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Choosing from a Tournament

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Abstract. A tournament is any complete asymmetric relation over a finite set A of outcomes describing pairwise comparisons. A choice correspondence assigns to every tournament on A a subset of winners. Miller's uncovered set is an example for which we propose an axiomatic characterization. The set of Copeland winners (outcomes with maximal scores) is another example; it is a subset of the uncovered set: we note that it can be a dominated subset. A third example is derived from the sophisticated agenda algorithm; we argue that it is a better choice correspondence than the Copeland set.

1. Introduction

Given a finite set A of outcomes (candidates, decisions) a tournament T on A expresses decisive preference judgments for all pairs of outcomes: aTb reads a beats b (no indifferences are allowed). When a tournament T has no Condorcet winner (i.e., no outcome beats every other outcome) there is no straightforward notion of winner(s) for T. This paper explores some such notions and compares them normatively.

The above question has applications to many decision problems far more general than sport competitions; see, for example, the monograph by Moon (1968) and references therein to the psychometric and biometric literature. Here we concentrate on solution concepts that were originally inspired by the theory of collective choice, in particular the strategic analysis of voting rules based upon majority comparisons: hence our terminology of Condorcet consistency, sophisticated agenda and so on. Of course we deal with abstract tournaments so that the potential applications are wider.

Summary of the Results

A choice correspondence associates to any tournament T a subset $S(T)$ of outcomes. The simplest example is the *top cycle* of T, made up of those outcomes that beat directly or indirectly every other outcome (a beats b indirectly if a beats c

and c beats b, or a beats c, c beats d, d beats b, and so on). A minimal rationality requirement of any choice correspondence is that it produces a *subset* of the top cycle (Moon 1968; Schwartz 1972).

Miller's *uncovered set* is made up of those outcomes a which, when opposed to any other b, either beat b or beat some c that beat b (thus a beats every other outcome in at most two steps). It was introduced independently by Miller (1977) and Fishburn (1977), and further explored by Miller (1980) and Shepsle and Weingast (1982) – see also McKelvey (1983) in the context of spatial voting. We propose our axiomatic characterization of the uncovered set (Theorem 1) based upon two familiar axioms from the theory of rationalizable choice functions (see Bordes (1983) and Moulin (1984)).

Next we turn (Sect. 5) to another choice correspondence, namely the *Copeland set* made up of those outcomes that beat the greatest number of opponents (Copeland 1951). Miller observed that the Copeland set is contained in the uncovered set; however, it might be a dominated subset of the uncovered set, at least when there are 13 outcomes or more (Theorem 2).

In Sect. 6 we explore a family of deterministic choice correspondences using a fixed binary tree (as in a single elimination tournament). These solutions generalize to the context of tournaments the voting by binary choice methods introduced by Farqharson (1969) and further studied by Miller (1977, 1980), McKelvey and Niemi (1978), and Moulin (1979). We prove that they always choose one outcome from the top cycle, but may display perverse nonmonotonicity features. We also prove that no selection of the Copeland set can be obtained by such a binary tree (at least when there are 8 outcomes or more, Corollary to Lemma 10).

The last section (Sect. 7) is devoted to one choice correspondence derived from a specific binary tree, the multistage elimination tree (Miller 1977; Moulin 1979). Its outcome is given by a remarkable algorithm called the sophisticated agenda algorithm (Shepsle and Weingast 1982). This algorithm produces an undominated outcome of the uncovered set (Banks 1985), thus is arguably superior to any selection of the Copeland set. When there are no more than four undominated outcomes in the uncovered set, all of them can be reached by an appropriate agenda (Banks 1985). If there are at least five such outcomes, some might be out of reach (Lemma 14).

Open Problems

Ranking the participants of a given tournament is another, related, problem. The Copeland scores (assigning to each outcome the number of outcomes it beats) induce such a preordering: Rubinstein (1980) and Henriet (1985) provide two distinct characterizations of it (the former based on a weak version of Arrow's Independence of Irrelevant Alternatives, the latter based on its invariance when the orientation of a cycle is reversed). Yet the arguments developed above raise doubts about its reasonableness. Much more appealing is the idea (proposed independently by Kemeny (1959) and Slater (1961) to approximate a given tournament by its closest orderings (the distance being measured by the number of binary comparisons that must be reversed). Yet little is known about the top elements of the closest orderings: they are in the uncovered set and can be all Copeland nonwinners (Bermond 1972); but are they undominated in the uncovered set? is there an easy algorithm to compute one of them?

Another widely open question is the generalization of choice correspondences to any complete relation on A (not necessarily asymmetric: indifferences are allowed). Already the definition of the top cycle and/or the uncovered set is far from simple if one wants to preserve some version of Condorcet consistency: on this see Miller (1984). Also the Copeland score can be defined in several nonequivalent ways (by counting one half point or zero point in case of a tie) among which the choice is not straightforward; see Henriet (1985).

2. Choice Correspondence and Condorcet Consistency

The finite set A contains all feasible *outcomes*. A *tournament* on A is a complete and asymmetric binary relation T: for each pair a, b of distinct outcomes, exactly one of aTb, bTa holds (and aTa for all a). We read aTb as: a beats b in the pairwise comparison a, b. Notice that tournaments allow cycles of arbitrary length (e.g., aTb, bTc, cTa is a cycle of length 3). A tournament with no cycles is an ordering of A (complete, transitive, asymmetric). We denote by τ the set of tournaments on A.

A *choice correspondence* (c.c.) is any multivalued mapping S from τ into A. To any tournament T, a choice correspondence associates a nonempty subset $S(T)$ of "best" outcomes, called the *choice set* at T.

If an outcome beats every other outcome in pairwise comparisons we call it a *Condorcet winner*. Our first axiom states that a Condorcet winner should be uniquely chosen:

Condorcet Consistency. For all $T \in \tau$, $a \in A\{aTb \quad \text{all} \quad b \in A\} \Rightarrow \{S(T) = \{a\}\}$

For instance, when tournament T is an ordering (i.e., has no cycles) its top outcome should be uniquely chosen. A much stronger statement than Condorcet's consistency is:

Condorcet Transitivity. For all $T \in \tau$, $\quad a, b \in A\{[a \in S(T), bTa] \Rightarrow b \in S(T)\}$. (1)

In words, any outcome that defeats some chosen outcome must be chosen as well. Observe that Condorcet transitivity is equivalently formulated as:

$$\text{for all} \quad T \in \tau \ , \quad a, b \in A\{[a \in S(T)\ b \notin S(T)] \Rightarrow aTb\} \ .$$

In words, any chosen outcome beats any nonchosen outcome.

Lemma 1 *(Schwartz 1972). There is a unique smallest (w.r.t inclusion) choice correspondence satisfying Condorcet's transitivity. It is called the top cycle and defined as*

$$tc(T) = \{a \in A \mid \textit{for all } b \in A, \textit{ there is an integer } n \textit{ and a sequence}$$

$$a = a_0, a_1, \ldots, a_n = b \quad \text{s.t.} \quad a_i T a_{i+1}, \textit{ all } i = 0, \ldots, (n-1) \ . \qquad (2)$$

Proof. Pick any c.c. S satisfying (1) and let tc be the c.c. defined by (2). Choose $a \in tc(T)$ and $b \in S(T)$. Since there is a sequence $a = a_0, a_1, \ldots, a_n = b$ such as in (2) we have

$$\{a_{n-1} Tb, b \in S(T)\} \Rightarrow a_{n-1} \in S(T)$$

$$\{a_{n-2} Ta_{n-1}, a_{n-1} \in S(T)\} \Rightarrow a_{n-2} \in S(T)$$

$$\{aTa_1, a_1 \in S(T)\} \Rightarrow a \in S(T) \ .$$

This proves $tc \subseteq S$. It remains to check that tc satisfies Condorcet transitivity. This is obvious from (2). Q.E.D.

An equivalent definition of tc goes by setting T^* to be the transitive closure of T:

aT^*b iff there is an integer n and a sequence

$$a = a_0, a_1, \ldots, a_n = b \quad \text{s.t.} \quad a_i T a_{i+1} \ , \quad \text{all} \quad i = 0, \ldots, (n-1) \ .$$

Then $tc(T)$ is the set of maximal elements of T^*.

The top cycle is too big a choice correspondence, as the following example shows: T coincides with the ordering $1 > 2 > 3 \ldots > n$ *except* for $nT1$. Then $tc(T) = \{1, \ldots, n\}$: the top cycle does not discriminate at all among outcomes, despite their asymmetry. All subsequent choice correspondences will actually choose $\{1, 2, n\}$ thus eliminating $3, \ldots, n-1$.

Another drawback of the top cycle arises when the tournament T is derived from binary majority comparisons (among finitely many agents, each endowed with a preference preordering over A). Then the top cycle may contain Pareto dominated outcomes. The standard example (Fishburn 1977, p. 89) has three agents and four outcomes

agent 1: $a > b > c > d$

agent 2: $d > a > b > c$

agent 3: $c > d > a > b$.

Here the Pareto inferior outcome b is in the top cycle.

The following weakening of Condorcet's transitivity was introduced by Smith (1973). It is satisfied by all choice correspondences below:

Smith's Consistency. For all $T \in \tau$, all $B \subset A \{bTa$ for all $b \in B$ and all $a \in A \setminus B\} \Rightarrow \{S(T) \subset B\}$. (3)

If a subset B is ahead of $A \setminus B$, in the sense that any outcome in B beats any one in $A \setminus B$, then the choice set does not contain any outcome of $A \setminus B$. This property is stronger than Condorcet's consistency, but weaker than Condorcet's transitivity.

Just like the c.c. satisfying Condorcet's transitivity are stable by intersection, those satisfying Smith's consistency are stable by union and thus have a largest element.

Lemma 2. *There is a unique largest (w.r.t inclusion) choice correspondence satisfying Smith's consistency. It is the top cycle.*

The proof is straightforward.

The following graph theoretical result is proven, for instance, by Miller (1977).

Lemma 3. *The top cycle of a tournament T is a singleton only if it is the Condorcet winner of T. Otherwise it is a cycle (with size at least 3); one can order it as $tc(T)=\{a_1,\ldots,a_p\}$ where*

$$a_iTa_{i+1}\ ,\quad i=1,\ldots,p-1 \quad \text{and} \quad a_pTa_1\ .$$

Proof. If T has no Condorcet winner, every outcome in $tc(T)$ is beaten by at least one other outcome in $tc(T)$. Hence one can find a T-cycle within $tc(T)$. Pick one which is inclusion maximal, say $C=\{a_1,\ldots,a_q\}$, and assume, per absurdum, that C is strictly smaller than $tc(T)$. Pick a in $tc(T)\setminus C$. If for some j, $1\leqq j\leqq q-1$, we have a_jTaTa_{j+1}, then $\{a_1,\ldots,a_j,a,a_{j+1},\ldots,a_q\}$ makes a bigger cycle, a contradiction. The same argument shows that a_qTaTa_1 is impossible. Thus only two cases are possible: i) for all $j=1,\ldots,q$, a_jTa *or* ii) for all $j=1,\ldots,q$, aTa_j. Then partition $tc(T)\setminus C$ as C_{i} (where i holds) and C_{ii} (where ii holds). If any one of C_{i} or C_{ii} is empty we have an easy contradiction by Smith's consistency (if $tc(T)=C\cup C_{\mathrm{i}}$ then C is ahead of $A\setminus C$; if $tc(T)=C\cup C_{\mathrm{ii}}$, then C_{ii} is ahead of $A\setminus C_{\mathrm{ii}}$). Now we can assume C_{i} contains some a and C_{ii} contains some b. Consider a T-path from a to b; such a path remains within $tc(T)$. Let b' be the first outcome on this path outside C_{i}. Then b' must be in C_{ii} so that $b'Ta_1Ta_2\ldots Ta_qTa$. We have just constructed a bigger T-cycle than C which is the desired contradiction. Q.E.D.

3. The Uncovered Set

For any tournament T on A define its covering relation as: b covers a iff

$$a\neq b\ ,\quad bTa \quad \text{and} \quad \text{for all} \quad c\in A\colon aTc\Rightarrow bTc\ . \tag{4}$$

The covering relation is transitive, yet not complete. Its maximal elements define the uncovered set.

The next Definition and Lemma are from Miller (1977). Lemma 4 is called the "two step principle" by Shepsle and Weingast (1982).

Definition 1. Given a tournament T, its uncovered set $uc(T)$ is defined by

$$a\in uc(T) \quad \text{iff for no} \quad b\in A\ ,\quad b \quad \text{covers} \quad a\ .$$

Lemma 4. *The uncovered set $uc(T)$ contains outcome a if and only if:*

$$\textit{for all}\quad b\in A\ ,\quad b\neq a\ ,\quad aTb \quad \textit{and/or for some} \quad c\in A\ ,\quad aTcTb\ . \tag{5}$$

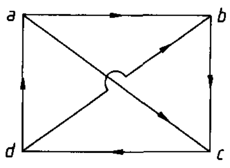
Proof. Suppose a satisfies (5) and that b covers a. Then bTa so there must exist c such that aTc and cTb. By (4) bTc so $b=c$ thus aTb, a contradiction (since $b\neq a$). Conversely, suppose a does not satisfy (5). Then for some $b\neq a$, we have bTa and, for all c, aTc and cTb cannot both be true. This is just statement (4). Q.E.D.

From Lemma 4 and Definition 1 it follows that the uncovered set of a tournament is a subset of its top cycle:

$uc(T) \subseteq tc(T)$ all tournament T .

Furthermore, if subset B is ahead of $A \setminus B$ (bTa for all $b \in B$ and all $a \in A \setminus B$) then any $b \in B$ cover$\not{s}$ any $a \in A \setminus B$. Therefore uc satisfies Smith's consistency (3).

As a first example, the tournament



has top cycle $\{a, b, c, d\}$ but uncovered set $\{a, c, d\}$ since a covers b (or equivalently, to go from b to a along a T-path, one needs 3 steps).

Consider next the example given above: T is the ordering $1 > 2 > \ldots > n$ except for $nT1$. Here 2 covers 3, 3 covers $4, \ldots, (n-2)$ covers $(n-1)$, therefore $uc(T) = \{1, 2, n\}$.

The uncovered set eliminates from the top cycle some bad outcomes: in particular if our tournament is deduced from a majority relation, its uncovered set contains only Pareto optimal outcomes (if b Pareto dominates a, then b covers a). To compute it is not too difficult either: say that $\boldsymbol{\theta}$ is the $|A| \times |A|$-matrix representing tournament T.

$\theta_{ab} = 1$ if aTb (in particular $\theta_{aa} = 1$)

0 otherwise .

Then the rows of $\boldsymbol{\theta}^2$ which contain no zero determine the outcomes in the uncovered set (this is an easy consequence of Lemma 4).

The uncovered set is not necessarily a T-cycle (contrary to the top cycle). However, if it is not a singleton, it cannot contain a winner, i.e., an outcome beating every other uncovered outcome.

Lemma 5 *(Miller 1977). The uncovered set of a tournament T is a singleton if and only if T has a Condorcet winner a (in this case $uc(T) = \{a\}$). If $uc(T)$ is not a singleton, then it contains at least three outcomes and the restriction of T to $uc(T)$ has no Condorcet winner: there is no outcome a in $uc(T)$ that beats every other b in $uc(T)$.*

Proof. Since the covering relation is transitive and A is finite, any outcome outside $uc(T)$ must be covered by an outcome inside $uc(T)$. Thus if $uc(T) = \{a\}$, outcome a covers any other outcome, so it is a Condorcet winner. Suppose now that T has no Condorcet winner, yet $a \in uc(T)$ beats every other $b \in uc(T)$. Set $B = \{b \in A \setminus a | bTa\} \subset A \setminus uc(T)$. We pick a^* in the uncovered set of the restriction of T to B and prove $a^* \in uc(T)$ (hence a contradiction) by means of (5). For all $b \in B$ there is a T-path within B from a^* to b of length at most 2. For all $b \in A \setminus B$, either $b = a$ or aTb. Since a^*Ta there is a T-path of length at most 2 from a^* to b.

This establishes that the restriction of T to $uc(T)$ has no Condorcet winner, which implies that its size is at least 3. Q.E.D.

The structure of $uc(T)$ as described in Lemma 5 cannot be specified more generally; this is the meaning of our next result.

Lemma 6. *Let A_0 contain at least three outcomes and let T_0 be any tournament on A_0 without a Condorcet winner. Then there exists a superset A of A_0 and a tournament T on A such that*

i) $uc(T) = A_0$

ii) the restriction of T to A_0 is T_0 .

Proof. Given A_0, T_0 as above, choose a set A_1 disjoint from A_0 and with the same size; also pick a bijection denoted $a \to a'$ from A_0 into A_1. Define T on $A = A_0 \cup A_1$ as follows

$$\begin{cases} \text{on } A_1: & T \text{ copies } T_0: \quad a'Tb' \quad \text{iff} \quad aT_0b \\ \text{on } A_0: & T \text{ is } T_0 \end{cases}$$

$$\begin{cases} \text{if } a, b \in A_0 \quad \text{and} \quad a \neq b: & aTb' \\ \text{if } \quad a \in A_0 & : \quad a'Ta \end{cases}$$

We claim that $uc(T) = A_0$. We use Lemma 4. Pick $a \in A_0$ and prove it satisfies (5): for all $b \in A_0 \setminus a$ we have aTb', $b'Tb$ hence (5); for all $b' \in A_1$, $b' \neq a'$, we have aTb'; finally if b is any outcome in $A_0 \setminus a$ s.t. bT_0a (by assumption a is no Condorcet winner) then aTb', $b'Ta'$.

Next pick $a' \in A_1$ and prove it violates (5): by assumption there is $b \in A_0 \setminus a$ s.t. bT_0a. Careful inspection reveals that a T-path from a' to b has length at least 3. Q.E.D.

4. The Uncovered Set for Variable Issues

This section is conceptually more elaborate than the rest of the paper. It can be skipped without affecting the understanding of any subsequent section.

We think of the subset B of feasible outcomes (the issue) as varying within a given set A and we seek to compare the uncovered set for the restrictions of our tournament to each particular issue. Thus we consider the mapping

$(T, B) \to uc(T, B) =$ uncovered set of the restriction of T to B

defined for all tournament $T \in \tau$ and all issue $B \subseteq A$.

Two essential invariance properties of uc are easy to check:

Neutrality (Nondiscrimination Among Outcomes). If σ is a permutation of A and T^σ is defined by $aT^\sigma b \Leftrightarrow \sigma^{-1}(a) T \sigma^{-1}(b)$, then:

$$\text{for all} \quad T, \quad \text{all} \quad B, \quad S(T^\sigma, B) = \sigma[S(T, B)] .$$

Arrow's Independence of Irrelevant Alternatives (AIIA). If T and T' have the same restriction on B, then

$$S(T,B)=S(T',B)\ , \quad (\text{all} \quad T,T',B)\ .$$

Now come the conditions relating the choice sets over some issue B and over some other issue B'. The following axiom is Sen's condition γ (Sen 1977).

Expansion. For all T,B,B': $S(T,B)\cap S(T,B')\subset S(T,B\cup B')$.

Theorem 1. *The uncovered set (viewed as the mapping $(T,B)\rightarrow uc(T,B)$) satisfies Neutrality, Arrow's IIA and Expansion. Conversely any mapping $(T,B)\rightarrow S(T,B)$ satisfying Neutrality, Arrow's IIA, Expansion and Condorcet consistency (if T has a Condorcet winner a on A, then $S(T,A)=a$) must contain the uncovered set ($uc(T,B)\subset S(T,B)$ all T,B).*

Proof. The first statement is proved straightforwardly. For instance, Expansion follows from Lemma 4. Let us prove the converse property. First we observe that the Expansion axiom is equivalently written as

for all sequence $B_1,\ldots,B_K$ of subsets of A , and all tournament T:

$$\bigcap_{1\leq k\leq K} S(T,B_k)\subset S\left(T,\bigcup_{1\leq k\leq K} B_k\right)\ . \tag{6}$$

Next observe that Arrow's IIA and Condorcet consistency together imply Condorcet consistency on every restricted issue, namely:

$$\text{for all} \quad T,B:\{aTb \quad \text{for all} \quad b\in B\}\Rightarrow\{S(T,B)=a\}\ . \tag{7}$$

Next we deduce from Neutrality and Arrow's IIA that if B is a triple and the restriction of T to B is a cycle (say $B=\{abc\}$, $aTbTcTa$) then $S(T,B)=B$. To see this, consider any permutation σ of A that permutes a, b, c. Then $S(T^{\sigma},B)=S(T,B)$ by Arrow's IIA whence by Neutrality $S(T,B)=\sigma(S(T,B))$.

Finally we prove the inclusion $uc\subset S$. Pick T, B and $a\in uc(T,B)$. Partition $B\setminus\{a\}$ as $B_+=\{b|aTb\}$ and $B_-=\{b|bTa\}$. For any $b\in B_+$ we have by (7) $S(T,\{ab\})=\{a\}$. For any $b\in B_-$ there exists (by Lemma 4) some $c\in B_+$ such that $aTcTb$. Thus the restriction of T to abc is a cycle so $S(T,\{abc\})=\{abc\}$. We invoke now (6):

$$a\in\left[\bigcap_{b\in B_+} S(T,\{ab\})\right]\cap\left[\bigcap_{b\in B_-} S(T,\{abc\})\right]$$

$$\Rightarrow a\in S\left(T,\left(\bigcup_{b\in B_+}\{ab\}\right)\cup\left(\bigcup_{b\in B_-}\{abc\}\right)\right)=S(T,B)\ .$$

This concludes the proof of Theorem 1. □

In the study of a single agent's choice function, two more axioms, called Chernoff and Aizerman play, together with Expansion, a prominent role:

Chernoff: for all B, B', all T: $\{B \subset B'\} \Rightarrow \{S(B') \cap B \subset S(B)\}$

Aizerman: for all B, B', all T: $\{S(B') \subset B \subset B'\} \Rightarrow \{S(B) \subset S(B')\}$.

We let the reader check that the uncovered set satisfies Aizerman but violates Chernoff. In fact we have the following negative result.

Lemma 7. *If A contains at least 3 outcomes, no mapping $(T, B) \rightarrow S(T, B)$ satisfies together Condorcet consistency, Arrow's IIA and Chernoff.*

Proof. Take $B' = \{a, b, c\}$ and T such that its restriction on B is a cycle: $aTbTcTa$. From Chernoff and aTb follows $S(B') \cap \{ab\} \subset S(ab) = \{a\} \Rightarrow b \notin S(B')$. Similar arguments from bTc and cTa show successively $c \notin S(B')$ and $a \notin S(B')$, a contradiction. Q.E.D.

The above result is a mini-impossibility theorem: indeed, Chernoff's axiom is necessary to represent the choice function $B \rightarrow S(B)$ as the maximal element of some fixed acyclic relation on A (in fact Chernoff and Expansion together are necessary and sufficient to such rationalization; see, e.g., Moulin 1984, Theorem 2). Thus Lemma 7 states that any choice correspondence for tournaments which selects a Condorcet winner when there is one, either violates Arrow's IIA axiom or is not rationalizable. In fact, the same proof yields a slightly more general statement, namely: if S is Condorcet consistent then it violates Arrow's IIA or it contains no rationalizable choice correspondence (the latter property is called subrationalizability; see Moulin (1984, Sect. 4)).

Remark: The top cycle choice correspondence over variable issues (namely the mapping $(T, B) \rightarrow tc(T, B)$) is amenable to similar characterizations. Just add one more axiom, namely

Sen. For all T, B, B': $\{B \subset B' \text{ and } S(B) \cap S(B') \neq \emptyset\} \Rightarrow \{S(B) \subset S(B')\}$.

Then the top cycle is the smallest mapping $(T, B) \rightarrow S(T, B)$ satisfying Neutrality, Arrow's IIA, Expansion, Sen and Condorcet consistency. The proof is similar to that of Theorem 1. Check first that the combination of Expansion axiom and Sen's axiom is equivalent to the following strong Expansion property: $\{S(B) \cap S(B') \neq \emptyset\} \Rightarrow \{S(B) \cup S(B') \subset S(B \cup B')\}$. Then deduce from strong Expansion that if the restriction of T to B is a cycle, namely $tc(T, B) = B$, then $S(T, B) = B$.

5. Copeland Winners

We are back to a fixed issue A and look for the choice correspondences $T \rightarrow S(T)$ that behave "better" than the uncovered set. If by better we mean more deterministic and easier to compute, then a very simple choice correspondence emerges.

Given a tournament T on A, the *Copeland score* (c-score) of outcome a is the number of outcomes it beats:

$$c(a) = |\{b \in A \setminus a \mid aTb\}| \ .$$

A *Copeland winner* is an outcome with maximal Copeland score. Their set is denoted $C(T)$,

$$C(T)=\{a\in A\,|\,c(a)\geqq c(b)\quad \text{all}\quad b\in A\}\ .$$

If a covers b then the c-score of a is strictly greater than the c-score of b. Thus

Lemma 8. *A Copeland winner belongs to the uncovered set*

$$C(T)\subset uc(T)\quad \textit{all tournaments}\quad T\ .$$

To compute the set $C(T)$ of a given tournament from its matrix $\boldsymbol{\theta}$, one needs only to compute $\boldsymbol{\theta}e$ (where e is the vector with all components 1) which is just $c+e$ (c is the vector of Copeland scores). This compares favorably to the computation of uc (which requires us to compute $\boldsymbol{\theta}^2$, see Sect. 2).

However, the Copeland winners may be very poor outcomes *in the restriction of the tournament to the uncovered set itself.* Consider the following example with 13 outcomes. Here we have four strong outcomes $abcd$ and nine weak outcomes β_i, γ_i, δ_i, $i=1,2,3$. Tournament T is depicted in Fig. 1.

It is understood that any arrow which is not explicitly depicted goes down. Thus a is beaten by b, c, d and beats β_i, γ_i, δ_i, $i=1,2,3$. Also, the arrow from the circle around β_1, β_2, β_3 upward to b means that each β_i beats b. Similarly each β_i beats each γ_i and so on.

The Copeland scores are:

$$\begin{cases} c(a)=9 \text{ (unique Copeland winner)} \\ c(b)=c(c)=c(d)=8 \\ c(\beta_i)=c(\gamma_i)=c(\delta_i)=5 \end{cases}$$

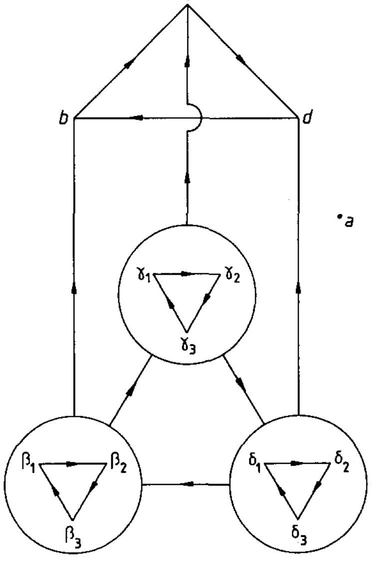


Fig. 1

The uncovered set is $\{abcd\}$: namely, β_i is covered by d, δ_i is covered by c and γ_i is covered by b. Thus in the uncovered set, a is a Condorcet *loser*. In particular, a is not *in the top cycle* of the restriction of our tournament to $\{abcd\}$.

Let us define $tc[uc(T)]$ to be the top cycle of the restriction of tournament T to its own uncovered set $uc(T)$. Then we have just given an example where

$$C(T) \cap tc[uc(T)] = \emptyset \ .$$

Theorem 2. *If A contains no more than* 9 *outcomes, then*

$$C(T) \subset tc[uc(T)] \quad \textit{all tournament} \quad T \ .$$

If A contains no more than 12 *outcomes, then*

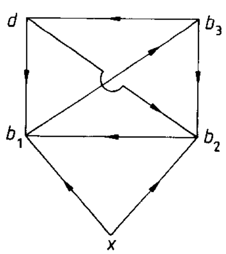
$$C(T) \cap tc[uc(T)] \neq \emptyset \ .$$

Proof. We prove the first claim. Suppose A is such that $|A| \leqq 9$ and some Copeland winner a does not belong to $B = tc[uc(T)]$. Since the restriction of T to $uc(T)$ has no Condorcet winner (Lemma 5), then B has size at least 3. Also it contains a cycle of size 3, say b_1, b_2, b_3 ($b_1 T b_2 T b_3 T b_1$). Let C be the complement of $\{a\} \cup B$ in A, with size p, $p \leqq 5$. Each b_i, $i = 1, 2, 3$ can beat at most $p-2$ outcomes in C, otherwise its c-score would be at least $(p-1)+2$ (since b_i beats a and another b_j) whereas the score of a is at most p (every b in B beats a). Thus for all $i = 1, 2, 3$ there are at least two outcomes in C that beat b_i. Since $p \leqq 5$, this implies the existence of some x in C that beat *two* among b_1, b_2, b_3.

We distinguish now three cases.

Case 1. $|B| = 3$ so $B = \{b_1, b_2, b_3\}$. Since $x \notin B$ it must be covered by some $y \in uc(T)$. This y beats two among b_1, b_2, b_3, therefore it is in the top cycle of $uc(T)$, namely in B, a contradiction.

Case 2. $|B| = 4$, so $B = \{b_1, b_2, b_3, d\}$ and x beats b_1, b_2. Now $p = |C| = 4$ so set $C = \{x, e_1, e_2, e_3\}$. By the same argument as in Case 1 we know that x must be covered by d: thus d beats x, b_1, b_2 and a; since the c-score of a is at most 4 we deduce that each e_i, $i = 1, 2, 3$ beats d. Next invoke Lemma 5. d is not a Condorcet winner in B so b_3 beats d. In fact, b_3 beats x as well (otherwise d would not cover x); it also beats a and one of b_1, b_2. Its c-score cannot be greater than 4 so each e_i, $i = 1, 2, 3$ beats b_3. So the situation is as follows:



Nondepicted arrows go downward

Take in e_1, e_2, e_3 an uncovered outcome (w.r.t. $\{e_1, e_2, e_3\}$). That outcome beats d, b_3 so it cannot be covered by b_1 or b_2 (dTb_i, $i = 1, 2$) nor x ($b_3 T x$) nor a (B beats a).

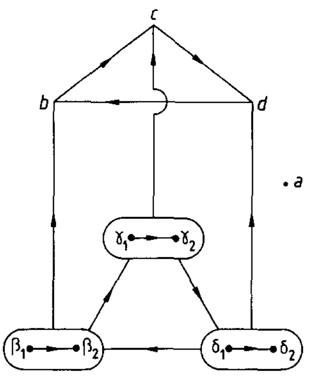


Fig. 2

So it is uncovered, but then it must be in B (since it beats some outcome in tc, see Lemma 1), a contradiction.

Case 3, $|B| \geqq 5$. Then the c-score of a is at most 3, so every outcome in B beats at most two other outcomes in B. When $|B| \geqq 6$ this is impossible, when $|B| = 5$ this implies that they all beat *exactly* two other outcomes in B. So every outcome in C beats every one in B, having then a c-score of at least 5, a contradiction.

The proof of the second claim in Lemma 9 exactly parallels the above argument and is therefore omitted. Q.E.D.

The example in Fig. 1 shows that the second statement in Theorem 2 cannot be improved. That in Fig. 2 shows that neither can the first statement. We have 10 outcomes, the four strong outcomes *abcd* together with six weak, β_i, γ_i, δ_i, $i = 1, 2$. The Copeland scores are 6 for each of a, b, c, d; 4 for each β_1, γ_1, δ_1; and 3 for each β_2, γ_2, δ_2. The uncovered set is $\{abcd\}$ with top cycle $\{bcd\}$.

6. Solving Tournaments by Binary Trees

The familiar method for choosing a unique outcome from a tournament is by means of a (finite) binary tree where to each terminal node is attached one outcome:

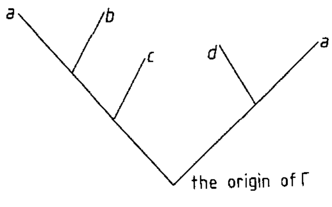


Fig. 3. A binary tree on $A = a, b, c, d$

Given A, *a binary tree on A* is a finite tree, where each node has zero or two successors, and to each terminal node is attached one outcome of A, *each outcome of A appearing at least once*. To any binary tree Γ on A we associate the following single-valued choice correspondence that *solves* every tournament T, i.e., selects a unique outcome $\Gamma(T) \in A$ by the following algorithm:

i) Pick a nonterminal node n whose two successors m, m' are terminal nodes (such a node exists: consider a path with maximal length going upward from the origin and take the node next to the last)
ii) if x, x' are the outcomes attached to m, m' use T to determine the winner outcome y ($y=x$ if xTx' or $x=x'$, otherwise $y=x'$).
iii) Chop the two branches nm and nm', thus making n a terminal node with attached outcome y.
iv) Repeat this operation until a tree with one single node (the origin of Γ) is left: its attached outcome is $\Gamma(T)$.

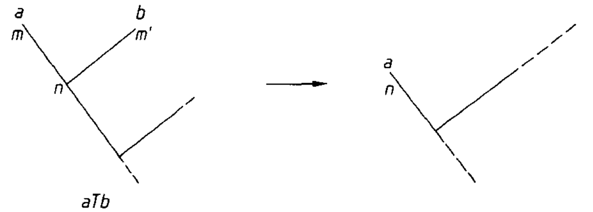


Fig. 4. The reduction algorithm

For instance, if Γ is the tree of Fig. 3 and T is the tournament (aTb, aTc, bTc, bTd, cTd, dTa) we get $\Gamma(T)=d$.

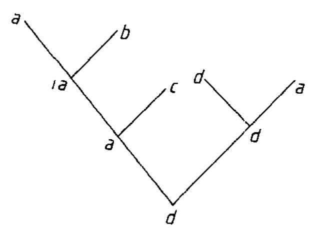


Fig. 5

The class of choice functions $T \to \Gamma(T)$ derived from arbitrary binary trees deserves attention. Where tournaments are derived from a majority relation, these choice functions, and only these, can be achieved by sophisticated voting using binary majority votes (Farqharson 1969).

Ideally one would like to find a few axiomatic properties characterizing this class. This goal, however, seems presently out of reach. More modestly we will exhibit two properties common to all choice functions in this class, allowing us in particular to prove that no one of them can select a Copeland winner for all tournaments (Corollary to Lemma 10). We shall emphasize also their potentially bad features, notably the lack of monotonicity.

Lemma 9. *(McKelvey and Niemi 1978). For any binary tree Γ and any tournament T, the outcome $\Gamma(T)$ is in the top cycle $tc(T)$.*

Proof. Paint all outcomes of $tc(T)$ in red and those in $A \setminus tc(T)$ in blue. By definition of the top cycle any red outcome beats any blue one. Since there is at least one red terminal node (each outcome appears at least once), the final winner must be red.

Q.E.D.

Another way of stating Lemma 9 is this: all choice functions $T \to \Gamma(T)$ satisfy Smith's consistency axiom.

For a fixed tournament T the set of outcomes $\Gamma(T)$ that are selected along some binary tree Γ is actually the whole top cycle $tc(T)$. To see this consider the *successive elimination* trees Γ_σ: let $\sigma = \{a_1, \dots, a_q\}$ be an ordering of A, and define Γ_σ as:

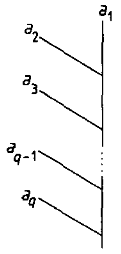
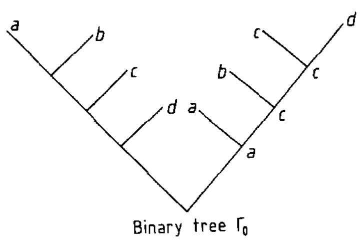


Fig. 6

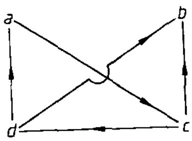
Fix an outcome $a \in tc(T)$; there exists an ordering σ of A such that $a = \Gamma_\sigma(T)$ (Miller 1977). Indeed, $tc(T)$ is a cycle (Lemma 3) so we can order it as $tc(T) = \{a_1, a_2, \dots, a_p = a\}$ where $a_p T a_{p-1} T \dots T a_2 T a_1 T a_p$. Pick some σ starting with $a_1, \dots, a_p$: then $a_1, a_2, \dots, a_{p-1}$ will be successively eliminated whereafter a_p eliminates $a_{p+1}, \dots, a_q$.

We noticed earlier that the top cycle is too big (see Sect. 2); this is clear as well on the tree of Fig. 6: outcome a can win by beating only one (well chosen) outcome. This poor feature is common to all binary trees where each outcome appears in exactly one terminal node (or single elimination tournaments). In those there is always at least one outcome that can win by beating no more than $\log_2 n$ other outcomes (since any binary tree with n terminal nodes has at least one node from which the path to the origin has length no more than $\log_2 n$). Thus we want to consider binary trees where each outcome is attached to many terminal nodes, so that it needs to beat on average "many" outcomes to be elected.

However, by multiplying the terminal nodes where the same outcome appears, another perverse feature might emerge. Consider the following game tree and tournaments:



Binary tree Γ_0



tournaments T_1, T_2

In T_1, we have bT_1a, so c is the winner in the left branch of Γ_0 and the overall winner is a. In T_2, we have aT_2b. Now d is the winner in the left branch of Γ_0, and the overall winner as well. So switching the comparison of b, *a in favor of a* is actually fatal to a!

Say that a choice correspondence $T \to S(T)$ (single-valued or not) is *monotonic* if for all $a \in A$, and T, $T' \in \tau$

$$\{a \in S(T) \quad and \quad [aTb \Rightarrow aT'b, \text{ all } b \in A \setminus a] \quad and \quad [bTc \Leftrightarrow bT'c, \text{ all } b, c \in A \setminus a]\} \Rightarrow \{a \in S(T')\} .$$

In words, if the only change from T to T' is to switch some arrows in favor of a, then a cannot be harmed. Notice that if each outcome of A is attached to exactly one terminal node of Γ, the associated choice function is monotonic.

In Sect. 7 we study a specific binary tree (the multi stage elimination process) which overcomes both difficulties mentioned above: its choice function $T \rightarrow \Gamma(T)$ is monotonic, and a selection of the uncovered set $uc(T)$. One way to do so is to look for a tree that would always elect a Copeland winner. However, such a tree does not exist if we have eight outcomes or more. To prove this somewhat surprising fact, we introduce the concept of adjacent set.

Given a tournament T on A and a proper subset $B \subsetneqq A$, $|B| \geqq 2$, we say that B is *an adjacent set* of T if we have

$$\text{for all} \quad b, b' \in B \ , \quad \text{for all} \quad a \in A \setminus B \colon aTb \Leftrightarrow aTb' .$$

We denote by τ_B the subset of tournaments which have B as an adjacent set. If $T \in \tau_B$ (and only in that case), we can merge all outcomes of B into a single outcome and define unambiguously a restricted tournament, denoted T_B, on $(A \setminus B) \cup \{B\}$ (the set with cardinality $|A| - |B| + 1$ where all of B is a single outcome).

Lemma 10. *Let $T \rightarrow \Gamma(T)$ the choice function derived from a binary tree Γ. Then for all proper subsets $B \subsetneqq A$ we have:*

$$\textit{for all} \quad T, T' \in \tau_B \ \{T_B = T'_B\} \Rightarrow \{\Gamma(T) \in B \Leftrightarrow \Gamma(T') \in B\} \tag{8}$$

and also

$$\textit{for all} \quad T, T' \in \tau_B \ \{T_B = T'_B \textit{ and } \Gamma(T) \in A \setminus B\} \Rightarrow \{\Gamma(T') = \Gamma(T)\} . \tag{9}$$

Proof. Fix a binary tree Γ on A. Consider the binary tree on $(A \setminus B) \cup \{B\}$ obtained by identifying all outcomes in B to a single one, denoted $\{B\}$. Denote Γ_B this binary tree (the tree is the same but there are fewer distinct nodes). Property (8) is implied by the following:

$$\text{for all} \quad T \in \tau_B \colon \Gamma(T) \in B \Leftrightarrow \Gamma_B(T_B) = \{B\} . \tag{10}$$

Now consider the reduction algorithm used to compute $\Gamma(T)$ and let $\Delta(\Gamma, T)$ be the mapping associating to *each* node of the tree Γ the corresponding provisional winner (Fig. 5 describes such a mapping). A straightforward induction argument (omitted for the sake of brevity) reveals that $\Delta(\Gamma, T)$ and $\Delta(\Gamma_B, T_B)$ are consistent in the following sense: to any given node of the tree, $\Delta(\Gamma, T)$ associates an outcome in B if and only if $\Delta(\Gamma_B, T_B)$ associates (to that node) the outcome $\{B\}$. Applying this to the origin node of Γ yields (10). Similarly, the same downward induction argument shows: to any given node of the tree, $\Delta(\Gamma, T)$ associates $a \in A \setminus B$ if and only if $\Delta(\Gamma_B, T_B)$ associates to that node outcome a as well. Applying this to the origin yields

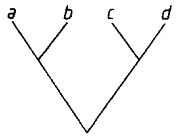
$$\text{for all} \quad T \in \tau_B \ , \quad \text{all} \quad a \in A \setminus B \ \{\Gamma(T) = a\} \Leftrightarrow \{\Gamma_B(T_B) = a\} .$$

This in turn establishes (9). Q.E.D.

Corollary to Lemma 10. *Suppose the size of A is at least 8. Then there is no binary tree Γ of which the choice function always selects a Copeland winner:*

$$\text{for all} \quad T \in \tau: \Gamma(T) \in C(T) \ .$$

Proof. First we observe that when A contains four outcomes or less, there *is* a binary tree of which the choice function always picks a Copeland winner, for instance

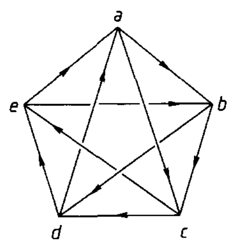


Thus in the statement of the Corollary, some lower bound on the size of A is needed. We do not know, however, the minimal lower bound.

For $|A| \geqq 8$ we prove that no selection of the Copeland winners correspondence satisfies (8). It is enough to prove the claim for $|A| = 8$ (for $|A| > 8$ consider tournaments with a top cycle of size 8). Set $A = \{a, b, c, d, e, x, y, z\}$ and $B = \{a, b, c, d, e\}$. We construct two tournaments T, T' both in τ_B such that $T_B = T'_B$ and yet $C(T) = \{a\}$, $C(T') = \{z\}$. The tournament $T_B = T'_B$ is like the ordering $\{B\} > x > y > z$ except for $zT\{B\}$. On B, the restriction of T and T' are respectively

T: ordering $a > b > c > d > e$

T': tournament where each outcome has Copeland score 2, e.g.,



The Copeland scores in T and T' are then

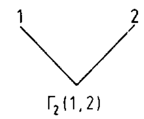
	a	b	c	d	e	x	y	z	
T	6	5	4	3	2	2	1	5	winner: a
T'	4	4	4	4	4	2	1	5	winner: z

Q.E.D.

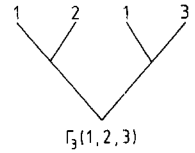
7. The Algorithm: Sophisticated Agenda

We describe first the *multi-stage elimination* tree. There is (up to permuting the outcomes) exactly one such tree for each size of A.

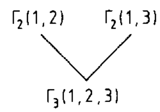
$A = \{1, 2\}$



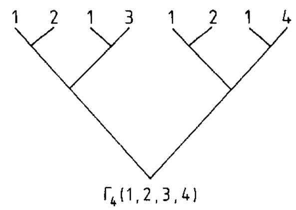
$A = \{1, 2, 3\}$



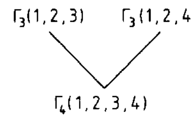
i.e.,



$A = \{1, 2, 3, 4\}$

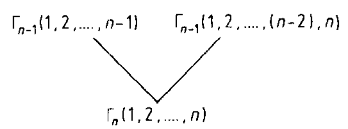


i.e.,



and so on:

$A = \{1, 2, \ldots, n\}$



(11)

This tree was introduced by Miller (1977) and Moulin (1979). Consider the induction formula (11) and suppose we start from the origin and move upward on the tree: the first decision is a choice between $\Gamma_{n-1}(1, 2, \ldots, n-1)$ – thus eliminating n for good – and $\Gamma_{n-1}(1, 2, \ldots, (n-2), n)$ – thus eliminating $(n-1)$ for good. Hence reading the tree upward (i.e., viewing it as an extensive game form, see Moulin (1979)) amounts to a successive elimination procedure: first $n-1$ or n, then $(n-1)$ or whichever survived from $(n-1)$, n, and so on. Of course, solving the tree downward (to compute its associated choice function $T \to \Gamma(T)$) is quite another story:

Theorem 3 *(Shepsle and Weingast 1982). The choice function $T \to \Gamma_n(T)$ achieved by the multi-stage elimination tree on $A = \{1, 2, \ldots, n\}$ is given by the following algorithm called sophisticated agenda.*

$$\begin{cases} \alpha_1 = 1\,; \ \textit{for} \ \ i = 2, \ldots, n\colon \ \alpha_i = \begin{cases} i & \textit{if} \ \ iT\alpha_{i-1} \ \textit{and} \ iT\alpha_{i-2} \ \textit{and} \ \ldots iT\alpha_1 \\ \alpha_{i-1} & \textit{if for some} \ \ j,\ 1 \leqq j \leqq i-1\colon \alpha_j T i \end{cases} \\ \alpha_n = \Gamma_n(T)\ . \end{cases} \tag{12}$$

Proof. By induction on n. For $n = 1, 2$ the claim is obvious. Suppose it is true up to $n-1$ and consider a tournament T on $\{1, 2, \ldots, n\}$. Denote T^{-n} (resp. $T^{-(n-1)}$) the restriction of T to $\{1, 2, \ldots, (n-1)\}$ (resp. to $\{1, 2, \ldots, (n-2), n\}$). Set $\beta_{n-1} = \Gamma_{n-1}(T^{-n})$: by the induction assumption, it is computed by (12), which raises a sequence $\beta_1, \ldots, \beta_{n-1}$. Similarly $\gamma_{n-1} = \Gamma_{n-1}(T^{-(n-1)})$ is computed by (12) along a sequence $\gamma_1, \ldots, \gamma_{n-1}$. Next by the inductive definition of Γ_N, (11), outcome $a = \Gamma_n(T)$ is worth:

$$a = \beta_{n-1} \quad \text{if} \quad \beta_{n-1} T \gamma_{n-1} \ ; \qquad a = \gamma_{n-1} \quad \text{if} \quad \gamma_{n-1} T \beta_{n-1} \ . \tag{13}$$

Now consider the sequence $\alpha_1, \ldots, \alpha_n$ defined by (12) from $\{1, 2, \ldots, n\}$. By construction we have

$$\{\text{for} \quad i = 1, \ldots, n-2 \colon \alpha_i = \beta_i = \gamma_i\} \quad \text{and} \quad \{\alpha_{n-1} = \beta_{n-1}\} \ .$$

In order to prove $\alpha_n = a$ we distinguish three cases.

Case 1. $\gamma_{n-1} = \gamma_{n-2}$.
Then for some i, $1 \leqq i \leqq n-2$, $\gamma_i T n$. Since $\alpha_i = \gamma_i$ this implies $\alpha_n = \alpha_{n-1}$. On the other hand, $\beta_{n-1} T \beta_{n-2}$ (by (12) again; remember the convention xTx), so from $\gamma_{n-1} = \beta_{n-2}$ we get $\beta_{n-1} T \gamma_{n-1}$. Therefore by (13) $a = \beta_{n-1}$ so $a = \alpha_{n-1} = \alpha_n$ and we are home.

Case 2. $\gamma_{n-1} = n$ *and* $\beta_{n-1} = \beta_{n-2}$.
In this case n beats any α_i, $1 \leqq i \leqq n-2$ (definition of γ_{n-1}) *and* β_{n-1} is one of them. So n beats $\alpha_1, \ldots, \alpha_{n-1}$ and $\alpha_n = n$. On the other hand, $\gamma_{n-1} = n$ beats $\beta_{n-1} = \alpha_{n-2}$ so by (13) we get $a = n$.

Case 3. $\gamma_{n-1} = n$ *and* $\beta_{n-1} = n-1$.
Here both n and $(n-1)$ beat $\{\alpha_1, \ldots, \alpha_{n-2}\}$. Thus $\alpha_n = n$ if $nT(n-1)$ and $(n-1)$ if $(n-1)Tn$. By (13), on the other hand, a is just the same. Q.E.D.

Algorithm (12) was first introduced in the context of strategic voting, whence its name (Shepsle and Weingast 1982). It describes the sophisticated equilibrium of the voting by successive elimination game form (described above) with agenda $n, n-1, \ldots, 1$.

Whenever A is ordered by an arbitrary permutation $\sigma = \{a_1, \ldots, a_n\}$ we also denote by $\alpha_1, \ldots, \alpha_n$ the corresponding sequence in (12) (where $\alpha_1 = a_1$, $\alpha_i = a_i$ or α_{i-1} etc.) and we set $\alpha_n = \Gamma_\sigma(T)$.

We study now the choice functions $T \to \Gamma_\sigma(T)$ in the perspective suggested by Sect. 6. First a straightforward argument shows that Γ_σ is monotonic. Next we observe that the sophisticated agenda algorithm ends up within the uncovered set.

Lemma 11. *Fix an outcome set A and a tournament T. Let σ be any ordering of A and consider the set PW_σ of provisional winners $\{\alpha_1, \ldots, \alpha_n\}$ in algorithm (12). Then α_n beats every other element in PW_σ. Moreover, no outcome (outside PW_σ) can beat all elements of PW_σ. In particular, $\Gamma_\sigma(T) \in uc(T)$ all $T \in \tau$.*

Proof. By the very definition of the algorithm, if $\alpha_{i+1} \neq \alpha_i$ then α_{i+1} must beat every outcome in the set $\{\alpha_1, \ldots, \alpha_i\}$, hence our first statement. To prove the second statement suppose b outside PW_σ beats every element in PW_σ. Say $b = a_k$ for some i, $1 \leqq k \leqq n$. Then b beats $\alpha_1, \ldots, \alpha_{k-1}$ and so equals α_k, a contradiction. The last statement holds true by definition of the covering relation. Q.E.D.

Strikingly enough, we can say more.

Theorem 4 *(Banks 1985). The sophisticated agenda algorithm ends up in the top cycle of the uncovered set:*

$$\Gamma_\sigma(T) \in tc[uc(T)] \quad \textit{all ordering } \sigma \textit{ of } A, \textit{ all tournament } T \ .$$

Proof. We state first an auxiliary result. Let T be any tournament and a an outcome outside $B = tc[uc(T)]$. Then

$$\{a \text{ is covered by some } b \in B\} \quad \text{and/or} \quad \{\text{for all } b \in B,\ bTa\} \ . \tag{14}$$

To prove the claim, suppose a is not covered by any $b \in B$. If a itself is uncovered, then bTa holds by definition of the top cycle. If a is covered, it is by some $d \in uc(T) \setminus B$. Now take any $b \in B$, then bTd (definition of tc) so bTa (by definition of the covering relation (4)).

Now to the proof of Theorem 2. Fix an ordering σ and a tournament T. Supposing α_n is not in B, we derive a contradiction. Suppose first that some α_i, $\alpha_i \neq \alpha_n$ belongs to B. Since $\alpha_n T \alpha_i$, (12), we derive from (14) that α_n is covered by some $b \in B$, which contradicts Lemma 11. Thus all α_i, $i = 1, \ldots, n$, are outside B. Suppose next no one of $\alpha_1, \ldots, \alpha_n$ is covered by an outcome in B. Then by (14) any $b \in B$ beats all $\alpha_1, \ldots, \alpha_n$, a contradiction of Lemma 11. So there is a largest integer i, $i \leqq n-1$, such that α_i is covered by some outcome b in B. Then b beats $\alpha_1, \ldots, \alpha_i$ (since α_i beats all its previous winners) as well as $\alpha_{i+1}, \ldots, \alpha_n$ (by (14)), a contradiction of Lemma 11 again. Q.E.D.

The algorithm of sophisticated voting requires *at most* to use the $n(n-1)/2$ pairwise comparisons of tournament T: this happens if, and only if, tournament T is an ordering and $\sigma = \{a_1, a_2, \ldots, a_n\}$ is exactly the reverse ordering. Thus, in general, it is cheaper to compute the outcome of this algorithm than the Copeland winners.

Viewed as a (single-valued) choice correspondence, the mapping $T \to \Gamma_\sigma(T)$ lacks *Neutrality*: the particular choice of σ breaks the symmetry across outcomes. We can restore this by considering all orderings of T and their corresponding winners. Formally,

$$S^*(T) = \{a \mid \text{for some ordering } \sigma \text{ of } A\colon a = \Gamma_\sigma(T)\} \ .$$

Lemma 12 *(Banks 1985). An outcome belongs to $S^*(T)$ if and only if there exists a subset $B \subseteq A$ such that*

i) the restriction of T to B is an ordering, with a as its top outcome;
ii) no outcome outside B beats every outcome in B.

Proof. If: Given B that satisfies i) and ii) order it according to T, say $B = \{a_1, a_2, \ldots, a_p = a\}$ (where $i < j \Rightarrow a_j T a_i$). Then for any ordering σ of A starting with $\{a_1, \ldots, a_p\}$ the sophisticated agenda algorithm has a_p for final winner.

Only if: If $a = \Gamma_\sigma(T)$ then $B = PW_\sigma$ satisfies i) and ii) (Lemma 11). Q.E.D.

The uncovered set $uc(T)$ can be defined in a similar way as $S^*(T)$: outcome a belongs to $uc(T)$ if and only if there exists a subset $B \subseteq A$ such that i) $a \in B$ and a beats every outcome in $B \setminus a$, ii) no outcome outside B beats every outcome in B.

Reinforcing Lemma 5 we have now

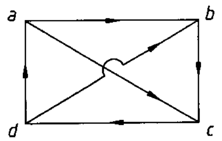
Lemma 13 *(Banks 1985). The set $S^*(T)$ is a singleton if and only if T has a Condorcet winner. If $S^*(T)$ is not a singleton then its size is at least three.*

Proof. Suppose T has no Condorcet winner. For all $a \in B$ there is b that beats a: for any ordering σ starting with $a_1 = a$, $a_2 = b, \ldots$ the output $PW_\sigma(T)$ cannot be a. Hence $S^*(T)$ contains at least two elements, say x, y. We prove now that it contains also a third one. Say that xTy. Since $y \in S^*(T)$ is not covered by x, there is z such that yTz, zTx (Lemma 4). Consider an ordering σ starting with $a_1 = x$, $a_2 = z, \ldots$. The algorithm of provisional winner, then, cannot produce x (by zTx we have $\alpha_2 = z$) nor y (since $x = \alpha_1$ beats y). Q.E.D.

To emphasize the power of Theorem 4, one needs to be convinced that the inclusion $S^*(T) \subset tc(uc(T))$ need not be an equality. Actually, we have

Lemma 14. *If $|tc(uc(T))| \leqq 4$ then $S^*(T) = tc(uc(T))$. If $|tc(uc(T))| \geqq 5$ then $S^*(T) \subsetneqq tc(uc(T))$ is possible.*

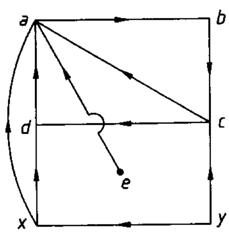
Proof. If T has no Condorcet winner, the size of $tc(uc(T))$ is at least 3 (Lemma 5) and so is the size of $S^*(T)$ (Lemma 13). Suppose now $tc(uc(T)) = \{a, b, c, d\}$. Up to relabeling the outcomes, the restriction of T to $tc(uc(T))$ must be:



Suppose a is not in $S^*(T)$. Since the restriction of T to $\{a, b, c\}$ is transitive with top outcome a, there exists (Lemma 12) an outcome x beating a, b, and c. We can even choose x uncovered (if it is not, take some uncovered outcome x' that covers x). Since x beats some outcome in the top cycle of $uc(T)$, it is itself in $tc(uc(T))$ so $x = d$, a contradiction.

Suppose next b is not in $S^*(T)$: since b is not covered by a in T, there is a z (outside $tc(uc(T))$) such that bTz, zTa. Now $\{b, c, z\}$ is transitive for T, so (Lemma 12) there exists x beating b, c, and z. Again x can be chosen uncovered, hence in $tc(uc(T))$. But none of a, d can do the job. Similar arguments show that c and d both belong to $S^*(T)$.

To prove the second statement of Lemma 14, we exhibit an example:



Every nonmarked arrow goes down

Here $uc(A)=\{abcde\}$ (x is covered by c, y by b)

$tc(uc(A))=\{abcde\}$ (a cycle is $aTbTcTdTeTa$)

yet $S^*(T)=\{abcd\}$. Indeed $e \notin S^*(T)$ since the subsets B over which T is transitive and with e on top are:

e, x, a: beaten by c

e, a, y: beaten by d

e, y, x: beaten by b .

Remark. The first statement in Lemma 14 and its proof are taken from Banks (1985).

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