On the Motive Mechanism of Snakes and Fish

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I. Introduction

While observing the behavior of fish one can easily notice that in its onward movement a fish moves not only its tail and fins but also the whole of its body. Snakes move in water and on land only by oscillatory bendings of the entire body. The bending of the whole body appears to be an essential factor of the tractive force in the motion of this kind of living being. In their paper "On a Principle for Creating a Tractive Force of Motion", M. A. LAVRENTIEV and M. M. LAVRENTIEV have suggested a simple method of discribing this kind of motion by taking into consideration the energetic aspect of the phenomenon. The crux of the method can be traced in the following simple example. At a certain moment of time let there be an elastic bar with constant longitudinal properties in a hard-walled channel with a monotonically changing curvature, and let there be no friction. As the elastic energy of the bent bar in each of its cross-sections is proportional to the square of the curvature, the position of the bar is obviously unstable, and at the next moment of time it has to shift along the channel in the direction of the decrease of the curvature. It is equally evident that the force shifting the bar along the channel is the resultant of all normal binding forces acting on the bar from the channel walls.

The motion of fish-like or snake-like living organisms can be described on this principle in the following way. The role of the hard walls of the channel is to a certain extent played by the medium surrounding the body. For example, in the case of these organisms moving in a liquid medium the role of the channel walls is played by the fluid, which due to its inertness (when acted upon rapidly enough) does not practically shift from the initial position within the time during which there occurs a substantial shift of the body proper. In the case of snakes moving on land the appropriate analogue of the channel is more complicated. First, the role of the channel walls can be played by the grass stems if the motion occurs in thick grass. Secondly, it is possible that when a snake is moving along its body axis the friction is much smaller than when it is moving perpendicularly to it. There are also a number of other hypotheses, but they have little bearing on the main task of this paper.

We shall now consider how the motion proceeds in such hypothetically built "channels". With its muscles relaxed, a grass-snake or any other similar organism can repose within a curved channel of any shape and experience no action from without. However, when contracting its muscles in a certain manner the snake becomes an elastic body moving in a channel in just the same way as an elastic bar. By regulating its muscular contractions, a living organism of this type can thus adjust itself to the shape of the channel so as to move uninterruptedly along it.

From this point on, we shall call all living organisms moving like snakes, grass-snakes, serpents or fish, *i.e.*, by cross-bendings of the whole body, simply *snakes* for short. Leaving aside for the present the physiological peculiarities of these organisms, we shall formulate our principal assumptions about their mechanical properties.

1. The snake is not extensible. This assumption does not impose any essential restrictions on the motion under consideration and at the same time helps to simplify calculations.

2. In the state of muscle relaxation the snake is ideally pliable, *i.e.*, it can deform in every way with no strain or stress.

3. In the state of muscle contraction the snake is an elastic body, whose properties, generally speaking, change lengthwise in an arbitrary way.

The elastic properties of the snake have to be restricted from the physiological point of view. Along with the highest admissible pressure, it seems natural to restrict the value of the maximum bending moment acting in the snake's cross-section. Thus it is required that the pressure p and the bending moments M should satisfy the inequalities

$$|p| \leqq p_0, \tag{1.1}$$

$$|M| \leqq M_0. \tag{1.2}$$

Of these conditions the first, as we shall see later, is not essential, for the action of the concentrated force can be replaced by that of a distributed load satisfying condition (1.1). The second restriction is essential, for it implies the existence of optimal regimes of motion. Generally speaking, condition (1.2) expresses a physiological property of the organism. In paper [*1*] this property is expressed more generally as the requirement that a definite integral determining the total effort of the organism be constant:

$$\int_0^l f(M)\,ds = \text{Const}. \tag{1.3}$$

When the snake is moving in a hard-walled channel, all useful work of its muscles is spent in imparting onward motion to the body and in overcoming the forces of friction. When the snake is moving in a fluid, a certain part of its muscular work is performed in communicating kinetic energy to the fluid. In this sense the efficiency of the motion in water is less than in a channel with hard walls (the friction being taken the same in each case). The two cases are discussed individually below.

II. Motion of a Snake in a Hard-Walled Channel (Planar Case)

1. Deduction of Principal Equations

Let the shape of the channel be given by the equation $y = y(x)$, and let the snake occupy in this channel the segment from $s_1 = \xi(t)$ to $s_2 = \xi(t) + l$, where s is arc length. We adopt the following notations:

$\vec{\mathscr{F}}$ = cross force,	m = mass of the snake per unit length,
$\vec{M}$ = bending moment,	κ = friction coefficient,
K = curvature of a curve,	$\vec{n}$ and $\vec{t}$ = normal and tangent vectors to a curve.

Let us agree to choose the direction of a normal so that its projection on the y-axis is positive, *i.e.*, $n_y > 0$, and consider the pressure p to be positive if it acts in the direction opposite to that of the normal.

In these notations the equation of motion of the elastic bar takes the form [2]:

$$\frac{d\vec{\mathscr{F}}}{ds} = p\vec{n} + m\dot{\xi}^2 K\vec{n} + m\ddot{\xi}\vec{t} + \kappa|p|\vec{t} \tag{2.1}$$

(the dot above ξ designates differentiation with respect to time), while the bending moment $\vec{M}$ is connected with the cross force by the equation

$$\frac{d\vec{M}}{ds} = [\vec{\mathscr{F}}, \vec{t}\,]. \tag{2.2}$$

We take the scalar product of each side of equation (2.1) by the vector $\vec{t}$:

$$\vec{t}\,\frac{d\vec{\mathscr{F}}}{ds} = m\ddot{\xi} + \kappa|p| \tag{2.3}$$

and transform the left-hand side of this equation in the following way:

$$\vec{t}\,\frac{d\vec{\mathscr{F}}}{ds} = \frac{d\mathscr{F}_t}{ds} - \vec{\mathscr{F}}\frac{d\vec{t}}{ds} = \frac{d\mathscr{F}_t}{ds} - \vec{\mathscr{F}}\vec{n}K. \tag{2.4}$$

Since

$$\vec{\mathscr{F}}\vec{n} = +[\vec{\mathscr{F}}, \vec{t}\,]_z,$$

we obtain from (2.2)

$$\vec{\mathscr{F}}\vec{n} = \left(\frac{d\vec{M}}{ds}\right)_z \tag{2.5}$$

(the vector $\vec{M}$ is perpendicular to the plane of the channel). Substituting (2.5) and (2.4) into (2.3), we get

$$m\ddot{\xi} = \frac{d\vec{\mathscr{F}}_t}{ds} - M'K - \kappa|p|. \tag{2.6}$$

We integrate this expression with respect to s from $s_1 = \xi(t)$ to $s_2 = \xi(t) + l$, taking into account the fact that at the snake's ends the tangential component of the cross force is reduced to zero:

$$m\,l\,\ddot{\xi} = -\int_{\xi(t)}^{\xi(t)+l} (M'K + \kappa|p|)\,dS \tag{2.7}$$

(the prime indicates differentiation with respect to S).

We shall now consider how expression (2.7) for the tractive force is obtained by the energetic principle mentioned above. As is known, the bending moment M is connected with the curvature K of the bent elastic bar by the relation

$$M = B \cdot K, \tag{2.8}$$

where B is a value characterizing the elastic properties of the bar. The elastic energy of the bar per unit length is equal to $\frac{1}{2}BK^2$, so the potential energy U of the entire body is expressed by the formula

$$U=\tfrac{1}{2}\int_{\xi}^{\xi+l} B(s)\,K^2(s,\xi)\,dS. \tag{2.9}$$

Hence the tractive force is

$$T=-\frac{\partial U}{\partial \xi},$$

which, when (2.8) is taken into account, gives an expression coinciding with the first term of the right-hand side of (2.7). The second term in (2.7) is the force of friction.

2. Motion without Friction

We shall show now that when there is no friction the maximum tractive force is obtained if the absolute value of the bending moment is constant. Indeed, in this case formula (2.7) has the form

$$T=-\int_{\xi}^{\xi+l} M'K\,ds.$$

We integrate this expression by parts, assuming the moment at the snake's ends to equal zero:

$$T=\int_{\xi}^{\xi+l} MK'\,ds. \tag{2.10}$$

According to (1.1) the moment is restricted in its absolute value by M_0. It is evident that the maximum value of the tractive force is obtained if the absolute value of the moment all over the snake's length is equal to M_0 and is the moment changes sign where the derivative of the curvature does.

Specifically, if the channel has the shape of a sinusoid, then the optimal bending moment must have the shape represented in Fig. 1. Thus the maximum amount of the tractive force is given by

$$T_{\max}=M_0\int_{\xi}^{\xi+l} |K'|\,ds. \tag{2.11}$$

Let the shape of the channel be determined by the quation

$$y=A\sin kx. \tag{2.12}$$

We will assume that a whole number n of wave lengths of a given sinusoid make up the snake's length l. A simple calculation shows that in this case the maximum tractive force has the form

$$T_{\max}=4M_0Ak^2n, \tag{2.13}$$

while the snake's length l is related to the parameters of the sinusoid by the equation

$$l = n \cdot \frac{4E\left(\sin\alpha, \frac{\pi}{2}\right)}{k\cos\alpha}, \tag{2.14}$$

where

$$E\left(\sin\alpha, \frac{\pi}{2}\right)$$

is the Euler function of the second kind, and $\operatorname{tg}\alpha = Ak$. Eliminating n between (2.13) and (2.14), we get

$$T_{\max} = \frac{M_0\, l\, k^2 \sin\alpha}{E\left(\sin\alpha, \frac{\pi}{2}\right)}. \tag{2.15}$$

The tractive force is directed along the tangent to the channel, and the efficiency of the snake's advance is determined by the X-component of this force:

$$T_{\max} \cdot \cos\alpha = \frac{M_0\, l\, k^2 \sin 2\alpha}{2E\left(\sin\alpha, \frac{\pi}{2}\right)}.$$

It is easy to see that with a fixed length l and wave number k this expression has its maximum at

$$Ak \approx 1. \tag{2.16}$$

The result obtained can be formulated as follows: when friction is excluded, the optimal regime of motion is obtained when the bends of the snake's body are such that the amplitude and the wave number are related by (2.16). From the optimal form of the moment, which is represented in Fig. 1, it follows that at the

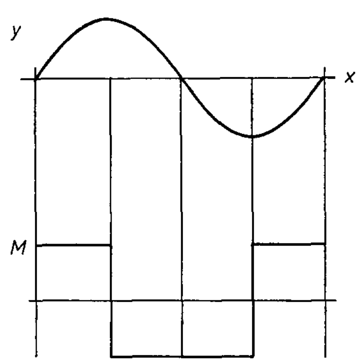

Fig. 1

vertices of the sinusoid there are concentrated moments and hence infinitely large pressures. This contradicts condition (1.1). We shall show that introducing a distributed load does not make any essential changes in the results obtained. For simplicity, let the snake be bent so as to occupy only one wave lentgh. The pressure and the moments are distributed as shown in Fig. 2. Consider a case of small sags, so the linear theory applies, *i.e.*, when the relations

$$p = M'', \quad M = B y'', \quad K = y'' \tag{2.17}$$

hold. In the linear case expression (2.10) is reduced, after a double integration by parts, to the form

$$T = \int_{\xi}^{\xi+l} p\, y'\, dx. \tag{2.18}$$

In the segments where pressure is applied, its absolute value is equal to p_0, and the length of the segment is chosen so that the maximum value of the moment can be equal to M_0. The calculations yield

$$T = 4\frac{M_0 A}{d^2}\left(1 - \cos k d \sqrt{2} + \frac{\sin 2 k d - 2 \sin k d}{2}\right)$$

where $d^2 p_0 = M_0$. As is seen, for $kd \ll 1$ this expression coincides with (2.13) to within terms of the first order of smallness.

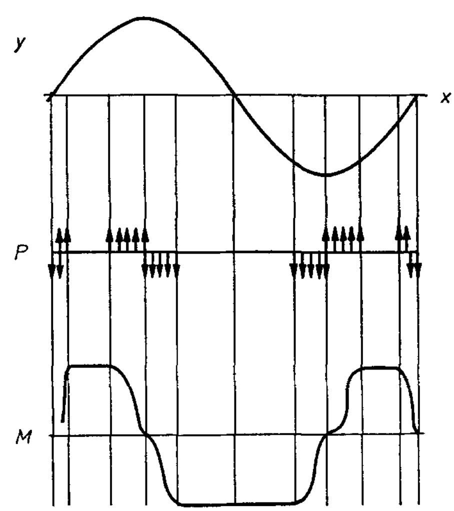

Fig. 2

The dependence of the tractive force on time is determined, naturally, by changing the organism's muscular efforts in time. If, for example, a snake is to gain the maximum speed possible it must evidently move with a constant value of the bending moment. In this case the motion will be uniformly accelerated. The rate of the snake's progress is proportional to the value $T \cdot \tau$ where τ is the time during which the organism can maintain a constant bending moment M_0. Evidently, this time τ depends on the value of the bending moment.

For a man's biceps this dependence is fairly well described by the formula

$$\tau = \tau_* e^{-\frac{M_0}{M_*}}.$$

If we assume that this formula is equally applicable to other arganisms, including the ones under consideration, then the velocity gained by the snake is determined by the expression

$$v=\frac{T}{m\,l}\,\tau_{*}\,e^{-\frac{M_0}{M_*}}.$$

Hence it can easily be shown, by taking into account (2.13), that the velocity has its maximum when $M_0=M_*$.

3. Motion with Friction

Let the snake's length consist in a whole number of half-waves of the sinusoid. Projecting both sides of equation (2.1) on the x-axis and carying out an integration, we obtain

$$\mathscr{F}_x=m\,l_x\,\ddot{\xi}=\int_{\xi_x}^{\xi_x+l_x}(p\,y'-\kappa\,|p|)\,dx, \tag{2.19}$$

where the index "x" stands for the projection of the corresponding values on the x-axis. As is plain, this expression nearly coincides with the linear expression (2.18). The only difference is that in the linear case this formula determines the amount of the entire force of traction, while in the nonlinear case it determines the projection of this force on the x-axis. Its physical sense is obvious: the first term can be called the force of repulsion; the second, that of friction. From (2.19) an important conclusion follows immediately: the segments where $y'<\kappa$ are unfavorable for motion in the sense that in these parts friction exceeds propulsion. Therefore, in those segments of the channel where $y'<\kappa$, the snake must represent a perfectly pliable body.

Thus, in the presence of friction the snake's muscular system has to operate so that pressures can act only in those segments of the channel where

$$y'\geqq\kappa. \tag{2.20}$$

Motion with a constant moment contradicts condition (2.20), so this regime is impossible in the presence of friction.

Let us consider two types of motion with friction.

It is apparant that the most economical regime of motion is realized when the ratio of the force of propulsion to that of friction is a maximum one. This ratio is reflected in the inequality

$$\frac{\int_{\xi_x}^{\xi_x+l} p\,y'\,dx}{\int_{\xi_x}^{\xi_x+l} \kappa\,|p|\,dx}\leqq\frac{y'_{\max}}{\kappa} \tag{2.21}$$

where $y'_{\max}$ is the maximum value of the derivative.

If the channel has the shape of sinusoid (2.12), then it can easily be shown that the sign of equality in (2.21) corresponds to a state where concentrated forces

$$p=f_n\,\delta\left(x-\frac{n\,\pi}{k}\right)$$

are acting at the points with maximum y', see Fig. 3. Indeed, in this case

$$\frac{\int_{\xi_x}^{\xi_x+l} p\, y'\, dx}{\int_{\xi_x}^{\xi_x+l} \kappa\, |p|\, dx} = \frac{A k \sum f_n}{\kappa \sum f_n} = \frac{A k}{\kappa}. \tag{2.22}$$

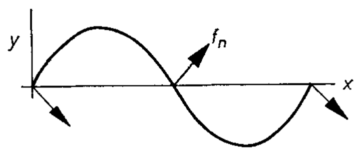

Fig. 3

The second regime of motion corresponds to the maximum value of the resultant force of traction (2.19). Consider the pressure distribution near the points with the maximum values of the derivative. We shall assume the channel to be sinusoidal as before. The form of the pressures, concentrated forces, and moments are approximately represented in Fig. 4. Consider a linear case when $p = M''$,

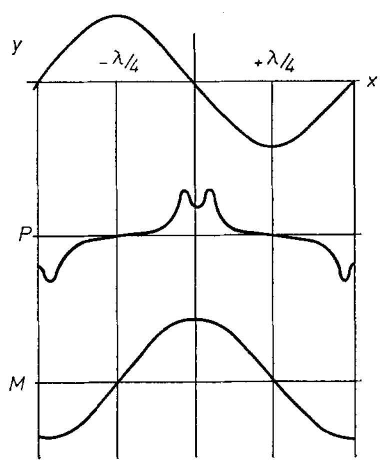

Fig. 4

and require the maximum value of the bending moment to be restricted: $M_{\max} = M_0$. Look for a solution in the form of concentrated forces:

$$p = \sum_n f_n\, \delta(x - x_n). \tag{2.23}$$

Substituting (2.23) into (2.19), we get

$$\mathscr{F}_x = \sum_n f_n\, y'(x_n) - \kappa f_n \tag{2.24}$$

(the sign of modulus in the second term can be omitted if we consider a segment where $f_n > 0$).

Further, we have (see Fig. 4)

$$M'(0)=0, \quad M\left(-\frac{\lambda}{4}\right)=0.$$

Since $p=M''$

$$M(x)=\int_{-\frac{\lambda}{4}}^{x} dx \int_{0}^{x} p(t)\,dt, \quad M(0)=M_0.$$

Hence, after substituting p into (2.23) and integrating by parts, we obtain

$$M_0=\sum_n \left(x_n+\frac{\lambda}{4}\right) f_n. \tag{2.25}$$

The maximum of the function (2.24) under condition (2.25) is determined in the usual manner by the relations:

$$\frac{\partial}{\partial f_n}(\mathscr{F}_x-\mu M_0)=0, \quad \frac{\partial}{\partial x_n}(\mathscr{F}_x-\mu M_0)=0,$$

which leads to the equation

$$y''(x_n)\left(x_n+\frac{\lambda}{4}\right)+y'(x_n)=0. \tag{2.26}$$

Thus, in a segment of the curve of length $\lambda/4$, a concentrated force is acting at the point determined by the equation (2.26). If there is no friction, $\kappa=0$, then the solution of (2.26) is

$$x=-\frac{\lambda}{4}, \quad y'=0, \tag{2.27}$$

i.e., concentrated moments are applied to the vertices of the sinusoid, and we obtain a motion with a constant value of the moment. If the friction coefficient has the value $y'=\kappa$, then from (2.26) it follows that

$$x=0, \quad y''=0,$$

i.e., we have the case represented in Fig. 3.

Consider the following simple example. Let the snake's body fit exactly into one length of the wave and suppose, the forces act at the points with the maximum

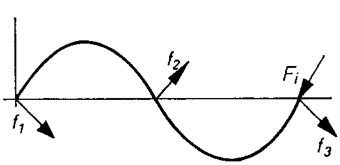

Fig. 5

values of the derivative. Assume that the entire mass of the snake is concentrated in its "head". This enables us to introduce into consideration a force of inertia $\mathscr{F}_i$ applied to the beginning of the curve and directed along the tangent to the channel (Fig. 5). From the condition of the body's equilibrium under the action of the system of forces represented in Fig. 5 we obtain the following relations

$$f_2=2f_1, \quad f_3=0, \quad \mathscr{F}_i=2f_1 \sin 2\alpha \tag{2.28}$$

where

$$\operatorname{tg} \alpha = y'_{\max} = A\,k.$$

Now use the restriction (1.2) imposed on the value of the bending moment. To do so one has to calculate the maximum value of the moment with respect to the curve and to equate it to M_0. Simple calculations yield the following result:

$$\mathscr{F}_i=\begin{cases} \dfrac{4M_0 k}{\pi} \sin\alpha, & A\,k<1 \\ \dfrac{4M_0 k}{\pi} \dfrac{\sin\alpha}{\sqrt{\operatorname{tg}^4\alpha-1}+\dfrac{\pi}{2}+\arcsin \operatorname{ctg}^2\alpha}, & A\,k>1. \end{cases} \tag{2.29}$$

The value k for a snake of length l is determined by the expression

$$k=\frac{4E\left(\sin\alpha, \dfrac{\pi}{2}\right)}{l\cos\alpha}. \tag{2.30}$$

It can be shown that for a fixed length l, the tractive force $\mathscr{F}_i$ determined by formula (2.29) and the value of its projection on the x-axis have a maximum value for curves whose amplitude and wave number are bound by the correlation

$$A\,k \approx 1. \tag{2.31}$$

Thus in Section 2 we have shown that in the presence of dry friction the maximum force of traction is attained when the forces are applied to those segments of the curve where the derivative is larger than the coefficient of friction. The body's bending under the optimal regime of motion, as well as in the case when there is no friction, must satisfy the conditions (2.16) and (2.31).

III. Motion of a Snake in Water

We shall now consider how a body which is able to change its shape under the action of its internal forces can move in a fluid. It is well known that in a real fluid, at the expense of viscosity forces, vortices can stream off the body, resulting in a force of traction that sets the body in motion. Such a mechanism of creating a tractive force is considered in the flapping wing theory [*3*, *4*] in the scheme of an ideal incompressible fluid which is known to admit the existence of tangential velocity breaks. The force of traction in the thin wing theory is thus created at the expense of a trailing vortex sheet whose length increases with time. The possibility that a deformable body can create a tractive force without forming whirls has been noted by some authors [*3*], but as far as we know it has not been investigated in detail. According to the above principle, the motion of a snake in water can certainly be accompanied by the appearance and shedding of vortices. Nevertheless, it is of interest to consider precisely those regimes of movement where the force of traction arises without vortex formation.

1. Some General Observations on Vortex-free Motion of a Deformable Body in an Ideal Fluid

Let a body which can arbitrarily change its shape be placed in an unbounded, weightless, ideal fluid. If a potential function φ which is regular everywhere outside the body corresponds to the flow of the liquid, then the hydrodynamic reactions acting on the body, as is known [3], are expressed by the resultant force vector $\vec{R}$ and the resultant momment vector $\vec{L}$:

$$\vec{R}=\frac{d}{dt}\rho\int_s \varphi\,\vec{n}\,ds, \tag{3.1}$$

$$\vec{L}=\frac{d}{dt}\rho\int_s \varphi(\vec{r}\times\vec{n})\,ds, \tag{3.2}$$

where ρ = fluid density,

$\vec{n}$ = external normal to the body's surface S,

$\vec{r}$ = position vector of fluid particles.

These expressions hold for a fixed co-ordinate system, the vector $\vec{L}$ being calculated with respect to the origin. Hence for a body with a mass m, moving at a velocity $\vec{v}$, we obtain the equations of motion

$$\frac{d\,m\,\vec{v}}{dt}=\vec{R}, \tag{3.3}$$

$$\frac{d\vec{Q}}{dt}=\vec{L}, \tag{3.4}$$

where $\vec{Q}$ is a moment of momentum relative to the fixed origin. Integrating these equations with respect to time yields

$$m\,\vec{v}=\rho\int_s \varphi\,\vec{n}\,ds+\vec{c}_1, \tag{3.5}$$

$$\vec{Q}=\rho\int_s \varphi(\vec{r}\times\vec{n})\,ds+\vec{c}_2, \tag{3.6}$$

where

$$\vec{c}_1=\left(m\,\vec{v}-\rho\int\varphi\,\vec{n}\,ds\right)_{t=t_0},\qquad \vec{c}_2=\left(\vec{Q}-\rho\int\varphi(\vec{r}\times\vec{n})\,ds\right)_{t=t_0}.$$

For the further description it is convenient to introduce into equation (3.6) the moment of momentum Q_0 and an impulse of the moment of forces relative to the body's center of inertia. We shall denote the position vector of the body's center inertia by $\vec{r}_0$ and the position vector of the particles relative to the centre of inertia by $\vec{r}_1$. Then

$$\vec{Q}=\vec{Q}_0+\vec{r}_0\times m\,\vec{v}, \tag{3.7}$$

and

$$\rho\int_s \varphi(\vec{r}\times\vec{n})\,ds=\rho\int_s \varphi(\vec{r}_0\times\vec{n})\,ds+\rho\int_s \varphi(\vec{r}_1\times\vec{n})\,ds. \tag{3.8}$$

Combining (3.5) – (3.8) we obtain

$$\vec{Q}_0=\rho\int_s \varphi(\vec{r}_1\times\vec{n})\,ds+\text{const}. \tag{3.9}$$

It is convenient to express the potential φ in a co-ordinate system whose origin at a certain moment of time t is the center of inertia and whose axes are parallel to those of the fixed co-ordinate system. The boundary condition for φ has the form

$$\frac{\partial \varphi}{\partial n} = v_i n_i, \tag{3.10}$$

where v_i = velocity components of the points of the surface in the fixed co-ordinate system, and

n_i = components of a unit normal vector.

At infinity the velocity of the fluid $\partial \varphi / \partial x_i$ tends to zero. One can expand the motion of the surface into: a) translation at the velocity of the centre of inertia, b) rotation at an angular velocity $\vec{\omega}$ and c) surface deformation proper:

$$v_i = V_i + (\vec{\omega} \times \vec{r}_1)_i + u_i \tag{3.11}$$

where u_i are surface motion velocity components in a co-ordinate system moving with the center of inertia and rotating at the angular velocity ω.

We look for a potential in the form

$$\varphi = V_i \varphi_i + \omega_i \varphi_{3+i} + \bar{\varphi}, \tag{3.12}$$

where φ_i, φ_{3+i} are harmonic functions which on the boundary of the body satisfy the conditions

$$\begin{aligned} \frac{\partial \varphi_i}{\partial n} &= n_i \\ \frac{\partial \varphi_{3+i}}{\partial n} &= (\vec{r}_1 \times \vec{n})_i . \end{aligned} \tag{3.13}$$

Due to (3.10)–(3.13) the boundary condition for $\vec{\varphi}$ has the form

$$\frac{\partial \bar{\varphi}}{\partial n} = u_i n_i, \tag{3.14}$$

while at infinity

$$\frac{\partial \bar{\varphi}}{\partial x_i} \to 0 .$$

The equations of motion (3.5), (3.6), after the substitution of (3.12), (3.13) into them, have the form

$$m V_i = \rho \int_s V_k \varphi_k \frac{\partial \varphi_i}{\partial n} ds + \rho \int_s \omega_k \varphi_{3+k} \frac{\partial \varphi_i}{\partial n} ds + \rho \int \bar{\varphi} n_i ds + c_{1i}, \tag{3.15}$$

$$Q_{0i} = \rho \int_s V_k \varphi_k \frac{\partial \varphi_{3+i}}{\partial n} ds + \rho \int_s \omega_k \varphi_{3+k} \frac{\partial \varphi_{3+i}}{\partial n} ds + \rho \int \bar{\varphi} \frac{\partial \varphi_{3+i}}{\partial n} ds + c_{2i} . \tag{3.16}$$

For further discussion it is necessary to define more exactly the moment of momentum $\vec{Q}_0$. We shall condier motions in which at the initial instant t_0 and the final instant t_1 the body changes its shape as a solid. For a solid body,

$$Q_{0\,i}=\mathscr{I}_{ik}\,\omega_k,$$

where $\mathscr{I}_{ik}$ is a tensor of inertia relative to the centre of inertia. We shall introduce the notations:

$$V_4=\omega_1, \quad V_5=\omega_2, \quad V_6=\omega_3,$$

and

$$-\rho\int_s \varphi_i\frac{\partial\varphi_k}{\partial n}\,ds=\lambda_{ik}, \tag{3.17}$$

$$I_{ik}=\begin{cases} m\,\delta_{ik} & i\leqq 3,\ k\leqq 3\\ \mathscr{I}_{ik} & i\geqq 3,\ k\geqq 3\\ 0 & \text{for other values of } i \text{ and } k.\end{cases}$$

Then, taking into account that $u_i=0$ for $t=t_0$ and $t=t_1$, we can write (3.15) and (3.16) in the form

$$[(I_{ik}+\lambda_{ik})\,V_k]_{t=t_1}=[(I_{ik}+\lambda_{ik})\,V_k]_{t=t_0}. \tag{3.18}$$

Hence

$$V_k|_{t=t_1}=[(I_{ik}+\lambda_{ik})_{t=t_0}]^{-1}\,[(I_{ik}+\lambda_{ik})\,V_k]_{t=t_0}. \tag{3.19}$$

Note that in this case the matrix $I_{ik}+\lambda_{ik}$ is positive-definite since $\frac{1}{2}(I_{ik}+\lambda_{ik})\,V_i\,V_k$ is the sum of the kinetic energies of the fluid and the body, and a solution of (3.18) always exists.

Hence, in particular, the following important conclusion holds: *if at the initial moment of time a body is at rest and after a certain period of time turns into a solid body, then at the moment of solidification the motion ceases.*

This conclusion is readily generalized to the case of the motion of a body having a hard core and an inertialess deformable envelope.

Indeed, in this case, from (3.15), (3.16) and (3.19), we obtain

$$\begin{aligned} V_k|_{t=t_1}=\Bigg\{&[(I_{ik}+\lambda_{ik})\,V_k]_{t=t_0}+\left[\int_s\bar\varphi\frac{\partial\varphi_i}{\partial n}\,ds\right]_{t=t_1}-\\ &-\left[\int_s\bar\varphi\frac{\partial\varphi_i}{\partial n}\,ds\right]_{t=t_0}\Bigg\}\,[(I_{ik}+\lambda_{ik})_{t=t_1}]^{-1}.\end{aligned} \tag{3.20}$$

If at the initial moment t_0 and at the final moment t_1, the form and the motion of the envelope in a co-ordinate system moving progressively together with the centre of inertia coincide, then

$$\left[\int_s\bar\varphi\frac{\partial\varphi_i}{\partial n}\,ds\right]_{t=t_0}=\left[\bar\varphi\frac{\partial\varphi_i}{\partial n}\,ds\right]_{t=t_1}$$

and hence

$$V_k|_{t=t_1}=[(I_{ik}+\lambda_{ik})\,V_k]_{t=t_0}\,[(I_{ik}+\lambda_{ik})_{t=t_1}]^{-1}=V_k|_{t=t_0}. \tag{3.21}$$

Thus, for a body with a hard core and an inertialess deformable envelope, the following assertion holds: *if, in a co-ordinate system connected with the center of inertia, the body performs periodic changes of shape, then the velocity of its motion also changes periodically.*

2. *A Partial Case of Vortex-free Motion*

We shall consider a case of motion in a fluid without vortex formation, when the deformation of the body in its own co-ordinate system is represented by a travelling wave.

Let the body be a cylindrical surface parallel to the z-axis. In its own co-ordinate system the projection of this surface has the form

$$y=y(\tau), \quad \tau=x+f(t). \tag{3.22}$$

If the motion is plane, the resultant vector $\vec{R}$ has the form

$$\vec{R}=i\frac{d}{dz}\int_s \rho\varphi\, dz \tag{3.23}$$

where

$$z=x+iy.$$

Hence the force acting along the x-axis is expressed by the formula

$$R_x=-\frac{d}{dt}\int_s \rho\varphi\, dy.$$

Denoting by m the body's mass falling on a band with a width h, we shall write the equation of motion

$$m\frac{dV}{dt}=-h\frac{d}{dt}\int_s \rho\varphi\, dy$$

which, after integration, gives

$$mV=-h\int_s \rho\varphi\, dy \tag{3.24}$$

if the body was initially at rest.

In this case on the body's surface y depends on x and t in conformity with (3.22). Lettring a prime denote differentiation with respect to x, we obtain from (3.24)

$$mV=-h\int_0^l \rho\varphi y'\, dx. \tag{3.25}$$

On the surface of the body the potential φ satisfies the boundary condition

$$\frac{\partial\varphi}{\partial n}=\dot{y}-Vy' \tag{3.26}$$

(the dot stands for differentiation with respect to time). Since, due to (3.22),

$$y'=\frac{dy}{d\tau}, \quad \dot{y}=\dot{f}\frac{dy}{d\tau},$$

we obtain from (3.25) and (3.26):

$$mV=-\frac{\rho h}{\dot{f}-V}\int_0^l \varphi\frac{\partial\varphi}{\partial n}\, dx. \tag{3.27}$$

The kinetic energy E_1 of the body's progressive motion is equal to $(mV^2)/2$, and that of the fluid is expressed by the equation

$$E_2 = -\frac{\rho h}{2} \int_0^l \varphi \frac{\partial \varphi}{\partial n} dx.$$

Consequently, from (3.27) we have

$$E_1 = \tfrac{1}{2} m V \dot{f} - E_2. \tag{3.28}$$

When, as we assume throughout this chapter, there is no friction, the total work performed by the body turns into the kinetic energy of the fluid and the body. Since in addition to its progressive motion, the body is also moving in its own co-ordinate system, its kinetic energy is the sum of the energy of the progressive motion and that of the proper motion.

The latter is evidently calculated from the formula

$$E_3 = \int_0^m \tfrac{1}{2} \dot{y}^2 dm.$$

If the mass is distributed uniformly along the body, then this becomes

$$E_3 = \frac{m}{2l} \int_0^l \dot{y}^2 dx. \tag{3.29}$$

The efficiency η is defined as the ratio of the kinetic energy of progressive motion to the total work performed by the body,

$$\eta = \frac{E_1}{E_1 + E_2 + E_3}. \tag{3.30}$$

Substituting (3.28) and (3.29) into this expression, we get

$$\eta = \left[\frac{\dot{f}}{V} + \frac{1}{l V^2} \int_0^l \dot{y}^2 dx\right]^{-1}. \tag{3.31}$$

We try to obtain the potential φ in the form

$$\varphi = [\dot{f}(t) - V]\, \varphi_1. \tag{3.32}$$

The boundary condition (3.26) can be satisfied if we assume that

$$\frac{\partial \varphi_1}{\partial n} = \frac{dy}{d\tau}. \tag{3.33}$$

Substituting (3.32) into (3.25), we obtain

$$m V = -\rho h \int_0^l (\dot{f} - V)\, \varphi_1 \frac{\partial \varphi_1}{\partial n} dx, \tag{3.34}$$

whence, denoting the virtual mass of fluid by

$$\mu = -\rho h \int_0^l \varphi_1 \frac{\partial \varphi_1}{\partial n} dx, \tag{3.35}$$

we have

$$V = \dot{f}\frac{\mu}{\mu+m}. \tag{3.36}$$

Taking into account (3.36) and (3.22), we can write expression (3.31) for the efficiency in the form

$$\eta = \left\{1+\frac{m}{\mu}+\left(1+\frac{m}{\mu}\right)^2 \cdot \frac{1}{l}\int_0^l \left(\frac{dy}{d\tau}\right)^2 dx\right\}^{-1}. \tag{3.37}$$

We shall carry through the calculation of the efficiency in one concrete case. Let

$$y(\tau) = A \sin k[x+f(t)]. \tag{3.38}$$

The given form of the travelling wave corresponds to the motion in a hard-walled channel of a sinusoidal form. To simplify the problem we assume the body to be infinitely long and the amplitude of the bends to be much smaller than the length of the wave. Thus the boundary condition can be regarded as satisfied on the real axis.

Let us fix a point on the body and consider the motion of the fluid in a fixed co-ordinate system whose origin at a given moment of time coincides with this point. In this case, condition (3.33) has the form

$$\left(\frac{\partial \varphi_1}{\partial y}\right)_{y=0} = A k \cos k\tau. \tag{3.39}$$

It can easily be shown that the complex potential

$$W_1 = \varphi_1 + i\psi_1 = -A e^{kz}, \qquad z = \tau + iy \tag{3.40}$$

satisfies this boundary condition. Hence we obtain

$$\varphi_1 = A e^{-ky} \cos k\tau. \tag{3.41}$$

Substituting (3.39) and (3.41) for $y=0$ into (3.35), we calculate the amount of the virtual mass falling on one wave length:

$$\mu_1 = -\rho h \int_0^{\frac{2\pi}{k}} \varphi_1 \frac{\partial \varphi_1}{\partial y} d\tau = \pi \rho A^2 h. \tag{3.42}$$

The calculation of the integral in (3.37) over one wave length leads to the expression

$$\int_0^{\frac{2\pi}{k}} \left(\frac{dy}{d\tau}\right)^2 d\tau = \pi k A^2. \tag{3.43}$$

Substituting (3.42) and (3.43) into (3.37), we obtain

$$\eta = \left[1+\frac{m}{\mu_1}+\left(1+\frac{m}{\mu_1}\right)^2 \frac{\mu_1 k}{\rho h l}\right]^{-1} \tag{3.44}$$

where

$$\mu_1 = \pi \rho A^2 h.$$

From (3.44) it follows that with a fixed wave number k the efficiency reaches its maximum value when

$$\mu_1^2 = m^2 \left(1 + \frac{\rho\, l\, h}{m\, k}\right). \tag{3.45}$$

Since the legnth of the sinusoid is bound with A and k by relation (2.30), we obtain

$$l = \frac{4E\left(\sin\alpha, \frac{\pi}{2}\right)}{k \cos\alpha}, \qquad \operatorname{tg}\alpha = A\,k \tag{3.46}$$

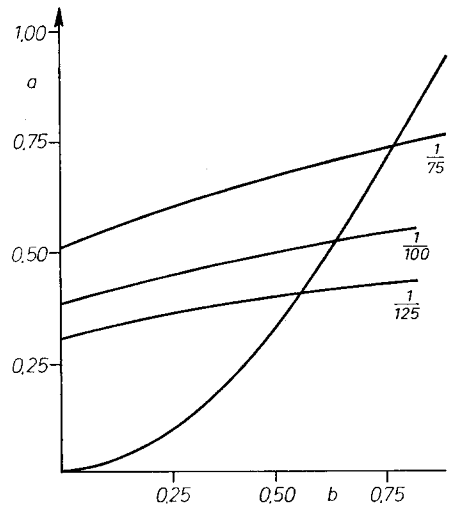

Fig. 6

whence expression (3.45) is an equation relative to A and k. With a fixed length of the body l the second equation gives correlation (3.46). After introducing dimensionless values we get

$$a = \frac{m\,k^2}{\rho\, h}, \qquad b = A^2 k^2, \qquad \beta = \frac{m}{\rho\, h\, l^2}. \tag{3.47}$$

These equations take the forms

$$\begin{aligned} a &= 2E\cdot(1+b)\left[\sqrt{1+\frac{\pi^2 b^2}{4E(1+b^2)}} - 1\right] \\ a &= 16\beta\, E^2 (1+b), \end{aligned} \tag{3.48}$$

respectively.

The corresponding graphs are shown in Fig. 6 for the values of β equal to $\frac{1}{75}$, $\frac{1}{100}$ and $\frac{1}{125}$. For a real snake the value of β is close to $\frac{1}{75}$ ($l = 75$ sm, $ml = 75$ gm, $h \approx 1$ sm). In this case, as is shown in Fig. 6, $b = 0.77$, *i.e.*,

$$A\,k = 0.88. \tag{3.49}$$

This value of Ak is close to that found for the hard-walled channel (see (2.16), (2.31)).

The value a, defined from the same graph, is equal to 0.73. The added mass, calculated from these data equals 3.3 m, while the efficiency

$$\eta = 0.54. \tag{3.50}$$

It is of interest to compare this value of efficiency with the corresponding value for the hard-walled channel. The case of the hard-walled channel is obtained from formula (3.44) within the limit when $\rho \to \infty$, i.e.

$$\eta' = \frac{1}{1 + \frac{\pi A^2 k}{l}}. \tag{3.51}$$

The correctness of this expression can also be proved by direct calculation for the hard-walled channel. Putting here the values of A and κ obtained above, we get

$$\eta' = 0.75. \tag{3.52}$$

Thus the kinetic energy of the fluid surrounding the snake makes up 21% of the snake's entire energy output.

3. *Impulse Formulation of the Problem*

Let us suppose that the snake's motion in water proceeds rather slowly, quasi-statically, the snake's body assumes in a fluid a certain bent form. In this state an impulsive straining of the organism's muscles results in an onward movement.

The further motion of the body can proceed, generally speaking, in two different ways. If no tangential discontinuity of velocities (vortex sheet) occurs behind the back edge of the body, then, in conformity with the conclusions of §1 of this chapter, the body has to perform bending oscillations. If the body stiffens and thus turns into a straight bar, it stops. In the second case, when a vortex sheet is left after the body, the snake can straighten and move some distance by its own momentum. If this distance is equal to a few of its lengths, the snake enters a region where the velocities of fluid particles are small. In this region the snake can assume its initial bent state, after which the whole cycle is repeated.

We consider only the initial stage of the motion, and in this stage we shall estimate that part of energy which is imparted to the fluid. The snake's motion will be considered in a linear approximation. We can accordingly assume the boundary conditions in the problem for a fluid to be given on a certain straight segment.

We designate by P the impulse pressure created by the snake

$$P = \int_0^\tau p(t)\, dt. \tag{3.53}$$

Let the snake occupy a segment of a real axis $0 \leqq x \leqq l$. On this segment the following boundary conditions are given:

$$\begin{aligned} \varphi_+ - \varphi_- &= \frac{P}{\rho}, \\ \psi_+ - \psi_- &= 0, \end{aligned} \tag{3.54}$$

where ρ = fluid density, φ = potential, and ψ = flow function. The indices "+" and "−" stand for the function values on the upper and lower sides of the segment, respectively. The first of conditions (3.54) is a representation of the Bernoulli integral [3], which is usual for impulse problems, the second expresses the continuity of the velocity on either side of the segment. Let

$$P = P_0\, \omega(x). \tag{3.55}$$

Here and below x, y and $z = x + i\, y$ stand for the dimensionless values x/e, y/e, z/e.

The solution of the boundary value problem (3.54) is given by the Sohotsky formula [5]:

$$w(z) = \varphi + i\, \psi = \frac{P_0}{2\pi\, i\, \rho} \int_0^1 \frac{\omega(t)\, dt}{t - z}. \tag{3.56}$$

Hence, on the segment (0.1), we have

$$\varphi(x, 0) = \pm \frac{1}{2}\, \frac{P_0}{\rho}\, \omega(x), \tag{3.57}$$

$$\frac{\partial \varphi}{\partial y} = \frac{P_0}{2\pi\, \rho\, l} \operatorname{Re} \frac{d}{dz} \int_0^1 \frac{\omega(t)\, dt}{t - z}. \tag{3.58}$$

The kinetic energy of the fluid E_2 is determined by the formula

$$E_2 = \frac{h\, \rho}{l} \int_0^1 \varphi\, \frac{\partial \varphi}{\partial y}\, dx = \frac{h\, P_0^2}{4\pi\, \rho\, l^2} \int_0^1 \omega(s) \operatorname{Re} \frac{d}{ds} \int_0^1 \frac{\omega(t)\, dt}{t - s}\, ds \tag{3.59}$$

where h is the snake's width.

In the linear approximation the tractive force is given by

$$T = h \int_0^1 p\, y'\, dx, \tag{3.60}$$

and the impulse $\mathscr{I}^0$ received by the body has the form

$$\mathscr{I} = \int_0^\tau T\, dt = h\, P_0 \int_0^1 \omega(x)\, y'\, dx. \tag{3.61}$$

The kinetic energy E_1 of the progressive motion of a snake with a mass m is determined by the formula

$$E_1 = \frac{\mathscr{I}^2}{2m} = \frac{h^2 P_0^2}{2m} \left[\int_0^1 \omega(x)\, y'\, dx\right]^2. \tag{3.62}$$

The function $\omega(x)$ must be chosen so that the energy of the fluid is finite. This condition is satisfied, for example, by the function

$$\omega(x) = \frac{d^2}{dx^2}\, x^4 (x - 1)^4.$$

For this function the value of the integral in the formula for the fluid's kinetic energy is equal to 0.531. The integral in the expression for the body's kinetic energy, after integration by parts, has the form

$$\int_0^1 \omega(x)\, y'\, dx = \int_0^1 x^4 (x-1)^4\, y'''\, dx .$$

Specifically, if the body is bent along the curve $y''' = \varepsilon$ where ε is a small constant characterizing the ratio of the maximum bend to the body's length, then the value of this integral is equal to 0.955 ε.

Thus the ratio of the amount of the fluid's kinetic energy to that of the body's progressive kinetic energy is determined by the formula

$$\frac{E_2}{E_1} = \frac{0{,}093}{\varepsilon} \frac{m}{\rho h l^2} . \tag{3.63}$$

If the value of the parameter

$$\beta = \frac{m}{\rho h l^2}$$

is taken to be the same as in the preceding section, *i.e.*, to be equal to 1/75, and ε is assumed to equal 0.1, then we have

$$\frac{E_2}{E_1} = 0.012 . \tag{3.64}$$

In the preceding section this same value was equal to the ratio of the body's mass to the added mass of the fluid, *i.e.*, to 0.30. Thus, impulsive motion under proves to be more efficient than motion as a travelling wave. It should be noted, however, that the relative value of the energy imparted to the fluid in the case of the impulse formulation depends on the relative value of the body's maximum bend. Moreover, at the succeeding moments of time the portion of energy communicated to the fluid may even increase.

IV. Conclusions

1. Living organisms capable of developing elastic effort by means of muscular contractions can move in any hard-walled channel with a varying curvature and a width equal to the thickness of the body.

2. In the absence of friction a maximum force of traction is created by the body in the case when the bending moment is constant in value, but changes its sign at points where the channel curvature also changes sign.

3. In the presence of (dry) friction, a body placed in a hard-walled channel should co-operate with the channel walls only in the places where the value of the first derivative exceeds the friction coefficient. For a sinusoid-shaped channel the maximum value of the resultant tractive force is gained in the case when in each quarter of a wave one concentrated force acts on the body. The maximum ratio of the force of propulsion to that of friction is obtained if the concentrated forces are applied at the points with the greatest value of the modulus of the first derivative.

4. The motion of a deformable body in a fluid can proceed under two different regimes — with vortices forming and streaming off the back edge of the body,

and without any vortex formation. In the latter case the study of motion analogous to that in a hard-walled sinusoid-shaped channel leads to the conclusion that 21 % of an oscillating body's total energy is imparted to the water. This portion of energy is less in an impulsive of motion.

5. When a body moves in either a hard-walled channel or in a fluid, the maximum efficiency is obtained in those sinusoidal oscillations wherein the product of the amplitude by the wave number approaches unity.

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