Thermodynamics and the Stability of Fluid Motion

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Contents

Introduction

Since the entropy of a body isolated from all external influences tends to increase in time, it seemed natural to GIBBS* to assert that an equilibrium state of such a body is *stable* if and only if the entropy of the body decreases in all variations of state compatible with isolation of the body. Out of this simple idea grew the rich science of thermostatics. In spite of the success of thermostatics in rationalizing many aspects of chemistry and physics, today, ninety years after the appearance of GIBBS' treatise, we may still ask: *What is the* ***dynamical*** *significance of Gibbs' concept of stability under isolation?*

If an equilibrium state of an isolated body is stable according to GIBBS' definition, is it then true that thermodynamic processes in the body which pass near (in some sense) to the equilibrium state at one time must remain near (in perhaps some other sense) to that state at all future times?

Consider the case of a fluid body free from variations in chemical composition. In GIBBS' theory, the variations "compatible with isolation" are taken to be those which preserve the volume and internal energy of the body. Thus, the kinetic energy which should occur in the variations appears to be ignored. Furthermore, in the calculation of the entropy of the varied states, one uses constitutive assumptions which, in general, can be expected to hold only in equilibrium, *i.e.* only when the body is at rest at present and may be presumed to have been at rest with its present energy field and in its present configuration at all times in the past, a presumption which is clearly violated in dynamical processes. While these criticisms of thermostatics are relatively easy to overcome,** there remains another which goes deeper: Even if we grant that, in actual thermodynamic processes, the total entropy H does tend to approach its value H_0 in a Gibbs stable equilibrium state, to claim any sort of dynamical stability we must *prove* that proximity of H to H_0 implies that the distribution of mass, energy, and velocity in the body is, in some sense, close to the equilibrium distribution.

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* [1873, *1*, *2*], [1875, *1*].

** See our Theorem 2.

In this essay we attempt to imbed classical thermostatics in the general framework of modern thermodynamics and in so doing find dynamical consequences of the thermostatic concept of stability. For simplicity we consider materially homogeneous fluids. Our theory is not, however, restricted to fluids that are "perfect" or "elastic". The materials we consider, called *regular fluids*, need obey the constitutive equations of thermostatics only in states of permanent equilibrium; in general thermodynamic processes a regular fluid can exhibit nonlinear viscosity, heat conduction, and stress relaxation. The class of regular fluids includes simple fluids with fading memory*, linearly viscous fluids (with or without heat conduction)**, and gases with relaxing internal degrees of freedom.

In Section 1 we briefly review some basic results from the classical thermostatics of chemically uniform fluids. The emphasis is laid on materials for which the Gibbs-stable equilibrium state is uniform, *i.e.* contains a single homogeneous phase. In Section 2 we give a mathematical meaning, suitable for fluid bodies, to the concept of a thermodynamic process. After a brief discussion of the limitations which the second law places on the constitutive equations of elastic fluids, linearly viscous fluids, and simple fluids with fading memory, we introduce there the more general concept of a regular fluid. The class of processes possible in an *isolated body* is also defined in Section 2. Basic properties of these processes are treated in Sections 3 and 4. Our Theorem 2 shows, in effect, that when states of uniform equilibrium maximize the entropy in the class of variations of state considered by GIBBS, they also maximize the entropy achievable in the dynamical processes compatible with isolation***. Our main result is Theorem 3, which asserts that if the uniform states of a regular fluid body $\mathscr{B}$ are stable in the thermostatic sense, then in the thermodynamic processes compatible with isolation of $\mathscr{B}$ the fields giving the spatial distributions of specific volume, internal energy, and kinetic energy will remain close, in $\mathscr{L}_1$, to the corresponding uniform equilibrium fields, provided that the initial total entropy of $\mathscr{B}$ is set sufficiently close to its value at uniform equilibrium. It is a corollary to Theorem 3 that if the uniform states of a regular fluid body $\mathscr{B}$ are Gibbs stable, then when $\mathscr{B}$ is isolated in every process for which the initial total entropy is close to its equilibrium value, the total volume, internal energy, and the kinetic energy of all the parts of $\mathscr{B}$ remain close to their equilibrium values. Although the concept of a thermodynamic process used in Sections 2–4 presupposes that the entropy of a body is the integral of the entropy of its parts and that multipolar interactions are absent, in Section 5 we show that it is easy to extend our results beyond such a framework.

We believe that our Theorem 3 supplies, at least for fluids with Gibbs stable uniform equilibrium states, one answer to the questions which have been raised about the dynamical significance of the concepts used in thermostatics. Other anwers are possible; indeed in some subjects, such as the theory of heat conduction in immobile bodies, thermodynamic inequalities imply a much stronger type of

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* For the foundations of the theory of fading memory see COLEMAN & NOLL [1960, *1*], [1961, *1*]. The thermodynamics of materials with fading memory was developed by COLEMAN [1964, *2*].

** And also the more general viscous fluids discussed by COLEMAN & MIZEL [1964, 2].

*** Actually, the proof of Theorem 2 is easily generalized to show that the entropy of any Gibbs stable state of an isolated body, even if the state contains more than one phase, maximizes the entropy achievable in dynamical processes.

stability than our present statement about entropies and $\mathscr{L}_1$ norms. Our goal here, however, has not been to obtain strong results for a small class of systems, but rather to obtain results which, because they cover a broad class of materials, are necessarily weak but yet may yield some insight into the meaning of the postulates of thermodynamics.

Our present study indicates that although regular fluids need obey the constitutive relations of classical thermostatics only in states of permanent rest, *the stability of dynamical processes in regular fluids is governed mainly by the properties of the equilibrium caloric equation of state.* For those who believe that the entire content of the second law of thermodynamics may someday be deduced from an as yet obscure postulate of stability,* the observation just made may suggest a new interpretation for the opening lines of GIBBS' second paper on geometric methods in thermodynamics:

"The *leading* thermodynamic properties of a fluid are determined by the relations which exist between the volume, pressure, temperature, energy, and entropy of a given mass of the fluid in a *state of thermodynamic equilibrium.*"**

1. Rudiments of Thermostatics

In the *thermostatics of simple fluids* it is assumed that the specific*** entropy η, the pressure p, and the temperature ϑ, at a material point X, are determined when the specific volume υ and the specific internal energy ε are specified at X:

$$\begin{aligned} \eta &= \bar{\eta}(\upsilon, \varepsilon), \\ p &= \bar{p}(\upsilon, \varepsilon), \\ \vartheta &= \bar{\vartheta}(\upsilon, \varepsilon) > 0. \end{aligned} \tag{1.1}$$

The functions $\bar{\eta}$, $\bar{p}$, $\bar{\vartheta}$, which have $(0, \infty) \times (0, \infty)$ for their domain of definition, are *the equilibrium response functions* for the fluid under consideration#. In particular, we call $\bar{\eta}$ the *equilibrium entropy function.* The partial derivatives $\bar{\eta}_\upsilon$ and $\bar{\eta}_\varepsilon$ of $\bar{\eta}$ determine $\bar{p}$ and $\bar{\vartheta}$ through the relations

$$\bar{\vartheta} = \frac{1}{\bar{\eta}_\varepsilon}, \qquad \bar{p} = \frac{\bar{\eta}_\upsilon}{\bar{\eta}_\varepsilon}. \tag{1.2}$$

Of course, a fluid body $\mathscr{B}$ is a bounded smooth manifold, of material points X, imbedded in a fixed three-dimensional Euclidean space $\mathscr{E}$. A body is called *materially homogeneous* if its response functions, such as $\bar{\eta}$, do not vary with X. Each fluid body $\mathscr{B}$ has a finite, positive, mass measure $\mathscr{M}$, which is defined on all Borel subsets of $\mathscr{B}$. A *static state* $\{\upsilon, \varepsilon\}$ of $\mathscr{B}$ is a pair of $\mathscr{M}$-measurable fields υ and ε over $\mathscr{B}$ with the property that the following three integrals are finite:

$$\int_{\mathscr{B}} \upsilon(X)\,dm, \qquad \int_{\mathscr{B}} \varepsilon(X)\,dm, \quad \text{and} \quad \int_{\mathscr{B}} \bar{\eta}(\upsilon(X), \varepsilon(X))\,dm;$$

here dm is the element of mass. These integrals are called the *total volume*, the *total internal energy*, and the *total equilibrium entropy* of $\mathscr{B}$ for the state $\{\upsilon, \varepsilon\}$.

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* *Cf.* COLEMAN & MIZEL [1967, *2*].

** GIBBS [1873, *2*], the italics are ours.

*** "Specific" here means "per unit mass".

\# The bar "‾" in the notation $\bar{\eta}$, $\bar{p}$, $\bar{\vartheta}$ serves to distinguish these functions from their values.

Two arbitrary states $\{\upsilon, \varepsilon\}$ and $\{\upsilon', \varepsilon'\}$ are said to be *equivalent* if there exists a measure-preserving transformation* f of $\mathscr{B}$ onto $\mathscr{B}$ such that $\upsilon'(X)=\upsilon(f(X))$, and $\varepsilon'=\varepsilon(f(X))$ for all X in $\mathscr{B}$. A state of $\mathscr{B}$ not equivalent to $\{\upsilon, \varepsilon\}$ is said to be *distinct* from $\{\upsilon, \varepsilon\}$.

Basic to thermostatics is GIBBS' [1873, *2*], [1875, *1*] concept of *stable equilibrium of an isolated system*, which in our present context may be rendered mathematical as follows.

Definition 1.1. *A static state $\{\upsilon, \varepsilon\}$ of a fluid body $\mathscr{B}$ is* ***Gibbs stable*** *if every static state $\{\upsilon', \varepsilon'\}$ of $\mathscr{B}$ which is distinct from $\{\upsilon, \varepsilon\}$ and for which*

$$\int_{\mathscr{B}} \upsilon'(X)\,dm=\int_{\mathscr{B}} \upsilon(X)\,dm \quad \textit{and} \quad \int_{\mathscr{B}} \varepsilon'(X)\,dm=\int_{\mathscr{B}} \varepsilon(X)\,dm \tag{1.3}$$

has the property that

$$\int_{\mathscr{B}} \bar{\eta}(\upsilon'(X),\varepsilon'(X))\,dm<\int_{\mathscr{B}} \bar{\eta}(\upsilon(X),\varepsilon(X))\,dm. \tag{1.4}$$

Thus, a static state is Gibbs stable if and only if every distinct state with the same total volume and the same total internal energy gives rise to a smaller total equilibrium entropy.

Using brilliant geometrical analogies, GIBBS [1873, *2*] showed how to determine the stable states of arbitrary materially homogeneous fluid bodies. Our goal is not to develop further GIBBS' theory, but rather to find a dynamical interpretation for his concept of stability. Although thermostatics, as a discipline divorced from dynamical interpretations, becomes a mathematically rich subject only when some of the stable states of a body are non-uniform, we shall here focus our attention on the stability of uniform states.

A static state $\{\upsilon_0, \varepsilon_0\}$ of $\mathscr{B}$ is called *uniform* if its component fields υ_0 and ε_0 are constant over $\mathscr{B}$, *i.e.* if

$$\upsilon_0(X)\equiv\upsilon_0=\text{const}, \qquad \varepsilon_0(X)\equiv\varepsilon_0=\text{const}. \tag{1.5}$$

Let us introduce the abbreviation

$$\omega=(\upsilon,\eta) \tag{1.6}$$

and review some properties of concave functions. If a real-valued function $\bar{\eta}$ obeys the inequality

$$\bar{\eta}(\alpha\,\omega_1+\beta\,\omega_2)>\alpha\,\bar{\eta}(\omega_1)+\beta\,\bar{\eta}(\omega_2) \tag{1.7}$$

for all pairs ω_1, ω_2 with $\omega_1\neq\omega_2$ and all numbers α, β with $\alpha>0$, $\beta>0$, and $\alpha+\beta=1$, then $\bar{\eta}$ is said to be *strictly concave*. If $\bar{\eta}$ is differentiable, then $\bar{\eta}$ is strictly concave if and only if the inequality

$$\bar{\eta}(\omega_2)<\bar{\eta}(\omega_1)+(\omega_2-\omega_1)\cdot\bar{\eta}_\omega(\omega_1), \tag{1.8}$$

i.e.

$$\bar{\eta}(\upsilon_2,\varepsilon_2)<\bar{\eta}(\upsilon_1,\varepsilon_1)+(\upsilon_2-\upsilon_1)\,\bar{\eta}_\upsilon(\upsilon_1,\varepsilon_1)+(\varepsilon_2-\varepsilon_1)\,\bar{\eta}_\varepsilon(\upsilon_1,\varepsilon_1), \tag{1.9}$$

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* *A measure-preserving transformation f of $\mathscr{B}$* is a one-to-one mapping of $\mathscr{B}$ into itself such that measurable subsets $\mathscr{S}$ correspond to measurable subsets and the mass is preserved under f in the sense that $\mathscr{M}(f(\mathscr{S}))=\mathscr{M}(\mathscr{S})$.

holds whenever $\omega_2 \neq \omega_1$. The graphical considerations of GIBBS make the following theorem obvious.

Theorem 1. *Let $\bar{\eta}$ be the equilibrium entropy function for a materially homogeneous fluid body. In order that every* ***uniform*** *static state of $\mathscr{B}$ be Gibbs stable, it is necessary and sufficient that $\bar{\eta}$ be strictly concave over its entire domain* $(0, \infty)\times(0, \infty)$.

Proof*. To prove sufficiency, let $\bar{\eta}$ be strictly concave, let $\{\upsilon_0, \varepsilon_0\}$ be a given uniform state of $\mathscr{B}$, and let $\{\upsilon', \varepsilon'\}$ be a state of $\mathscr{B}$ distinct from $\{\upsilon_0, \varepsilon_0\}$ with

$$\int_{\mathscr{B}} \upsilon'(X)\, dm = \int_{\mathscr{B}} \upsilon_0\, dm = M\upsilon_0\,, \qquad \int_{\mathscr{B}} \varepsilon'(X)\, dm = \int_{\mathscr{B}} \varepsilon_0\, dm = M\varepsilon_0\,, \tag{1.10}$$

where $M=\mathscr{M}(\mathscr{B})$ is the total mass of $\mathscr{B}$. It follows from (1.9) that at each $X\in\mathscr{B}$

$$\bar{\eta}\big(\upsilon'(X), \varepsilon'(X)\big) \leq \bar{\eta}(\upsilon_0, \varepsilon_0) + \big(\upsilon'(X)-\upsilon_0\big)\,\bar{\eta}_\upsilon(\upsilon_0, \varepsilon_0) + \big(\varepsilon'(X)-\varepsilon_0\big)\,\bar{\eta}_\varepsilon(\upsilon_0, \varepsilon_0)\,. \tag{1.11}$$

Since $\{\upsilon', \varepsilon'\}$ is distinct from $\{\upsilon_0, \varepsilon_0\}$, (1.11) is a strict inequality on some set $\mathscr{P}\in\mathscr{B}$ with $\mathscr{M}(\mathscr{P})>0$. Integrating (1.11) over $\mathscr{B}$ and using (1.10), we obtain

$$\int_{\mathscr{B}} \bar{\eta}\big(\upsilon'(X), \varepsilon'(X)\big)\, dm < \int_{\mathscr{B}} \bar{\eta}(\upsilon_0, \varepsilon_0)\, dm = M\bar{\eta}(\upsilon_0, \varepsilon_0)\,, \tag{1.12}$$

which shows that $\{\upsilon_0, \varepsilon_0\}$ is Gibbs stable.

To prove necessity, we assume that every uniform state of $\mathscr{B}$ is Gibbs stable, we let ω_1 and ω_2 be distinct arbitrary points in $(0, \infty)\times(0, \infty)$, and we let α and β be arbitrary numbers obeying $\alpha>0$, $\beta>0$, and $\alpha+\beta=1$. If we put

$$(\upsilon_0, \varepsilon_0) = \omega_0 = \alpha\,\omega_1 + \beta\,\omega_2\,, \tag{1.13}$$

then the uniform state $\{\upsilon_0, \varepsilon_0\}$ is Gibbs stable. We now let $\mathscr{P}_1$ and $\mathscr{P}_2$ be two disjoint measurable subsets of $\mathscr{B}$ with

$$\mathscr{M}(\mathscr{P}_1)=\alpha M\,, \qquad \mathscr{M}(\mathscr{P}_2)=\beta M\,, \quad \text{and hence} \quad \mathscr{M}(\mathscr{B}-[\mathscr{P}_1\cup\mathscr{P}_2])=0\,. \tag{1.14}$$

For the state $\{\upsilon', \eta'\}$ obeying

$$\big(\upsilon'(X), \varepsilon'(X)\big) = \omega'(X) = \begin{cases} \omega_1\,, & \text{for } X\in\mathscr{P}_1 \\ \omega_2\,, & \text{for } X\in\mathscr{P}_2\,, \end{cases} \tag{1.15}$$

we have, by (1.13) and (1.14),

$$\int_{\mathscr{B}} \omega'(X)\, dm = M\alpha\,\omega_1 + M\beta\,\omega_2 = M\omega_0\,; \tag{1.16}$$

i.e. (1.10) holds, and since $\{\upsilon_0, \varepsilon_0\}$ is stable it follows that

$$\int_{\mathscr{B}} \bar{\eta}\big(\omega'(X)\big)\, dm < \int_{\mathscr{B}} \bar{\eta}(\omega_0)\, dm\,. \tag{1.17}$$

Since ω_0 is a constant, the right side of (1.17) is just $M\bar{\eta}(\omega_0)$, and using (1.14) and (1.15) to evaluate the left side of (1.17) we find that

$$M\alpha\,\bar{\eta}(\omega_1) + M\beta\,\bar{\eta}(\omega_2) < M\,\bar{\eta}(\omega_0)\,. \tag{1.18}$$

In view of (1.13) and the fact that M is strictly positive, (1.18) tells us that (1.7) holds, and by the arbitrariness in our choice of ω_1, ω_2, α, and β, this means that the function $\bar{\eta}$ is strictly concave throughout its domain; q.e.d.

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* This proof, which we quote from unpublished notes prepared by B.D. COLEMAN & W. NOLL in 1958, does no more than give an obvious, but analytical interpretation to geometric ideas found in GIBBS' second paper [1873, *2*].

2. Thermodynamics, Regular Fluids, and Isolated Bodies

In continuum physics a *thermodynamic process* $\mathscr{C}$ in a body $\mathscr{B}$ is a collection of time dependent fields over $\mathscr{B}$ that are compatible with the law of balance of momentum and the law of balance of energy. When dealing with fluids, we can consider $\mathscr{C}$ to be composed of eight fields whose values have the following physical interpretations:

(1) the *velocity* $\boldsymbol{v}=\boldsymbol{v}(X,t)$,

(2) the symmetric Cauchy *stress tensor* $\boldsymbol{T}=\boldsymbol{T}(X,t)$,

(3) the specific *body force* $\boldsymbol{b}=\boldsymbol{b}(X,t)$ exerted on $\mathscr{B}$ at X by the external world, *i.e.* by other bodies which do not intersect $\mathscr{B}$,

(4) the specific *internal energy* $\varepsilon=\varepsilon(X,t)$,

(5) the *heat flux* vector $\boldsymbol{q}=\boldsymbol{q}(X,t)$,

(6) the *heat supply* $r=r(X,t)$ absorbed by $\mathscr{B}$ at X and furnished by radiation from the external world,

(7) the *specific entropy* $\eta=\eta(X,t)$,

(8) the *absolute temperature* $\vartheta=\vartheta(X,t)>0$.

When these eight fields suffice to describe a thermodynamic process, the laws of *balance of momentum* and *balance of energy* take the forms

$$\frac{d}{dt}\int_{\mathscr{P}}\boldsymbol{v}\,dm=\int_{\mathscr{P}}\boldsymbol{b}\,dm+\int_{\partial\mathscr{P}}\boldsymbol{T}\boldsymbol{n}\,da, \tag{2.1}$$

$$\frac{d}{dt}\int_{\mathscr{P}}(\varepsilon+\tfrac{1}{2}v^2)\,dm=\int_{\mathscr{P}}(\boldsymbol{v}\cdot\boldsymbol{b}+r)\,dm+\int_{\partial\mathscr{P}}(\boldsymbol{v}\cdot\boldsymbol{T}\boldsymbol{n}-\boldsymbol{q}\cdot\boldsymbol{n})\,da. \tag{2.2}$$

These integral relations must hold at all times t, $-\infty<t<\infty$, and in all parts* $\mathscr{P}$ of $\mathscr{B}$; here $\boldsymbol{n}$ is the exterior unit normal vector to the surface $\partial\mathscr{P}$ of $\mathscr{P}$ in the configuration at time t, da is the element of surface area in the configuration at time t, and $v=\sqrt{\boldsymbol{v}\cdot\boldsymbol{v}}$ is the magnitude of $\boldsymbol{v}$.

A *material* is defined by *constitutive assumptions* which place limitations on the processes that are admissible in a body consisting of the material. For example, a *materially homogeneous elastic fluid* (with heat conduction) is characterized by four response functions $\bar{\eta}$, $\bar{p}$, $\bar{\vartheta}$, and $\bar{\kappa}$. A process is said to be admissible in such a fluid if the following constitutive equations hold at each material point X and at all times t:

$$\begin{gathered}\eta=\bar{\eta}(\upsilon,\varepsilon),\\ \boldsymbol{T}=-p\boldsymbol{1},\quad p=\bar{p}(\upsilon,\varepsilon),\\ \vartheta=\bar{\vartheta}(\upsilon,\varepsilon),\\ \boldsymbol{q}=-\kappa\boldsymbol{g},\quad \kappa=\bar{\kappa}(\upsilon,\varepsilon,g).\end{gathered} \tag{2.3}$$

Here $\boldsymbol{1}$ is the unit tensor, $\boldsymbol{g}=\operatorname{grad}\vartheta$, and $g=\sqrt{\boldsymbol{g}\cdot\boldsymbol{g}}$ is the magnitude of grad ϑ.**

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* A part $\mathscr{P}$ of $\mathscr{B}$ is a subset of the closure of $\mathscr{B}$ with certain properties of regularity which we need not enumerate here; *cf.* NOLL [1959, *1*], [1966, *2*]. Of course, $\mathscr{B}$ itself is a member of the set of parts of $\mathscr{B}$.

** It is usually assumed that κ is independent of g, but there is no need to do so here. Of course, our present theory remains valid when $\boldsymbol{q}\equiv\boldsymbol{0}$, *i.e.* when the fluid does not conduct heat.

The ***second law of thermodynamics*** is the assertion that *for every admissible process in a body $\mathscr{B}$ consisting of a given material, the Clausius-Duhem inequality*[*]

$$\frac{d}{dt}\int_{\mathscr{P}} \eta\, dm \geqq \int_{\mathscr{P}} \frac{r}{\vartheta}\, dm - \int_{\partial\mathscr{P}} \frac{1}{\vartheta}\, \boldsymbol{q}\cdot\boldsymbol{n}\, da \tag{2.4}$$

must hold at all times t and in all parts $\mathscr{P}$ of $\mathscr{B}$.

The second law places restrictions on constitutive assumptions. For example, the equations (2.3), defining an elastic fluid, are compatible with the second law if and only if

$$\vartheta = [\bar{\eta}_\varepsilon(v,\varepsilon)]^{-1}, \qquad p = \vartheta\,\bar{\eta}_v(v,\varepsilon), \qquad \bar{\kappa}(v,\varepsilon,\boldsymbol{g}) \geqq 0 \tag{2.5}$$

for all v, ε, and $\boldsymbol{g}$. Of course, the first two of these relations are the equations (1.2) assumed in thermostatics.

If $\mathscr{B}$ is composed of a *linearly viscous fluid*, all the constitutive equations listed in (2.3) still hold at each X in $\mathscr{B}$ with the exception of $(2.3)_2$, which is replaced by the more general equation

$$\boldsymbol{T} = -p\boldsymbol{1} + 2\mu\boldsymbol{D} + \lambda(\operatorname{tr}\boldsymbol{D})\boldsymbol{1}, \qquad \mu = \bar{\mu}(v,\varepsilon), \qquad \lambda = \bar{\lambda}(v,\varepsilon), \tag{2.6}$$

where p is still given by $(2.3)_3$ and $\boldsymbol{D}$ is the *stretching tensor* (*i.e.* the symmetric part $\frac{1}{2}(\boldsymbol{L}+\boldsymbol{L}^T)$ of the velocity gradient $\boldsymbol{L} = \operatorname{grad}\boldsymbol{v}$). The constitutive equations $(2.3)_{1,3,4,5\&6}$ and (2.6) defining a linearly viscous fluid are compatible with the second law if and only if the relations (2.5) hold and, in addition,

$$\bar{\mu}(v,\varepsilon) \geqq 0, \qquad \bar{\lambda}(v,\varepsilon) + \tfrac{2}{3}\bar{\mu}(v,\varepsilon) \geqq 0 \tag{2.7}$$

for all v and ε. Thus, neither the shear viscosity μ nor the bulk viscosity $\lambda + \frac{2}{3}\mu$ can be negative, and heat cannot flow from cold to hot (*i.e.* by $(2.5)_3$, $\boldsymbol{q}\cdot\boldsymbol{g}$ must be $\leqq 0$). Furthermore, when a viscous fluid is at rest with $\boldsymbol{g} = \boldsymbol{0}$, the stress reduces to a hydrostatic pressure, and the basic equations (1.1) and (1.2) of thermostatics hold.[**]

Let a motion of a body $\mathscr{B}$ be given, and let $\boldsymbol{x}$ be the position in Euclidean space $\mathscr{E}$ of the material point X of $\mathscr{B}$ at time t, which we interpret as the present time. Suppose that at time $\tau \leqq t$ the same material point X occupied the position $\boldsymbol{\xi}$ in $\mathscr{E}$. For the dependence of $\boldsymbol{\xi}$ on $\boldsymbol{x}$, t, and τ, we write

$$\boldsymbol{\xi} = \boldsymbol{\chi}_t(\boldsymbol{x},\tau). \tag{2.8}$$

It is easy to show that the function $\boldsymbol{\chi}_t$ is determined when the velocity field $\boldsymbol{v}$ is specified at all times.[***] We call the function $\boldsymbol{C}_r^t$ defined by

$$\boldsymbol{C}_r^t(s) = \boldsymbol{F}^T(s)\boldsymbol{F}(s), \quad \text{with} \quad \boldsymbol{F}(s) = \operatorname{grad}_{\boldsymbol{x}}\boldsymbol{\chi}_t(\boldsymbol{x}, t-s), \qquad 0 < s < \infty, \tag{2.9}$$

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[*] The form (2.4) of the Clausius-Duhem inequality is given by TRUESDELL & TOUPIN [1960, *1*, p. 644]; in our interpretation of (2.4) as an inequality which must hold for *all admissible processes* we are following COLEMAN & NOLL [1963, *1*].

[**] The details of the arguments used to derive (2.5) and (2.7) for classical elastic and linearly viscous fluids are given by COLEMAN & NOLL [1963, *1*]. The relation of the classical constitutive equations (2.4) and (2.6) to constitutive hypotheses reflecting TRUESDELL's principle of equipresence is discussed by COLEMAN [1964, *1*] and COLEMAN & MIZEL [1964, *2*].

[***] For details see COLEMAN & NOLL [1961, *2*].

the *past history* (up to time t) *of the strain* at the material point X. This function $\boldsymbol{C}_r^t$ maps $(0, \infty)$ into the set of symmetric positive definite tensors. The value of $\boldsymbol{C}_r^t(s)$ is the right Cauchy-Green tensor at X at time $t-s$ computed using the configuration at time t as reference.

The *past history* (up to time t) *of the specific internal energy* at the material point X is the function ε_r^t mapping $(0, \infty)$ into $(0, \infty)$ and given by

$$\varepsilon_r^t(s) = \varepsilon(X, t-s), \qquad 0 < s < \infty. \tag{2.10}$$

A *general simple fluid* is defined by four constitutive functionals $\mathfrak{h}$, $\mathfrak{T}$, $\mathfrak{t}$, and $\mathfrak{q}$; a process is *admissible* in a homogeneous body $\mathscr{B}$ consisting of the fluid if at each point X and each time t the present values of η, $\boldsymbol{T}$, ϑ, and $\boldsymbol{q}$ are given by the equations

$$\begin{aligned} \eta &= \mathfrak{h}(\upsilon, \varepsilon; \boldsymbol{C}_r^t, \varepsilon_r^t), \\ \boldsymbol{T} &= \mathfrak{T}(\upsilon, \varepsilon; \boldsymbol{C}_r^t, \varepsilon_r^t), \\ \vartheta &= \mathfrak{t}(\upsilon, \varepsilon; \boldsymbol{C}_r^t, \varepsilon_r^t), \\ \boldsymbol{q} &= \mathfrak{q}(\upsilon, \varepsilon, \boldsymbol{g}; \boldsymbol{C}_r^t, \varepsilon_r^t), \end{aligned} \tag{2.11}$$

where the number $\upsilon = \upsilon(X, t)$, the number $\varepsilon = \varepsilon(X, t)$, and the vector $\boldsymbol{g} = \boldsymbol{g}(X, t)$ are the present values of υ, ε, and grad ϑ, while the functions $\boldsymbol{C}_r^t$ and ε_r^t are the past histories defined in (2.9) and (2.10).[*]

Assuming that the functionals $\mathfrak{h}$, $\mathfrak{T}$, $\mathfrak{t}$, and $\mathfrak{q}$ obey the principle of fading memory[**], COLEMAN [1964, *1*] has found the restrictions which the second law places on general constitutive equations of the type (2.11). Rather than enumerate all these restrictions, we shall state here only those consequences of the Clausius-Duhem inequality which have a direct bearing on theorems to be proven in Section 3.

Let $1^\dagger$ and $\mathbf{1}^\dagger$ be the constant functions on $(0, \infty)$ equal, respectively, to the number 1 and the unit tensor $\mathbf{1}$:

$$1^\dagger(s) \equiv 1, \qquad \mathbf{1}^\dagger(s) \equiv \mathbf{1}, \qquad 0 < s < \infty. \tag{2.12}$$

If a region containing the point X has remained in its present configuration at all times $\tau \leqq t$, or has been subjected only to rigid rotations, then $\boldsymbol{C}_r^t = \mathbf{1}^\dagger$ at X; if the specific internal energy at X has remained constant at its present value ε for all $\tau \leqq t$, then $\varepsilon_r^t = \varepsilon 1^\dagger$ at X. Thus, we may call the function $\bar{\eta}$ defined by

$$\bar{\eta}(\upsilon, \varepsilon) \overset{\text{def}}{=} \mathfrak{h}(\upsilon, \varepsilon; \mathbf{1}^\dagger, \varepsilon 1^\dagger) \tag{2.13}$$

the *equilibrium entropy function* for the simple fluid under consideration. Among the consequences of the second law is the following inequality which functional $\mathfrak{h}$ obeys throughout its domain[***]:

$$\mathfrak{h}(\upsilon, \varepsilon; \boldsymbol{C}_r^t, \varepsilon_r^t) \leqq \mathfrak{h}(\upsilon, \varepsilon; \mathbf{1}^\dagger, \varepsilon 1^\dagger) = \bar{\eta}(\upsilon, \varepsilon). \tag{2.14}$$

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[*] The concept of a general simple fluid was introduced by NOLL [1958, *2*] in a framework which ignores thermodynamic influences. The full set of equations (2.11) was studied by COLEMAN [1964, *1*, pp. 36 and 43].

[**] This is a postulate of smoothness introduced by COLEMAN & NOLL [1960, *1*], [1961, *1*] and recently developed from a set of elementary axioms by COLEMAN & MIZEL [1966, *2*].

[***] COLEMAN [1964, *1*, p. 38, Remark 31].

In words: For a simple fluid with fading memory, of all strain-energy histories ending with given values of υ and ε, that corresponding to perpetual rest at the present strain and energy has the maximum entropy.

In analogy to (2.13) we may define *equilibrium response functions for the stress, the temperature, and the heat flux:*

$$\begin{aligned}\bar{\boldsymbol{T}}(\upsilon,\varepsilon)&\overset{\text{def}}{=}\mathfrak{T}(\upsilon,\varepsilon;\mathbf{1}^{\dagger},\varepsilon\mathbf{1}^{\dagger}),\\ \bar{\vartheta}(\upsilon,\varepsilon)&\overset{\text{def}}{=}\mathfrak{t}(\upsilon,\varepsilon;\mathbf{1}^{\dagger},\varepsilon\mathbf{1}^{\dagger}),\\ \bar{\boldsymbol{q}}(\upsilon,\varepsilon,\boldsymbol{g})&\overset{\text{def}}{=}\mathfrak{q}(\upsilon,\varepsilon,\boldsymbol{g};\mathbf{1}^{\dagger},\varepsilon\mathbf{1}^{\dagger}).\end{aligned}\tag{2.15}$$

Using the principle of material frame indifference[*], one can easily show that the functions $\bar{\boldsymbol{T}}$ and $\bar{\boldsymbol{q}}$ must have the forms

$$\begin{aligned}\bar{\boldsymbol{T}}(\upsilon,\varepsilon)&=-p\mathbf{1}, & p&=\bar{p}(\upsilon,\varepsilon),\\ \bar{\boldsymbol{q}}(\upsilon,\varepsilon,\boldsymbol{g})&=-\kappa\boldsymbol{g}, & \kappa&=\bar{\kappa}(\upsilon,\varepsilon,\boldsymbol{g}).\end{aligned}\tag{2.16}$$

It is a consequence of the second law that the functional $\mathfrak{h}$ must determine the functionals $\mathfrak{T}$ and $\mathfrak{t}$ through a formula called the "generalized stress relation". For our present purposes it is not necessary to assemble the apparatus required to state the generalized stress relation; we need merely note that the relation implies that the equilibrium response functions $\bar{\eta}$, $\bar{\vartheta}$ and $\bar{p}$ must obey (1.2). In fact, although the dynamical behavior of simple fluids with fading memory differs markedly from that of elastic fluids, in purely statical situations the two types of fluids are indistinguishable.

Definition 2.1. *A body $\mathscr{B}$ is said to be in permanent rest if for all t in $(-\infty,\infty)$ and all X in $\mathscr{B}$*

$$\dot{\varepsilon}(X,t)=0 \quad \textit{and} \quad \boldsymbol{v}(X,t)=\mathbf{0}.\tag{2.17}$$

Definition 2.2. *We say that the material comprising $\mathscr{B}$ is a* ***regular fluid*** *if*

(i) *when $\mathscr{B}$ is in permanent rest,* grad $\vartheta=\mathbf{0}$ *implies* $\boldsymbol{q}=\mathbf{0}$,

(ii) *there exist equilibrium response functions $\bar{\eta}$, $\bar{p}$, and $\bar{\vartheta}$, obeying* (1.2) *and such that when $\mathscr{B}$ is in permanent rest at each point X in $\mathscr{B}$ we have*

$$\begin{aligned}\eta&=\bar{\eta}(\upsilon,\varepsilon),\\ \boldsymbol{T}=-p\mathbf{1} \quad &\textit{with} \quad p=\bar{p}(\upsilon,\varepsilon),\\ \vartheta&=\bar{\vartheta}(\upsilon,\varepsilon),\end{aligned}\tag{2.18}$$

and

(iii) *in every admissible process of $\mathscr{B}$ we have*

$$\eta(X,t)\leqq\bar{\eta}(\upsilon(X,t),\varepsilon(X,t))\tag{2.19}$$

for all t and all X in $\mathscr{B}$, where $\bar{\eta}$ is the ***static entropy function*** *appearing in* (2.18).

We have observed that elastic fluids, linearly viscous fluids, and simple fluids with fading memory are indistinguishable in states of permanent rest and obey the conditions (i) and (ii). The condition (iii) is of importance in processes that

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[*] Noll [1958, *2*].

depart from permanent rest. For elastic fluids and viscous fluids, (iii) holds trivially. For simple fluids with fading memory, (iii) is not trivial but is nevertheless a consequence of the second law of thermodynamics.*
Hence, *every elastic fluid, every linearly viscous fluid, and every simple fluid with fading memory is a regular fluid.***

We now render mathematical the physical concept of "an isolated body".

Definition 2.3. *If an admissible thermodynamic process $\mathscr{C}$ in a body $\mathscr{B}$ is such that at each time t*

$$\begin{gathered} \boldsymbol{b}=\mathbf{0} \quad \textit{and} \quad r=0 \quad \textit{for all } X\in\mathscr{B}, \\ \boldsymbol{v}\cdot\boldsymbol{n}=0, \quad \boldsymbol{v}\cdot\boldsymbol{T}\boldsymbol{n}=0 \quad \textit{and} \quad \boldsymbol{q}\cdot\boldsymbol{n}=0 \quad \textit{for all } X\in\partial\mathscr{B}, \end{gathered} \tag{2.20}$$

then $\mathscr{C}$ is said to be ***compatible with isolation of*** $\mathscr{B}$.

Occasionally, to indicate that $\mathscr{C}$ obeys (2.20), we say, less formally, that $\mathscr{C}$ is a *process in an isolated body.*

In the special case of an elastic fluid we have $\boldsymbol{T}=-p\mathbf{1}$, and hence $\boldsymbol{v}\cdot\boldsymbol{T}\boldsymbol{n}=-p\,\boldsymbol{v}\cdot\boldsymbol{n}$. Therefore, *for an elastic fluid, an admissible process with $p\neq0$ is compatible with isolation if and only if for all t*

$$\begin{gathered} \boldsymbol{b}=\mathbf{0} \quad \text{and} \quad r=0 \text{ for all } X\in\mathscr{B}, \\ \boldsymbol{v}\cdot\boldsymbol{n}=0 \quad \text{and} \quad \boldsymbol{q}\cdot\boldsymbol{n}=0 \text{ for all } X\in\partial\mathscr{B}. \end{gathered} \tag{2.21}$$

For more general fluids the stress need not reduce to a hydrostatic pressure $-p\mathbf{1}$. However, we can assert that *a process admissible in a body comprised of a general fluid is compatible with isolation of the body if for all t*

$$\begin{gathered} \boldsymbol{b}=\mathbf{0} \quad \text{and} \quad r=0 \text{ for all } X\in\mathscr{B}, \\ \boldsymbol{v}=\mathbf{0} \quad \text{and} \quad \boldsymbol{q}\cdot\boldsymbol{n}=0 \text{ for all } X\in\partial\mathscr{B}. \end{gathered} \tag{2.22}$$

The conditions (2.22), though sufficient, are not necessary for isolation.

Returning now to our original definition (2.20), we may observe that the condition $\boldsymbol{v}\cdot\boldsymbol{n}=0$ on $\partial\mathscr{B}$, implies that, for all t,

$$\frac{d}{dt}\int_{\mathscr{B}} \upsilon\, dm=0, \tag{2.23}$$

while the remaining conditions, when combined with the equation (2.2) of balance of energy, with $\mathscr{P}$ put equal to $\mathscr{B}$, yield

$$\frac{d}{dt}\int_{\mathscr{B}} (\varepsilon+\tfrac{1}{2}v^2)\, dm=0 \tag{2.24}$$

for all t. Thus, *the total volume and total energy of an isolated body are constant in time.*

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* Although (2.19) reduces to an equation for elastic and linearly viscous fluids, the inequality in (2.19) can be strict for a fluid with memory that has not been held in permanent rest.

** Fluids with internal state variables are also regular fluids, provided that the evolution equation for the internal state obeys a stability postulate discussed in § 10 of COLEMAN & GURTIN'S [1967, *1*] article.

In the next section we study processes which occur in an isolated body composed of a regular fluid. We assume the second law of thermodynamics, *i.e.* that the constitutive equations of the fluid are such that the Clausius-Duhem inequality holds for all processes. Therefore, by (2.20) and (2.4) with $\mathscr{P}=\mathscr{B}$, for all admissible processes compatible with isolation of $\mathscr{B}$ we have

$$\frac{d}{dt}\int_{\mathscr{B}}\eta\,dm\geqq 0 \tag{2.25}$$

for all t. Or, as expected: *The total entropy of an isolated body does not decrease in time.*

3. A Basic Property of Equilibrium

When the body force $\boldsymbol{b}$ and the heat supply r vanish, the balance laws (2.1) and (2.2) may be cast into the differential forms

$$\begin{aligned}\rho\dot{\boldsymbol{v}}&=\operatorname{div}\boldsymbol{T},\\ \rho\dot{\varepsilon}&=\boldsymbol{T}\cdot\operatorname{grad}\boldsymbol{v}+\operatorname{div}\boldsymbol{q}.\end{aligned} \tag{3.1}$$

Using (3.1) and the Definitions 2.1 – 2.3 one may easily verify the following remark which shows that each uniform static state $\{\upsilon_0,\varepsilon_0\}$ of a regular fluid body is realizable as an admissible process which is compatible with isolation and which keeps the body in permanent rest.

Remark 3.1. *Let $\mathscr{B}$ be a regular fluid body with total mass $\mathscr{M}(\mathscr{B})=M$. To each pair $V>0$, $E>0$ there corresponds a unique admissible process $\mathscr{C}_0(V,E)$ of $\mathscr{B}$ such that*

(i) *$\mathscr{C}_0(V,E)$ is compatible with isolation of $\mathscr{B}$, and*

(ii) *in the process $\mathscr{C}_0(V,E)$, $\mathscr{B}$ is in permanent rest with*

$$\upsilon(X,t)\equiv\upsilon_0\overset{\text{def}}{=}V/M,\qquad \varepsilon(X,t)\equiv\varepsilon_0\overset{\text{def}}{=}E/M. \tag{3.2}$$

This process has the following properties:

$$\begin{gathered}\boldsymbol{v}\equiv\boldsymbol{0},\quad \boldsymbol{b}\equiv\boldsymbol{0},\quad r\equiv 0,\quad \boldsymbol{q}\equiv\boldsymbol{0},\\ \eta(X,t)\equiv\eta_0\overset{\text{def}}{=}\bar{\eta}(\upsilon_0,\varepsilon_0),\\ \boldsymbol{T}(X,t)\equiv-p_0\boldsymbol{1}\quad\text{with}\quad p_0\overset{\text{def}}{=}\bar{p}(\upsilon_0,\varepsilon_0),\\ \vartheta(X,t)\equiv\vartheta_0\overset{\text{def}}{=}\bar{\vartheta}(\upsilon_0,\varepsilon_0).\end{gathered} \tag{3.3}$$

We call $\mathscr{C}_0(V,E)$ the ***uniform equilibrium process*** *of $\mathscr{B}$ with total volume V and total energy E.*

In other words, the uniform static states $\{\upsilon_0,\varepsilon_0\}$ mentioned in Theorem 1, do have a thermo*dynamic* significance: They can occur in isolated bodies as *equilibrium processes.* We now show that the hypothesis of Theorem 1 implies that, of all the processes which can occur in an isolated regular fluid body, the uniform equilibrium process gives the body the greatest entropy at all times.

We have seen, in equations (2.23) and (2.24), that during a thermodynamic process $\mathscr{C}$ in an isolated regular fluid body $\mathscr{B}$ the total volume

$$V=\int_{\mathscr{B}} \upsilon(X,t)\,dm \tag{3.4}$$

and the total energy

$$E=\int_{\mathscr{B}} [\varepsilon(X,t)+\tfrac{1}{2}v^2(X,t)]\,dm \tag{3.5}$$

are constant in time.

Definition 3.1. *Let $\mathscr{C}$ be a thermodynamic process admissible in an isolated fluid body $\mathscr{B}$ of total mass M. The unique uniform equilibrium process $\mathscr{C}_0(V,E)$ which gives $\mathscr{B}$ the same total volume V and total energy E as $\mathscr{C}$ is called the* ***uniform equilibrium process associated with*** $\mathscr{C}$. *Putting*

$$\upsilon_0=V/M, \qquad \varepsilon_0=E/M, \tag{3.6}$$

we call the number

$$\Delta(\mathscr{C},t)=\int_{\mathscr{B}} |\upsilon(X,t)-\upsilon_0|\,dm+\int_{\mathscr{B}} |\varepsilon(X,t)-\varepsilon_0|\,dm+\tfrac{1}{2}\int_{\mathscr{B}} v^2(X,t)\,dm \tag{3.7}$$

the departure of $\mathscr{C}$ ***from uniform equilibrium at time*** t.

We shall regard $\Delta(\mathscr{C},t)$ as a measure of the distance at time t between the process $\mathscr{C}$ and its associated uniform equilibrium process. It follows from (3.3) that during the equilibrium process $\mathscr{C}_0(V,E)$ the total entropy of $\mathscr{B}$ is constant in time and is given by the formula

$$H_0=\int_{\mathscr{B}} \bar{\eta}(\upsilon_0,\varepsilon_0)\,dm=M\bar{\eta}(\upsilon_0,\varepsilon_0), \tag{3.8}$$

where $\bar{\eta}$ is the equilibrium entropy function for the regular fluid comprising $\mathscr{B}$.

Theorem 2. *Let $\mathscr{C}$ be an admissible thermodynamic process in a regular fluid body $\mathscr{B}$. Suppose that $\mathscr{C}$ is compatible with isolation of $\mathscr{B}$ and that the equilibrium entropy function $\bar{\eta}$ for $\mathscr{B}$ is strictly concave. If the departure of $\mathscr{C}$ from uniform equilibrium at time t is not zero, then the true total entropy*

$$H(t)=\int_{\mathscr{B}} \eta(X,t)\,dm \tag{3.9}$$

of $\mathscr{B}$ at time t in the process $\mathscr{C}$ is strictly less than the total entropy H_0 of $\mathscr{B}$ in the uniform equilibrium process associated with $\mathscr{C}$; i.e.

$$\Delta(\mathscr{C},t)>0 \;\Rightarrow\; H(t)<H_0 \tag{3.10}$$

with $\Delta(\mathscr{C},t)$ and H_0 given by (3.4)–(3.8).

Proof. Since $\mathscr{B}$ is comprised of a regular fluid, it follows from (2.19) that in the process $\mathscr{C}$ at each point X in $\mathscr{B}$ and at each time t we have

$$\eta(X,t)\leqq\bar{\eta}(\upsilon(X,t),\varepsilon(X,t)). \tag{3.11}$$

Since $\bar{\eta}$ is assumed to be strictly concave, we also have, by (1.11),

$$\begin{aligned}\bar{\eta}(\upsilon(X,t),\varepsilon(X,t))\leqq\bar{\eta}(\upsilon_0,\varepsilon_0)&+[\upsilon(X,t)-\upsilon_0]\,\bar{\eta}_\upsilon(\upsilon_0,\varepsilon_0)+\\ &+[\varepsilon(X,t)-\varepsilon_0]\,\bar{\eta}_\varepsilon(\upsilon_0,\varepsilon_0),\end{aligned} \tag{3.12}$$

and this is a strict inequality whenever either $\upsilon(X,t)\neq\upsilon_0$ or $\varepsilon(X,t)\neq\varepsilon_0$. Using the fact that $\bar{\eta}_\varepsilon(\upsilon_0,\varepsilon_0)^{-1}=\bar{\vartheta}(\upsilon_0,\varepsilon_0)>0$, we may observe that the inequalities (3.11) and (3.12) imply that

$$\begin{aligned}\eta(X,t)\leqq\bar{\eta}(\upsilon_0,\varepsilon_0)+[\upsilon(X,t)-\upsilon_0]\,\bar{\eta}_\upsilon(\upsilon_0,\varepsilon_0)+[\varepsilon(X,t)-\varepsilon_0]\,\bar{\eta}_\varepsilon(\upsilon_0,\varepsilon_0)+&\\ +\tfrac{1}{2}v^2(X,t)\,\bar{\eta}_\varepsilon(\upsilon_0,\varepsilon_0)\overset{\text{def}}{=}\zeta&\end{aligned}\tag{3.13}$$

with $\upsilon(X,t)$, $\varepsilon(X,t)$, and $v^2(X,t)$ evaluated in the process $\mathscr{C}$. The inequality in (3.13) is strict whenever

$$\upsilon(X,t)\neq\upsilon_0,\qquad \varepsilon(X,t)\neq\varepsilon_0,\quad\text{or}\quad v^2(X,t)\neq 0. \tag{3.14}$$

Of course, (3.13) holds for any pair $\upsilon_0,\varepsilon_0>0$. We, however, choose υ_0 and ε_0 as in (3.6) with V and E given by (3.4) and (3.5). Therefore, for the integral over $\mathscr{B}$ of the right side of (3.13) we have

$$\int_{\mathscr{B}}\zeta\,dm=M\bar{\eta}(\upsilon_0,\varepsilon_0). \tag{3.15}$$

Furthermore, since, by hypothesis, $\Delta(\mathscr{C},t)$ is nonzero, it follows from (3.7) that there exists a subset $\mathscr{S}_t$ of $\mathscr{B}$, with $\mathscr{M}(\mathscr{S}_t)>0$, such that at least one of the inequalities (3.14) holds for all X in $\mathscr{S}_t$. Since (3.13) is a strict inequality for all X in $\mathscr{S}_t$, integration of (3.13) over $\mathscr{B}$ yields

$$\int_{\mathscr{B}}\eta(X,t)\,dm<\int_{\mathscr{B}}\zeta\,dm, \tag{3.16}$$

and this inequality, by (3.9), (3.15), and (3.8) implies that $H(t)$ is strictly less than H_0; q.e.d.

4. Dynamical Stability of Uniform Equilibrium

Our investigation of the stability of uniform equilibrium processes requires a sharpening of the result obtained in Theorem 2; to this end we make the following preliminary observation.

Remark 4.1. *Let* $\bar{\eta}$ *be the equilibrium entropy function of a regular fluid, let* $\upsilon_0>0$ *and* $\varepsilon_0>0$ *be arbitrarily assigned, and let the function* $\varphi_{\upsilon_0,\varepsilon_0}$ *be defined by*

$$\begin{aligned}\varphi_{\upsilon_0,\varepsilon_0}(\upsilon,\varepsilon,v^2)=\bar{\eta}(\upsilon_0,\varepsilon_0)+(\upsilon-\upsilon_0)\,\bar{\eta}_\upsilon(\upsilon_0,\varepsilon_0)+(\varepsilon-\varepsilon_0)\,\bar{\eta}_\varepsilon(\upsilon_0,\varepsilon_0)+&\\ +\tfrac{1}{2}v^2\,\bar{\eta}_\varepsilon(\upsilon_0,\varepsilon_0)-\bar{\eta}(\upsilon,\varepsilon),&\\ \upsilon>0,\qquad \varepsilon>0,\qquad v^2\geqq 0.&\end{aligned}\tag{4.1}$$

If $\bar{\eta}$ *is strictly concave over* $(0,\infty)\times(0,\infty)$, *then given any number* $d>0$ *there exists another number* $A(d,\upsilon_0,\varepsilon_0)$ *such that*

$$\inf_{(\upsilon,\varepsilon,v^2)\in\mathscr{T}_d}\frac{\varphi_{\upsilon_0,\varepsilon_0}(\upsilon,\varepsilon,v^2)}{|\upsilon-\upsilon_0|+|\varepsilon-\varepsilon_0|+\frac{1}{2}v^2}\geqq A(d,\upsilon_0,\varepsilon_0), \tag{4.2}$$

where

$$\mathscr{T}_d=\{(\upsilon,\varepsilon,v^2)\mid \upsilon>0,\ \varepsilon>0,\ v^2>0,\ |\upsilon-\upsilon_0|+|\varepsilon-\varepsilon_0|+\tfrac{1}{2}v^2\geqq d\}. \tag{4.3}$$

Proof. Since we assume that $\bar{\eta}$ is strictly concave, the function f, defined by

$$f(\upsilon,\varepsilon)=\bar{\eta}(\upsilon_0,\varepsilon_0)+(\upsilon-\upsilon_0)\,\bar{\eta}_\upsilon(\upsilon_0,\varepsilon_0)+(\varepsilon-\varepsilon_0)\,\bar{\eta}_\varepsilon(\upsilon_0,\varepsilon_0)-\bar{\eta}(\upsilon,\varepsilon), \tag{4.4}$$

is strictly convex over $(0, \infty)\times(0, \infty)$, and

$$(\upsilon, \varepsilon)\neq(\upsilon_0, \varepsilon_0) \Rightarrow f(\upsilon, \varepsilon)>0. \tag{4.5}$$

Furthermore, if

$$\bar{\eta}_\varepsilon(\upsilon_0, \varepsilon_0)>0, \tag{4.6}$$

as is always the case for a regular fluid, we have, trivially,

$$v^2\neq 0 \Rightarrow \tfrac{1}{2}v^2\bar{\eta}_\varepsilon(\upsilon_0, \varepsilon_0)>0. \tag{4.7}$$

Since

$$\varphi_{\upsilon_0, \varepsilon_0}(\upsilon, \varepsilon, v^2)=f(\upsilon, \varepsilon)+\tfrac{1}{2}v^2\bar{\eta}_\varepsilon(\upsilon_0, \varepsilon_0), \tag{4.8}$$

$\varphi_{\upsilon_0, \varepsilon_0}(\upsilon, \varepsilon, v^2)$ is a convex function over $(0, \infty)\times(0, \infty)\times[0, \infty)$, and

$$(\upsilon, \varepsilon, v^2)\neq(\upsilon_0, \varepsilon_0, 0) \Rightarrow \varphi_{\upsilon_0, \varepsilon_0}(\upsilon, \varepsilon, v^2)>0. \tag{4.9}$$

Now, since it is also true that

$$\varphi_{\upsilon_0, \varepsilon_0}(\upsilon_0, \varepsilon_0, 0)=0, \tag{4.10}$$

it follows from a known theorem on convex functions* that for each triplet (a, b, c),

$$\frac{1}{\alpha}\,\varphi_{\upsilon_0, \varepsilon_0}(\varepsilon_0+\alpha a, \upsilon_0+\alpha b, \alpha c)$$

is an increasing function of α for all $\alpha>0$ with $\varepsilon_0+\alpha a>0$, $\upsilon_0+\alpha b>0$, and $\alpha c>0$, and this observation, when combined with (4.9), proves the remark.

We use Remark 4.1 to establish

Remark 4.2. *For each regular fluid with a strictly concave equilibrium entropy function* $\bar{\eta}$, *there exists a function* $\delta(\omega, M, V, E)$ *possessing the following properties:*

(i) $\delta(\omega, M, V, E)>0$ *for all* $\omega>0$, $M>0$, $V>0$, *and* $E>0$.

(ii) *If* $\mathscr{C}$ *is a process admissible in a body* $\mathscr{B}$ *comprised of the regular fluid, then given any* $\omega>0$

$$H_0-H(t)\leqq\delta(\omega, M, V, E) \Rightarrow \Delta(\mathscr{C}, t)\leqq\omega, \tag{4.11}$$

with M *the total mass of* $\mathscr{B}$, *and* $H(t)$, *given by* (3.9), *the true total entropy of* $\mathscr{B}$ *at time* t *in the process* $\mathscr{C}$. *The numbers* H_0 *and* $\Delta(\mathscr{C}, t)$ *are given by* (3.7) *and* (3.8), *with* $V, E, \upsilon_0, \varepsilon_0$ *obtained from* (3.4)–(3.6) *using the values of* $\upsilon, \varepsilon, \mathbf{v}$ *in the process* $\mathscr{C}$ *at time* t.

It will be noticed that (4.11) is a statement about *any* process admissible in the regular fluid, not only processes compatible with isolation. Of course, in a general admissible process, the numbers $V, E, \upsilon_0, \varepsilon_0$, and H_0, obtained from (3.4), (3.5), (3.6), and (3.8), will depend on t, but the proof we have given for Theorem 2 still can be used to show that $H_0-H(t)$ is not negative.

Proof of Remark 4.2. Since we are dealing with a regular fluid, it follows from (2.19) that in an admissible process $\mathscr{C}$ we have

$$\int_{\mathscr{B}}\eta(X, t)\,dm\leqq\int_{\mathscr{B}}\bar{\eta}(\upsilon(X, t), \varepsilon(X, t))\,dm; \tag{4.12}$$

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* See, for example, Eggleston [1958, p. 47].

therefore, by (3.8) and (3.9),

$$\begin{aligned} H_0 - H(t) &\overset{\text{def}}{=} \int_{\mathscr{B}} \bar{\eta}(\upsilon_0, \varepsilon_0)\, dm - \int_{\mathscr{B}} \eta(X,t)\, dm \\ &\geqq \int_{\mathscr{B}} \bar{\eta}(\upsilon_0, \varepsilon_0)\, dm - \int_{\mathscr{B}} \bar{\eta}\big(\upsilon(X,t), \varepsilon(X,t)\big)\, dm . \end{aligned} \tag{4.13}$$

Of course, by (3.4) – (3.6),

$$\begin{aligned} \int_{\mathscr{B}} \big([\upsilon(X,t)-\upsilon_0]\, \bar{\eta}_\upsilon(\upsilon_0, \varepsilon_0) + [\varepsilon(X,t)-\varepsilon_0]\, \bar{\eta}_\varepsilon(\upsilon_0, \varepsilon_0) + {} \\ + \tfrac{1}{2} v^2(X,t)\, \bar{\eta}_\varepsilon(\upsilon_0, \varepsilon_0)\big)\, dm = 0 . \end{aligned} \tag{4.14}$$

Adding (4.14) to (4.13) and using (4.1) we obtain

$$H_0 - H(t) \geqq \int_{\mathscr{B}} \varphi_{\upsilon_0, \varepsilon_0}\big(\upsilon(X,t), \varepsilon(X,t), v^2(X,t)\big)\, dm . \tag{4.15}$$

Now, let $\omega > 0$ be given and let the sets $\mathscr{S}_\omega(t)$ and $\tilde{\mathscr{S}}_\omega(t)$ be defined as follows:

$$\begin{gathered} \mathscr{S}_\omega(t) = \left\{ X \in \mathscr{B} \;\middle|\; |\upsilon(X,t) - \upsilon_0| + |\varepsilon(X,t) - \varepsilon_0| + v^2(X,t) < \frac{\omega}{2M} \right\}, \\ \tilde{\mathscr{S}}_\omega(t) = \mathscr{B} - \mathscr{S}_\omega(t) . \end{gathered} \tag{4.16}$$

It follows from (3.7) that

$$\Delta(\mathscr{C}, t) = \int_{\mathscr{B}} \big[|\upsilon - \upsilon_0| + |\varepsilon - \varepsilon_0| + \tfrac{1}{2} v^2\big]\, dm = \mathscr{I}_1 + \mathscr{I}_2 , \tag{4.17}$$

where

$$\begin{aligned} \mathscr{I}_1 &= \int_{\mathscr{S}_\omega(t)} \big[|\upsilon - \upsilon_0| + |\varepsilon - \varepsilon_0| + \tfrac{1}{2} v^2\big]\, dm \\ \mathscr{I}_2 &= \int_{\tilde{\mathscr{S}}_\omega(t)} \frac{|\upsilon - \upsilon_0| + |\varepsilon - \varepsilon_0| + \tfrac{1}{2} v^2}{\varphi_{\upsilon_0, \varepsilon_0}(\upsilon, \varepsilon, v)}\, \varphi_{\upsilon_0, \varepsilon_0}(\upsilon, \varepsilon, v^2)\, dm ; \end{aligned} \tag{4.18}$$

here we have written υ for $\upsilon(X,t)$, ε for $\varepsilon(X,t)$, and v for $v(X,t)$. By (4.16),

$$\mathscr{I}_1 \leqq \frac{\omega}{2} . \tag{4.19}$$

By Remark 4.1,

$$\sup_{X \in \tilde{\mathscr{S}}_\omega(t)} \frac{|\upsilon - \upsilon_0| + |\varepsilon - \varepsilon_0| + \tfrac{1}{2} v^2}{\varphi_{\upsilon_0, \varepsilon_0}(\upsilon, \varepsilon, v)} \leqq \frac{1}{A\left(\dfrac{\omega}{2M}, \upsilon_0, \varepsilon_0\right)} < \infty , \tag{4.20}$$

and hence,

$$\begin{aligned} \mathscr{I}_2 &\leqq \frac{1}{A\left(\dfrac{\omega}{2M}, \upsilon_0, \varepsilon_0\right)} \int_{\tilde{\mathscr{S}}_\omega(t)} \varphi_{\upsilon_0, \varepsilon_0}(\upsilon, \varepsilon, v^2)\, dm \\ &\leqq \frac{1}{A\left(\dfrac{\omega}{2M}, \upsilon_0, \varepsilon_0\right)} \int_{\mathscr{B}} \varphi_{\upsilon_0, \varepsilon_0}(\upsilon, \varepsilon, v^2)\, dm . \end{aligned} \tag{4.21}$$

Substituting (4.19), (4.21), and (4.15) into (4.17), we find that

$$\Delta(\mathscr{C},t) \leqq \frac{\omega}{2} + \frac{H(t)-H_0}{A\left(\dfrac{\omega}{2M}, \upsilon_0, \varepsilon_0\right)}, \tag{4.22}$$

from which it follows that (4.11) holds with

$$\delta(\omega, M, V, E) = \frac{\omega}{2} A\left(\frac{\omega}{2M}, \frac{V}{M}, \frac{E}{M}\right); \tag{4.23}$$

q.e.d.

We have now assembled the apparatus required to state and prove our main theorem.

Theorem 3. *Let $\mathscr{B}$ be a regular fluid body, and suppose that every uniform static state of $\mathscr{B}$ is Gibbs stable. Let $V>0$ and $E>0$ be arbitrarily assigned, and let $\mathfrak{C}(V,E)$ be the class of admissible processes in $\mathscr{B}$ which are compatible with isolation of $\mathscr{B}$ and which give $\mathscr{B}$ the (constant) total volume V and total energy E. Of course, all the processes in $\mathfrak{C}(V, E)$ are associated with the same uniform equilibrium process $\mathscr{C}_0(V, E)$; we denote by $H_0(V, E)$ the total entropy of $\mathscr{B}$ in the process $\mathscr{C}_0(V, E)$. Now, given any $\omega>0$, there exists a δ_ω such that if a process $\mathscr{C}$ in $\mathfrak{C}(V, E)$ satisfies*

$$H_0(E,V) - H(\tau) \leqq \delta_\omega, \quad \text{with} \quad H(\tau) = \int_{\mathscr{B}} \eta(X,\tau)\, dm, \tag{4.24}$$

*at any one time τ, then for **all** $t \geqq \tau$, $\mathscr{C}$ satisfies*

$$\Delta(\mathscr{C},t) \leqq \omega \tag{4.25}$$

with $\Delta(\mathscr{C}, t)$ defined in (3.7).

In words: If all uniform static states are Gibbs stable, and if we know that the total entropy H of an isolated body $\mathscr{B}$ is at present close to its value H_0 in uniform equilibrium, then we can be sure that the departure of $\mathscr{B}$ from uniform equilibrium, as measured by the metric (3.7), will be small at all future times.

Proof of Theorem 3. Since we assume that every uniform state of $\mathscr{B}$ is Gibbs stable, it follows from Theorem 1 that $\bar{\eta}$ is strictly concave. Now, we let $\omega>0$ be given, and we put

$$\delta_\omega = \delta(\omega, M, V, E), \tag{4.26}$$

where the function on the right is that given to us by Remark 4.2, M is the mass of $\mathscr{B}$, and V and E are the total energy and total volume of $\mathscr{B}$ for the processes in $\mathfrak{C}(V, E)$. It then follows from (4.11) that for all processes in $\mathfrak{C}(V, E)$

$$H_0 - H(t) \leqq \delta_\omega \;\Rightarrow\; \Delta(\mathscr{C},t) < \omega. \tag{4.27}$$

But, since the processes in $\mathfrak{C}(V, E)$ are compatible with isolation of $\mathscr{B}$, for these processes, by (2.25),

$$\frac{d}{dt} H(t) \geqq 0.$$

Hence, if $\mathscr{C}$ is in $\mathfrak{C}(V, E)$ and obeys (4.24) at some time τ, then $H_0 - H(t) \leqq \delta_\omega$ at all times $t \geqq \tau$, and, by (4.27), $\mathscr{C}$ also obeys (4.25); q.e.d.

The reader will notice that although (4.27) holds, it does not follow that $H_0 - H(t)$ must be small whenever $\Delta(\mathscr{C}, t)$ is small. Theorem 3 tells us that uniform equilibrium processes are dynamically stable in the sense that no perturbation which keeps the entropy at time τ close to its equilibrium value can yield a large departure $\Delta(\mathscr{C}, t)$ from uniform equilibrium at a later time t. Since the hypothesis of Theorem 3 does *not* imply that if $\Delta(\mathscr{C}, \tau)$ is small, then $\Delta(\mathscr{C}, t)$ must be small for all $t > \tau$, our theorem deals with a type of stability weaker than the Lyapunov stability considered in the theory of differential equations.

Theorem 3 has the following immediate corollary.

Remark 4.3. *Under the hypothesis of Theorem* 3, *for each* $\omega > 0$ *there exists a* $\delta_\omega > 0$ *such that if a process* $\mathscr{C}$ *in* $\mathfrak{C}(V, E)$ *satisfies* (4.24) *at any one time* τ, *then* $\mathscr{C}$ *also satisfies*

$$\Big|\int_{\mathscr{P}} \upsilon(X,t)\,dm - V_0(\mathscr{P})\Big| \leqq \omega \quad \textit{with} \quad V_0(\mathscr{P}) = V\frac{\mathscr{M}(\mathscr{P})}{\mathscr{M}(\mathscr{B})}, \tag{4.28}$$

$$\Big|\int_{\mathscr{P}} \varepsilon(X,t)\,dm - E_0(\mathscr{P})\Big| \leqq \omega \quad \textit{with} \quad E_0(\mathscr{P}) = E\frac{\mathscr{M}(\mathscr{P})}{\mathscr{M}(\mathscr{B})}, \tag{4.29}$$

and

$$\tfrac{1}{2}\int_{\mathscr{P}} v^2(X,t)\,dm \leqq \omega, \tag{4.30}$$

at each time $t \geqq \tau$ *and for all parts* $\mathscr{P}$ *of* $\mathscr{B}$.

In words: When all the uniform static states of a regular fluid body $\mathscr{B}$ are Gibbs stable, to insure that the volume, the internal energy, and the kinetic energy of all the parts of $\mathscr{B}$ remain close to their values at uniform equilibrium, it suffices to isolate $\mathscr{B}$ and to set the initial total entropy of $\mathscr{B}$ close to its value at uniform equilibrium.

Although we know no simpler proof of (4.28) and (4.29), restrictions on the kinetic energy of the parts of $\mathscr{B}$ can be derived by arguments more direct than that which we used to prove Theorem 3. To show this we now establish

Theorem 4. *Let* $\mathscr{B}$ *be a regular fluid body with the property that every uniform static state of* $\mathscr{B}$ *is Gibbs stable. In each process* $\mathscr{C}$ *that is admissible in* $\mathscr{B}$ *and compatible with isolation of* $\mathscr{B}$, *we have, for each time* τ *and for all parts* $\mathscr{P}$ *of* $\mathscr{B}$,

$$t \geqq \tau \;\Rightarrow\; \tfrac{1}{2}\int_{\mathscr{P}} v^2(X,t)\,dm \leqq [H_0 - H(\tau)]\,\vartheta_0, \tag{4.31}$$

where $H(\tau)$ *is the true total entropy of* $\mathscr{B}$ *at time* τ *in the process* $\mathscr{C}$, *and* ϑ_0 *and* H_0 *are the temperature and total entropy of* $\mathscr{B}$ *in the uniform equilibrium process associated with* $\mathscr{C}$.

Proof. Let υ_0 and ε_0 be defined by (3.6) with V and E the (constant) total volume and energy of $\mathscr{B}$ in the process $\mathscr{C}$. The argument used in the proof of Remark 4.2 may be applied here to obtain the relation (4.15), and we may use the equations (4.8), $(3.3)_9$, and $(1.2)_1$ to write (4.15) in the form

$$H_0 - H(t) \geqq \int_{\mathscr{B}} f(\upsilon(X,t), \varepsilon(X,t))\,dm + \frac{1}{2\vartheta_0}\int_{\mathscr{B}} v^2(X,t)\,dm \tag{4.32}$$

with the function f defined by (4.4). From Theorem 1 and our hypothesis that every uniform static state of $\mathscr{B}$ is Gibbs stable, we deduce that $\bar{\eta}$ is strictly concave, which implies that (4.5) holds. Hence (4.32) yields

$$H_0-H(t)\geqq\frac{1}{2\vartheta_0}\int_{\mathscr{B}} v^2(X,t)\,dm\geqq\frac{1}{2\vartheta_0}\int_{\mathscr{P}} v^2(X,t)\,dm \tag{4.33}$$

for all parts $\mathscr{P}$ of $\mathscr{B}$. Of course, since the process $\mathscr{C}$ is compatible with isolation of $\mathscr{B}$, it follows from (2.25) that

$$t\geqq\tau \Rightarrow H_0-H(\tau)\geqq H_0-H(t), \tag{4.34}$$

and this with (4.33) clearly yields (4.31); q.e.d.

It is evident from the proof just given that by making use of (2.24) and the strict inequality in (4.5), one can strengthen (4.31) as follows:

Remark 4.4. *Under the hypothesis of Theorem 4,*

$$t\geqq\tau, \quad \int_{\mathscr{B}} v^2(X,t)\,dm\neq 0 \Rightarrow \tfrac{1}{2}\int_{\mathscr{P}} v^2(X,t)\,dm<\vartheta_0[H_0-H(\tau)] \tag{4.35}$$

for all parts $\mathscr{P}$ of $\mathscr{B}$.

An interesting corollary to our proof of Theorem 3 is

Remark 4.5. *If in addition to the hypothesis of Theorem 3 we suppose that the constitutive equations of the regular fluid are such that for processes in $\mathfrak{C}(V, E)$*

$$H(t)\neq H_0(V,E) \Rightarrow \frac{d}{dt}H(t)\neq 0, \tag{4.36}$$

then for each $\mathscr{C}$ in $\mathfrak{C}(V, E)$[*]

$$\lim_{t\to\infty} \Delta(\mathscr{C},t)=0. \tag{4.37}$$

Proof. For each $\mathscr{C}$ in $\mathfrak{C}(V, E)$ we have, by (2.25) and Theorem 2,

$$\frac{d}{dt}H(t)\geqq 0 \quad \text{and} \quad H(t)\leqq H_0(V,E) \tag{4.38}$$

for all t; hence

$$\lim_{t\to\infty} H(t) \overset{\text{def}}{=} H(\infty) \tag{4.39}$$

exists. But, (4.36) implies that $H(\infty)$ cannot be different from $H_0(V, E)$; hence

$$\lim_{t\to\infty} H(t)=H_0(V,E). \tag{4.40}$$

It follows immediately from this observation and Remark 4.2 that (4.37) holds for each $\mathscr{C}$ in $\mathfrak{C}(V, E)$; q.e.d.

Of course, in applications it will not be a trivial matter to check whether (4.36) holds for a given set of constitutive equations. It is clear, however, that (4.36) does not hold for a perfectly elastic non-conducting fluid, i.e. for a material which always obeys (2.3) with $\kappa\equiv 0$.

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[*] It will be recalled that, by definition, the time-dependent fields comprising a *process* $\mathscr{C}$ of $\mathscr{B}$ exist for *all* t in $(-\infty, \infty)$.

5. On Non-Simple Fluids

The definition of a thermodynamic process given in Section 2 presupposes (i) that the force system acting on $\mathscr{B}$ may be resolved into a stress field $\boldsymbol{T}$ and a body force field $\boldsymbol{b}$, (ii) that all transfers of non-mechanical energy can be expressed with a heat supply field r and a heat flux field $\boldsymbol{q}$, and (iii) that the entropy H of $\mathscr{B}$ is an integral of the entropies of the parts of $\mathscr{B}$. Here we shall observe that the theorems of Sections 3 and 4 can be generalized to cover materials for which the suppositions (i) – (iii) fail.

Let us continue to suppose that the list of fields comprising a thermodynamic process contains the specific internal energy ε. Any thermodynamic theory presupposes some set of constitutive assumptions, a law of balance of energy, and a law of positive production of entropy. Here we need not state these principles explicitly. It suffices to assume that for the processes in $\mathscr{B}$ that are compatible with isolation of $\mathscr{B}$ we have

$$\frac{d}{dt}V=0, \quad \frac{d}{dt}E=0, \tag{5.1}$$

and

$$\frac{d}{dt}H(t)\geqq 0, \tag{5.2}$$

where V, E, and H are the total volume, energy, and entropy of $\mathscr{B}$. Of course, V is the integral of υ:

$$V=\int_{\mathscr{B}} \upsilon(X,t)\,dm. \tag{5.3}$$

The formula (3.5) for E may be generalized to permit expressions for the kinetic energy other than the classical:

$$E=\int_{\mathscr{B}} [\varepsilon(X,t)+\beta(X,t)]\,dm, \qquad \beta(X,t)\geqq 0. \tag{5.4}$$

It is not necessary to assume that $H(t)$ is always given by (3.9). However, we say that the material comprising $\mathscr{B}$ is a *generalized regular fluid* if (i) there exists a function $\bar{\eta}$ over $(0, \infty)\times(0, \infty)$ such that in every admissible process of $\mathscr{B}$

$$H(t)\leqq \int_{\mathscr{B}} \bar{\eta}\big(\upsilon(X,t),\varepsilon(X,t)\big)\,dm, \tag{5.5}$$

and (ii) for each pair $V>0$, $E>0$ there exists an admissible process $\mathscr{C}_0(V,E)$ of $\mathscr{B}$ that is compatible with isolation of $\mathscr{B}$ and such that

$$\beta(X,t)\equiv 0, \quad \upsilon(X,t)\equiv\upsilon_0 \overset{\text{def}}{=} V/M, \quad \varepsilon(X,t)\equiv\varepsilon_0 \overset{\text{def}}{=} E/M, \tag{5.6}$$

$$H(t)=\int_{\mathscr{B}} \bar{\eta}(\upsilon_0,\varepsilon_0)\,dm=M\bar{\eta}(\upsilon_0,\varepsilon_0). \tag{5.7}$$

The function $\bar{\eta}$ is again called the *equilibrium entropy function* for the regular fluid; we assume that the number ϑ_0, given by

$$\vartheta_0=[\eta_\varepsilon(\upsilon_0,\varepsilon_0)]^{-1} \tag{5.8}$$

is strictly positive.

Using the function $\bar{\eta}$ we can repeat the theory of Section 1. For the small part of classical thermostatics described there, we may take the functions $\bar{p}$ and $\bar{\vartheta}$ to be *defined* by (1.2).

It will be noticed that our present definition of a generalized regular fluid assumes the existence, but not the uniqueness,* of an equilibrium process of the type described in Remark 3.1. We here call any admissible process in $\mathscr{B}$ obeying (5.6) and (5.7) a *uniform equilibrium process.*

For an arbitrary admissible process $\mathscr{C}$ in $\mathscr{B}$ we now define $\Delta(\mathscr{C}, t)$ by

$$\Delta(\mathscr{C},t)=\int_{\mathscr{B}}|\upsilon(X,t)-\upsilon_0|\,dm+\int_{\mathscr{B}}|\varepsilon(X,t)-\varepsilon_0|\,dm+\int_{\mathscr{B}}\beta(X,t)\,dm, \tag{5.9}$$

where

$$\upsilon_0=V/M, \quad \varepsilon_0=E/M, \tag{5.10}$$

with V and E given by (5.3) and (5.4). If $\mathscr{B}$ is comprised of a generalized regular fluid and if $\mathscr{C}$ is compatible with isolation of $\mathscr{B}$, then $\Delta(\mathscr{C}, t)$ gives a measure of the departure of $\mathscr{C}$ from the (possibly several) uniform equilibrium processes $\mathscr{C}_0(V, E)$ having the same total volume V and energy E as $\mathscr{C}$.

Using the few definitions and hypotheses just stated it is now easy to generalize Theorems 2, 3, and 4. As a generalization of Theorem 2 we have

Remark 5.1. *Let $\mathscr{B}$ be a generalized regular fluid body with a strictly concave equilibrium entropy function $\bar{\eta}$. If $\mathscr{C}$ is an admissible process compatible with isolation of $\mathscr{B}$ then, for any t in* $(-\infty, \infty)$,

$$\Delta(\mathscr{C},t)>0 \;\Rightarrow\; H(t)<H_0, \tag{5.11}$$

where $H(t)$ is the true entropy of $\mathscr{B}$ in the process $\mathscr{C}$ at time t, the number $\Delta(\mathscr{C}, t)$ is given by (5.9), *and H_0 is the constant*

$$H_0=M\bar{\eta}(\upsilon_0,\varepsilon_0) \tag{5.12}$$

with υ_0 and ε_0 given by (5.10), (5.3), *and* (5.4).

Proof. By (5.5), the concavity of $\bar{\eta}$, and the positivity of $\beta(X, t)$ and η_ε, we have

$$\begin{aligned} H(t) &\leqq \int_{\mathscr{B}} \bar{\eta}(\upsilon(X,t),\varepsilon(X,t))\,dm \\ &\leqq \int_{\mathscr{B}} \{\eta_0(\upsilon_0,\varepsilon_0)+[\upsilon(X,t)-\upsilon_0]\,\bar{\eta}_\upsilon(\upsilon_0,\varepsilon_0)+ \\ &\qquad\qquad +[\varepsilon(X,t)-\varepsilon_0]\,\bar{\eta}_\varepsilon(\upsilon_0,\varepsilon_0)+\beta\,\bar{\eta}_\varepsilon(\upsilon_0,\varepsilon_0)\}\,dm. \end{aligned} \tag{5.13}$$

When υ_0 and ε_0 are chosen as in (5.10) with V and E given by (5.3) and (5.4), the right side of (5.13) reduces to $M\eta_0(\upsilon_0, \varepsilon_0)$. Furthermore, since $\bar{\eta}$ is strictly concave and $\bar{\eta}_\varepsilon(\upsilon_0, \varepsilon_0)>0$, the inequality $(5.13)_2$ is *strict* whenever $\Delta(\mathscr{C}, t)\neq 0$; q.e.d.

Theorems 3 and 4 are generalized in

Remark 5.2. *Let $\mathscr{B}$ be a generalized regular fluid body with $\bar{\eta}$ for its equilibrium entropy function, and suppose that every uniform static state of $\mathscr{B}$ is Gibbs stable.*

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* In applications we expect the conditions (5.6) to determine a unique admissible process compatible with isolation, as in Remark 3.1, but such uniqueness is not necessary to the proofs of our theorems.

Let $V>0$ and $E>0$ be given, put

$$H_0 = M\bar{\eta}(\upsilon_0, \varepsilon_0), \qquad \vartheta_0 = [\eta_\varepsilon(\upsilon_0, \varepsilon_0)]^{-1}, \qquad \upsilon_0 = V/M, \qquad \varepsilon_0 = E/M, \tag{5.14}$$

and let $\mathfrak{C}(V, E)$ be the set of admissible processes in $\mathscr{B}$ which are compatible with isolation of $\mathscr{B}$ and which give $\mathscr{B}$ the total volume V and total energy E. Now, given any $\omega>0$, there exists a δ_ω such that for processes in $\mathfrak{C}(V, E)$

$$H_0 - H(\tau) \leqq \delta_\omega \;\Rightarrow\; \Delta(\mathscr{C}, t) \leqq \omega \qquad \textit{for all } t \geqq \tau; \tag{5.15}$$

here $H(\tau)$ is the entropy of $\mathscr{B}$ in $\mathscr{C}$ at time τ, and $\Delta(\mathscr{C}, t)$ is given by (5.9). *It is also true that for processes in $\mathfrak{C}(V, E)$,*

$$t \geqq \tau \;\Rightarrow\; \int_{\mathscr{P}} \beta(X, t)\, dm \leqq [H_0 - H(\tau)]\, \vartheta_0, \tag{5.16}$$

where $\mathscr{P}$ is any part of $\mathscr{B}$.

Proof. Use the proofs of Theorems 3 and 4 with $\frac{1}{2} v^2$ and $\int_{\mathscr{B}} \eta(X, t)\, dm$ replaced by β and $H(t)$, respectively, wherever they occur.

It is now clear that our Remarks 4.3 – 4.5 are also easily extended to generalized regular fluids.

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References

1873 [*1*] Gibbs, J.W., Trans. Connecticut Acad. **2**, 309 – 342.

[*2*] Gibbs, J.W., Trans. Connecticut Acad. **2**, 382 – 404.

1875 [*1*] Gibbs, J.W., Trans. Connecticut Acad. **3**, 108 – 248, 343 – 524 (1875 – 1878).

1958 [*1*] Eggleston, H.G., Convexity, Cambridge Tracts in Mathematics and Mathematical Physics, No. 47.

[*2*] Noll, W., Arch. Rational Mech. Anal. **2**, 197 – 226.

1959 [*1*] Noll, W., In Proceedings of the Berkely Symposium on the Axiomatic Method with Special Reference to Geometry and Physics. Amsterdam: North Holland Co. pp. 266 – 281.

1960 [*1*] Coleman, B.D., & W. Noll, Arch. Rational Mech. Anal. **6**, 355 – 370.

[*2*] Truesdell, C., & R. Toupin, The Classical Field Theories. In Encyclopedia of Physics, vol. III/1, ed. by S. Flügge. Berlin-Göttingen-Heidelberg: Springer.

1961 [*1*] Coleman, B.D., & W. Noll, Reviews Mod. Phys. **33**, 239 – 249; erratum: *ibid.* **36**, 1103 (1964).

[*2*] Coleman, B.D., & W. Noll, Ann. N.Y. Acad. Sci. **89**, 672 – 714.

1963 [*1*] Coleman, B.D., & W. Noll, Arch. Rational Mech. Anal. **13**, 167 – 178.

1964 [*1*] Coleman, B.D., Arch. Rational Mech. Anal. **17**, 1 – 46, 230 – 254.

[*2*] Coleman, B.D., & V.J. Mizel, J. Chem. Phys. **40**, 1116 – 1125.

1966 [*1*] Coleman, B.D., & V.J. Mizel, Arch. Rational Mech. Anal. **23**, 87 – 123.

[*2*] Noll, W., Report 66-2, Department of Mathematics, Carnegie Institute of Technology.

1967 [*1*] Coleman, B.D., & M.E. Gurtin, J. Chem. Phys. (in press).

[*2*] Coleman, B.D., & V.J. Mizel, Arch. Rational Mech. Anal. **24**, 87 – 123.

Note added in proof. We should like to bring to the reader's attention J. Serrin's article "On the Uniqueness of Compressible Fluid Motions". Arch. Rational Mech. Anal. **3**, 271 – 288 (1959), which reflects an outlook very similar to ours, although there is no overlap of results.

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