(x_{i-1}, y_{j-1})),$$

from Sections 2 and 3, one has, for example, that

$$\begin{aligned} &|f_{i-1,j-1} - f(\xi, \eta, u(\xi,\eta), p(\xi,\eta), q(\xi,\eta))| \\ &\quad \leqq |f(x_{i-1}, y_{j-1}, u_{m_r n_r}(x_{i-1}, y_{j-1}), p_{m_r n_r}(x_{i-1}, y_{j-1}), q_{m_r n_r}(x_{i-1}, y_{j-1})) - \\ &\qquad - f(x_{i-1}, y_{j-1}, u(x_{i-1}, y_{j-1}), p(x_{i-1}, y_{j-1}), q(x_{i-1}, y_{j-1}))| + \\ &\qquad + |f(x_{i-1}, y_{j-1}, u(x_{i-1}, y_{j-1}), p(x_{i-1}, y_{j-1}), q(x_{i-1}, y_{j-1})) - \\ &\qquad - f(\xi, \eta, u(\xi,\eta), p(\xi,\eta), q(\xi,\eta))| < 2\varepsilon, \end{aligned}$$

whenever $x_{i-1} \leqq \xi \leqq x_i$ and $y_{j-1} \leqq \eta \leqq y_j$. Consequently, the absolute value of the difference $u_{m_r n_r} - [\ldots]$ is less than (see Section 2 for the definition of the constants $C$ and $D$)

$$\begin{aligned} &|\sigma(x) - \sigma(x_k)| + |\tau(y) - \tau(y_l)| + C \cdot (x - x_k) + D \cdot (y - y_l) + \\ &\qquad + 2\varepsilon (x_k - x_0)(y_l - y_0) + 2\varepsilon (x - x_k) \Big( \sum_{j=1}^{l} (y_j - y_{j-1}) \Big) + \\ &\qquad + 2\varepsilon (y - y_l) \Big( \sum_{i=1}^{k} (x_i - x_{i-1}) \Big) + 2\varepsilon (x - x_k)(y - y_l) \\ &\quad \leqq 2\varepsilon (C + D + ab + \varepsilon b + \varepsilon a + \varepsilon^2) \end{aligned}$$

whenever $m_r > N_\varepsilon$ and $n_r > N_\varepsilon$, and hence

$$u(x, y) = \sigma(x) + \tau(y) - \sigma(0) + \int_{x_0}^{x} \int_{y_0}^{y} f(\xi, \eta, u(\xi,\eta), p(\xi,\eta), q(\xi,\eta)) \, d\xi \, d\eta$$

for any $(x, y)$ in $R$. From this last equality it follows that $\partial u/\partial x$, $\partial u/\partial y$, and $\partial^2 u/\partial x\,\partial y$ $(= \partial^2 u/\partial y\,\partial x)$ exist and are continuous throughout the rectangle $R$. As a matter of fact

$$\frac{\partial u}{\partial x}(x, y) = \sigma'(x) + \int_{y_0}^{y} f(x, \eta, u(x,\eta), p(x,\eta), q(x,\eta)) \, d\eta,$$

$$\frac{\partial u}{\partial y}(x, y) = \tau'(y) + \int_{x_0}^{x} f(\xi, y, u(\xi, y), p(\xi, y), q(\xi, y)) \, d\xi,$$

while

$$\frac{\partial^2 u}{\partial x\,\partial y}(x, y) = \frac{\partial^2 u}{\partial y\,\partial x}(x, y) = f(x, y, u(x, y), p(x, y), q(x, y)),$$

for any $(x, y)$ of $R$.

The proofs of Theorems 2 and 3 will be complete once it is shown that $\partial u/\partial x \equiv p$ and $\partial u/\partial y \equiv q$. It suffices to consider only $\partial u/\partial x$. Let $\varepsilon > 0$ be given, and the numbers $\varepsilon > \delta_\varepsilon > \delta_\varepsilon^* > 0$ and $m_r > N_\varepsilon$, $n_r > N_\varepsilon$ be as in the argument just carried out. Let $(x, y)$ be a point of $R$. There are two cases to consider: either $x_k \leqq x < x_{k+1}$ and $y_l \leqq y \leqq y_{l+1}$ for suitable integers $k$ and $l$, with $0 \leqq k \leqq m_r - 2$ and $0 \leqq l \leqq n_r - 1$ or $x_{m_r - 1} \leqq x \leqq x_{m_r} \equiv x_0 + a$, and $y_l \leqq y \leqq y_{l+1}$ with $0 \leqq l \leqq n_r - 1$. Consider the difference

$$p_{m_r n_r}(x, y) - \Big[\sigma'(x) + \int_{y_0}^{y} f(x, \eta, u(x,\eta), p(x,\eta), q(x,\eta)) \, d\eta\Big],$$

which may be written (in either of the two cases mentioned, with $k=m_r-1$ in the second case)

$$p_{m_r n_r}(x,y)-\Big[\sigma'(x)+\sum_{j=1}^{l}\int_{y_{j-1}}^{y_j} f\big(x,\eta,u(x,\eta),p(x,\eta),q(x,\eta)\big)\,d\eta+ \\ +\int_{y_l}^{y} f\big(x,\eta,u(x,\eta),p(x,\eta),q(x,\eta)\big)\,d\eta\Big].$$

Recalling the definition of $p_{m_r n_r}(x,y)$ from Section 5, and the fact that from Sections 2 and 3

$$f_{i-1,j-1}=f\big(x_{i-1},y_{j-1},u_{m_r n_r}(x_{i-1}y_{j-1}),p_{m_r n_r}(x_{i-1},y_{j-1}),q_{m_r n_r}(x_{i-1},y_{j-1})\big),$$

one has again, for example, that

$$|f_{i-1,j-1}-f(\xi,\eta,u(\xi,\eta),p(\xi,\eta),q(\xi,\eta))|<2\varepsilon,$$

whenever $x_{i-1}\leqq\xi\leqq x_i$ and $y_{j-1}\leqq\eta\leqq y_j$. Besides, the mean value theorem of the differential calculus and the definition of the constant $C$ of Section 2 imply that

$$\left|\frac{u_{k+1,0}-u_{k,0}}{x_{k+1}-x_k}-\sigma'(x)\right|=|\sigma'(x^*)-\sigma'(x)|\leqq C|x^*-x|\leqq C\,\alpha_{n_r}.$$

Consequently, the absolute value of the difference $p_{m_r n_r}(x,y)-[\ldots]$ is less than

$$\varepsilon C+2\varepsilon\Big(\sum_{i=1}^{k}(x_i-x_{i-1})\Big)+2\varepsilon(x-x_k)\leqq\varepsilon(C+2a),$$

whenever $m_r>N_\varepsilon$ and $n_r>N_\varepsilon$, and hence

$$p(x,y)=\sigma'(x)+\int_{y_0}^{y} f\big(x,\eta,u(x,\eta),p(x,\eta),q(x,\eta)\big)\,d\eta$$

for any $(x,y)$ in $R$. Since the right hand side of the last equation is already known to be equal to $\frac{\partial u}{\partial x}(x,y)$, it follows that $\frac{\partial u}{\partial x}\equiv p$, as desired. By symmetry one has also that

$$q(x,y)=\tau'(y)+\int_{x_0}^{x} f\big(\xi,y,u(\xi,y),p(\xi,y),q(\xi,y)\big)\,d\xi$$

for any $(x,y)$ in $R$, from which it follows that $\partial u/\partial y\equiv q$, and the proof is complete.

Under the hypotheses of Theorem 3, the preceding argument shows that any singly infinite subsequence $\{u_{m_r n_r}(x,y)\}$, where $\lim_{r\to\infty} m_r=\infty$ and $\lim_{r\to\infty} n_r=\infty$, contains a subsequence which converges uniformly on $R$ to a solution. On the other hand, under the hypotheses of Theorem 2 (in which case there is but one solution) the preceding argument implies that the whole double sequence $\{u_{mn}(x,y)\}$ converges to the solution, *i.e.* that

$$\lim_{\substack{m\to\infty\\ n\to\infty}} u_{mn}(x,y)$$

is the solution, the convergence being uniform on $R$.

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