On the Uniqueness and Non-existence of Stokes Flows

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1. Introduction

It has been known since the time of STOKES that under suitable assumptions on the behavior of the velocity field at infinity, there exists no steady two-dimensional flow of a viscous fluid past an obstacle, in which the velocity components are infinitesimals of the first order. On the other hand, flows of this type in three dimensions are explicitly known (*cf.* [2]). An explanation sometimes advanced for the discrepancy in results is that in the known flow past a sphere [2], the assumption on the ratio of inertial to viscous forces which is used to derive the equations of motion becomes violated in a neighborhood of infinity. In the view of the authors, this phenomenon brings into serious question both the physical and the mathematical validity of the known results, and makes it imperative to investigate the sense in which the boundary problem is correctly set. More precisely, it is natural to ask whether there exist two-dimensional flows in which the velocity tends to its limit more slowly than has in the past been assumed, and whether a three-dimensional flow is unique in a (physically reasonable) class of flows for which the usual uniqueness proofs break down*. In this paper we prove that the answer to the first question is no, to the second yes. In fact, we show that there are no two-dimensional flows for which the velocity is bounded, and we prove the uniqueness of a three-dimensional flow under the single assumption that the velocity tends to a limit at infinity**. As a corollary, we obtain representations, valid in a neighborhood of infinity, for the velocity field of the most general such flow.

Once the uniqueness of a flow past an obstacle with prescribed velocity at infinity is established, the continuous dependence of the velocity field on the

* In [2], SOMMERFELD dismisses this question with the remark, „Daß die so gewonnenen Gln. (13) und (13a) die einzig möglichen Lösungen unseres Problems sind, haben wir zwar nicht bewiesen. Wir folgern es aber aus dem Axiom, daß jedes richtig gestellte Problem der mathematischen Physik nur eine Lösung haben kann.“ This reasoning is evidently circular. In order to establish that the problem is correctly set it is necessary to prove the uniqueness of a solution.

** After preparation of this manuscript our attention was directed to a paper of CHARNES & KRAKOWSKI, Carnegie Inst. Tech. Technical Report No. 37 (1953), in which a rigorous proof is given for the non-existence of a plane Stokes flow. The proof we present here seems simpler, and applies without essential change to the proof of uniqueness of a three-dimensional flow and to the derivation of asymptotic representations.

limiting speed is a direct consequence of the linearity of the equations of motion. The existence of such a flow, despite extensive studies in this direction, is yet to be proved.

2. Notation and definitions

Throughout this paper we use GIBBS' notation of vector analysis. The underlying space is of dimension $n=2$ or $n=3$ (plane or ordinary three-space). Points are denoted by the letters P, Q, *etc.*, vectors by bold face letters. By $P+\boldsymbol{r}$ we mean the end point of the vector $\boldsymbol{r}$ if it starts at P. The symbol $\boldsymbol{e}$ is reserved for a variable unit vector or, equivalently, for a variable point on the unit sphere (circle) ω. The surface element (element of arc) of the unit sphere (circle) will be denoted by $d\omega$. In the plane ($n=2$) $\boldsymbol{u}\times\boldsymbol{v}$ and curl $\boldsymbol{u}$ have to be regarded as scalars.

By a *slow flow*, or *Stokes flow*, in a region, we shall mean a vector field $\boldsymbol{q}(P)$ which satisfies the system

$$\Delta \boldsymbol{q} = \operatorname{grad} p, \qquad \operatorname{div} \boldsymbol{q} = 0 \tag{2.1}$$

for some scalar field $p(P)$. In the text we shall refer to a Stokes flow simply as a *flow*. We assume that within the flow region $\boldsymbol{q}$ is three times and p twice continuously differentiable. p is determined only up to an arbitrary additive constant.

By an *obstacle* $\mathfrak{B}$ we shall understand a finite number of piecewise smooth nonintersecting simple closed surfaces (curves). A solution of (2.1) defined throughout the exterior of an obstacle $\mathfrak{B}$ will be called a *flow past* $\mathfrak{B}$ if $\boldsymbol{q}$ is continuous up to $\mathfrak{B}$ and $\boldsymbol{q}=0$ on $\mathfrak{B}$ and if p and the first partial derivatives of $\boldsymbol{q}$ are bounded up to $\mathfrak{B}$.

The region exterior to a sphere (circle) of radius r_0 and center Q shall be denoted by $\mathfrak{E}$ and called a *neighborhood of infinity*. Let $v=v(P)$ be a vector or scalar field defined in $\mathfrak{E}$. Then we write $v=o(r^k)$ if $\lim_{r\to\infty} r^{-k}v(Q+r\boldsymbol{e})=0$ uniformly in $\boldsymbol{e}$, and we write $v=O(r^k)$ if $|r^{-k}v(Q+r\boldsymbol{e})|<\text{const.}<\infty$ for large r. These properties are evidently independent of the choice of Q.

A flow defined in a neighborhood of infinity will be called *uniform at infinity* if $\boldsymbol{q}=\boldsymbol{q}_0+o(1)$, and $\boldsymbol{q}_0$ is said to be its *velocity at infinity*.

3. The basic lemmas

Lemma 1. *Let h be a field which is harmonic in a neighborhood of infinity $\mathfrak{E}$. Then h has a unique decomposition $h=h_1+h_2$,*

where h_1 is harmonic in the entir space (plane) except at ∞, and

where h_2 is harmonic in $\mathfrak{E}$ and, for $n=3$, $h_2=O(r^{-1})$ while for $n=2$ it is of the form

$$h_2(Q+r\boldsymbol{e}) = a\log r + h_2^*(Q+r\boldsymbol{e}), \qquad h_2^* = O(r^{-1})\,^{*}. \tag{3.1}$$

h is harmonic at ∞ if and only if $h=O(r^{-n+2})$.

* h_1, h_2 and a are uniquely determined by h and independent of the choice of Q, but h_2^*, of course, is not.

Lemma 2. *Let h be as in lemma 1. Then the following three statements are fully equivalent:*

(a) $h = o(r^{-\nu}), \quad \nu = \begin{cases} 0, 1, 2, \dots & \text{if} \quad n = 2 \\ -1, 0, 1, 2, \dots & \text{if} \quad n = 3 \end{cases}$

(b) $h = O(r^{-\nu-1})$

(c) *h has an expansion of the form*

$$h = h(Q + r\mathbf{e}) = \sum_{k=\nu+1}^{\infty} r^{-k} s_{k-1}(\mathbf{e})\,^{\star}, \tag{3.2}$$

uniformly convergent in every compact subregion of $\mathfrak{E}$.

The series obtained from (3.2) *by taking the gradient term by term is also uniformly convergent in every compact subregion of* $\mathfrak{E}$ *and hence it represents an expansion for* grad h. *We have* grad $h = O(r^{-\nu-2})$ *and, conversely, if* grad $h = O(r^{-\nu-2})$, *then* $h = O(r^{-\nu-1}) + \text{const}$.

These two lemmas are corollaries of well known theorems on isolated singularities of harmonic functions (*cf.* [*1*] §§ 93—95).

Lemma 3. *Let h be as in lemma 1. Then the following two statements are fully equivalent:*

(a) $h = O(r^{-n+1})$,

(b) $\int\limits_{\omega} h(Q + r\mathbf{e})\, d\omega = 0$ *for any point Q and any* $r > r_0$**.

Proof. Assume first that (a) holds. Then by lemma 1 h is harmonic at ∞. Hence the function $\bar{h}(P) = \bar{h}(Q + s\mathbf{e}) = s^{-n+2} h(Q + s^{-1}\mathbf{e})$ is harmonic in the neighborhood of Q and at Q. Therefore by the mean value theorem for harmonic functions we have ($\omega = 2\pi$ or 4π)

$$\omega \bar{h}(Q) = \int\limits_{\omega} \bar{h}(Q + r^{-1}\mathbf{e})\, d\omega = r^{n-2} \int\limits_{\omega} h(Q + r\mathbf{e})\, d\omega.$$

On the other hand, (a) implies $\bar{h}(Q) = \lim\limits_{r\to\infty} r^{n-2} h(Q + r\mathbf{e}) = 0$, hence that (b) holds.

Assume now that (b) is true. We use the decomposition $h = h_1 + h_2$ of lemma 1. Since h_1 is harmonic everywhere the mean value theorem implies $\omega h_1(Q) = \int\limits_{\omega} h_1(Q + r\mathbf{e})\, d\omega$. Using the same method as before and lemma 1, we get

$$\int\limits_{\omega} h_2(Q + r\mathbf{e})\, d\omega = \begin{cases} \omega\, 2\pi a \log r & \text{for } n = 2 \\ \omega\, b\, r^{-1} & \text{for } n = 3, \end{cases}$$

where $b = \lim\limits_{r\to\infty} r\, h_2(Q + r\mathbf{e})$ for $n = 3$. Therefore (b) implies

$$-h_1(Q) = \begin{cases} 2\pi a \log r & \text{for } n = 2 \\ b\, r^{-1} & \text{for } n = 3. \end{cases}$$

* For $n = 2$ the $s_k(\mathbf{e})$ are trigonometric functions while for $n = 3$ they are spherical harmonics of order k.

** This lemma, as well as its proof, is valid also for $n > 3$.

These relations are possible for arbitrary Q and arbitrary large r only if $h_1(Q)\equiv 0$ and $a=b=0$. Hence $h=h_2$ and, for $n=3$, $h_2=o(r^{-1})$. Lemma 1 for $n=2$ and lemma 2 for $n=3$ show that (a) holds, *q. e. d.*

Lemma 4. *Let* $\boldsymbol{q}$ *be a three-dimensional flow defined in a neighborhood of infinity* $\mathfrak{E}$ *and uniform at* ∞. *Then*

$$\operatorname{curl}\boldsymbol{q}=O(r^{-2}). \tag{3.3}$$

Proof. We put $\boldsymbol{w}=\operatorname{curl}\boldsymbol{q}$. By (2.1) we have $\Delta\boldsymbol{w}=\operatorname{curl}\Delta\boldsymbol{q}=0$, *i.e.* $\boldsymbol{w}$ is harmonic in $\mathfrak{E}$. We denote the sphere of radius r around Q by $\mathfrak{S}_r$ and the volume between $\mathfrak{S}_{r_0}$ and $\mathfrak{S}_r$ by $\mathfrak{B}_{rr_0}$. Then Green's identity yields

$$0=\int\limits_{\mathfrak{B}_{rr_0}}\Delta\boldsymbol{w}\,dV=\int\limits_{\mathfrak{S}_{r_0}}\frac{\partial\boldsymbol{w}}{\partial n}\,dS+\int\limits_{\mathfrak{S}_r}\frac{\partial\boldsymbol{w}}{\partial n}\,dS,$$

where $\partial/\partial n$ denotes the normal derivative in the direction away from $\mathfrak{B}_{rr_0}$. The first integral on the right is independent of r and will be denoted by $\boldsymbol{a}$. For the second we have $\frac{\partial\boldsymbol{w}}{\partial n}=\frac{d}{dr}\boldsymbol{w}(Q+r\boldsymbol{e})$ and $dS=r^2d\omega$. Thus

$$\frac{d}{dr}\int\limits_{\omega}\boldsymbol{w}(Q+r\boldsymbol{e})\,d\omega=-\boldsymbol{a}r^{-2}. \tag{3.4}$$

Integration with respect to r yields

$$\int\limits_{\omega}\boldsymbol{w}(Q+r\boldsymbol{e})\,d\omega=\boldsymbol{a}r^{-1}+\boldsymbol{b}. \tag{3.5}$$

We multiply this by r^2 and integrate again with respect to r, obtaining

$$\int\limits_{\mathfrak{B}_{r_0r}}\boldsymbol{w}\,dV=\frac{r^2}{2}\boldsymbol{a}+\frac{r^3}{3}\boldsymbol{b}+\boldsymbol{c}, \tag{3.6}$$

where $\boldsymbol{c}$ is independent of r. Another Green's identity yields

$$\int\limits_{\mathfrak{B}_{r_0r}}\operatorname{curl}\boldsymbol{q}\,dV=\int\limits_{\mathfrak{S}_{r_0}}\boldsymbol{q}\times\boldsymbol{n}\,dS+\int\limits_{\mathfrak{S}_r}\boldsymbol{q}\times\boldsymbol{n}\,dS, \tag{3.7}$$

where $\boldsymbol{n}$ is the unit normal vector directed away from $\mathfrak{B}_{r_0r}$. Again, the first integral on the right is independent of r and will be denoted by $\boldsymbol{d}$, the second can be written in the form

$$\int\limits_{\mathfrak{S}_r}\boldsymbol{q}\times\boldsymbol{n}\,dS=r^2\int\limits_{\omega}\boldsymbol{q}(Q+r\boldsymbol{e})\times\boldsymbol{e}\,d\omega.$$

Hence from (3.6) and (3.7) we get

$$\int\limits_{\omega}\boldsymbol{q}(Q+r\boldsymbol{e})\times\boldsymbol{e}\,d\omega=\frac{1}{2}\boldsymbol{a}+\frac{r}{3}\boldsymbol{b}+(\boldsymbol{c}-\boldsymbol{d})\,r^{-2}. \tag{3.8}$$

By hypothesis, $\boldsymbol{q}$ is uniform at ∞, *i.e.* $\lim\limits_{r\to\infty}\boldsymbol{q}(Q+r\boldsymbol{e})=\boldsymbol{q}_0$ uniformly in $\boldsymbol{e}$. Hence, letting $r\to\infty$ in (3.8) yields $\boldsymbol{b}=0$ and $\boldsymbol{q}_0\times\int\limits_{\omega}\boldsymbol{e}\,d\omega=0=\frac{1}{2}\boldsymbol{a}$. From (3.5) and lemma 3 for $n=3$ follows then $\boldsymbol{w}=O(r^{-2})$, *q. e. d.*

Lemma 5. *Let* $\boldsymbol{q}$ *be a plane flow defined and bounded in a neighborhood of infinity* $\mathfrak{E}$. *Then*

$$\operatorname{curl} \boldsymbol{q} = O(r^{-1}) \, \star. \tag{3.9}$$

The *proof* of this lemma is analogous to that of lemma 4. Surface integrals are to be replaced by line integrals and volume integrals by area integrals. $w = \operatorname{curl} \boldsymbol{q}$ is now a scalar field. In place of (3.4) we get

$$\frac{d}{dr} \int\limits_{\omega} w(Q + r\boldsymbol{e})\, d\omega = -a r^{-1},$$

so that

$$\int\limits_{\omega} w(Q + r\boldsymbol{e})\, d\omega = -a \log r + b. \tag{3.10}$$

Instead of (3.8), we now have

$$\int\limits_{\omega} \boldsymbol{q}(Q + \boldsymbol{e}r) \times \boldsymbol{e}\, d\omega = -a r\left(\frac{1}{2}\log r - \frac{1}{4}\right) + \frac{r}{2} b + (c - d)\, r^{-1}.$$

It suffices here to assume that $\boldsymbol{q}$ is bounded in order to conclude that $a = 0$ and $b = 0$. From (3.10) and lemma 3 for $n = 2$ follows $w = O(r^{-1})$, *q. e. d.*

4. Uniqueness of three-dimensional flows

Theorem I. *For any obstacle* $\mathfrak{B}$ *there is at most one three-dimensional flow* $\boldsymbol{q}$ *past* $\mathfrak{B}$ *which is uniform at infinity and has a prescribed velocity at infinity* $\boldsymbol{q}_0$.

Proof. Since the equations (2.1) are linear, it is sufficient to prove that $\boldsymbol{q}_0 = 0$ implies $\boldsymbol{q} \equiv 0$.

By lemma 4 we have

$$\boldsymbol{w} = \operatorname{curl} \boldsymbol{q} = O(r^{-2}) \tag{4.1}$$

and hence, by lemma 2,

$$-\operatorname{curl} \boldsymbol{w} = -\operatorname{curl}\operatorname{curl} \boldsymbol{q} = \Delta \boldsymbol{q} = \operatorname{grad} p = O(r^{-3}).$$

From lemma 2 follows that $p = O(r^{-2}) + \text{const.}$, but since we are free to adjust p by an additive constant, we may assume

$$p = O(r^{-2}). \tag{4.2}$$

Consider now the Green's identity

$$\int\limits_{\mathfrak{B}} (\operatorname{curl} \boldsymbol{q})^2 dV = \int\limits_{\mathfrak{B}} \boldsymbol{q} \cdot \Delta \boldsymbol{q}\, dV + \int\limits_{\mathfrak{S}} \operatorname{curl} \boldsymbol{q} \cdot (\boldsymbol{q} \times \boldsymbol{n})\, dS, \tag{4.3}$$

⋆ Professor TRUESDELL has pointed out to us that our proof of non-existence of plane flows can be viewed as an application of a theorem of HAMEL and KAMPÉ DE FERIET, which asserts that in a plane motion adhering to a fixed boundary the vorticity is orthogonal to every harmonic function. But in a *slow* plane motion the vorticity is harmonic. Therefore the vorticity is zero, from which one concludes easily that the velocity is zero. To make this argument rigorous in our case it is necessary to find a suitable estimate on the behavior of the vorticity at infinity. This estimate is provided by lemma 5.

valid for any smooth vector field $\boldsymbol{q}$ with $\operatorname{div}\boldsymbol{q}=0$ defined in a finite region $\mathfrak{V}$ with piesewise smooth boundary $\mathfrak{S}$. We apply this identity to the flow $\boldsymbol{q}$ and the region $\mathfrak{V}_r$ between the obstacle $\mathfrak{B}$ and a sphere $\mathfrak{S}_r$ so large that $\mathfrak{B}$ lies interior to $\mathfrak{S}_r$. By (2.1) we have $\boldsymbol{q}\cdot\Delta\boldsymbol{q}=\boldsymbol{q}\cdot\operatorname{grad}p=\operatorname{div}(p\boldsymbol{q})$, so that (4.3) can be transformed into

$$\int_{\mathfrak{V}_r}(\operatorname{curl}\boldsymbol{q})^2\,dV=\int_{\mathfrak{S}_r}[p\,\boldsymbol{q}\cdot\boldsymbol{n}-\operatorname{curl}\boldsymbol{q}\cdot(\boldsymbol{q}\times\boldsymbol{n})]\,dS. \tag{4.4}$$

Here we have made use of the fact that $\boldsymbol{q}=0$ on $\mathfrak{B}$. It follows from (4.1), (4.2), and $\boldsymbol{q}=o(1)$ that the surface integral in (4.4) tends to zero as $r\to\infty$. Hence

$$\int_{\mathfrak{V}}(\operatorname{curl}\boldsymbol{q})^2\,dV=0, \tag{4.5}$$

where $\mathfrak{V}$ is the region exterior to $\mathfrak{B}$. Since $\operatorname{curl}\boldsymbol{q}$ is continuous, it follows that $\operatorname{curl}\boldsymbol{q}\equiv0$ throughout $\mathfrak{V}$. Together with $\operatorname{div}\boldsymbol{q}=0$, this implies $\Delta\boldsymbol{q}=0$. But $\boldsymbol{q}=0$ on $\mathfrak{B}$ and $\boldsymbol{q}=o(1)$; hence, by the maximum and minimum principle for harmonic functions, applied to each component of $\boldsymbol{q}$, it follows that $\boldsymbol{q}\equiv0$, *q.e.d.*

5. Non-existence of plane flows

Theorem II. *For any obstacle $\mathfrak{B}$, any bounded plane flow $\boldsymbol{q}$ past $\mathfrak{B}$ is the state of rest $\boldsymbol{q}\equiv0$.*

Proof. An argument analogous to that used in the proof of theorem I shows that from lemma 5 follows

$$w=\operatorname{curl}\boldsymbol{q}=O(r^{-1}),\qquad p=O(r^{-1}). \tag{5.1}$$

Also, we have

$$\int_{\mathfrak{A}_r}(\operatorname{curl}\boldsymbol{q})^2\,dA=\int_{\mathfrak{C}_r}[p\,\boldsymbol{q}\cdot\boldsymbol{n}-\operatorname{curl}\boldsymbol{q}\,(\boldsymbol{q}\times\boldsymbol{n})]\,ds, \tag{5.2}$$

where $\mathfrak{A}_r$ is the region between $\mathfrak{B}$ and a large circle $\mathfrak{C}_r$. But here we can conclude from (5.1) and the boundedness of $\boldsymbol{q}$ only that the integral

$$I=\int_{\mathfrak{A}}(\operatorname{curl}\boldsymbol{q})^2\,dA=\int_{\mathfrak{A}}w^2\,dA \tag{5.3}$$

over the exterior $\mathfrak{A}$ of $\mathfrak{B}$ is convergent. However, we know that w is harmonic in $\mathfrak{A}$ and that $w=O(r^{-1})$. Hence, by lemma 2, (3.2), w is of the form

$$w=w(Q+r\boldsymbol{e})=r^{-1}s_0(\boldsymbol{e})+O(r^{-2}), \tag{5.4}$$

and therefore

$$w^2=r^{-2}s_0^2+O(r^{-3}). \tag{5.5}$$

Let $\mathfrak{A}_{r_0r}$ be an area between two circles $\mathfrak{C}_{r_0}$ and $\mathfrak{C}_r$ both exterior to $\mathfrak{B}$, so that $\mathfrak{A}_{r_0r}$ is contained in $\mathfrak{A}$. We then have from (5.5)

$$\int_{\mathfrak{A}_{r_0r}}w^2\,dA=\log\frac{r}{r_0}\int_{\omega}s_0^2\,d\omega+\int_{\mathfrak{A}_{r_0r}}O(r^{-3})\,dA.$$

This must tend to a finite limit as $r\to\infty$, because the integral (5.3) converges. The second term on the right certainly converges and hence, since $\log\frac{r}{r_0}$ is not bounded as $r\to$, we can conclude that $\int_{\omega}s_0^2\,d\omega=0$ and hence $s_0=0$.

Therefore, by (5.4), $w = O(r^{-2})$, from which, as before, $p = O(r^{-2})$ follows. Going back to (5.2) we see now that the integral on the right in fact tends to zero as $r \to \infty$ and hence, as in the proof of theorem I, we conclude that $\operatorname{curl} \boldsymbol{q} = 0$, $\operatorname{div} \boldsymbol{q} = 0$ and $\Delta \boldsymbol{q} = 0$ throughout $\mathfrak{A}$. Thus $\boldsymbol{q}$ is harmonic in a neighborhood of infinity. Since $\boldsymbol{q}$ is also bounded, it follows from lemma 1 for $n = 2$ that $\boldsymbol{q}$ is harmonic at ∞ and hence uniform at infinity. Therefore, for some choice of a Cartesian coordinate system, we have $q_x \to q_0$, $q_y \to 0$, where q_x and q_y are the components of $\boldsymbol{q}$. As before, the maximum principle for harmonic functions implies $q_y \equiv 0$. Therefore $\operatorname{curl} \boldsymbol{q} = \frac{\partial q_x}{\partial y} = 0$, $\operatorname{div} \boldsymbol{q} = \frac{\partial q_x}{\partial x} = 0$, and hence $q_x = \text{const}$. But, since $q_x = 0$ on $\mathfrak{B}$, we must have $q_x \equiv 0$ everywhere, *q. e. d.*

6. Representation formulas

The following theorem establishes a one-to-one correspondence between three-dimensional flows with given velocity at infinity and vector fields which are harmonic at infinity.

Theorem III. *Let $\boldsymbol{q}$ be a flow defined in a neighborhood of infinity $\mathfrak{E}$ with velocity at infinity $\boldsymbol{q}_0$. Then there is a harmonic vector field $\boldsymbol{u}$, defined in $\mathfrak{E}$ and harmonic at infinity such that*

$$\boldsymbol{q} = \boldsymbol{q}_0 + \boldsymbol{u} + \operatorname{grad} \psi, \tag{6.1}$$

where

$$\psi = \psi(Q + r\boldsymbol{e}) = -\tfrac{1}{4} r^{\frac{1}{2}} \int_r^\infty s^{\frac{1}{2}} \operatorname{div} \boldsymbol{u}(Q + s\boldsymbol{e})\, ds. \tag{6.2}$$

Conversely, if $\boldsymbol{u}$ is an arbitrary harmonic vector field defined in $\mathfrak{E}$ and harmonic at infinity, then (6.1) *and* (6.2) *give a flow with velocity at infinity $\boldsymbol{q}_0$.*

Proof. Consider the differential equation

$$\Delta(r^2 \chi) - r^2 \Delta \chi = 6\chi + 4r \frac{\partial \chi}{\partial r} = p(Q + r\boldsymbol{e}). \tag{6.3}$$

A special solution is

$$\chi = \chi(Q + r\boldsymbol{e}) = \tfrac{1}{4} r^{-\frac{3}{2}} \int_r^\infty s^{\frac{1}{2}} p(Q + s\boldsymbol{e})\, ds. \tag{6.4}$$

The integral converges because of (4.2); moreover,

$$\chi = O(r^{-2}), \qquad \Delta \chi = O(r^{-4}), \qquad \operatorname{grad}(r^2 \chi) = O(r^{-1}). \tag{6.5}$$

Applying Laplace's operator to (6.3) and observing that $\Delta p = 0$ we get

$$14 \Delta \chi + 4r \frac{\partial}{\partial r} \Delta \chi = 0,$$

hence $\Delta \chi = c(\boldsymbol{e})\, r^{\frac{1}{2}-4}$. This is compatible with (6.5) only if $c(\boldsymbol{e}) = 0$. Thus $\Delta \chi = 0$ and, by (6.3), $\Delta(r^2 \chi) = p$. Hence $\Delta[\boldsymbol{q} - \boldsymbol{q}_0 - \operatorname{grad}(r^2 \chi)] = \operatorname{grad} p - \operatorname{grad} \Delta(r^2 \chi) = 0$ *i.e.* $\boldsymbol{u} = \boldsymbol{q} - \boldsymbol{q}_0 - \operatorname{grad}(r^2 \chi)$ is harmonic. From (6.5), $\boldsymbol{q} - \boldsymbol{q}_0 = o(1)$, and from lemma 1 it follows that $\boldsymbol{u}$ is harmonic at infinity. Also, we have

$$\operatorname{div} \boldsymbol{u} = -\operatorname{div} \operatorname{grad}(r^2 \chi) = -\Delta(r^2 \chi) = -p,$$

hence $\psi = r^2\chi$ in (6.2). Thus the first part of the theorem is proved. The second part can be verified immediately.

By use of the series expansion (3.2) for $\boldsymbol{u}$, *i.e.*

$$\boldsymbol{u} = \sum_{k=1}^{\infty} r^{-k} \boldsymbol{s}_{k-1}(\boldsymbol{e}), \tag{6.6}$$

it follows easily as a corollary of theorem III that any flow which is uniform at infinity can be represented by an expansion

$$\boldsymbol{q} = \boldsymbol{q}_0 + \sum_{k=1}^{\infty} r^{-k} \left[\boldsymbol{s}_{k-1}(\boldsymbol{e}) - \frac{1}{2(2k-1)} \boldsymbol{s}^*_{k+1}(\boldsymbol{e}) \right], \tag{6.7}$$

where the spherical harmonic vectors $\boldsymbol{s}^*_k(\boldsymbol{e})$ are defined in terms of the $\boldsymbol{s}_k(\boldsymbol{e})$ by

$$\boldsymbol{s}^*_k(\boldsymbol{e}) = r^{k+1} \operatorname{grad} \operatorname{div} [r^{-k+1} \boldsymbol{s}_{k-1}(\boldsymbol{e})], \qquad k = 2, 3, \ldots. \tag{6.8}$$

Conversely, for any set of spherical harmonic vectors $\boldsymbol{s}_k(\boldsymbol{e})$ such that (6.6) converges uniformly in some neighborhood of infinity, (6.7) and (6.8) define a flow with velocity at infinity $\boldsymbol{q}_0$. The classical Stokes flow past a sphere of radius r_0 [2] corresponds to

$$\boldsymbol{s}_0 = -\tfrac{3}{4} r_0 \boldsymbol{q}_0, \quad \boldsymbol{s}_2 = \tfrac{1}{8} r_0^3 [\boldsymbol{q}_0 - 3\boldsymbol{e}(\boldsymbol{q}_0 \cdot \boldsymbol{e})], \quad \boldsymbol{s}_1 = \boldsymbol{s}_3 = \boldsymbol{s}_4 = \cdots = 0.$$

Theorem IV. *Let* $\boldsymbol{q}$ *denote a flow defined in a neighborhood of infinity* $\mathfrak{E}$ *with velocity at infinity* $\boldsymbol{q}_0$. *Then there are harmonic vector fields* $\boldsymbol{\Omega}$ *and* $\boldsymbol{H}$, *defined in* $\mathfrak{E}$ *and harmonic at infinity, such that*

$$\begin{gathered} \operatorname{div} \boldsymbol{H} = 0 \\ \operatorname{div} \Delta (r^2 \boldsymbol{\Omega}) = 0, \qquad \boldsymbol{\Omega} = O(r^{-2}), \end{gathered} \tag{6.9}$$

and such that

$$\boldsymbol{q} = \boldsymbol{q}_0 + \operatorname{curl}(r^2 \boldsymbol{\Omega}) + \boldsymbol{H}. \tag{6.10}$$

Conversely, if $\boldsymbol{H}$ *and* $\boldsymbol{\Omega}$ *are harmonic vector fields defined in* $\mathfrak{E}$, *harmonic at infinity and satisfying* (6.9), *then* (6.10) *represents a flow with velocity at infinity* $\boldsymbol{q}_0$.

Proof. Define

$$\boldsymbol{\Omega} = -\tfrac{1}{4} r^{-\frac{3}{2}} \int_r^{\infty} s^{\frac{1}{2}} \operatorname{curl} \boldsymbol{q}(Q + s\boldsymbol{e})\, ds. \tag{6.11}$$

Then, as in the proof of theorem III, $\Delta \boldsymbol{\Omega} = 0$, $\Delta (r^2 \boldsymbol{\Omega}) = -\operatorname{curl} \boldsymbol{q}$,

$$\Delta (\boldsymbol{q} - \operatorname{curl} r^2 \boldsymbol{\Omega}) = \Delta \boldsymbol{q} + \operatorname{curl} \operatorname{curl} \boldsymbol{q} = 0,$$

hence

$$\boldsymbol{q} = \boldsymbol{q}_0 + \operatorname{curl} r^2 \boldsymbol{\Omega} + \boldsymbol{H}$$

for a harmonic vector $\boldsymbol{H}$. Since $\operatorname{curl} \boldsymbol{q} = O(r^{-2})$, we have $\boldsymbol{\Omega} = O(r^{-2})$ and $\operatorname{curl} r^2 \boldsymbol{\Omega} = O(r^{-1})$. Hence $\boldsymbol{H} = o(1)$. By lemma 2, $\boldsymbol{\Omega}$ and $\boldsymbol{H}$ are harmonic at infinity.

If, conversely, $\boldsymbol{H}$ and $\boldsymbol{\Omega}$ are harmonic in $\mathfrak{E}$ and at infinity and satisfy (6.9), then if we set

$$\boldsymbol{q} = \boldsymbol{q}_0 + \operatorname{curl} r^2 \boldsymbol{\Omega} + \boldsymbol{H},$$

then $\operatorname{div} \boldsymbol{q} = 0$ and $-\operatorname{curl} \Delta \boldsymbol{q} = \Delta \Delta r^2 \boldsymbol{\Omega} = 0$, hence there is a scalar p such that $\Delta \boldsymbol{q} = \operatorname{grad} p$. Also, one can see that $\boldsymbol{q} = \boldsymbol{q}_0 + O(r^{-1})$, *q. e. d.*

7. Representation of a flow past an obstacle

If the flow is defined throughout the exterior $\mathfrak{V}$ of an obstacle $\mathfrak{B}$ and $\boldsymbol{q} = 0$ on $\mathfrak{B}$ the results of the preceding section can be made more specific. For we may set

$$(7.1) \qquad \boldsymbol{\varphi}(Q) = -\frac{1}{4\pi} \int_{\mathfrak{V}} r^{-1} \operatorname{curl} \boldsymbol{q}(P)\, dV$$

where r is the distance between Q and P. This integral in general does not converge absolutely, but it exists in the sense of a principal value as the limit of integrals over concentric spheres with center at Q or at any arbitrary point. This is the case because by (6.1) and (6.6) we have

$$r^{-1} \operatorname{curl} \boldsymbol{q} = r^{-1} \operatorname{curl} \boldsymbol{u} = r^{-3} \mathbf{s}_0 \times \boldsymbol{e} + O(r^{-4}), \qquad \mathbf{s}_0 = \text{const.},$$

and $\int_\omega \boldsymbol{e}\, d\omega = 0$. Since $\boldsymbol{n} \cdot \operatorname{curl} \boldsymbol{q} = 0$ on $\mathfrak{B}$, one can show that $\Delta \boldsymbol{\varphi} = \operatorname{curl} \boldsymbol{q}$ and $\operatorname{div} \boldsymbol{\varphi} = 0$, hence $-\operatorname{curl} \operatorname{curl} \boldsymbol{\varphi} = \Delta \boldsymbol{\varphi} = \operatorname{curl} \boldsymbol{q}$, which implies the existence of a scalar h such that $\boldsymbol{q} = \boldsymbol{q}_0 - \operatorname{curl} \boldsymbol{\varphi} + \operatorname{grad} h$. The function h may be multi-valued if $\mathfrak{V}$ is not simply connected, but $\operatorname{grad} h$ is single-valued in $\mathfrak{V}$. Evidently, $\operatorname{div} \operatorname{grad} h = \Delta h = 0$. Also, simple estimates on (7.1), using $\operatorname{curl} \boldsymbol{q} = O(r^{-2})$, show that $\operatorname{curl} \boldsymbol{\varphi} = O(r^{-1})$. Hence $\operatorname{grad} h = O(r^{-1})$, and, for any given determination of h in a (simply connected) neighborhood $\mathfrak{E}$ of infinity, $h = O(\log r)$. By lemma 2, h tends to a limit at infinity in $\mathfrak{E}$. Thus follows

Theorem V. *Every flow past $\mathfrak{B}$ with velocity at infinity $\boldsymbol{q}_0$ admits the representation, valid in the exterior $\mathfrak{V}$ of $\mathfrak{B}$,*

$$(7.2) \qquad \boldsymbol{q} = \boldsymbol{q}_0 - \operatorname{curl} \boldsymbol{\varphi} + \operatorname{grad} h,$$

where $\boldsymbol{\varphi}$ is defined by (7.1), and where h is harmonic in $\mathfrak{V}$. If $\mathfrak{V}$ is simply connected, h is single valued and may be chosen to be harmonic at infinity. Otherwise, any determination of h tends to a finite limit at infinity.

We may combine this result with the results of § 6 to obtain still other representations.

Theorem VI. *If $\mathfrak{B}$ is star-shaped relative to a point Q, every flow past $\mathfrak{B}$ which is uniform at infinity admits the representation*

$$(7.3) \qquad \boldsymbol{q} = \boldsymbol{q}(Q + \boldsymbol{e} r) = \boldsymbol{q}_0 + \operatorname{curl} r^2 \boldsymbol{\Omega} + \operatorname{curl} \boldsymbol{H} + \operatorname{grad} h,$$

where $\boldsymbol{H}$ and h are harmonic in $\mathfrak{V}$ and at infinity and $\boldsymbol{\Omega}$ is defined by (6.11). If $\mathfrak{B}$ is not star-shaped, (7.3) are valid in any star-shaped neighborhood of infinity in the flow.

Proof. If $\boldsymbol{\varphi}$ is defined by (7.1), $\boldsymbol{\Omega}$ by (6.11), then $\Delta(\boldsymbol{\varphi} + r^2 \boldsymbol{\Omega}) = 0$. Hence $\boldsymbol{\varphi} = -(r^2 \boldsymbol{\Omega} + \boldsymbol{H})$ for a harmonic vector $\boldsymbol{H}$. Using (7.2),

$$\boldsymbol{q} = \boldsymbol{q}_0 + \operatorname{curl} r^2 \boldsymbol{\Omega} + \operatorname{curl} \boldsymbol{H} + \operatorname{grad} h.$$

From (6.9) and (7.1) we find $r^2 \boldsymbol{\Omega} = O(1)$, $\boldsymbol{\varphi} = O(\log r)$. Hence $\boldsymbol{H} = O(\log r)$ and, using lemma 2, we may assume $\boldsymbol{H}$ is harmonic at infinity. This yields $(7.3)_1$.

For the classical Stokes flow past a unit sphere with center Q we obtain for $\boldsymbol{q}(Q + \boldsymbol{e} r)$

$$\boldsymbol{q} = \boldsymbol{q}_0 - \frac{3}{4} \operatorname{curl}\left(r^2 \operatorname{curl} \frac{\boldsymbol{q}_0}{r}\right) - \frac{1}{4} \operatorname{curl}\left(\operatorname{curl} \frac{\boldsymbol{q}_0}{r}\right),$$

$$\boldsymbol{q} = \boldsymbol{q}_0 - \frac{3}{4} \operatorname{curl}\left(r^2 \operatorname{curl} \frac{\boldsymbol{q}_0}{r}\right) - \frac{1}{4} \operatorname{grad}\left(\frac{\boldsymbol{e} \cdot \boldsymbol{q}_0}{r^2}\right).$$

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