An Existence Theorem for the von Kármán Equations

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In this paper we obtain *a priori* estimates and a global existence theorem for solutions of the non-linear equations governing deflections in a thin, elastic plate which is clamped at the edges and subjected to combined normal and edge loading. For small data there is a unique solution.

Let the region G, constituting the middle plane of the undeflected plate, lie in the xy-plane and let Γ be its boundary. The equilibrium stress function f^*+F^* and deflection w^* of the plate out of its plane satisfy the following version of the VON KÁRMÁN equations [*14*]:

$$\Delta^2 f^* = -(E/2)[w^*, w^*],$$
$$\Delta^2 w^* = (h/D)\{[f^*, w^*]+[F^*, w^*]+(p^*/h)\}$$

where Δ^2 is the biharmonic operator and

$$[f, g] = f_{xx}\, g_{yy} + f_{yy}\, g_{xx} - 2 f_{xy}\, g_{xy}.$$

Here $p^* = p^*(x, y)$ denotes the lateral load on the plate and D is the flexural rigidity constant depending upon the modulus of elasticity E, the plate thickness h, and Poisson's ratio. $F^*(x, y)$ is the (known) stress function which would arise from a given combination of any or all of a variety of sources (e.g. compression or tension on the edges, thermal stresses) if the plate were not allowed to bend.

Now let

$$|\nabla\alpha| = \{\alpha_x^2 + \alpha_y^2\}^{\frac{1}{2}},$$
$$|\nabla\nabla\alpha| = \{\alpha_{xx}^2 + 2\alpha_{xy}^2 + \alpha_{yy}^2\}^{\frac{1}{2}},$$
$$\mu = (h/D)\max\{\sup_G |F^*|, \sup_G |\nabla F^*|, \sup_G |\nabla\nabla F^*|\},$$
$$\nu = D^{-1}(\tfrac{1}{2} h E D^{-1})^{\frac{1}{2}} \sup_G |p^*|,$$

and set

$$f^* = D h^{-1} f,$$
$$w^* = (2 D h^{-1} E^{-1})^{\frac{1}{2}} w,$$
$$F^* = \mu D h^{-1} F,$$
$$p^* = \nu D (2 D h^{-1} E^{-1})^{\frac{1}{2}} p.$$

Then the von Kármán equations become

$$\Delta^2 f = -[w, w], \tag{1a}$$

$$\Delta^2 w = [f, w] + \mu [F, w] + \nu p. \tag{1b}$$

The system (1a, 1b) together with the boundary conditions

$$f = |\nabla f| = w = |\nabla w| = 0 \qquad \text{on } \Gamma \tag{1c}$$

for the clamped plate shall constitute problem (1).

Much of the existing work on the von Kármán equations has been concerned with problems involving radial or rectangular symmetry, e.g. [*1*, *5*, *7*, *8*]. In this paper we consider regions of a general class defined in §1. References [*2*, *4*, *11*] also treat problems in general regions. In [*11*] MOROSOV has solved problem (1) for $\mu = 0$ and arbitrary $\nu \geqq 0$. BERGER & FIFE [*2*] consider the case $\nu = 0$ and prove the existence of non-trivial solutions (bifurcations of the solution $w = f = 0$ which are shown to occur at certain eigenvalues, μ, of the linearized problem). In [*4*], FIFE deals with a different problem involving the von Kármán equations.

For arbitrary $\mu \geqq 0$, $\nu \geqq 0$ we shall prove the existence of a classical solution of problem (1) satisfying (for a wide class of regions G)

$$\sup |w| + \sup |f| \leqq C \{\mu^2 + \nu^2 + \exp(C\mu)\}^{\frac{1}{2}}. \tag{2}$$

(Here, and throughout the paper, the letter C denotes constants which depend only upon G.) These results rely strongly upon the non-linear character of the problem. In fact, at an eigenvalue, μ, the linearized problem has solutions of arbitrary norm. Hence no bound of the type (2) is possible in the linear case.

In §1, after the introduction of some notation and basic inequalities, a precise definition of a classical solution of problem (1) is given. Then the problem is restated in generalized and abstract forms which are convenient for proving (2) and the existence theorem. The equivalence of these formulations of problem (1) follows from two theorems of BERGER & FIFE quoted from [*2*].

The existence theorem is proved in § 2. For the proof the Leray-Schauder fixed-point theorem [*10*] is employed in a somewhat simpler form due to SCHAEFER [*12*]. Application of the Schaefer theorem requires *a priori* estimates for which the inequality (2) suffices.

Derivation (§ 2) of the estimate (2) constitutes the chief technical difficulty of the paper. The result is established by exploiting the divergence structure of the nonlinearity in dealing with the terms involving F. In this maneuver a modification due to EDWARDS [*3*] of an auxiliary function of HOPF [*6*] plays an important role.

In § 3 we examine the case of small data (i.e. μ, ν small). If only $\mu < 1$, the estimate

$$\sup_G |w| + \sup_G |f| \leqq C \nu (1-\mu)^{-1}$$

is proved without difficulty. For v sufficiently small, uniqueness of the solution of problem (1) then follows in a standard way. The uniqueness question for large data remains unsettled. However, the above mentioned study of bifurcations by BERGER & FIFE [2] as well as earlier work by FRIEDRICHS & STOKER [5] and KELLER, KELLER & REISS [7] show that uniqueness is not, in general, to be expected.

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1. Preliminaries

The following notation, in addition to that set forth in the introduction, will be used in this paper:

$C^k(G)$: functions whose derivatives up to order k are continuous in G.

$C_0^k(G)$: functions in $C^k(G)$ having compact support in G.

$Cl\,G$: closure of G.

$$\|\varphi\| = \{\int_G (\varphi^2 + |\nabla\varphi|^2 + |\nabla\nabla\varphi|^2)\, dA\}^{\frac{1}{2}},$$

W: the Hilbert space obtained by completion of $C_0^\infty(G)$ in the norm $\|\varphi\|$.

H: $W \times W$; the Hilbert space of vector functions $\bar{\varphi} = (\varphi_1, \varphi_2)$ with components φ_1 and φ_2 in W and norm $|||\bar{\varphi}||| = \{\|\varphi_1\|^2 + \|\varphi_2\|^2\}^{\frac{1}{2}}$.

Note, for functions φ in W, that

$$\int_G |\nabla\nabla\varphi|^2\, dA = \int_G |\Delta\varphi|^2\, dA; \tag{3}$$

for example, if f and g lie in W then

$$\int_G |[f, g]|\, dA \leqq \int_G (|\nabla\nabla f|\, |\nabla\nabla g|)\, dA \leqq (\int_G |\Delta f|^2\, dA)^{\frac{1}{2}} (\int_G |\Delta g|^2\, dA)^{\frac{1}{2}}. \tag{4}$$

It follows fom the SOBOLEV embedding theorem (see, e.g., [13, p. 56]) that the elements, φ, of W are continuous functions and satisfy

$$\sup_G |\varphi| \leqq C\, \|\varphi\|\,. \tag{5}$$

Lemma 1 below is established for $C_0^\infty(G)$ functions by repeated application of the Poincaré inequality and a use of (3). The result then follows for W by a standard limiting argument.

Lemma 1. *There exists a constant C, depending only on G, such that, for all φ in W,*

$$\|\varphi\| \leqq C(\int_G |\nabla\nabla\varphi|^2\, dA)^{\frac{1}{2}} = C(\int_G |\Delta\varphi|^2\, dA)^{\frac{1}{2}}.$$

Next we state a result of LERAY [9] (see also [6]).

Lemma 2. *Let the function h belong to* $C^1([0, a])$ *and let* $h(0)=0$. *Then*

$$\int_0^a s^{-2} h^2(s)\, ds \leqq 4 \int_0^a |h'(s)|^2\, ds.$$

Assumptions on the region G. G shall be a bounded region. The boundary Γ shall consist of a finite number of disjoint simple, closed curves. There shall exist a subregion, G_δ, of G and a non-singular coordinate system (s, t), defined on $Cl G_\delta$, with the properties:

i) The function $s=s(x, y)$ is in $C^2(G_\delta) \cap C^0(Cl G_\delta)$.

ii) G_δ is the annular region $0<s<2\delta$, having Γ as the boundary curve given by $s=0$; the remaining boundary given by $s=2\delta$.

iii) There are positive constants m and K such that

(6a) $$x_s^2 + y_s^2 \leqq K \qquad \text{on } G_\delta,$$

and the Jacobian

$$J = \frac{\partial(s, t)}{\partial(x, y)}$$

satisfies the bounds

(6b) $$0 < m \leqq J \leqq m K \qquad \text{on } G_\delta.$$

iv) For any annular region $0<s_1 \leqq s \leqq 2\delta$ there is a constant $M(s_1)$, possibly depending on s_1, such that

$$|\nabla s| \leqq M(s_1), \qquad |\Delta s| \leqq M(s_1).$$

If Γ is a C^3 curve we may choose s as distance from Γ measured along normals. We choose t as arc length on Γ and constant on normals. In this case $|\nabla s|$ and $|\Delta s|$ are bounded independently of s_1. The stated class of regions is, however, wide enough to allow "corners" on Γ.

The functions F and p appearing in (1b) are *admissible* if F is in $C^2(G)$, p is in $C^0(G)$, and if

(7) $$|p| \leqq 1, \qquad |F| \leqq 1, \qquad |\nabla F| \leqq 1, \qquad |\nabla\nabla F| \leqq 1.$$

These bounds impose no real restriction upon the problem in view of the definitions of μ and ν. We assume throughout the paper that F and p are admissible functions.

Definition. A *classical solution* of problem (1) is a pair of functions f, w which lie in the class $C^4(G) \cap C^1(Cl G)$ and satisfy the equations (1a, b, c) at each point.

Proceeding now to reformulate problem (1), we let φ and ψ be smooth functions in W. Then upon multiplying equation (1a) by φ, equation (1b) by ψ, and integrating by parts over G, we find

(8a) $$a(\varphi, f) = -b(\varphi, w; w),$$

(8b) $$a(\psi, w) = b(\psi, f; w) + \mu\, b(\psi, F; w) + \nu\, b_1(\psi, p)$$

where

$$
\begin{aligned}
a(\alpha,\beta) &= \int_G \Delta\alpha\,\Delta\beta\,dA \\
b(\alpha,\beta;\gamma) &= \int_G \{\alpha_x(\beta_y\gamma_{xx}-\beta_x\gamma_{yy})+\alpha_y(\beta_x\gamma_{xy}-\beta_y\gamma_{xx})\}\,dA \qquad (9)\\
b_1(\alpha,\beta) &= \int_G \alpha\,\beta\,dA.
\end{aligned}
$$

Definition. A *generalized solution* of problem (1) is a pair of functions f, w in W satisfying (8a, b) for all φ, ψ in W.

The problem may also be expressed in terms of an operator equation defined on the Hilbert space H. This is the point of Theorem 2 below. We outline the ideas briefly. Considering $u=(f, w)$ and $v=(\varphi, \psi)$ in H we may add (8a) and (8b) to obtain an expression of the form $\mathfrak{A}(u, v)=\mathfrak{B}(u, v)$. For fixed u in H, $\mathfrak{A}(u, v)$ and $\mathfrak{B}(u, v)$ are bounded, linear functionals of v in H, hence are associated respectively with elements Au and Bu in H. Since A is invertible the problem is to find u in H such that $Iu-Tu=0$, where I is the identity operator and $T=A^{-1}B$ is a compact operator from H into H.

For the case $\nu=0$, Berger & Fife [2] have established the following two theorems. The proofs may be amended in a straightforward way to obtain the results in case $\nu>0$.

Theorem 1. *Every classical solution of problem* (1) *is a generalized solution. Conversely, every generalized solution is a classical solution in G and at all sufficiently smooth portions of Γ.*

Theorem 2. *The generalized solutions f, w of problem* (1) *are identical with the solutions $u=(f, w)$ of an operator equation $Iu=Tu$ defined on the separable Hilbert space H. The operator T is compact.*

2. Existence; A Priori Estimates

The question of existence of classical solutions of problem (1) is, in view of Theorems 1 and 2, reduced to the existence question for the operator equation

$$Iu=\lambda Tu \qquad (10)$$

with λ set equal to unity, u in H and T compact. We shall use the Schaefer version [12] of the Leray-Schauder fixed-point theorem [10] to establish the existence of at least one solution of (10). The Schaefer theorem requires, beyond the compactness of T, only that there exist a sphere in H containing all possible solutions of (10) for $0\leqq\lambda\leqq 1$. To this end we shall prove

Theorem 3. *Let f, w be a solution of problem* (1). *Then there are constants C, depending only on the region G, such that*

$$
\begin{aligned}
\sup_G|f|+\sup_G|w| &\leqq C\,|||(f,w)||| = C[\|f\|^2+\|w\|^2]^{\frac{1}{2}} \qquad (11)\\
&\leqq C\{\mu^2+\nu^2+\exp(C\mu)\,M^2[\exp(-C\mu)]\}^{\frac{1}{2}}.
\end{aligned}
$$

Note that if $M(s_1) \leqq C s_1^{-\gamma}$ for some $\gamma \geqq 0$ then the estimates (11) reduce to the form (2).

For general λ, $0 \leqq \lambda \leqq 1$, the proof (given below) of Theorem 3 remains valid. Thus for fixed μ and ν all solutions of (10) in H lie within a sphere of radius

$$\operatorname{const}\{\mu^2 + \nu^2 + \exp(C\mu) M^2[\exp(-C\mu)]\}^{\frac{1}{2}}.$$

The Schaefer theorem now assures the existence of a solution of (10) for each λ, $0 \leqq \lambda \leqq 1$. In particular, when $\lambda = 1$ we conclude

Theorem 4. *For any non-negative numbers μ and ν, problem* (1) *possesses at least one classical solution.*

We now turn to a proof of Theorem 3. We shall proceed from the generalized form of the equations and follow the "energy" method. The terms involving F cause the principle difficulty as the non-linear terms drop out of the energy expression because of the property

$$b(\alpha, \beta; \gamma) = b(\beta, \alpha; \gamma). \tag{12}$$

It is this structure that enables us to control the F terms by absorbing part of them into the corresponding f terms.

Let ξ be in $C^2(G)$ and satisfy $\xi = 1$, $\nabla \xi = 0$ on Γ. Given that f, w is a solution of problem (1), introduce the modified pair g, w where $g = f + \mu(1-\xi)F$. Then g, w satisfy the boundary conditions (1c) and the modified equations (in generalized form)

$$a(\varphi, g) = a(\varphi, \mu(1-\xi)F) - b(\varphi, w; w),$$

$$a(\psi, w) = b(\psi, g; w) + \mu\, b(\psi, \xi F; w) + \nu\, b_1(\psi, p)$$

for all φ, ψ in W. Since g and w are in W, set $\varphi = g$, $\psi = w$ and use (12) to obtain

$$a(g, g) + a(w, w) = \mu\, a(g, (1-\xi)F) + \mu\, b(w, \xi F; w) + \nu\, b_1(w, p). \tag{13}$$

With reference to the integral forms (9) of the quantities we may estimate the first and third terms on the right in (13) using the Schwarz inequality and, in the b_1 estimate, Lemma 1. We find

$$|\mu\, a(g, (1-\xi)F)| \leqq \mu\Big(\int_G |\Delta g|^2\, dA\Big)^{\frac{1}{2}} \Big(\int_G |\Delta\{(1-\xi)F\}|^2\, dA\Big)^{\frac{1}{2}},$$

$$|\nu\, b_1(w, p)| \leqq \nu\, C\Big(\int_G p^2\, dA\Big)^{\frac{1}{2}} \Big(\int_G |\Delta w|^2\, dA\Big)^{\frac{1}{2}}.$$

To bound the remaining term, $\mu b(w, \xi F; w)$, we shall require additional properties of the function ξ. It is convenient to introduce the subregion G_δ and associated coordinate system (s, t) as described in §1. It is clear that we may assume $0 < \delta \leqq \frac{1}{2}$. Then the following version of a lemma of HOPF [*6*] (see also EDWARDS [*3*]) provides the function ξ with the needed properties:

Lemma 3. *Given $\varepsilon>0$ there exists a function $\xi(x, y)$ in $C^2(G)\cap C^1(ClG)$ with the properties:*

i) $\xi=1$, $|\nabla\xi|=0$ *in the strip* $0\leqq s\leqq s_1=\delta 4^{-\frac{1}{2}}\exp(-2\varepsilon^{-1})$,

ii) $\xi=0$ *outside* G_δ,

iii) $|\xi|\leqq\varepsilon s^{-1}$, $|\nabla\xi|\leqq\varepsilon s^{-1}$ *throughout* G_δ,

iv) $|\xi|\leqq 1$, $|\nabla\xi|\leqq C\varepsilon\exp(\varepsilon^{-1})\, M(s_1)$, $|\Delta\xi|\leqq C\varepsilon\exp(2\varepsilon^{-1})\, M(s_1)$,
throughout G with constants C depending only on G and δ (but not on ε).

One such ξ, a function of s alone, is suggested by HOPF and EDWARDS:

$$\xi(s)=\tfrac{1}{2}\,\varepsilon\int_{\alpha}^{1}\sigma^{-1}\,\psi[(s-s_1)\,\sigma^{-1}]\,d\sigma$$

where $\alpha=\exp(-2\varepsilon^{-1})$ and

$$\psi(\beta)=\begin{cases}1 & \text{if } \beta\leqq 0\\ (1-\beta^3\delta^{-3})^3 & \text{if } 0\leqq\beta\leqq\delta\\ 0 & \text{if } \beta\geqq\delta.\end{cases}$$

Statements i), ii) and iv) of the lemma follow easily. The first of the estimates iii) may be proved by considering the cases $s\leqq\alpha$, $s\geqq\alpha$; here $\delta\leqq\frac{1}{2}$ is useful. To establish the remaining inequality one first shows that the point s_0, where $|\xi'(s)|$ is a maximum, satisfies $s_0-s_1>4^{-\frac{1}{2}}\alpha\delta$, then considers the cases $s\leqq s_1$, $s_1\leqq s\leqq s_0$, $s_0\leqq s$.

We return to the bound for

$$Q=b(w,\xi F, w)$$

where ξ is given by Lemma 3. Then from the integral representation for $b(\alpha,\beta;\gamma)$ and standard inequalities there follows

$$Q\leqq\int_{G_\delta}|\nabla w|\,|\nabla\nabla w|\,|\nabla(\xi F)|\,dA.$$

From this inequality and the estimates (7) and iii) of Lemma 3 we find

$$Q\leqq 2\varepsilon\Big(\int_{G_\delta}|\nabla\nabla w|^2\,dA\Big)^{\frac{1}{2}}\Big(\int_{G_\delta}(s^{-1}|\nabla w|)^2\,dA\Big)^{\frac{1}{2}}.$$

As an application of Lemma 2, with $h(s)=|\nabla w|$, we have (the properties (6a, b) are also used here)

$$\int_0^\delta(s^{-1}|\nabla w|)^2 J\,ds\leqq 4mK^2\int_0^\delta|\nabla\nabla w|^2 J\,m^{-1}\,ds,$$

and we conclude

$$\int_{G_\delta}(s^{-1}|\nabla w|)^2\,dA\leqq 4K^2\int_{G_\delta}|\nabla\nabla w|^2\,dA.$$

Together with this inequality and (3), the last estimate for Q implies

$$Q\leqq 4K\varepsilon\int_G|\Delta w|^2\,dA.$$

If we collect estimates for the terms on the right in (13), set $\varepsilon=(8\mu K)^{-1}$ and use the integral form (9) of each expression, then we find

$$\begin{aligned}\int_G (2|\Delta g|^2+|\Delta w|^2)\,dA \leqq 2\mu\Big[\int_G |\Delta g|^2\,dA\Big]^{\frac{1}{2}}\Big[\int_G |\Delta\{(1-\xi)F\}|^2\,dA\Big]^{\frac{1}{2}}\\ +\nu C\Big[\int_G |\Delta w|^2\,dA\Big]^{\frac{1}{2}}\Big[\int_G p^2\,dA\Big]^{\frac{1}{2}}.\end{aligned} \tag{14}$$

The bounds (7) for F and p may be used together with iv) of Lemma 3 to show there are constants C, depending only on G, such that

$$\int_G p^2\,dA \leqq C \quad\text{and}\quad \int_G |\Delta\{(1-\xi)F\}|^2\,dA \leqq D(\varepsilon)$$

where

$$D(\varepsilon)=C\big[1+\varepsilon^2\exp(C\varepsilon^{-1})\,M^2(s_1)\big].$$

If we insert these bounds in (14), it then follows that

$$\int_G (2|\Delta g|^2+|\Delta w|^2)\,dA \leqq \mu^2 D(\varepsilon)+\nu^2 C$$

and, since $g=f+\mu(1-\xi)F$, $s_1=C\exp(-C\varepsilon^{-1})$, $\varepsilon=C\mu^{-1}$, that

$$\int_G (|\Delta f|^2+|\Delta w|^2)\,dA \leqq C\{\mu^2+\nu^2+\exp(C\mu)\,M^2[C\exp(-C\mu)]\}.$$

Theorem 3 follows from this last inequality, (5), and Lemma 1.

3. Small Data; Uniqueness

In this section we examine the case in which μ and ν are small, *i.e.* the applied forces F^* and p^* are sufficiently small. First, if only $\mu<1$, we derive an *a priori* estimate in the method of the preceeding section, but without the use of the Hopf function.

Let f, w be a solution of problem (1) and set $\varphi=f$ in (8a), $\psi=w$ in (8b). Then add the integral forms of the equations to find, after integration by parts over G,

$$\int_G (|\Delta f|^2+|\Delta w|^2)\,dA=\mu\int_G [w,w]\,F\,dA+\nu\int_G w\,p\,dA.$$

From (4), (7), and Lemma 1 it follows that

$$\int_G (|\Delta f|^2+|\Delta w|^2)\,dA \leqq \mu\int_G |\Delta w|^2\,dA+\nu C\Big(\int_G |\Delta w|^2\,dA\Big)^{\frac{1}{2}}$$

and, consequently,

$$\int_G (|\Delta f|^2+|\Delta w^2|)\,dA \leqq \nu^2(1-\mu)^{-2}C.$$

Now (5) and Lemma 1 imply

$$\sup_G |f|+\sup_G |w| \leqq \nu(1-\mu)^{-1}C. \tag{15}$$

In considering the question of uniqueness, first observe that if f, w and $f+u$, $w+v$ are two classical solution pairs corresponding to the same data, then the difference pair u, v satisfies the boundary conditions (1c) and the equations

$$\Delta^2 u = -[v, v] - 2[v, w],$$
$$\Delta^2 v = [u, v] + [u, w] + [f, v] + \mu[F, v].$$

If the first of these equations is multiplied by u, the second by v, and if the resulting equations are added, we obtain after integration by parts over G,

$$\int_G (|\Delta u|^2 + |\Delta v|^2)\, dA = \int_G \{(f+\mu F)[v, v] + w[u, v]\}\, dA.$$

From this equation and the relations (4) and (7) we get

$$\int_G (|\Delta u|^2 + |\Delta v|^2)\, dA \leqq (\mu + \sup_G |f|) \int_G |\Delta v|^2\, dA + \sup_G |w| \Big(\int_G |\Delta u|^2\, dA\Big)^{\frac{1}{2}} \Big(\int_G |\Delta v|^2\, dA\Big)^{\frac{1}{2}}$$
$$\leqq (\mu + \sup_G |f| + \sup_G |w|) \int_G (|\Delta u|^2 + |\Delta v|^2)\, dA.$$

Clearly $u = v = 0$ if

$$\mu + \sup_G |f| + \sup_G |w| < 1.$$

Thus, in view of (15), the uniqueness follows if $\mu + C\nu(1-\mu)^{-1} \leqq 1$, *i.e.* if $\mu < 1$ and if ν is sufficiently small depending only on μ and G.

We collect the results of this section.

Theorem 5. *Let f, w be a solution of problem* (1) *with $\mu < 1$. Then* (15) *holds with constant C depending only on G. If, in addition, ν is sufficiently small depending only on μ and G, then f, w is the only solution of problem* (1).

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