On a Boundary Layer Problem for the Nonlinear Boltzmann Equation

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Abstract

This article deals with a boundary-layer problem arising in the kinetic theory of gases when the mean free path of molecules tends to zero. The model considered here is the stationary, nonlinear Boltzmann equation in one dimension with a slightly perturbed reflection boundary condition. We restrict our attention to the case of hard spheres collisions, with GRAD's cutoff assumption. Existence, uniqueness and asymptotic behavior are derived by means of energy estimates.

1. Introduction

This work is devoted to the boundary-layer problem for the Boltzmann equation

$$\xi_1 \, \partial_x f = Q(f, f) + S_1 \qquad (x > 0, \xi \in \mathbb{R}^3) \tag{1.1}$$

where S_1 is a small source. Such a problem arises when the boundary layer for the full Boltzmann equation as the mean free path tends to zero is considered; *cf.* GRAD [10]. The distribution is supposed to vary rapidly in the direction perpendicular to the boundary; therefore, spatial dependence is supposed one-dimensional while the three-dimensional velocity is retained.

We seek a solution of (1.1) satisfying a specular reflection condition at the boundary $x = 0$:

$$f(0, \xi) = f(0, R\xi) + h_1(R\xi), \qquad \xi_1 > 0 \tag{1.2}$$

with

$$\xi = (\xi_1, \xi_2, \xi_3) \qquad \text{and} \qquad R\xi = (-\xi_1, \xi_2, \xi_3)$$

and the condition at infinity

$$\lim_{x \to +\infty} f(x, \xi) = M(\xi) = (2\pi)^{-3/2} \exp\left(-|\xi|^2/2\right). \tag{1.3}$$

The collision operator $Q(f,f)$ is defined by (*cf.* CERCIGNANI [5], GRAD [11], and TRUESDELL & MUNCASTER [14]):

$$Q(f,f) = \int_{\mathbb{R}^3_\zeta \times S^2_\omega} [f(\zeta')f(\xi') - f(\zeta)f(\xi)]\, q(\zeta - \xi, \omega)\, d\zeta\, d\omega \tag{1.4}$$

with

$$\begin{aligned} \zeta' &= \zeta - ((\zeta - \xi)\cdot\omega)\,\omega, \\ \xi' &= \xi + ((\zeta - \xi)\cdot\omega)\,\omega. \end{aligned} \tag{1.5}$$

We consider collision kernels for hard sphere gas satisfying the angular cutoff assumption as proposed in GRAD [10, 11]:

$$q(\zeta - \xi, \omega) = \sigma(\zeta - \xi)\cdot\omega \quad (\sigma \text{ constant} > 0). \tag{1.6}$$

In view of condition (1.3), we linearize Q around the Maxwellian distribution $M(\xi)$ and take

$$f = M + M^{1/2} u$$

so that (1.1) becomes

$$\begin{aligned} &\xi_1\, \partial_x u + Lu = \nu\Gamma(u,u) + S && \text{(a)} \\ &u(0,\xi) = u(0, R\xi) + h(R\xi), \quad \xi_1 \geqq 0 && \text{(b)} \\ &\lim_{x\to\infty} u(x,\xi) = 0 && \text{(c)} \end{aligned} \tag{1.7}$$

with

$$\begin{aligned} Lu &= -2M^{1/2}\, Q(M, M^{1/2} u), \\ \nu\Gamma(u,u) &= M^{-1/2}\, Q(M^{1/2} u, M^{1/2} u). \end{aligned} \tag{1.8}$$

The main idea in proving the well-posedness of (1.7) is to consider the linear associated problem (by discarding $\nu\Gamma(u,u)$ in (1.7a)) and to use the results of BARDOS, CAFLISCH, & NICOLAENKO [2] and CERCIGNANI [6]. Notice that the linear problem associated to (1.7a–b) admits a four-dimensional affine space of solutions since an element of the kernel of L having a zero mass flux in the x-direction is a solution of the homogeneous problem. This allows us to choose the solution of (1.7) that vanishes at infinity. Then $\nu\Gamma(u,u)$ also vanishes at infinity and can be considered as a small perturbation. The solution of (1.7) is obtained by means of the Banach fixed point theorem in the space

$$E = \left\{u, \int e^{2\gamma x}\, dx \int (1 + |\xi|) f(x,\xi)^2\, d\xi + \int \sup_x |e^{\gamma x} f(x,\xi)|^2\, d\xi < \infty\right\}$$

(for small enough γ), which seems well adapted to our problem. We thus prove new estimates which provide the main difference from the work in [2] and [6]. Let us point out that this paper does not answer the question of the existence of a solution for the problem with a given incoming distribution at the wall.

Linear boundary-layer problems have motivated a large literature, see ARTHUR & CERCIGNANI [1] for the BGK model, MASLOVA [12], GOLSE & POUPAUD [9], the latter giving a precise description of the rate of convergence for hard and soft potentials. In [4], CAFLISCH has studied a weakly nonlinear problem with a Dirac mass at infinity. In [13], VAN DER MEE discusses representations of these boundary layer equations as integral equations. In [15], UKAI & ASANO give theorems of existence, uniqueness and stability for the full nonlinear Boltzmann equation for a three-dimensional flow past an obstacle with specular or reverse reflection conditions at the boundary and a (non absolute) Maxwellian distribution at infinity. The main result of the present work is the following:

Theorem. *Consider the half-space problem*

$$\begin{gathered} \xi_1 \partial_x f = Q(f,f) + S_1, \quad x > 0, \\ f(0,\xi) = f(0, R\xi) + h_1(R\xi), \quad \xi_1 > 0, \\ f(x,\xi) \to M(\xi) \quad \text{as } x \to +\infty \end{gathered} \tag{1.9}$$

where $M(\xi) = (2\pi)^{-3/2} e^{-|\xi|^2/2}$. *Assume that* h_1 *is such that*

$$\int \xi_1 h_1 M^{1/2}\, d\xi = \int \xi_1\xi_2 h_1 M^{1/2}\, d\xi = \int \xi_1\xi_3 h_1 M^{1/2}\, d\xi = \int \xi_1 |\xi|^2 h_1 M^{1/2}\, d\xi = 0$$

and that S_1 *satisfies orthogonal properties that will be listed below. Then, for small* h_1 *and* S_1 *(in a sense that involves in particular exponential decay of* $S_1(x,\xi)$ *as* x *goes to infinity), there is a unique solution* f *of the above problem, decaying exponentially as* x *goes to infinity.*

Remark. To simplify the subsequent technicalities, we will assume that h_1 satisfies moreover:

$$\int \xi_1^2 h_1 M^{1/2}\, d\xi = 0. \tag{1.10}$$

This additional assumption does not restrict the generality of the above statement; we refer to Remark 3.1 for further comments.

This paper is organized as follows: in Section 2, we recall the basic properties of the operators L and Γ and state the assumptions and the main results. Section 3 is devoted to the linear problem with a source term. The nonlinear problem is solved in Section 4.

2. Notations and main results

The properties of the operators L and $\nu\Gamma$ defined in (1.8) have been thoroughly analyzed in [5, 10]. In particular, the linearized collision operator L is a nonnegative self-adjoint unbounded operator on $L^2(\mathbb{R}^3_\xi)$.

Because of the cutoff assumption (1.6), L can be split as

$$L = \nu(\xi) - K \tag{2.1}$$

where $\nu(\xi)$ is the frequency of collisions satisfying the hard sphere condition

$$\nu_0(1 + |\xi|) \leqq \nu(\xi) \leqq \nu_1(1 + |\xi|) \tag{2.2}$$

and K is a compact operator on $L^2(\mathbb{R}^3_\xi)$ which can be written in the form

$$Kv(\xi) = \int k(\xi, \zeta)\, v(\zeta)\, d\zeta, \tag{2.3}$$

with [10], [3]

$$\int k(\xi, \zeta)^2\, d\zeta \leqq C(1 + |\xi|)^{-1}. \tag{2.4}$$

The domain of L is

$$D(L) = \{u \in L^2(\mathbb{R}^3_\xi),\ \nu(\xi)^{1/2}\, u \in L^2(\mathbb{R}^3_\xi)\} \tag{2.5}$$

and its nullspace $N(L)$ is spanned by $(\psi_\alpha)_{\alpha=0,\dots,4}$, where

$$\begin{aligned} \psi_0(\xi) &= M(\xi)^{1/2}, \\ \psi_i(\xi) &= \xi_i M(\xi)^{1/2}, \quad i = 1, 2, 3, \\ \psi_4(\xi) &= (|\xi|^2/3 - 1)\, M(\xi)^{1/2}. \end{aligned} \tag{2.6}$$

Any function u can be split uniquely as follows:

$$u = w_u + q_u,$$

where $q_u \in N(L)$ is called the hydrodynamic part and $w_u \in R(L) = N(L)^\perp$ the kinetic part. We have, for some $\mu > 0$:

$$\forall\, u \in D(L) \int_{\mathbb{R}^3} uLu\, d\xi \geqq \mu \int_{\mathbb{R}^3} \nu(\xi)\, w_u^2\, d\xi. \tag{2.7}$$

The nonlinear term $\nu\Gamma$ has also particular properties described in [5, 10]. We shall mainly use the orthogonality relations

$$\int \nu\Gamma(u, u)\, \psi_\alpha\, d\xi = 0, \quad \forall\, \alpha = 0, \dots, 4 \quad \text{and} \quad \forall\, u \in L^2(\mathbb{R}^3_\xi) \tag{2.8}$$

and some estimates proved in the Appendix.

Finally, we introduce the notations

$$\|f\|^2 = \int_{\mathbb{R}^3} f(\xi)^2\, d\xi \quad \text{for } f \in L^2(\mathbb{R}^3_\xi), \tag{2.9}$$

$$|||f|||^2 = \int_{\mathbb{R}^+\times\mathbb{R}^3} f(x, \xi)^2\, dx\, d\xi \quad \text{for } f \in L^2(\mathbb{R}^+ \times \mathbb{R}^3_\xi) \tag{2.10}$$

and

$$\|f\|^2_{2,\gamma} = \int_{\mathbb{R}^3} \sup_{x\geqq 0} (e^{2\gamma x}\, |f(x, \xi)|^2)\, d\xi. \tag{2.11}$$

We shall denote by $L^2(\mathbb{R}^3, L^\infty_\gamma(\mathbb{R}^+))$ (respectively $L^2(\nu\, d\xi)$) the space of functions equipped with the norm $\|\cdot\|_{2,\gamma}$ (respectively $\|\nu^{1/2}\cdot\|$).

Theorem 2.1. *Let S and h be given such that*

$$\nu S \in N(L)^\perp \quad \text{for a.e. } x,\ \xi_1 h \in N(L)^\perp, \tag{2.12}$$

$$\|S\|_{2,\gamma_0} < \infty, \quad |||e^{\gamma_0 x}\nu^{1/2} S||| < \infty \quad \text{for some } \gamma_0 > 0, \tag{2.13}$$

and

$$\|\nu^{1/2}\, h\| < \infty.$$

Then, for θ small enough, the problem

$$\begin{gathered}\xi_1\,\partial_x u + Lu = \nu\Gamma(u,u) + \theta\nu S,\\ u(0,\xi) = u(0,R\xi) + \theta h(R\xi), \qquad \xi_1 > 0,\\ \nu^{1/2}\,u \in L^2(\mathbb{R}^+\times\mathbb{R}^3)\end{gathered} \tag{2.15}$$

has a unique solution satisfying the estimate

$$\|u\|_{2,\gamma} + |||e^{\gamma x}\nu^{1/2}\,u||| \leqq C_\gamma(\|\nu^{1/2}\,h\| + \|S\|_{2,\gamma} + |||e^{\gamma x}\,\nu^{1/2}S|||) \tag{2.16}$$

for any γ such that $0 < \gamma < \inf(\nu_0, \gamma_0)$.

In this statement, the function h is given for $\xi_1 \leqq 0$, and we have assumed that it is extended by 0 for $\xi_1 > 0$. This convention will be used throughout the paper.

3. The linear problem

This section is devoted to the study of the linear problem with a source term

$$\xi_1\,\partial_x u + Lu = \nu S, \tag{3.1}$$

$$u(0,\xi) = u(0,R\xi) + h(R\xi), \qquad \xi_1 \geqq 0, \tag{3.2}$$

$$\nu^{1/2}\,u \in L^2(\mathbb{R}^+\times\mathbb{R}^3). \tag{3.3}$$

We prove the existence of a unique solution satisfying some estimates which will be used in Section 4 for the nonlinear problem.

Proposition 3.1. *Let S and h satisfy* (2.12)–(2.14). *Then the system* (3.1)–(3.3) *has a unique solution, and it satisfies the inequality*

$$|||e^{\gamma x}\,\nu^{1/2}\,u||| + \|u\|_{2,\gamma} \leqq C_\gamma(|||e^{2\gamma x}\,\nu^{1/2}\,S||| + \|S\|_{2,2\gamma} + \|\nu^{1/2}\,h\|) \tag{3.4}$$

for $0 < \gamma < \inf(\nu_0, \gamma_0)$.

Remark. This result is close to that of [2, 5, 9]. The proof is also inspired by those papers. The main new point is this estimate in $L^2(\mathbb{R}^3_\xi, L^\infty_\gamma(\mathbb{R}^+_x))$. It ensures uniqueness for the nonlinear problem. As mentioned in the introduction, one can add any linear combination of $(\psi_\alpha)_{\alpha=0,2,3,4}$ and still get a solution of (3.1)–(3.2).

Proof of Proposition 3.1. Let $\chi(x)$ be a C^2 nonnegative function defined for $x \geqq 0$ such that $\chi(0) = 1$ and $\chi(x) = 0$ for $x \geqq 1$, and let us write

$$v(x,\xi) = u(x,\xi) + \chi(x)\,h(\xi).$$

Eqs. (3.1)–(3.3) read, when restricted to a slab

$$\xi_1\,\partial_x v + Lv = \nu S' = \nu S + \xi_1 h\,\partial_x\chi + Lh\chi(x), \tag{3.6}$$

$$v(0,\xi) = v(0,R\xi); \qquad v(B,\xi) = v(B,R\xi). \tag{3.7}$$

We introduce the penalized system

$$\xi_1 \, \partial_x v^\varepsilon + L v^\varepsilon + \varepsilon v^\varepsilon = \nu S', \tag{3.8$_\varepsilon$}$$

$$v^\varepsilon(0, \xi) = v^\varepsilon(0, R\xi), \quad v^\varepsilon(B, \xi) = v^\varepsilon(B, R\xi) \quad \xi_1 \geqq 0. \tag{3.9$_\varepsilon$}$$

The proof is organized as follows: we first have uniform estimates in ε for u^ε defined for $x \in (0, B)$ and pass to the limit in ε. We then write uniform estimates in B and pass to the limit as $B \to \infty$ in a weak sense. We then prove estimates (3.4)–(3.5) and finally the uniqueness theorem.

One easily verifies that S' has the same properties as S. Classical theorems suffice to prove that the system $(3.8)_\varepsilon$–$(3.9)_\varepsilon$ has a unique solution [5] such that $\nu^{1/2} v^\varepsilon \in L^2$. To prove uniform estimates with respect to ε, one first writes

$$\int_{[0,B]\times\mathbb{R}^3} \nu(\xi) \, w_{v^\varepsilon}(x, \xi)^2 \, dx \, d\xi \leqq (1/\mu) \int_{[0,B]\times\mathbb{R}^3} \nu(\xi) \, S'(x, \xi)^2 \, dx \, d\xi. \tag{3.10}$$

To show that q_{v^ε} remains bounded in $L^2([0, B]\times\mathbb{R}^3)$, one uses a contradiction argument: suppose that $A^\varepsilon = \int_{[0,B]\times\mathbb{R}^3} q_{v^\varepsilon}(x, \xi)^2 \, dx \, d\xi$ tends to ∞ when $\varepsilon \to 0$ and write $g^\varepsilon = v^\varepsilon / A^\varepsilon$. Then g^ε satisfies

$$\xi_1 \, \partial_x g^\varepsilon = (1/A_\varepsilon) \, (-L w_{v^\varepsilon} + \nu S') - \varepsilon g^\varepsilon \tag{3.11}$$

and

$$M \xi_1 \, \partial_x g^\varepsilon \to 0 \quad \text{in } L^2([0, B]\times\mathbb{R}^3) \quad \text{when } \varepsilon \to 0. \tag{3.12}$$

Moreover, g^ε is bounded in $L^2([0, B]\times\mathbb{R}^3)$. Then use of the compactness theorem of GOLSE, PERTHAME & SENTIS [8] shows that there is a subsequence of g^ε such that q_{g^ε} (and therefore g^ε, from the fact that $w_{g^\varepsilon} \to 0$) converges strongly $L^2([0, B]\times\mathbb{R}^3)$ to a function $Q \in N(L)$ for a.e. x with Q independent of x. On the other hand, multiplying $(3.8)_\varepsilon$ by ψ_α $(\alpha = 0, 2, 3, 4)$ and integrating over x and ξ shows that

$$\int_0^B \int_\xi v^\varepsilon \psi_\alpha \, dx \, d\xi = 0, \quad \alpha = 0, 2, 3, 4, \tag{3.13}$$

and

$$\int \psi_1 v^\varepsilon(z, \xi) \, d\xi + \varepsilon \int_{[0,z]} \psi_0 v^\varepsilon(s, \xi) \, d\xi \, ds = 0. \tag{3.14}$$

Therefore $Q = 0$, which contradicts $\|g^\varepsilon\| \geqq 1$. Finally, it is easy to pass to the limit in $(3.8)_\varepsilon$ and so obtain a unique solution of

$$\xi_1 \, \partial_x v_B + L v_B = \nu S', \quad 0 < x < B, \tag{3.15}$$

$$v_B(0, R\xi) = v_B(0, \xi), \quad v_B(B, R\xi) = v_B(B, \xi), \quad \xi_1 \geqq 0, \tag{3.16}$$

$$\int_0^B \int v_B(x, \xi) \, \psi_\alpha \, d\xi \, dx = 0, \quad \alpha = 0, 2, 3, 4. \tag{3.17}$$

(*i*) *Uniform estimates with respect to B for the solution v of* (3.15)–(3.17)

One first has

$$\int_0^B \int_{\mathbb{R}^3} \nu(\xi)\, w_{v_B}(x,\xi)^2\, dx\, d\xi \leqq (1/\mu) \int_0^B \int_{\mathbb{R}^3} \nu(\xi)\, S'(x,\xi)^2\, dx\, d\xi. \tag{3.18}$$

Multiplying (3.15) by ψ_α $(2 \leqq \alpha \leqq 4)$, and integrating $\mathbb{R}^3_\xi$, one obtains

$$\int \xi_1\, \psi_\alpha v_B(x,\xi)\, d\xi = 0, \quad \forall\, x \in [0,B], \quad \alpha = 2,3,4. \tag{3.19}$$

Multiplying by ψ_0, one has

$$\int \psi_1 v_B(x,\xi)\, d\xi = 0, \quad \forall\, x \in [0,B]. \tag{3.20}$$

Consequently, q_{v_B} has the special form

$$q_{v_B}(x) = (b_0^B(x) + b_2^B(x)\,\xi_2 + b_3^B(x)\,\xi_3 + b_4^B(x)\,(|\xi|^2/3 - 1))\, M^{1/2}. \tag{3.21}$$

Combining (3.19) and (3.21) delivers

$$\int \xi_1 q_{v_B}(x,\xi)^2 = 0, \quad \forall\, x \in [0,B], \tag{3.22}$$

$$\int \xi_1 q_{v_B}(x,\xi)\, w_{v_B}(x,\xi)\, d\xi = 0, \tag{3.23}$$

$$\int \xi_1 v_B(x,\xi)^2\, d\xi = \int \xi_1 w_{v_B}(x,\xi)^2\, d\xi. \tag{3.24}$$

Now we need to prove that q_{v_B} is uniformly bounded with respect to B in some space of distributions. First multiply (3.6) by ψ_1, and integrate over $\mathbb{R}^3_\xi$:

$$\partial_x \int \xi_1 \psi_1 v_B(x,\xi)\, d\xi = 0. \tag{3.25}$$

Then, as in [6], multiply (3.15) by $L^{-1}(\xi_1\psi_\alpha)$, $2 \leqq \alpha \leqq 4$, and integrate over $\mathbb{R}^3_\xi$:

$$\partial_x \int \xi_1 L^{-1}(\xi_1\psi_\alpha)\, v_B(x,\xi)\, d\xi = \int \nu(\xi)\, S'(x,\xi)\, L^{-1}(\xi_1\psi_\alpha)\, d\xi \tag{3.26}$$

(notice that $L^{-1}(\xi^1\psi_\alpha)$ decays exponentially as $|\xi| \to \infty$, which will be of constant use in the sequel) since

$$\int L^{-1}(\xi_1\psi_\alpha)\, L w_{v_B}(x,\xi)\, d\xi = \int \xi_1 \psi_\alpha w_{v_B}(x,\xi)\, d\xi = 0$$

by (3.19) and (3.21). Introduce then the functions

$$\Phi_1^B(x) = \int \xi_1 \psi_1 v_B(x,\xi)\, d\xi, \tag{3.27}$$

$$\Phi_\alpha^B(x) = \int \xi_1 L^{-1}(\xi_1\psi_\alpha)\, v_B(x,\xi)\, d\xi, \quad \alpha = 2,3,4;$$

these Φ_α^B $(\alpha = 1, \dots, 4)$ are defined in $[0,B]$ and satisfy the relation

$$\operatorname*{Sup}_{1 \leqq \alpha \leqq 4,\, 0 < B < \infty} \| e^{\gamma x}\, \partial_x \Phi_\alpha^B \|_{L^2[0,B]} < \infty. \tag{3.28}$$

Remark 3.1. Hypothesis (1.10) has been used to obtain (3.25). If (1.10) is not satisfied, an additional term of the form $(\partial_x\chi) \int \xi_1\, h\psi_1\, d\xi$ appears in (3.25), but because χ has compact support, Φ_1^B still satisfies (3.28).

Now, we split Φ_α^B as follows:

$$\Phi_1^B(x) = \int \xi_1\psi_1\, q_{v_B}\, d\xi + \int \xi_1\psi_1\, w_{v_B}\, d\xi \tag{3.29}$$

$$\Phi_\alpha^B(x) = \int \xi_1 L^{-1}(\xi_1\psi_\alpha)\, q_{v_B}\, d\xi + \int \xi_1 L^{-1}(\xi_1\psi_\alpha)\, w_{v_B}\, d\xi, \quad \alpha = 2, 3, 4. \tag{3.30}$$

From (3.18), we know that

$$w_{v_B} \to w \quad \text{in } L^2(dx \otimes \nu^{1/2}\, d\xi) \text{ weakly.} \tag{3.31}$$

In order to carry this fact from w_{v_B} to q_{v_B}, we shall use the lemma

Lemma 3.2. *The matrix of entries*

$$m_{11} = \int \xi_1\psi_1\psi_0\, d\xi, \quad m_{1\beta} = \int \xi_1\psi_1\psi_\beta\, d\xi \quad \text{for } \beta = 2, 3, 4;$$

$$m_{\alpha 1} = \int \xi_1 L^{-1}(\xi_1\psi_\alpha)\, \psi_0\, d\xi, \quad m_{\alpha\beta} = \int \xi_1 L^{-1}(\xi_1\psi_\alpha)\, \psi_\beta\, d\xi \quad \text{for } \alpha, \beta = 2, 3, 4;$$

is invertible.

Proof. Notice that $m_{1\beta} = 0$ for $\beta = 2, 3, 4$, and that the block $(m_{\alpha\beta}; \alpha, \beta = 2, 3, 4)$ is invertible because it is the matrix of the following scalar product

$$f, g \to \int fLg\, d\xi,$$

defined on $\operatorname{span}(L^{-1}(\omega\xi_1\psi_\alpha),\ \alpha = 2, 3, 4)$.

From (3.17) and (3.28), one can extract from Φ_α^B a subsequence that converges in $L^2(dx \otimes \nu\, d\xi)$ when B goes to $+\infty$. Use of Lemma 3.2 with (3.29) and (3.30), (3.31) ensures that

$$q_{v_B} \to q \quad \text{in } L^2(dx \otimes \nu\, d\xi) \text{ weakly}.$$

Taking $v = w + q$, we obtain a weak solution of

$$\begin{aligned} &\xi_1\, \partial_x v + Lv = \nu S', \quad x > 0, \\ &v(0, \xi) = v(0, R\xi), \quad \xi_1 \geqq 0 \end{aligned} \tag{3.32}$$

satisfying

$$\int_0^\infty \int \nu(\xi)\, w_v^2(x, \xi)\, d\xi\, dx \leqq C \int_0^\infty \int \nu S'^2(x, \xi)\, d\xi\, dx, \tag{3.33}$$

$$\int \xi_1\psi_\alpha w_v\, d\xi = \int \xi_1\psi_\alpha v\, d\xi = 0, \quad \alpha = 0, 2, 3, 4, \tag{3.34}$$

$$\int \xi_1 w_v^2\, d\xi = \int \xi_1 v^2\, d\xi \geqq 0, \tag{3.35}$$

$$q_v \in L^2_{\text{loc}}(dx; L^2(\nu\, d\xi)). \tag{3.36}$$

(ii) L^2-estimates on the solution v

First we proceed as in [2] to obtain that

$$\int_0^\infty \int \nu(\xi)\, w_v(x,\xi)^2\, e^{2\gamma x}\, d\xi\, dx \leqq (C/(\nu_0-\gamma)) \int_0^\infty \int \nu(\xi)\, S'^2\, e^{2\gamma x}\, d\xi\, dx. \tag{3.37}$$

Then we multiply (3.32) by $L^{-1}(\xi_1\psi_\alpha)$, $\alpha = 2, 3, 4$, and integrate over $\mathbb{R}^3_\xi \times [x, y]$

$$\int \xi_1 L^{-1}(\xi_1\psi_\alpha)\, v(y,\xi)\, d\xi - \int \xi_1 L^{-1}(\xi_1\psi_\alpha)\, v(x,\xi)\, d\xi$$
$$= \int_x^y \int L^{-1}(\xi_1\psi_\alpha)\, \nu S'(z,\xi)\, d\xi\, dz. \tag{3.38}$$

Also, multiplying (3.32) by ψ_1, we obtain the invariance relation

$$\int \xi_1 \psi_1 v(x,\xi)\, d\xi = C_1. \tag{3.39}$$

Observe that the integral on the right-hand side of (3.38) converges (uniformly in x) when y goes to infinity following assumption (2.13). Then we can write

$$\lim_{y\to\infty} \int \xi_1 L^{-1}(\xi_1\psi_\alpha)\, v(y,\xi)\, d\xi = C_\alpha, \quad 2 \leqq \alpha \leqq 4 \tag{3.40}$$

and since we know from (3.37) that $w_v(y_n,\xi) \to 0$ in $L^2(\nu\, d\xi)$ for some sequence y_n going to infinity, (3.40) reads

$$\lim_{y_n\to\infty} \int \xi_1 L^{-1}(\xi_1\psi_\alpha)\, q_v(y_n,\xi)\, d\xi = C_\alpha, \quad 2 \leqq \alpha \leqq 4. \tag{3.41}$$

From (3.34), (3.39), (3.41)

$$q_v(y_n,\xi) \to q_\infty \quad \text{in } L^2(\nu\, d\xi)$$

when y_n goes to infinity, where q_∞ has the form (3.21). Defining $v_1 = v - q_\infty$, we obtain a solution of

$$\begin{gathered} \xi_1\, \partial_x v_1 + L v_1 = \nu S', \quad 0 < x, \\ v_1(0, R\xi) = v_1(0,\xi), \\ q_{v_1}(y_n,\xi) \to 0 \quad \text{in } L^2(\nu\, d\xi) \quad \text{when } y_n \to \infty \end{gathered} \tag{3.42}$$

that satisfies (3.33)-(3.42) with $C_\alpha = 0$ for $\alpha = 1, \dots, 4$. In what follows we shall use the notation v for v_1 and S for S'.

Proposition 3.3. *The solution v of* (3.42) *satisfies*

$$|||e^{\gamma x}\, \nu(\xi)^{1/2}\, v||| \leqq C_\gamma\, |||e^{2\gamma x}\, \nu(\xi)^{1/2}\, S||| \tag{3.43}$$

for any $0 < \gamma < \inf(\nu_0, \gamma_0)$.

Proof. We go back to (3.38) and rewrite it:

$$-e^{\gamma x} \int_{\mathbb{R}^3} \xi_1 L^{-1}(\xi_1\psi_\alpha)\, v(x,\xi)\, d\xi = -e^{-\gamma x} \int_x^\infty \int L^{-1}(\xi_1\psi_\alpha)\, \nu S e^{2\gamma y}\, e^{2\gamma(x-y)}\, d\xi\, dy;$$

thus

$$|\int \xi_1 L^{-1}(\xi_1 \psi_\alpha)\, e^{\gamma x}\, v(x, \xi)\, d\xi| \leqq Ce^{-\gamma x}\, |||e^{2\gamma y}\, \nu^{1/2}\, S|||. \tag{3.44}$$

Now (3.44) becomes

$$|\int \xi_1 L^{-1}(\xi_1 \psi_\alpha)\, e^{\gamma x}\, q_v(x, \xi)\, d\xi| \leqq Ce^{-\gamma x}\, |||e^{2\gamma y}\, \nu^{1/2}\, S||| + |\int \xi_1 L^{-1}(\xi_1 \psi_\alpha)\, e^{\gamma x}\, w_v(x, \xi)\, d\xi| \tag{3.45}$$

for $\alpha = 2, 3, 4$. In view of (3.45), (3.39) (which holds with $C_1 = 0$), (3.37) and Lemma 3.2, one obtains (3.43).

(iii) L^∞-estimates for the solution v

Proposition 3.4. *The solution v of* (4.42) *satisfies the* a priori *estimate*

$$||v||_{2,\gamma} \leqq C_\gamma\, (|||e^{\gamma x}\, \nu^{1/2}\, S||| + ||S||_{2,\gamma}) \tag{3.46}$$

for any $0 < \gamma < \inf(\nu_0, \gamma_0)$.

Proof. To obtain estimates of L^∞-type, one uses, following [6], the integral form of (3.42) and, with the notation $\lambda = \nu(\xi)/\xi_1$,

$$v(y, \xi)\, e^{\lambda y} - v(x, \xi)\, e^{\lambda x} = \int_x^y (1/\xi_1)\, (\nu S + Kv)\, (s, \xi)\, e^{\lambda s}\, ds. \tag{3.47}$$

First suppose that $\xi_1 \leqq 0$ and let x go to infinity. By use of the fact that v vanishes at infinity, we can rewrite (3.47) as follows:

$$e^{\gamma y}\, v(y, \xi) = \int_y^{+\infty} (1/|\xi_1|)\, \nu S(s, \xi)\, e^{\gamma y}\, e^{\lambda(s-y)}\, ds + \int_y^{+\infty} (1/|\xi_1|)\, Kv(s, \xi)\, e^{\gamma y}\, e^{\lambda(s-y)}\, ds. \tag{3.48}$$

On the right-hand side of (3.48) the first integral is bounded by

$$\left|\int_y^{\infty} (\nu/|\xi_1|)\, S(s, \xi)\, e^{\gamma y}\, e^{\lambda(s-y)}\, ds\right| \leqq (C\, |\lambda|/(|\lambda| + \gamma))\, \sup_x |S(x, \xi)\, e^{\gamma x}| \leqq C_\gamma \sup_x |S(x, \xi)\, e^{\gamma x}|.$$

For the second integral on the right-hand side of (3.48), one separates the cases $|\xi_1| > 1$ and $|\xi_1| < 1$. For $|\xi_1| > 1$,

$$\left|\int_y^{+\infty} (1/|\xi_1|)\, Kv(s, \xi)\, e^{\gamma y}\, e^{\lambda(s-y)}\, ds\right| \leqq (C/\nu(\xi))\, |\lambda|/|\lambda - \gamma|^{1/2}\, (\int (Kv)\, (s, \xi)^2\, e^{2\gamma s}\, ds)^{1/2} \tag{3.49}$$

$$\leqq C_\gamma (\int (Kv)\, (s, \xi)^2\, e^{2\gamma s}\, ds)^{1/2}.$$

For $|\xi_1| < 1$, one splits the integral into two parts:

$$\int_y^{+\infty} (1/|\xi_1|)\, Kv(s,\xi)\, e^{\gamma y} e^{\lambda(s-y)}\, ds = \int_{y+\varepsilon}^{\infty} + \int_y^{y+\varepsilon}. \tag{3.50}$$

Then

$$\begin{aligned} &\left|\int_{y+\varepsilon}^{\infty} (1/|\xi_1|)\, Kv(s,\xi)\, e^{\gamma y} e^{\lambda(s-y)}\, ds\right| \\ &\qquad \leqq C\,|\lambda|\, e^{(\lambda-\gamma)\varepsilon}/|\lambda-\gamma|^{1/2} \left(\int (Kv)(s,\xi)^2\, e^{2\gamma s}\, ds\right)^{1/2} \\ &\qquad \leqq (C/\varepsilon^{1/2}) \left(\int (Kv)(s,\xi)^2\, e^{2\gamma s}\, ds\right)^{1/2} \end{aligned} \tag{3.51}$$

and

$$\begin{aligned} &\left|\int_y^{y+\varepsilon} (1/|\xi_1|)\, Kv(s,\xi)\, e^{\gamma y} e^{\lambda(s-y)}\, ds\right| \\ &\qquad \leqq C \int_0^{\varepsilon} (Kv(y+s,\xi)\, e^{\gamma(s+y)})/(|\xi_1|^{\alpha}\, \nu(\xi)^{1-\alpha})\, (1/s^{1-\alpha})\, ds \\ &\qquad \leqq C\varepsilon^{\alpha} (1/|\xi_1|^{\alpha}\, \nu(\xi)^{1-\alpha}) \sup_y |Kv(y,\xi)\, e^{\gamma y}|. \end{aligned} \tag{3.52}$$

Thus

$$\begin{aligned} \int_{\xi_1 \leqq 0} \sup_y |e^{\gamma y} v(y,\xi)|^2\, d\xi &\leqq (C/\varepsilon) \int_0^{\varepsilon} \int_{\xi} Kv(s,\xi)^2\, e^{2\gamma s}\, ds \\ &\quad + C\varepsilon^{2\alpha} \int_{|\xi_1|<1} (1/|\xi_1|^{2\alpha}\, \nu^{2-2\alpha}) \sup_y e^{2\gamma y}\, |Kv(y,\xi)|^2\, d\xi \\ &\quad + C \int_{\xi} \sup_x |S(x,\xi)^2\, e^{2\gamma x}|\, d\xi. \end{aligned} \tag{3.53}$$

An upper bound for the second term appearing on the right-hand side of (3.53) is given by

$$\begin{aligned} &\int_{|\xi_1|<1} (1/|\xi_1|^{2\alpha}) \left(\int |k(\xi,\zeta)| \sup_y |v(y,\zeta)\, e^{\gamma y}|\, d\zeta\right)^2 d\xi \\ &\qquad \leqq C \int \left(\sup_y |v(y,\zeta)\, e^{\gamma y}|\right)^2 d\zeta \int (1/|\xi_1|^{2\alpha})\, (1/(1+|\xi|))\, d\xi \\ &\qquad \leqq C\, \|v\|_{2,\gamma}^2 \quad \text{when } \alpha < 1/2. \end{aligned} \tag{3.54}$$

Putting together estimates (3.53) and (3.54), one has

$$\|1_{\xi_1 \leqq 0}\, v\|_{2,\gamma} \leqq C\, |||v|||_{\gamma}/\varepsilon^{1/2} + C\varepsilon^{\alpha}\, \|v\|_{2,\gamma} + C\, \|S\|_{2,\gamma}. \tag{3.55}$$

When $\xi_1 \geqq 0$, one takes $x = 0$ in (3.47) and proceeds as above:

$$e^{\gamma y}\, |v(y,\xi)| \leqq e^{(\gamma-\lambda)y}\, |v(0,\xi)| + \left|\int_0^y (1/\xi_1)(\nu S + Kv)(s,\xi)\, e^{\gamma y} e^{\lambda(s-y)}\, ds\right|. \tag{3.56}$$

In (3.56),

$$\left|\int_0^y (1/\xi_1)\, \nu S\, e^{\gamma y}\, e^{\lambda(s-y)}\, ds\right|$$
$$\leqq \sup_x \left(|S(x,\xi)|\, e^{\gamma x}\right) \left(|\lambda|/|\lambda-\gamma|\right) e^{(\gamma-\lambda)y} \int_0^y e^{(\lambda-\gamma)s}\, ds \qquad (3.57)$$
$$\leqq C \sup_x \left(|S(x,\xi)|\, e^{\gamma x}\right)$$

and for $|\xi_1| > 1$

$$\left|\int_0^y (1/\xi_1)\, Kv(s,\xi)\, e^{\lambda(s-y)}\, e^{\gamma y}\, ds\right|$$
$$\leqq \left(|\lambda|/(\nu(\xi)\, |\lambda-\gamma|^{1/2})\right) \left(\int Kv(s,\xi)^2\, e^{2\gamma s}\, ds\right)^{1/2} \qquad (3.58)$$
$$\leqq C\left(\int Kv(s,\xi)^2\, e^{2\gamma s}\, ds\right).$$

For $|\xi_1| \leqq 1$, we write again:

$$\int_0^y = \int_0^{y-\varepsilon} + \int_{y-\varepsilon}^y, \qquad (3.59)$$

with

$$\left|\int_0^{y-\varepsilon} (1/\xi_1)\, Kv(s,\xi)\, e^{\lambda(s-y)}\, e^{\gamma y}\, ds\right|$$
$$\leqq C\left(|\lambda|/|\lambda-\gamma|^{1/2}\right) e^{(\lambda-\gamma)\varepsilon} \left(\int Kv(s,\xi)^2\, e^{2\gamma s}\, ds\right)^{1/2} \qquad (3.60)$$
$$\leqq (C/\varepsilon^{1/2}) \left(\int Kv(s,\xi)^2\, e^{2\gamma s}\, ds\right)^{1/2}$$

and

$$\left|\int_{y-\varepsilon}^y (1/\xi_1)\, Kv(s,\xi)\, e^{\lambda(s-y)}\, e^{\gamma y}\, ds\right|$$
$$\leqq \left|\int_0^\varepsilon (1/\xi_1)\, (Kv(y-s,\xi)\, e^{\gamma(y-s)})\, e^{(\gamma-\lambda)s}\, ds\right|$$
$$\leqq \sup_y |Kv(y,\xi)\, e^{\gamma y}| \int_0^\varepsilon \left((|\gamma-\lambda|\, s)^{1-\alpha}/(\xi_1\, |\gamma-\lambda|^{1-\alpha})\right) e^{(\gamma-\lambda)s}\, (ds/s^{1-\alpha}) \qquad (3.61)$$
$$\leqq C\varepsilon^\alpha \sup_y |Kv(y,\xi)\, e^{\gamma y}|\; 1/(\xi_1\, \nu(\xi)^{(1-\alpha)}).$$

Thus (using the reflection condition) we obtain

$$\|v\, 1_{\xi_1 \geqq 0}\|_{2,\gamma} \leqq C \int_\xi v(0,\xi)^2\, 1_{\xi_1 \leqq 0}\, d\xi + (C/\varepsilon)\, |||v|||_\gamma^2$$
$$+ C^2 \varepsilon^\alpha\, \|v\|_{2,\gamma}^2 + C\, \|S\|_{2,\gamma}. \qquad (3.61)$$

Putting together (3.55)–(3.61), we find that

$$\|v\|_{2,\gamma} \leqq (C/\varepsilon^{1/2})\, |||v|||_\gamma + C\varepsilon^\alpha\, \|v\|_{2,\gamma} + C\, \|S\|_{2,\gamma}. \qquad (3.62)$$

Choosing ε such that $C\varepsilon^\alpha < 1$ and using estimates (3.43) and (3.62), one has for the solution v of (3.42)

$$||v||_{2,\gamma} + |||e^{\gamma x} \nu^{1/2} v||| \leqq C(|||e^{2\gamma x} \nu^{1/2} S||| + ||S||_{2,\gamma}) \tag{3.63}$$

for small enough $\gamma \geqq 0$. Thus the solution u of (3.1)–(3.3) satisfies:

$$||u||_{2,\gamma} + |||e^{\gamma x} \nu^{1/2} u||| \leqq C(|||e^{2\gamma x} \nu^{1/2} S||| + ||S||_{2,2\gamma} + ||\nu^{1/2} h||). \tag{3.64}$$

(iv) The uniqueness theorem

To prove the uniqueness theorem, let us consider a solution v of (3.1)–(3.3) with $S = 0$ and $h = 0$ and prove that it is identically 0. It satisfies

$$(1/2) \int_{\mathbb{R}^3} \xi_1 w(x, \xi)^2 \, d\xi + \int_0^x \int_{\mathbb{R}^3} \nu(\xi) w_v(s, \xi)^2 \, d\xi \, ds \leqq 0 .$$

Since w vanishes at infinity,

$$\int_0^\infty \int_{\mathbb{R}^3} \nu(\xi) w_v(s, \xi)^2 \, d\xi \, ds = 0 .$$

Thus $w_v = 0$ a.e., q_v is independent of x and therefore is also 0. This concludes the proof of the uniqueness and of Prop. 3.1.

4. The nonlinear problem

We now have the main tools to study the nonlinear problem. We shall use the Banach fixed-point theorem to prove existence and uniqueness of a solution of (2.15). We work in the Banach space E of measurable functions equipped with the norm

$$||u||_E = |||\nu^{1/2} u e^{\gamma x}||| + ||u||_{2,\gamma}, \tag{4.1}$$

and consider the mapping T defined on E into E by $Tv = u$ where u is the unique solution of

$$\begin{aligned} &\xi_1 \, \partial_x u + Lu = \theta \nu S + \nu \Gamma(v, v), \quad x \geqq 0, \\ &u(0, \xi) = u(0, R\xi) + \theta h(R\xi), \quad \xi_1 \geqq 0, \\ &|||\nu^{1/2} u||| < \infty . \end{aligned} \tag*{(4.2)$_\theta$}$$

Lemma 4.1. *For* $v \in E$, *one has the estimates*

$$|||e^{2\gamma x} \nu^{1/2} \Gamma(v, v)||| \leqq C \, |||e^{\gamma x} \nu^{1/2} v|||^2, \tag{4.3}$$

$$||\nu^{1/2} \Gamma(v, v)||_{2,\gamma} \leqq C \, ||v||_{2,\gamma}^2 . \tag{4.4}$$

The proofs of the estimates are given in the Appendix in a more general form.

The orthogonality condition (2.12) on Γ is exactly (2.8), and using the lemma shows that problem $(4.2)_\theta$ is well posed for $v \in E$. Moreover, $u = Tv \in E$ and satisfies

$$\|u\|_E \leqq C(\theta H + \|v\|_E^2) \tag{4.5}$$

(see Proposition 3.1) with

$$H = \|S\|_E + \|\nu^{1/2} h\|, \tag{4.6}$$

Hence, if $v \in B_E(\theta) = \{v, \|v\|_E < \theta\}$, and θ small enough, then $Tv \in B_E(\theta)$. The mapping T is thus defined from $B_E(\theta)$ into itself, is continuous, and for small enough θ is a contraction. We may apply the Banach fixed-point theorem to prove Theorem 2.1.

Appendix

In this Appendix we prove some basic estimates on $\nu\Gamma(f,f)$. For the sake of generality, we shall not restrict our analysis to a gas of hard spheres but consider hard potentials in general. The operator $\nu\Gamma$ is defined by

$$\nu\Gamma(f,f) = M^{-1/2} Q(M^{1/2} f, M^{1/2} f), \tag{A.1}$$

where Q is given by (1.4)–(1.5). We assume that $\nu(\xi)$ satisfies, for $0 \leqq \beta \leqq 1$

$$\nu_0(1 + |\xi|)^\beta \leqq \nu(\xi) \leqq \nu_1(1 + |\xi|)^\beta \tag{A.2}$$

and that the cross-section q is given by

$$0 \leqq q(V, \omega) \leqq V^\beta \Lambda(\omega) \tag{A.3}$$

with $\Lambda(\omega) \in L^1(S^2)$.

Lemma A.1. *Under assumptions* (A.1) *and* (A.2), *we have, for any* $\alpha \in [0, 1]$,

$$\|\nu^\alpha \Gamma(f,f)\| \leqq C \|\nu^\alpha f\| \|f\|, \tag{A.4}$$

where

$$C = C_0 \int_{S^2} |\Lambda(\omega)| \, d\omega \quad \text{and } C_0 \text{ is a universal constant.}$$

Proof. $\nu\Gamma$ is rewritten in the form

$$\nu\Gamma(f,f) = A - B \tag{A.5}$$

with

$$A = \int q(\xi - \zeta, \omega) \, M(\zeta)^{1/2} f(\xi) f(\zeta) \, d\zeta \, d\omega \tag{A.6}$$

and

$$B = \int q(\xi - \zeta, \omega) \, M(\zeta)^{1/2} f(\xi') f(\zeta') \, d\zeta \, d\omega. \tag{A.7}$$

To have an upper bound for the contribution of A in $\nu^\alpha \Gamma$, we use the inequality

$$|\xi - \zeta| \leqq (1 + |\xi|)(1 + |\zeta|), \tag{A.8}$$

and thus we have:

$$\begin{aligned}
&\int (A(\xi)^2/(1+|\xi|)^{2\beta(1-\alpha)})\, d\xi \\
&\quad = \int (1/(1+|\xi|)^{2\beta(1-\alpha)}) \left(\int q(|\xi-\zeta|,\omega) f(\xi) f(\zeta)\, M(\zeta)^{1/2}\, d\zeta\, d\omega\right)^2 d\xi \\
&\quad \leqq \int (1/(1+|\xi|)^{2\beta\alpha}) f(\xi)^2 \left(\int_{\zeta,\omega} A(\omega)\, |f(\zeta)|\, (1+|\zeta|)^{\beta} M(\zeta)^{1/2}\, d\zeta\, d\omega\right)^2 d\xi \qquad \text{(A.9)} \\
&\quad \leqq C_0 \|\nu^\alpha f\|^2 \left(\int_{S^2} |A(\omega)|\, d\omega\right)^2 \|f\|^2.
\end{aligned}$$

To have the contribution of B in $\nu^\alpha \Gamma$, we use that:

$$|\zeta-\xi| = |\zeta'-\xi'|, \qquad \text{(A.10)}$$

and thus

$$\begin{aligned}
&\int (B(\xi)^2/(1+|\xi|)^{2\beta(1-\alpha)})\, d\xi \\
&\quad \leqq \int_{\xi} (d\xi/(1+|\xi|^{2\beta(1-\alpha)}) \\
&\qquad \times \left(\int_{\zeta,\omega} A(\omega)\, |\xi'-\zeta'|^{\beta\alpha}\, |\xi-\zeta|^{(1-\alpha)\beta}\, |f(\zeta')|\, |f(\xi')|\, M(\zeta)^{1/2}\, d\zeta\, d\xi\right)^2 \qquad \text{(A.11)} \\
&\quad \leqq C \int_{\xi} d\xi \left(\int_{\zeta,\omega} A(\omega)\, |\xi'-\zeta'|^{\beta\alpha} (1+|\zeta|)^{(1-\alpha)\beta}\, |f(\xi')|\, |f(\zeta')|\, M(\zeta)^{1/2}\, d\zeta\, d\omega\right)^2 \\
&\quad \leqq C \int_{\xi} d\xi \left(\int_{\zeta,\omega} f(\xi')^2 f(\zeta')^2 (|\xi'|^{2\alpha\beta} + |\zeta'|^{2\alpha\beta})\, A(\omega)\, d\zeta\, d\omega\right) \\
&\qquad \times \int_{\zeta,\omega} A(\omega)\, (|1+|\zeta|)^{2(1-\alpha)\beta} M(\zeta)\, d\zeta\, d\omega).
\end{aligned}$$

Since the mapping $(\xi,\zeta) \to (\xi',\zeta')$ is an isometry, we have

$$\int (B^2(\xi)/(1+|\xi|)^{2\beta(1-\alpha)})\, d\xi \leqq C\, |A|^2_{L^1(S^2)} \|\nu^\alpha f\|^2 \|f\|^2. \qquad \text{(A.12)}$$

Lemma A.2. Under assumptions (A.1) and (A.2), $\nu\Gamma$ satisfies

$$\|\nu^\alpha \Gamma(f,f)\|_{2,\gamma} \leqq C\, \|\nu^\alpha f\|_{2,\gamma}\, \|f\|_{2,\gamma}.$$

The proof is similar to that of Lemma A.1.

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