The Pi Theorem of Dimensional Analysis

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1. Introduction

The fundamental principle in dimensional analysis is known as the Pi Theorem. While the ideas involved were used by earlier authors, BUCKINGHAM [1] stated the theorem essentially as follows:

If an equation in n arguments is dimensionally homogeneous with respect to m fundamental units, it can be expressed as a relation between $n-m$ independent dimensionless arguments.

It is well known that this useful practical rule is not strictly correct [2]. We shall give a simple, *constructive* proof of the theorem in its rigorous form. This makes no use of partial differentiation [3] or abstract spaces [4, 5] and is based upon a simple idea already used in dealing with homogeneous functions. Although dimensional quantities may be regarded as elements of a vector space defined by a set of postulates [6], the Pi Theorem needs no elaborate logical setting. The proof given in § 4 depends only on matrix algebra. However we shall sketch two other recent treatments, due to BIRKHOFF [7] and DROBOT [8] respectively, at the end of this paper.

In applying the Pi Theorem to actual problems, considerations beyond the domain of pure mathematics may enter. Thus certain "paradoxes" have arisen which have occasioned much discussion [9]. These paradoxes are not due to any failure of the Pi Theorem; for this is a straightforward mathematical proposition of universal validity. The successful application of dimensional analysis nearly always depends on a real understanding of the essential variables involved in the problem. But for every physical assumption, the Pi Theorem gives a corresponding answer; and experiment alone can decide which of several answers most nearly matches the facts. We proceed therefore, without physics or metaphysics [9], to lay down a few definitions essential for a precise statement of the Pi Theorem.

2. Dimensional Matrix

We consider problems in which the physical quantities X_i involved have positive measures x_i which depend upon a system of m fundamental units $U_1, U_2, \ldots, U_m$. When these units are changed to

$$U_j' = U_j/t_j \qquad (t_j > 0), \tag{1}$$

the positive variables x_i also change. If the new value x_i' is related to the old by the equation

$$x_i' = t_1^{a_{i1}} t_2^{a_{i2}} \dots t_m^{a_{im}} x_i, \tag{2}$$

we say that x_i (strictly X_i) has the dimensions

$$(a_{i1}, a_{i2}, \dots, a_{im})$$

in the units $U_1, U_2, \dots, U_m$. If all $a_{ij} = 0$, x_i is said to be *dimensionless*. In any case the dimensions a_{ij} are real numbers.

The dimensions of n quantities X_i may be arranged in a rectangular $n \times m$ matrix:

	U_1	U_2	$\dots$	U_m
x_1	a_{11}	a_{12}	$\dots$	a_{1m}
x_2	a_{21}	a_{22}	$\dots$	a_{2m}
$\vdots$	$\vdots$	$\vdots$		$\vdots$
x_n	a_{n1}	a_{n2}	$\dots$	a_{nm}

We denote this *dimensional matrix* (a_{ij}) by A.

In mechanics the fundamental units are of length L, time T, and mass M, and $m = 3$. For example in dealing with the speed of sound v in a gas, the additional variables involved are the pressure p, density ϱ, and viscosity μ of the gas; and the dimensional matrix has the form:

	L	T	M
v	1	-1	0
p	-1	-2	1
ϱ	-3	0	1
μ	-1	-1	1

Lemma. *If $x_1, x_2, \dots, x_n$ have the dimensional matrix A and the products*

$$y_i = x_1^{b_{i1}} x_2^{b_{i2}} \dots x_n^{b_{in}}, \qquad (i = 1, 2, \dots, p) \tag{3}$$

have the exponential matrix $(b_{ij}) = B$, then $y_1, y_2, \dots, y_p$ have the dimensional matrix BA.

Proof. From (2) we have

$$y_i' = x_1'^{b_{i1}} x_2'^{b_{i2}} \dots x_n'^{b_{in}} = t_1^{c_{i1}} t_2^{c_{i2}} \dots t_m^{c_{im}} y_i$$

where $c_{ij} = b_{i1} a_{1j} + \dots + b_{in} a_{nj}$. Thus the dimensional matrix (c_{ij}) of the y's is $C = BA$.

A matrix is said to be of *rank r* if it contains at least one non-zero determinant of order r, while all determinants of higher order which the matrix may contain are zero. If the matrix contains *no* determinants of order $r+1$, it will be of rank r if it contains one non-zero r-rowed determinant. Thus if $m < n$, the $n \times m$ dimensional matrix A is of rank m when it contains a single non-zero m-rowed determinant. This is the usual situation in dimensional analysis.

3. Isobaric Functions

Let the measure x of the physical quantity be expressed as a function of $x_1, x_2, \ldots, x_n$:

$$x = f(x_1, x_2, \ldots, x_n).$$

If this equation holds for all changes of units, it becomes

$$x' = f(x_1', x_2', \ldots, x_n')$$

when $U_i' = U_i/t_i$. Hence if x has the dimensions $(a_1, a_2, \ldots, a_m)$, we have from (2)

$$t_1^{a_1} t_2^{a_2} \ldots t_m^{a_m} x = f(t_1^{a_{11}} t_2^{a_{12}} \ldots t_m^{a_{1m}} x_1, \ldots).$$

Thus under the given unit change the function satisfies the identity in $t_1, t_2, \ldots, t_m$:

$$f(t_1^{a_{11}} t_2^{a_{12}} \ldots t_m^{a_{1m}} x_1,\ t_1^{a_{21}} t_2^{a_{22}} \ldots t_m^{a_{2m}} x_2, \ldots) = t_1^{a_1} t_2^{a_2} \ldots t_m^{a_m} f(x_1, x_2, \ldots). \tag{4}$$

Such a function is said to be *isobaric* with the dimensions $(a_1, a_2, \ldots, a_m)$. We use the term "isobaric" (Greek *isos*, equal; *barus*, heavy) in preference to the more cumbersome phrase "dimensionally homogeneous".

We note that isobaric functions relative to m fundamental units $U_1, \ldots, U_m$ are also isobaric with respect to any subset of these units, say $U_1, U_2, \ldots, U_k$ $(k < m)$; for in the identity (4) we need only put $t_{k+1} = t_{k+2} = \cdots = t_m = 1$. Thus the isobaric functions relative to $U_1, U_2, \ldots, U_k$ form a larger class than the functions isobaric relative to $U_1, U_2, \ldots, U_m$ $(m > k)$. In brief, *the fewer the fundamental units, the larger the class of corresponding isobaric functions.* We shall later have occasion to use this fact in explaining an alleged paradox.

If the sum

$$f = c_1 x_1 + c_2 x_2 + \cdots + c_n x_n \tag{5}$$

with dimensionless coefficients is isobaric, each of its terms must have the same dimensions. For if we apply (4) and put $t_j = t$ and all other t's equal to 1, we get

$$t^{a_{1j}} c_1 x_1 + t^{a_{2j}} c_2 x_2 + \cdots + t^{a_{nj}} c_n x_n = t^{a_j} f$$

or

$$t^{(a_{1j} - a_j)} c_1 x_1 + t^{(a_{2j} - a_j)} c_2 x_2 + \cdots + t^{(a_{nj} - a_j)} c_n x_n = f.$$

Since the right-hand side of this identity in t is independent of t, the same must be true of the left-hand side; that is

$$a_{1j} = a_{2j} = \cdots = a_{nj} = a_j \qquad (j = 1, 2, \ldots, m).$$

All rows of the dimensional matrix are now identical.

The product

$$f = x_1^{k_1} x_2^{k_2} \ldots x_n^{k_n} \tag{6}$$

is an isobaric function whose dimension a_j in any unit U_j is obtained by multiplying the dimensions of its factors by their exponents and adding the results:

$$k_1 a_{1j} + k_2 a_{2j} + \cdots + k_n a_{nj} = a_j. \tag{7}$$

For if we change only U_j, say $U_j' = U_j/t$, we have

$$(t^{a_{1j}} x_1)^{k_1} (t^{a_{2j}} x_2)^{k_2} \cdots = t^{k_1 a_{1j} + k_2 a_{2j} + \cdots} f.$$

Since this is true for all $t > 0$, we see that f is isobaric with the dimension a_j given by (7) with respect to the unit in question.

4. The Pi Theorem

Let the function f in an equation

$$f(x_1, x_2, \ldots, x_n) = 0 \tag{8}$$

with n arguments be isobaric with respect to m fundamental units $U_1, U_2, \ldots, U_m$. Then if the $n \times m$ dimensional matrix of $x_1, \ldots, x_n$ is of rank $r = n - k$, the given equation is equivalent to

$$f(1, 1, \ldots, 1, \pi_1, \pi_2, \ldots, \pi_k) = 0 \tag{9}$$

in which the first r arguments are 1, and the π's are $n - r$ independent and dimensionless products formed from $x_1, \ldots, x_n$.

Proof. Let us first consider a special case. In ordinary analysis a function is said to be *homogeneous of degree a*, if for any $t > 0$,

$$f(t x_1, t x_2, \ldots, t x_n) = t^a f(x_1, x_2, \ldots, x_n) \tag{10}$$

is an identity in t. From our present point of view f is an isobaric function of dimension a with respect to a single unit U_1; and its arguments $x_1, x_2, \ldots, x_n$ have the dimensional matrix consisting of a column of n ones. Since (10) is an identity in t it will hold when $t = 1/x_1$; then we have

$$f\left(1, \frac{x_2}{x_1}, \ldots, \frac{x_n}{x_1}\right) = x_1^{-a} f(x_1, x_2, \ldots, x_n).$$

Hence the equation $f(x_1, x_2, \ldots, x_n) = 0$ is equivalent to the equation

$$f\left(1, \frac{x_2}{x_1}, \ldots, \frac{x_n}{x_1}\right) = 0$$

in the $n - 1$ dimensionless products $\pi_1 = x_2/x_1, \ldots, \pi_{n-1} = x_n/x_1$. Since the dimensional matrix is of rank 1, we have proved the Pi Theorem for this case. The proof in the general case consists in a natural extension of this reasoning.

By suitably numbering the arguments x_i and the units U_j we can bring a non-singular $r \times r$ matrix P into the upper left corner of the dimensional matrix $A = (a_{ij})$. Then A may be written as a matrix of matrices

$$A = \begin{pmatrix} P & R \\ Q & S \end{pmatrix}, \quad \text{where } \det P \neq 0. \tag{11}$$

If $n = r + k$, the sub-matrices P, Q, R, S are respectively $r \times r$, $k \times r$, $r \times k$, $k \times k$.

We first show that the sub-matrix

$$S = Q P^{-1} R. \tag{12}$$

Since A is of rank r, the last k rows of A, namely (Q, S), are linear combinations of the first r rows (P, R). Thus there exists a $k \times r$ matrix C such that $Q = CP$, $S = CR$; hence $C = Q P^{-1}$ and $S = Q P^{-1} R$.

After this preliminary, the proof in the general case is based on the same idea as that used in the preceding special case. In the defining equation (4) of an isobaric function we set the first r arguments equal to 1; if we put $t_{r+1} = \cdots = t_m = 1$ in these equations we can then determine $t_1, \ldots, t_r$ uniquely since det $P \neq 0$. These t-values then define a change of units which converts $f(x_1, x_2, \ldots, x_r, x_{r+1}, \ldots, x_n)$ into a multiple of $f(1, 1, \ldots, 1, \pi_1, \ldots, \pi_k)$. However we present the argument in a modified form which is easier to follow.

If the $r \times r$ matrix $P^{-1} = (b_{ij})$, consider the r quantities

$$(13) \qquad y_i = x_1^{b_{i1}} x_2^{b_{i2}} \ldots x_r^{b_{ir}}, \qquad i = 1, 2, \ldots, r.$$

Since $x_1, x_2, \ldots, x_r$ have the $r \times m$ dimensional matrix (P, R), the lemma shows that the quantities y_i have the $r \times m$ dimensional matrix

$$P^{-1}(P, R) = (I_r, P^{-1}R),$$

where I_r is the unit $r \times r$ matrix. Now take

$$t_1 = \frac{1}{y_1}, \quad t_2 = \frac{1}{y_2}, \ldots, \quad t_r = \frac{1}{y_r}, \quad t_{r+1} = \cdots = t_m = 1$$

in the defining equation (4) of an isobaric function. Equation (8) is then equivalent to

$$(14) \qquad f\left(\frac{x_1}{z_1}, \frac{x_2}{z_2}, \ldots, \frac{x_r}{z_r}, \frac{x_{r+1}}{z_{r+1}}, \ldots, \frac{x_n}{z_n}\right) = 0$$

where the n quantities

$$(15) \qquad z_i = y_1^{a_{i1}} y_2^{a_{i2}} \ldots y_r^{a_{ir}}, \qquad i = 1, 2, \ldots, n.$$

Since their $n \times r$ exponential matrix is $\begin{pmatrix} P \\ Q \end{pmatrix}$, the lemma shows that the z's have the dimensional matrix

$$\begin{pmatrix} P \\ Q \end{pmatrix} (I_r, P^{-1}R) = \begin{pmatrix} P & R \\ Q & QP^{-1}R \end{pmatrix} = \begin{pmatrix} P & R \\ Q & S \end{pmatrix}$$

in view of (12). *Thus $z_1, \ldots, z_n$ have the same dimensional matrix as $x_1, \ldots, x_n$; consequently all the arguments x_i/z_i in (14) are dimensionless.*

Finally, if we substitute y_i from (13) into (15) we have

$$(16) \qquad z_i = x_1^{c_{i1}} x_2^{c_{i2}} \ldots x_r^{c_{ir}}, \qquad i = 1, 2, \ldots, n.$$

To find the $n \times r$ exponential matrix $\begin{pmatrix} C_r \\ C_k \end{pmatrix}$ we have from the lemma

$$\begin{pmatrix} C_r \\ C_k \end{pmatrix} (P, R) = \begin{pmatrix} P & R \\ Q & QP^{-1}R \end{pmatrix};$$

hence

$$C_r P = P, \quad C_r = I_r \quad \text{and} \quad C_k P = Q, \quad C_k = QP^{-1}.$$

The z's in (16) therefore have the exponential matrix

$$(17) \qquad C = \begin{pmatrix} I_r \\ QP^{-1} \end{pmatrix};$$

that is, $z_i = x_i$ $(i = 1, 2, \ldots, r)$ whereas $z_{r+1}, \ldots, z_n$ have the exponential matrix QP^{-1}. Equation (14) now becomes

$$(18) \qquad f\left(1, 1, \ldots, 1, \frac{x_{r+1}}{z_{r+1}}, \ldots, \frac{x_n}{z_n}\right) = 0,$$

an equation with $n-r$ dimensionless arguments, which are obviously independent since each contains an x_i (in the numerator) that none of the others contain. On writing

$$\frac{x_{r+i}}{z_{r+i}} = \pi_i, \quad i = 1, 2, \ldots, k, \tag{19}$$

in (18) we obtain the equation (9) required by the theorem. These dimensionless arguments are readily found since $z_{r+1}, \ldots, z_{r+k}$ are given by (16) with the exponential matrix QP^{-1}.

The above proof now enables us to state the Pi Theorem in a still more specific form.

Let $x_1, x_2, \ldots, x_n$ have the $n \times m$ dimensional matrix of rank $r = n-k$:

$$A = \begin{pmatrix} P & R \\ Q & QP^{-1}R \end{pmatrix}, \tag{20}$$

where P is a non-singular $r \times r$ matrix. Then if $f(x_1, x_2, \ldots, x_n)$ is an isobaric function with respect to m fundamental units, the equation (8) *is equivalent to equation* (9) *in which*

$$\pi_i = x_1^{e_{i1}} x_2^{e_{i2}} \ldots x_n^{e_{in}}, \qquad i = 1, 2, \ldots, k,$$

are $k = n-r$ independent and dimensionless quantities with the $k \times n$ exponential matrix

$$E = (-QP^{-1}, I_k) \tag{21}$$

where I_k is the $k \times k$ unit matrix.

In brief, the Pi Theorem states that

$$EA = 0. \tag{22}$$

As an example, consider the 6×4 dimensional matrix of rank 3:

A:

	U_1	U_2	U_3	U_4
x_1	0	-1	0	-1
x_2	0	-4	3	2
x_3	-2	0	1	0
x_4	1	3	-1	2
x_5	1	2	-2	-1
x_6	-1	1	2	4

To verify that A is rank 3 we note that the vectors $[x_4]$, $[x_5]$, $[x_6]$ in the last three rows are linear combinations of $[x_1]$, $[x_2]$, $[x_3]$; for example

$$[x_4] = -\tfrac{7}{3}[x_1] - \tfrac{1}{6}[x_2] - \tfrac{1}{2}[x_3].$$

We may take the 3×3 matrix in the upper left corner as P since $\det P = 6$. Then Q is the matrix in the lower left corner and

$$QP^{-1} = \begin{pmatrix} 1 & 3 & -1 \\ 1 & 2 & -2 \\ -1 & 1 & 2 \end{pmatrix} \begin{pmatrix} -\frac{2}{3} & \frac{1}{6} & -\frac{1}{2} \\ -1 & 0 & 0 \\ -\frac{4}{3} & \frac{1}{3} & 0 \end{pmatrix} = \begin{pmatrix} -\frac{7}{3} & -\frac{1}{6} & -\frac{1}{2} \\ 0 & -\frac{1}{2} & -\frac{1}{2} \\ -3 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}.$$

The dimensionless products have the exponential matrix

$$E = (-QP^{-1}, I_3) = \begin{pmatrix} \frac{7}{3} & \frac{1}{6} & \frac{1}{2} & 1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 & 1 & 0 \\ 3 & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 & 1 \end{pmatrix};$$

hence

$$\pi_1 = x_1^{\frac{7}{3}} x_2^{\frac{1}{6}} x_3^{\frac{1}{2}} x_4, \qquad \pi_2 = x_2^{\frac{1}{2}} x_3^{\frac{1}{2}} x_5, \qquad \pi_3 = x_1^3 x_2^{-\frac{1}{2}} x_3^{-\frac{1}{2}} x_6.$$

If we wish to avoid fractional exponents, $\pi_1^6, \pi_2^2, \pi_3^2$ may be taken as the dimensionless products.

We may readily verify that $EA = 0$, thus checking the calculation. The fact that $S = QP^{-1}R$, also shows that A is of rank 3.

5. Fundamental Units

In a certain area of knowledge let the quantities X involved belong to a certain class $\mathscr{C}$ which includes the real numbers. The quantities $U_1, U_2, \ldots, U_m$ is said to form a set of *fundamental units* in the class $\mathscr{C}$ if every $X \in \mathscr{C}$ may be expressed uniquely in the form

(23) $$X = x\, U_1^{a_1} U_2^{a_2} \ldots U_m^{a_m}, \qquad x > 0;$$

here x is the measure of X and $a_1, a_2, \ldots, a_m$ are real numbers which give the *dimensions* of X in the units $U_1, U_2, \ldots, U_m$ respectively. The dimensions of X form a vector which in MAXWELL's notation is denoted by

(24) $$[X] = (a_1, a_2, \ldots, a_m).$$

It is customary, however, to write x for X in (24); then $[x]$ denotes the dimensions of the physical variable X whose positive measure is x. We shall adopt this convention.

When the units are fundamental $X = x$, a pure number, when and only when X is dimensionless, that is, when $[X] = 0$, the zero vector. Thus when the units are fundamental,

(25) $$[X] = 0 \quad \text{implies} \quad X = x, \ \text{a positive number.}$$

If $U_1, U_2, \ldots, U_m$ form a system of fundamental units, the system

$$V_i = U_1^{b_{i1}} U_2^{b_{i2}} \ldots U_m^{b_{im}}, \qquad i = 1, 2, \ldots, m,$$

is also fundamental provided $\det b_{ij} \neq 0$. For the system of linear, homogeneous equations which state that

$$X = x\, V_1^{\alpha_1} V_2^{\alpha_2} \ldots V_m^{\alpha_m}$$

is dimensionless in the units U_i has the solution $\alpha_1 = \alpha_2 = \cdots = \alpha_m = 0$ when and only when $\det b_{ij} \neq 0$; then $X = x$, a positive number.

In mechanics units of length, time and mass (L, T, M) are usually chosen as fundamental. But engineers often prefer to regard units of length, time and force (L, T, F) as fundamental. This is permissible since $F = L T^{-2} M$ and the determinant

$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & -2 & 1 \end{vmatrix} \neq 0.$$

To illustrate the importance of the choice of fundamental units let us consider the following problem [11] which leads to an alleged paradox in dimensional analysis.

A ball of diameter d is fixed in a stream of liquid and kept at a temperature ϑ above that of the liquid at a great distance from the ball. If the velocity of the stream is v and the liquid has the heat capacity c (per unit volume) and thermal conductivity k, find the rate r at which heat is transferred from the ball to the liquid.

Solution 1. Units of length, time, mass and temperature (L, T, M, H) taken as fundamental.

The dimensions of calories are those of energy, $L^2T^{-2}M$. Since c is given in calories per degree per cubic centimeter,

$$[c] = (L^2T^{-2}M)/HL^3 = L^{-1}T^{-2}MH^{-1}.$$

The thermal conductivity k is given as calories per second per square centimeter per unit of temperature gradient; hence

$$[k] = \frac{L^2T^{-2}M}{TL^2(HL^{-1})} = LT^{-3}MH^{-1}.$$

The rate r of heat transfer is given in calories per second; hence

$$[r] = L^2T^{-2}MT^{-1}.$$

Thus we have the dimensional matrix:

	L	T	M	H	
d	1	0	0	0	P
v	1	-1	0	0	P
k	1	-3	1	-1	P
ϑ	0	0	0	1	P
c	-1	-2	1	-1	Q
r	2	-2	1	-1	Q

This matrix has the rank 4; for the 4×4 matrix P has $\det P = -1$. Hence there are $6-4=2$ dimensionless products. Since

$$P^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 2 & -3 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}^{\star}, \qquad QP^{-1} = \begin{pmatrix} -1, & -1, & 1, & 0 \\ 1, & 0, & 1, & 1 \end{pmatrix},$$

$$E = (-QP^{-1}, I_2) = \begin{pmatrix} 1, & 1, & -1, & 0, & 1, & 0 \\ -1, & 0, & -1, & -1, & 0, & 1 \end{pmatrix},$$

$\star$ To find P^{-1} partition P into four 2×2 matrices $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$; then since $B=0$ we have $P^{-1} = \begin{pmatrix} A^{-1} & 0 \\ -D^{-1}CA^{-1} & D^{-1} \end{pmatrix}$.

they are

$$\pi_1 = d\, v\, k^{-1} c, \qquad \pi_2 = d^{-1} k^{-1} \vartheta^{-1} r.$$

Thus a relation between the six variables may be put in the form $\pi_2 = f(\pi_1)$, or

(i) $$r = f(\pi_1)\, d\, k\, \vartheta.$$

Solution 2. Units of length, time and mass (L, T, M) taken as fundamental.

If the temperature of a body is regarded as a measure of the kinetic energy of its molecules, we have $H = L^2 T^{-2} M$. Our dimensional matrix is now

	L	T	M	
d	1	0	0	P
k	-1	-1	0	
ϑ	2	-2	1	
c	-3	0	0	Q.
v	1	-1	0	
r	2	-3	1	

The rank is 3 since the 3×3 matrix P has det $P = -1$. The Pi Theorem now gives $6-3=3$ dimensionless products. Since

$$P^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & -1 & 0 \\ -4 & -2 & 1 \end{pmatrix}, \qquad Q P^{-1} = \begin{pmatrix} -3 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix},$$

$$E = (-QP^{-1}, I_3) = \begin{pmatrix} 3 & 0 & 0 & 1 & 0 & 0 \\ -2 & -1 & 0 & 0 & 1 & 0 \\ -1 & -1 & -1 & 0 & 0 & 1 \end{pmatrix},$$

they are

$$\pi_1 = d^3 c, \qquad \pi_2 = d^{-2} k^{-1} v, \qquad \pi_3 = d^{-1} k^{-1} \vartheta^{-1} r.$$

A relation between the six variables may be put in the form $\pi_3 = g(\pi_1, \pi_2)$ or

(ii) $$r = g(\pi_1, \pi_2)\, d\, k\, \vartheta.$$

Since functions of two variables form a much wider class than functions of one variable, the added knowledge that $H = L^2 T^{-2} M$ given by the molecular theory of heat seems, as Lord Rayleigh said, to "put us in a worse position than before" [*12*] when the desired relation was given by (i). The explanation of this apparent paradox is given in § 3; for solution 2 with three units admits a larger class of isobaric functions than solution 1 with four units.

The paradox may be explained also as follows [*13*]. The assumption that L, T, M, H are fundamental units implies that

$$X = x\, L^a T^b M^c H^d$$

is dimensionless when and only when

(iii) $$a = b = c = d = 0.$$

In solution 2 the assumption that L, T, M are fundamental units implies that

$$X = x L^a T^b M^c (L^2 T^{-2} M)^d$$

is dimensionless when and only when

(iv) $$a = -2d, \quad b = 2d, \quad c = -d.$$

Now conditions (iv) are included in (iii) when $d=0$, and are therefore less restrictive than conditions (iii). Thus the Pi Theorem gives formula (i) under one theory, formula (ii) under another; only experiment can decide which is correct.

6. Recent Developments

In 1914 BUCKINGHAM [1] proved the Pi Theorem for functions capable of being expanded in Maclaurin series. This restriction is not imposed in BRIDGMAN's proof, but the functions are assumed to be differentiable and the proof depends on the solution of a linear, partial differential equation of the first order [3]. The problem, however, is strictly algebraic and differentiability is not a relevant requirement. Purely algebraic proofs have been given by LANGHAAR [4] and BIRKHOFF [7]; but neither author gives the theorem in the specific form stated in § 4, where the precise function of the dimensionless arguments is given as well as their exponential matrix. BIRKHOFF's argument is difficult to follow in view of its extreme concision and a variety of misprints; moreover his definition of "unit-free" functions f as those for which "the locus defined by $f=0$ is invariant under all transformations [of units]" is not well adapted to a sharp proof.

S. DROBOT in his recent paper *On the Foundations of Dimensional Analysis* [5] aims "to construct the Dimensional Analysis by means of quite simple algebraic methods, namely using the theory of linear space". In part II of his paper the usual postulates for a linear (or vector) space Σ over the field of real numbers are given. The operations involved are the addition of elements of Σ and their multiplication by numbers. He then proves two theorems on the form of functions whose arguments, as well as the function itself, are elements of Σ. In part III this entire theory is carried over bodily to a "multiplicative form of linear space" Π whose postulates are precise analogues of those for Σ. The elements of Π are called "dimensional quantities", and the positive numbers are regarded as a subclass of Π. The operations involved are the multiplication of elements in Π and the raising them to real powers. If $A_1, A_2, \ldots, A_m$ are "dimensionally independent" elements (*i.e.* a system of units) any element P_j of Π can be uniquely represented in the form

(i) $$P_j = \pi_j A_1^{p_{j1}} \ldots A_m^{p_{jm}}$$

where $\pi_i, p_{j1}, \ldots, p_{jm}$ are real numbers and π_j is positive. The two theorems of part II are now combined and stated in the following multiplicative form appropriate to an n-dimensional space Π:

Let $\Phi(A_1, \ldots, A_m; P_1, \ldots, P_r)$ be a dimensionally invariant and homogeneous function. If $A_1, A_2, \ldots, A_m$ $(m \leq n)$ are dimensionally independent and the P_j have the form (i), *then*

(ii) $$\Phi(A_1, \ldots, A_m;\ P_1, \ldots, P_r) = \varphi A_1^{f_1} \ldots A_m^{f_m},$$

where the positive coefficient φ *does not depend on the* A's, *and the real exponents* f_j *depend neither on the* A's *nor on* $\pi_1, \ldots, \pi_r$.

This theorem, which the author regards as basic, gives the form of a function element of Π. It states what φ does *not* depend upon; and later the author casually remarks that φ *does* depend upon the positive coefficients $\pi_1, \ldots, \pi_r$ and the exponents p_{jk} of the elements $P_1, \ldots, P_r$ that appear in Φ. But since the nature of this dependence on p_{jk} is never stated, this theorem is no substitute for the Pi Theorem. Indeed in the examples cited, the coefficient φ is a function of the r dimensionless quantities $\pi_1, \pi_2, \ldots, \pi_r$ which appear in the P's. Since there are $m+r$ arguments in Φ and m units (the A's), this suggests the Pi Theorem in the form first given by BUCKINGHAM in which the rank of the dimensional matrix is not considered.

DROBOT concludes his paper with a number of illuminating examples. One of these disposes very clearly of Lord RAYLEIGH's paradox; and others deal with the theory of quality control by sampling and particular solutions of partial differential equations.

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