Rall, W.: Distinguishing theoretical synaptic potentials computed for different soma-dendritic distributions of synaptic input. J. Neurophysiol. 30, 1138–1168 (1967).
— Distributions of potential in cylindrical coordinates and time constants for a membrane cylinder. Biophys. J. 9, 1509–1541 (1969a).
— Time constants and electrotonic length of membrane cylinders and neurons. Biophys. J. 9, 1483–1508 (1969b).
— Burke, R. E., Smith, T. G., Nelson, P. G., Frank, K.: Dendritic location of synapses and possible mechanisms for the monosynaptic EPSP in motoneurons. J. Neurophysiol. 30, 1169–1193 (1967).
Sabah, N. H., Leibovic, K. N.: Subthreshold oscillatory responses of the Hodgkin-Huxley cable model for the giant squid axon. Biophys. J. 9, 1206–1222 (1969).
Shilov, G. E.: Generalized functions and partial differential equations. New York: Gordon and Breach 1968.
Smith, T. G., Wuerker, R. B., Frank, K.: Membrane impedance changes during synaptic transmission in cat spinal motoneurons. J. Neurophysiol. 30, 1072–1096 (1967).
Swigert, C. J.: A mode control model of a neuron's axon and dendrites. Kybernetik 7, 31–41 (1970).
Weinberger, H. F.: Partial differential equations. Waltham, Mass.: Blaisdell Publishing Co. 1965.

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On a Response Characteristic of a Mathematical Neuron Model

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Abstract. A mathematical neuron model in the form of a nonlinear difference equation is proposed and its response characteristic is investigated.

If a sequence of pulses with a fixed frequency is applied to the neuron model as an input, and the amplitude of the input pulses is progressively decreased, the firing frequency of the neuron model, regarded as the output, also decreases. The relationship between them is quite complicated, but a mathematical investigation reveals that it takes the form of an extended Cantor's function. This result explains the "unusual and unsuspected" phenomenon which was found by L. D. Harmon in experimental studies with his transistor neuron models.

Besides this, as an analogue of our mathematical neuron model, a very simple circuit composed of a delay line and a negative resistance element is presented and discussed.

Introduction

In the course of experimental studies with his artificial neurons (hardware neuron model using transistors), Harmon (1961) found an "unusual and unspected" phenomenon which is described below.

If an artificial neuron unit is used to drive another directly, and the pulse amplitude is sufficiently high, the firing of the second follows the first, pulse for pulse. However, if the output amplitude of the driving unit be monotonically decreased, firing in the second unit begins to skip. More precisely, as the driven unit receives progressively less excitation, there will be a critical point at which a given driving pulse is insufficient to fire the driven unit. However, it turns out that when the next driving pulse comes along, it is integrated with the preceding one to come firing, and the firing frequency of the driven unit is half that of the driver. One would expect that as the driving pulse amplitudes are reduced still more, every third pulse would be effective, then every fourth, and so on.

This is what one can reasonably expect to happen, but in fact it does not. What does occur is much more complicated. As the excitation of the second unit decreases, the integral steps expected show up, but considerably more complex behavior also appears. Thus other than the predicted steps such as 1:1, 1:2, 1:3, ..., 1:10, a much larger number of nonintegral steps, such as 3:5, 5:16, 3:19, etc., also appear. The relationship between the amplitude of driving pulses (let it be a) and the ratio of the pulse frequency of the

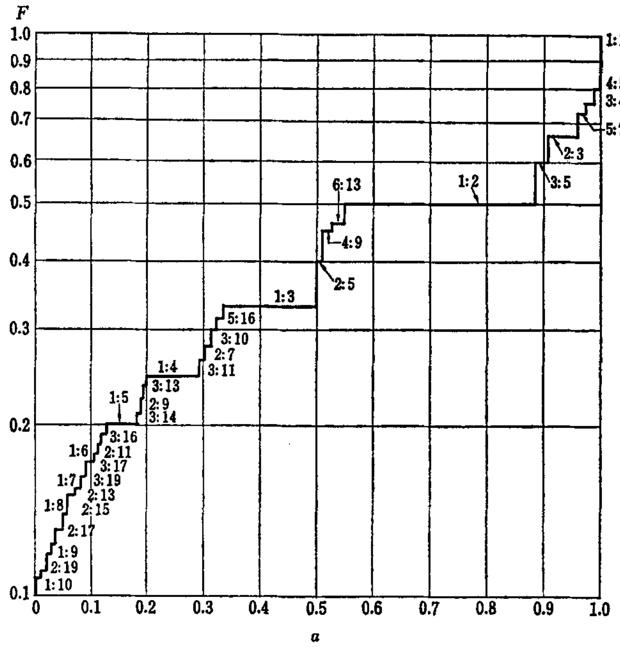

Fig. 1. Relationship between the amplitude of driving pulses (a) and the pulse frequency ratio of the driven unit to that of the driving unit (F). Reproduced from Harmon (1961) with a minor modification

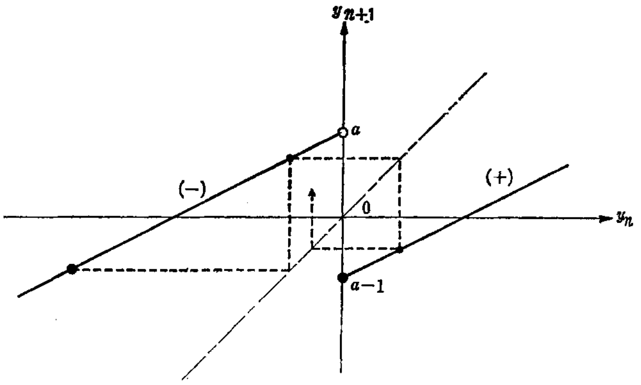

Fig. 2. Graphic display of the mathematical neuron model — nonlinear difference equation (9)

driven unit to that of the driving unit (let it be F) is shown in Fig. 1, which was reproduced from Harmon (1961) with a minor modification.

The purpose of this paper is to present a mathematical neuron model which can elucidate the above experimental results.

Mathematical Neuron Model and Average Firing Rate

From the functional point of view, a neuron can be regarded as a threshold element with a refractory period. In the present model, the refractoriness is so assumed that the inhibitory influence of a past firing upon the excitability of the neuron at the present instant decreases exponentially with time, and time is assumed to be discrete. Under these assumptions, the behavior of the neuron is expressed by a nonlinear difference equation (Caianiello, 1961):

$$x_{n+1} = 1\left[A_n - \alpha \sum_{r=0}^{n} b^{-r} x_{n-r} - \theta\right], \tag{1}$$

where $1[x] = 1\,(x \geqq 0)$, $= 0\,(x < 0)$,

x_n the state of the neuron at the instant n. The resting state is represented by 0, and the excited state by 1,

A_n magnitude of the input stimulus applied at the instant n,

θ threshold value,

$\alpha > 0$, $b > 1$.

Although our neuron model may be regarded as a discrete-time version of Caianiello and De Luca's continuous-time neuron model (1966):

$$x(t+\tau) = 1\left[A(t) - \alpha \int_0^t b^{-r} x(t-r)\,dr - \theta\right], \tag{2}$$

the former seems to have a much richer variety of forms of solution than the latter.

Now, introduction of a new variable y_n:

$$y_n = \alpha^{-1}(A_n - \theta) - \sum_{r=0}^{n} b^{-r} x_{n-r} \tag{3}$$

reduces (1) to

$$y_{n+1} = b^{-1} y_n + a_n - 1[y_n], \tag{4}$$

where

$$a_n = \frac{1}{\alpha}\left[\left(A_{n+1} - \frac{A_n}{b}\right) - \theta\left(1 - \frac{1}{b}\right)\right], \tag{5}$$

and

$$x_{n+1} = 1[\alpha y_n] = 1[y_n]. \tag{6}$$

In case the magnitude of the input stimulus is constant, that is,

$$A_n = A \tag{7}$$

for all n, it follows that

$$a_n = \frac{A-\theta}{\alpha}\left(1 - \frac{1}{b}\right) = a \quad \text{(constant)}, \tag{8}$$

and (4) becomes

$$\begin{cases} y_n \geqq 0: & y_{n+1} = b^{-1} y_n + a - 1, \\ y_n < 0: & y_{n+1} = b^{-1} y_n + a. \end{cases} \tag{9}$$

The former equation corresponds to the (+) branch, and the latter to the (−) branch in Fig. 2. Hereafter, the case of the constant input is considered, and a is regarded as representing the magnitude of the input stimulus.

Given the initial value y_0, a sequence of y:

$$y_0, y_1, y_2, y_3, \ldots \tag{10}$$

is determined from (9), and correspondingly a sequence of x:

$$x_1, x_2, x_3, \ldots \tag{11}$$

follows from (6).

In the sequel, our consideration is restricted to cases where (10) is a periodic sequence or a sequence which asymptotically approaches a periodic sequence. In such cases, (11), after a finite number of steps of n, becomes a periodic sequence iterating $x_1^*, x_2^*, x_3^*, \ldots, x_l^*$ infinitely, and denoted by

$$\{x_1^* x_2^* x_3^* \ldots x_l^*\}. \tag{12}$$

For such periodic sequences, definition of the average firing rate $F(a)$ is given by

$$F(a) = \frac{\text{number of 1 in } x_1^*, x_2^*, \ldots, x_l^*}{l}. \tag{13}$$

Simple considerations show that if $a \geqq 1$, $y_n \to y^* \geqq 0$ as $n \to \infty$. Hence $x_n = 1$ for $n > n_0$, and $F(a) = 1$. On the other hand, if $a < 0$, $y_n \to y^* < 0$. Hence $x_n = 0$ for $n > n_0$, and $F(a) = 0$. Thus our problem is to investigate the relationship between a and F for $0 < a < 1$.

It is worth mentioning that the replacements of a by $1-a$, y_n by $-y_n$ in (9) cause an interchange of the (+) branch and the (−) branch, so that F is replaced by $1-F$. In other words, the function $F(a)$ is symmetrical with respect to the point $(a = \frac{1}{2}, F = \frac{1}{2})$.

Set of Periodic Sequences S Having Some Special Forms

In what follows, our consideration will be further limited to such periodic sequences as have some special forms, and denote the whole of such periodic sequences by S. The set S is the totality of an infinite number of subsets $S_1, S_2, S_3, \ldots$, each S_i $(i = 1, 2, 3, \ldots)$ being a set of periodic sequences having the special form described below.

Denote a periodic sequence in which 0 appears consecutively n (a positive integer) times after 1 has appeared consecutively m (a positive integer) times by $\{1^m 0^n\}$. Then the set S_1 is the whole of periodic sequences in the form $\{1^m 0^n\}$ with $m=1$ or $n=1$.

The set S_2 is constructed from two neighboring elements of the set S_1 by the same method as above. To cite an example, all periodic sequences of the form $\{(01)^m(001)^n\}$ (where $m=1$ or $n=1$), which are derived from two elements $\{01\}$ and $\{001\}$ neighboring in S_1, belong to the set S_2.

In the same way, the set S_3 is set up from two neighboring elements of the set S_2. For example, all periodic sequences of the form $\{(01001)^m(01001001)^n\}$ (where $m=1$ or $n=1$), which are derived from two elements $\{01001\}$ and $\{01001001\}$ neighboring in S_2, belong to the set S_3.

Then the set S is defined as a union of all such subsets S_i ($i=1, 2, 3, \ldots$) by

$$S = S_1 \cup S_2 \cup S_3 \cup \cdots .$$

Obviously

$$S_i \cap S_j = \phi \quad \text{for } i \neq j.$$

Our next task then is to show that each element of the set S is a periodic solution of (9) having a particular value of a. Henceforth the correspondence between the periodic sequence of S and the value of a in (9) will be investigated.

A. Elements of Set S_1

As an example of the element of S_1, $\{0^m 1\}$ ($m \geqq 1$) is taken into consideration. In Fig. 3

$$\begin{aligned} &y_1 < 0, \\ &y_2 = b^{-1} y_1 + a < 0, \\ &y_3 = b^{-2} y_1 + a(1 + b^{-1}) < 0, \\ &\vdots \\ &y_{m+1} = b^{-m} y_1 + a(1 + b^{-1} + b^{-2} + \cdots + b^{-m+1}) \geqq 0. \end{aligned}$$

Define

$$y_{m+2} = b^{-m-1} y_1 + a(1 + b^{-1} + b^{-2} + \cdots + b^{-m}) - 1, \tag{14}$$

and put $y_{m+2} = y_1$. Then

$$y_1 = b\{a(b^m + b^{m-1} + \cdots + 1) - b^m\}(b^{m+1} - 1)^{-1}, \tag{15}$$

and

$$y_m = b\{a(b^m + b^{m-1} + \cdots + 1) - b\}(b^{m+1} - 1)^{-1}. \tag{16}$$

These y_1 and y_m must satisfy the inequalities

$$y_m < 0, \quad y_1 \geqq a - 1, \tag{17}$$

which are rewritten as

$$\frac{b}{b^m + b^{m-1} + \cdots + 1} > a \geqq \frac{1}{b^m + b^{m-1} + \cdots + 1}. \tag{18}$$

Conversely, if condition (18) is satisfied, it follows from (14) that $y_{m+2} \geqq a-1$ for $y_1 = a-1$. Moreover, when $y_m = 0$, or when $y_1 = \hat{y}$, $y_{m+2} < \hat{y}$, where $\hat{y} = -a(b^{m-1} + b^{m-2} + \cdots + b)$. Hence it is obvious from Fig. 4 that $y_{1+i(m+1)} \to y^*$ as $i \to \infty$, where $\hat{y} > y^* \geqq a-1$, independent of the initial value y_0. Consequently the corresponding sequence of x becomes the periodic sequence $\{0^m 1\}$ after a finite number of steps.

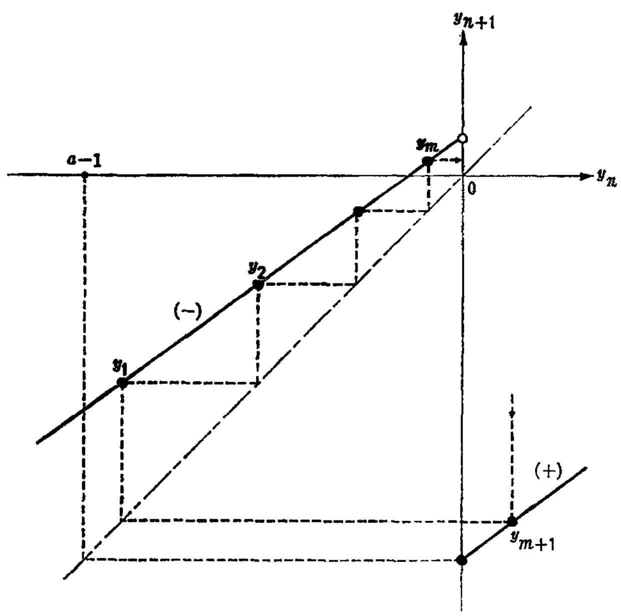

Fig. 3. Graphic display of periodic sequence $\{0^m 1\}$ ($m \geqq 1$), which is an element of the set S_1

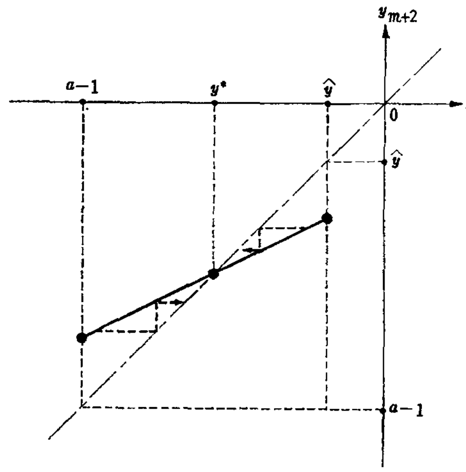

Fig. 4. If condition (18) is satisfied, the corresponding sequence of y approaches to a periodic sequence independent of the initial value

To summarize, it has been shown that (18) is a necessary and sufficient condition for the sequence of y to become a periodic sequence which corresponds to the periodic sequence $\{0^m 1\}$ of x. The value of a which satisfies (18) is called the value (or interval) of a which corresponds to the periodic sequence $\{0^m 1\}$. For this sequence it is apparent that $F = (m+1)^{-1}$.

It should be noted that the sequence of x becomes the periodic sequence $\{0^m 1\}$ even if

$$a = b(b^m + b^{m-1} + \cdots + 1)^{-1}.$$

Therefore, instead of (18),

$$\frac{b}{b^m + b^{m-1} + \cdots + 1} \geqq a \geqq \frac{1}{b^m + b^{m-1} + \cdots + 1} \tag{19}$$

is adopted as the interval of a which corresponds to the periodic sequence $\{0^m 1\}$.

Representation of (19) in the scale of b yields

$$\frac{00\ldots010}{11\ldots111} \geqq a(b) \geqq \frac{00\ldots001}{11\ldots111}, \tag{20}$$

Table 1. The values of a and F which correspond to periodic sequences $\{1^m0\}$ and $\{10^m\}$ of the set S_1 for cases $m=1$, 2 and 3

$\frac{b^3+b^2+b}{b^3+b^2+b+1} \geqq a \geqq \frac{b^3+b^2+1}{b^3+b^2+b+1}$ $\left(\frac{1110}{1111} \geqq a(b) \geqq \frac{1101}{1111}\right)$	$F=\frac{3}{4}$
$\frac{b^2+b}{b^2+b+1} \geqq a \geqq \frac{b^2+1}{b^2+b+1}$ $\left(\frac{110}{111} \geqq a(b) \geqq \frac{101}{111}\right)$	$F=\frac{2}{3}$
$\frac{b}{b+1} \geqq a \geqq \frac{1}{b+1}$ $\left(\frac{10}{11} \geqq a(b) \geqq \frac{01}{11}\right)$	$F=\frac{1}{2}$
$\frac{b}{b^2+b+1} \geqq a \geqq \frac{1}{b^2+b+1}$ $\left(\frac{010}{111} \geqq a(b) \geqq \frac{001}{111}\right)$	$F=\frac{1}{3}$
$\frac{b}{b^3+b^2+b+1} \geqq a \geqq \frac{1}{b^3+b^2+b+1}$ $\left(\frac{0010}{1111} \geqq a(b) \geqq \frac{0001}{1111}\right)$	$F=\frac{1}{4}$

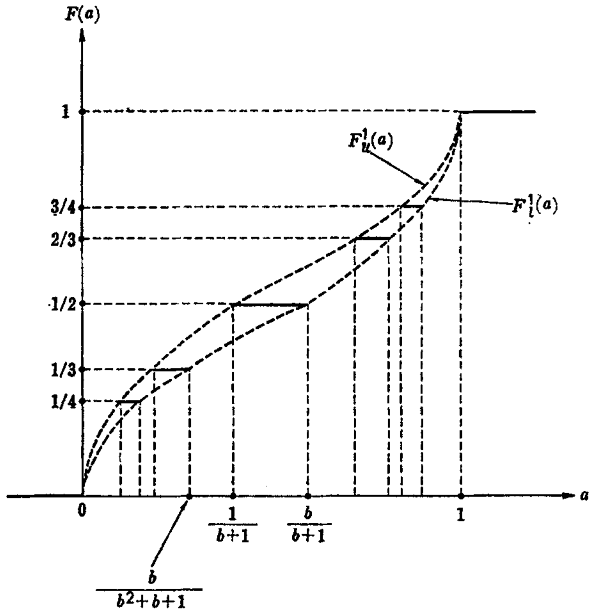

Fig. 5. The relation between a and F for periodic sequences of the set S_1. $F_u^1(a)$ and $F_l^1(a)$ are functions which give an upper bound and a lower bound, respectively

where $a(b)$ means the value of a represented in the scale of b. Numerators on the left and right-hand sides of (20) indicate the periodic sequence under consideration, and the numerator on the left is derived from that on the right by consecutive rotations of the numerals of the latter. Incidentally, the value of F is given by the ratio of the sum of the numerals in the numerator to that in the denominator.

Replacements of a by $1-a$, F by $1-F$ in (19) yield results for the periodic sequence $\{1^m0\}$. The values of a and F, which correspond to the periodic sequences $\{1^m0\}$ and $\{10^m\}$, are shown in Table 1 and in Fig. 5 for cases $m=1$, 2 and 3.

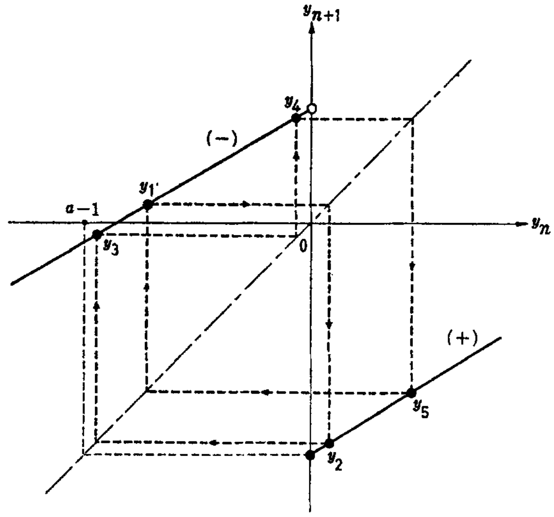

Fig. 6. Graphic display of periodic sequence {01001}, which is an element of the set S_2

In Fig. 5, $F_u^1(a)$ and $F_l^1(a)$ are functions which give an upper bound and a lower bound respectively, and

$$F_u^1(a)=\frac{1}{\log\left(1+\frac{b-1}{a}\right)} \qquad \left(\frac{1}{b+1} \geqq a>0\right),$$
$$=1-\frac{1}{\log\left(1+\frac{b(b-1)}{1-a}\right)} \qquad \left(1>a \geqq \frac{1}{b+1}\right), \tag{21}$$

$$F_l^1(a)=\frac{1}{\log\left(1+\frac{b(b-1)}{a}\right)} \qquad \left(\frac{b}{b+1} \geqq a>0\right),$$
$$=1-\frac{1}{\log\left(1+\frac{b-1}{1-a}\right)} \qquad \left(1>a \geqq \frac{b}{b+1}\right). \tag{22}$$

B. Elements of Set S_2

The values of a and F will be computed for $\{01001\}$ which was previously cited as an example of the element of the set S_2 (Fig. 6). By a calculation like that before, the values $y_1, y_2, \ldots, y_5$ are obtained as functions of a and b. The condition corresponding to (17) turns out

$$y_4<0, \qquad y_3 \geqq a-1, \tag{23}$$

from which

$$\frac{b^3+b}{b^4+b^3+\cdots+1} \geqq a \geqq \frac{b^3+1}{b^4+b^3+\cdots+1} \tag{24}$$

is derived corresponding to (19). Conversely, if condition (24) is satisfied, the corresponding sequence of x becomes the periodic sequence $\{01001\}$ after a finite number of steps. Clearly $F=\frac{2}{5}$.

Table 2 and Fig. 7 show the values of a and F which correspond to $\{(01)^m(001)\}$ and $\{(01)(001)^m\}$ for cases $m=1, 2$ and 3. In Fig. 7, functions giving an upper bound and a lower bound are

$$F_u^2(a)=\frac{1}{3}\left[1+\frac{1}{\log\left(1+\frac{b-1}{a(b^2+b+1)-b}\right)}\right]$$
$$\left(\frac{b^3+1}{b^4+b^3+\cdots+1}\geqq a>\frac{b}{b^2+b+1}\right),$$
$$=\frac{1}{2}\left[1-\frac{1}{\log\left(1+\frac{b^2(b-1)}{1-a(b+1)}\right)}\right] \tag{25}$$
$$\left(\frac{1}{b+1}>a\geqq\frac{b^3+1}{b^4+b^3+\cdots+1}\right),$$

$$F_l^2(a)=\frac{1}{3}\left[1+\frac{1}{\log\left(1+\frac{b^3(b-1)}{a(b^2+b+1)-b}\right)}\right]$$
$$\left(\frac{b^3+b}{b^4+b^3+\cdots+1}\geqq a>\frac{b}{b^2+b+1}\right),$$
$$=\frac{1}{2}\left[1-\frac{1}{\log\left(1+\frac{b-1}{1-a(b+1)}\right)}\right] \tag{26}$$
$$\left(\frac{1}{b+1}>a\geqq\frac{b^3+b}{b^4+b^3+\cdots+1}\right),$$

and the following inequalities hold for

$$\frac{1}{b+1}>a>\frac{b}{b^2+b+1}.$$

$$\tfrac{1}{2}>F_u^1(a)>F_u^2(a)>F(a)>F_l^2(a)>F_l^1(a)>\tfrac{1}{3}.$$

C. *Elements of Set S_3*

For the example $\{0100101001001\}$ of the element of S_3 previously cited, the condition corresponding to (17) is found to be

$$y_{12}<0,\quad y_8\geqq a-1, \tag{27}$$

or

$$\frac{b^{11}+b^8+b^6+b^3+b}{b^{12}+b^{11}+\cdots+1}\geqq a\geqq\frac{b^{11}+b^8+b^6+b^3+1}{b^{12}+b^{11}+\cdots+1} \tag{28}$$

(see Fig. 8). Obviously $F=\frac{5}{13}$.

The values of a and F are shown in Table 3 which correspond to periodic sequences

$\{(01001)^m(01001001)\}$ and $\{(01001)(01001001)^m\}$

for cases $m=1$ and 2.

Total Length of the Intervals of a which Correspond to the Elements of S

Our next task then is to calculate the total length of intervals of a which correspond to the elements of S.

For the element $\{10\}=\{01\}$ of S_1, the corresponding interval is

$$\frac{b}{b+1}\geqq a\geqq\frac{1}{b+1},$$

so that the length of the interval is $(b-1)^2(b^2-1)^{-1}$. Similarly, the length of the intervals for $\{110\}$ and $\{001\}$ is $(b-1)^2(b^3-1)^{-1}$, one for $\{1110\}$ and $\{0001\}$ is $(b-1)^2(b^4-1)^{-1}$. Hence the total length of intervals which correspond to the elements of S_1 is given by $(b-1)^2L_1$, where

$$L_1=\frac{1}{b^2-1}+2\left(\frac{1}{b^3-1}+\frac{1}{b^4-1}+\frac{1}{b^5-1}+\cdots\right).$$

Table 2. The values of a and F which correspond to periodic sequences $\{(01)^m(001)\}$ and $\{(01)(001)^m\}$ of the set S_2 for cases $m=1$, 2 and 3

$\frac{b^7+b^5+b^3+b}{b^8+b^7+\cdots+1}\geqq a\geqq\frac{b^7+b^5+b^3+1}{b^8+b^7+\cdots+1}$ $\left(\frac{010101010}{111111111}\geqq a(b)\geqq\frac{010101001}{111111111}\right)$	$F=\frac{4}{9}$
$\frac{b^5+b^3+b}{b^6+b^5+\cdots+1}\geqq a\geqq\frac{b^5+b^3+1}{b^6+b^5+\cdots+1}$ $\left(\frac{0101010}{1111111}\geqq a(b)\geqq\frac{0101001}{1111111}\right)$	$F=\frac{3}{7}$
$\frac{b^3+b}{b^4+b^3+\cdots+1}\geqq a\geqq\frac{b^3+1}{b^4+b^3+\cdots+1}$ $\left(\frac{01010}{11111}\geqq a(b)\geqq\frac{01001}{11111}\right)$	$F=\frac{2}{5}$
$\frac{b^6+b^3+b}{b^7+b^6+\cdots+1}\geqq a\geqq\frac{b^6+b^3+1}{b^7+b^6+\cdots+1}$ $\left(\frac{01001010}{11111111}\geqq a(b)\geqq\frac{01001001}{11111111}\right)$	$F=\frac{3}{8}$
$\frac{b^9+b^6+b^3+b}{b^{10}+b^9+\cdots+1}\geqq a\geqq\frac{b^9+b^6+b^3+1}{b^{10}+b^9+\cdots+1}$ $\left(\frac{01001001010}{11111111111}\geqq a(b)\geqq\frac{01001001001}{11111111111}\right)$	$F=\frac{4}{11}$

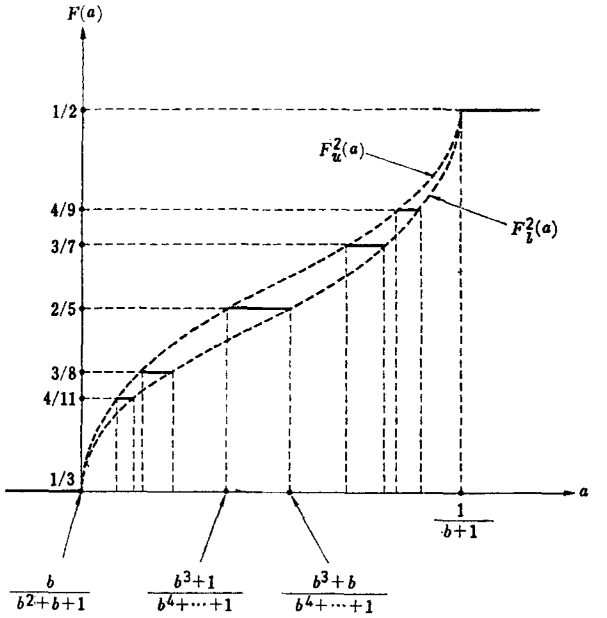

Fig. 7. The relation between a and F for periodic sequences $\{(01)^m(001)\}$ and $\{(01)(001)^m\}$ of the set S_2. $F_u^2(a)$ and $F_l^2(a)$ are functions which give an upper bound and a lower bound, respectively

Next, the total length of intervals of the elements of S_2 will be considered. The intervals for the elements of S_2, which lie between $\{01\}$ and $\{001\}$, are as follows. From Table 2, it is seen that the interval for $\{(01)(001)\}$ is

$$\frac{b^3+b}{b^4+b^3+\cdots+1}\geqq a\geqq\frac{b^3+1}{b^4+b^3+\cdots+1},$$

so that the length of the interval is given by $(b-1)^2(b^5-1)^{-1}$. The lengths of the intervals for

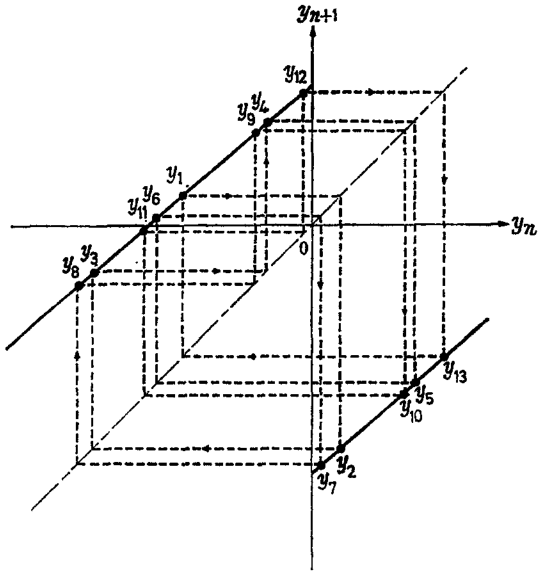

Fig. 8. Graphic display of periodic sequence {0100101001001}, which is an element of the set S_3

Table 3. The values of a and F which correspond to periodic sequences $\{(01001)^m(01001001)\}$ and $\{(01001)(01001001)^m\}$ of the set S_3 for cases $m=1$ and 2

$\dfrac{b^{16}+b^{13}+b^{11}+b^8+b^6+b^3+b}{b^{17}+b^{16}+\cdots+1} \geqq a \geqq \dfrac{b^{16}+b^{13}+b^{11}+b^8+b^6+b^3+1}{b^{17}+b^{16}+\cdots+1}$	$F=\dfrac{7}{18}$
$\left(\dfrac{010010100101001010}{111111111111111111} \geqq a(b) \geqq \dfrac{010010100101001001}{111111111111111111}\right)$	
$\dfrac{b^{11}+b^8+b^6+b^3+b}{b^{12}+b^{11}+\cdots+1} \geqq a \geqq \dfrac{b^{11}+b^8+b^6+b^3+1}{b^{12}+b^{11}+\cdots+1}$	$F=\dfrac{5}{13}$
$\left(\dfrac{0100101001010}{1111111111111} \geqq a(b) \geqq \dfrac{0100101001001}{1111111111111}\right)$	
$\dfrac{b^{19}+b^{16}+b^{14}+b^{11}+b^8+b^6+b^3+b}{b^{20}+b^{19}+\cdots+1} \geqq a \geqq \dfrac{b^{19}+b^{16}+b^{14}+b^{11}+b^8+b^6+b^3+1}{b^{20}+b^{19}+\cdots+1}$	$F=\dfrac{8}{21}$
$\left(\dfrac{010010100100101001010}{111111111111111111111} \geqq a(b) \geqq \dfrac{010010100100101001001}{111111111111111111111}\right)$	

$\{(01)^2(001)\}$ and $\{(01)^3(001)\}$ are $(b-1)^2(b^7-1)^{-1}$ and $(b-1)^2(b^9-1)^{-1}$, respectively. On the other hand, the lengths of the intervals for $\{(01)(001)^2\}$ and $\{(01)(001)^3\}$ are $(b-1)^2(b^8-1)^{-1}$ and $(b-1)^2(b^{11}-1)^{-1}$, respectively. Hence the total length of intervals for elements of S_2 which lie between $\{01\}$ and $\{001\}$ is given by $(b-1)^2 L_2^{(5)}$, where

$$L_2^{(5)}=\frac{1}{b^5-1}+\left(\frac{1}{b^7-1}+\frac{1}{b^9-1}+\frac{1}{b^{11}-1}+\cdots\right)+\left(\frac{1}{b^8-1}+\frac{1}{b^{11}-1}+\frac{1}{b^{14}-1}+\cdots\right).$$

The length of the intervals corresponding to the elements of S_2 which lie between $\{110\}$ and $\{10\}$ is also equal to $(b-1)^2 L_2^{(5)}$.

Likewise, the length of intervals corresponding to the elements of S_2, which lie between $\{001\}$ and $\{0001\}$ and between $\{1110\}$ and $\{110\}$, is given by $(b-1)^2 L_2^{(7)}$, where

$$L_2^{(7)}=\frac{1}{b^7-1}+\left(\frac{1}{b^{10}-1}+\frac{1}{b^{13}-1}+\frac{1}{b^{16}-1}+\cdots\right)+\left(\frac{1}{b^{11}-1}+\frac{1}{b^{15}-1}+\frac{1}{b^{19}-1}+\cdots\right).$$

In the same way, $L_2^{(9)}, L_2^{(11)}, L_2^{(13)}, \ldots$ are obtained, and the total length of intervals corresponding to the elements of S_2 is given by $(b-1)^2 L_2$, where

$$L_2=2\left(L_2^{(5)}+L_2^{(7)}+L_2^{(9)}+\cdots\right).$$

Next, the total length of intervals which correspond to elements of S_3 is calculated and denoted by $(b-1)^2 L_3$. To cite an example, the total length of intervals corresponding to the element of S_3 which lie between $\{(01)(001)\}$ and $\{(01)(001)^2\}$ is, referring to Table 3, given by $(b-1)^2 L_3^{(13)}$, where

$$L_3^{(13)}=\frac{1}{b^{13}-1}+\left(\frac{1}{b^{18}-1}+\frac{1}{b^{23}-1}+\frac{1}{b^{28}-1}+\cdots\right)+\left(\frac{1}{b^{21}-1}+\frac{1}{b^{29}-1}+\frac{1}{b^{37}-1}+\cdots\right).$$

In general, the length of intervals corresponding to the elements of S is given by $(b-1)^2 L$ with

$$L=\sum_{i=1}^{\infty} L_i,$$

where $(b-1)^2 L_i$ is the total length of intervals corresponding to the elements of S_i $(i=1, 2, 3, \ldots)$.

The above-mentioned results are summarized in Table 4, where the integers n $(n \geqq 2)$ indicate the terms $(b^n-1)^{-1}$, C_i represents the set of the integers n which correspond to the elements of S_i, and only a half of the whole is shown in this table because it is symmetrical. Table 5 contains numbers of the integer n which belongs to each C_i $(i=1, 2, 3)$, the sum of them Σ, and double the sum $\varphi(n)$ (except for the case $n=2$). Since $\varphi(n)$ is the number of terms $(b^n-1)^{-1}$ which are involved in L,

$$L=\sum_{n=2}^{\infty}\frac{\varphi(n)}{b^n-1}. \tag{29}$$

It is proved that $\varphi(n)$ is nothing but Euler's function (see Appendix I), namely, the number of positive integers not greater than and prime to n.

Now,

$$\sum_{n=1}^{\infty}\frac{\varphi(n)}{b^n-1}=\sum_{n=1}^{\infty}\varphi(n)\sum_{r=1}^{\infty}b^{-rn}=\sum_{m=1}^{\infty}b^{-m}\sum_{s\mid m}\varphi(s), \tag{30}$$

where the last sum is taken over every divisor s of m. But since (Hardy and Wright, 1960)

$$\sum_{s\mid m}\varphi(s)=m, \tag{31}$$

Table 4. Positive integer n implies the term $(b^n-1)^{-1}$, $(b-1)^2$ times of which is the length of the interval corresponding to an element of S. C_i is the set of integers n which correspond to the elements of S_i. Only a half is shown in this table because it is symmetrical

C_1	C_2	C_3	C_1	C_2	C_3
2			4		
	...			...	
	13			13	
	11			9	
	9			...	
	7		5		
		...		...	
		12		11	
		...		...	
	5		6		
		...		...	
		13		13	
		...		...	
	8		7		
	11		⋮		
	...				
3					
	...				
	13				
	10				
	7				
	11				
	...				
4					

Table 5. This table contains numbers of the integer n which belongs to each C_i $(i=1, 2, 3)$, the sum of them Σ, and double the sum $\varphi(n)$ (except when $n=2$)

n	C_1	C_2	C_3	Σ	$\varphi(n)$
2	1	0	0	1	1
3	1	0	0	1	2
4	1	0	0	1	2
5	1	1	0	2	4
6	1	0	0	1	2
7	1	2	0	3	6
8	1	1	0	2	4
9	1	2	0	3	6
10	1	1	0	2	4
11	1	4	0	5	10
12	1	0	1	2	4
13	1	4	1	6	12
⋮	⋮	⋮	⋮	⋮	⋮

it immediately follows that

$$\sum_{n=1}^{\infty} \frac{\varphi(n)}{b^n-1} = \sum_{m=1}^{\infty} m b^{-m} = b(b-1)^{-2}, \qquad (32)$$

and hence

$$L = b(b-1)^{-2} - (b-1)^{-1} = (b-1)^{-2}. \qquad (33)$$

Thus it is shown that the total length of the intervals which correspond to the elements of S is equal to unity.

Function $F(a)$

As has been seen in the previous section, the function $F(a)$ is defined over an enumerably infinite number of intervals which are everywhere dense in the interval $0 \leqq a \leqq 1$, and the total length of the intervals is equal to unity. It is clear that this function can be uniquely extended to a function which is defined throughout the interval $0 \leqq a \leqq 1$ in a natural way. The extended function (denoted by $F(a)$ anew) is continuous, nondecreasing, flat $(F'(a)=0)$ almost everywhere in $0 \leqq a \leqq 1$; nevertheless $F(0)=0$, $F(1)=1$. Thus the conclusion that the function $F(a)$ is an extended Cantor's function (Titchmarsh, 1968) is achieved.

Correspondence to Harmon's Experimental Results

The values of F of the periodic sequences which belong to the set S_2 and lie between $\{1^{m+1}0\}$ and $\{1^m 0\}$ are given by

$$F_1 = \frac{n(m+1)+m}{n(m+2)+(m+1)} \qquad (n \geqq 1),$$

or

$$F_2 = \frac{(m+1)+nm}{(m+2)+n(m+1)} \qquad (n \geqq 1),$$

and those between $\{0^m 1\}$ and $\{0^{m+1}1\}$ are given by

$$F_3 = \frac{n+1}{n(m+1)+(m+2)} \qquad (n \geqq 1),$$

or

$$F_4 = \frac{1+n}{(m+1)+n(m+2)} \qquad (n \geqq 1).$$

A detailed correspondence of these values of F to Harmon's experimental results is shown in Table 6. Since a in (9) is not exactly the same as that in Fig. 1, correspondence of the values of a can not be made.

Table 6. Comparison of the values of F between Harmon's experimental results and our theoretical results

Harmon	F	m	n	Harmon	F	m	n
1:1	F_1	∞	—	2:9	F_3	3	1
4:5	F_1	4	0	3:14	F_4	3	2
3:4	F_1	3	0	1:5	F_4	4	0
5:7	F_1	2	1	3:16	F_3	4	2
2:3	F_1	2	0	2:11	F_3	4	1
3:5	F_1	1	1	3:17	F_4	4	2
1:2	F_1	1	0	1:6	F_4	5	0
6:13	F_3	1	5	3:19	F_3	5	2
4:9	F_3	1	3	2:13	F_3	5	1
2:5	F_3	1	1	1:7	F_3	6	0
1:3	F_4	2	0	2:15	F_3	6	1
5:16	F_3	2	4	1:8	F_4	7	0
3:10	F_3	2	2	2:17	F_3	7	1
2:7	F_3	2	1	1:9	F_4	8	0
3:11	F_4	2	2	2:19	F_3	8	1
1:4	F_4	3	0	1:10	F_4	9	0
3:13	F_3	3	2				

An Analogue Circuit

It is shown that the very simple circuit in Fig. 9, which is composed of a delay line and a negative resistance element, is an analogue of our mathematical neuron model if the characteristic of the negative resistance element is chosen as shown in Fig. 10, where Z is the characteristic impedance of the delay line and $Z > R$ (see Appendix II).

It is known that the voltage $v(l, t)$ maintains a constant value during the time-period T, where T is double the delay time of the delay line. Then the sequence of x defined by

$$x_n = 1[v(l, nT)] \qquad (n = 1, 2, 3, \ldots) \qquad (34)$$

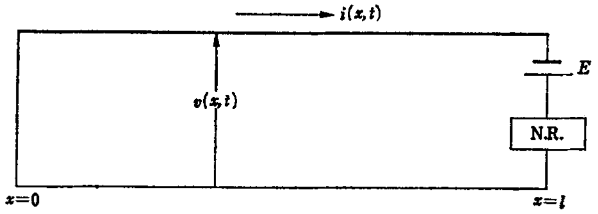

Fig. 9. An analogue circuit of the mathematical neuron model

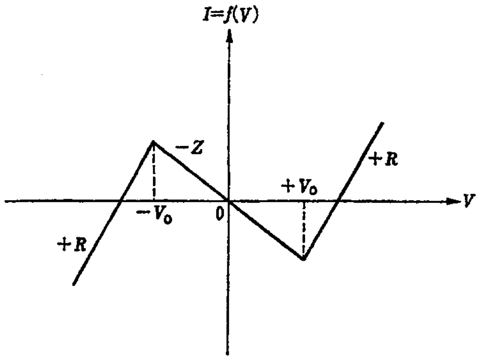

Fig. 10. The characteristic of the negative resistance element in the analogue circuit, where Z is the characteristic impedance of the delay line and $Z > R$

is proved to agree with that discussed in previous sections. Furthermore, a simple computation gives

$$a = \frac{1}{2}\left(1 + \frac{E}{V_0}\right), \tag{35}$$

$$b = \frac{Z+R}{Z-R} > 1. \tag{36}$$

Therefore the condition $-V_0 < E < V_0$ implies $0 < a < 1$.

Some experimental results of this circuit, when a tunnel diode is used as the negative resistance element, was given by Nagumo and Shimura (1961).

Appendix I

In this appendix, the constitution of Table 4 will be investigated. First of all,

$$C_1 = \{2, 3, \ldots\}.$$

The element of C_2 which lies between p_1 (positive integer not less than 2) and p_1+1 of C_1 takes the form $n = p_2 p_1 + q_2(p_1+1)$, where p_2 and q_2 are positive integers and $p_2 = 1$ or $q_2 = 1$.

In general, the positive integer of the form $n = pp_1 + q\,(p_1+1)$, where p, q and p_1 are positive integers and $p_1 \geqq 2$, is denoted by $[p, q]$. If p and q are prime each other, it is called a "prime pair".

It is obvious that every element of

$$C_2 = \{\ldots, [3,1], [2,1], [1,1], [1,2], [1,3], \ldots\}$$

is a prime pair. Since C_2 is symmetric, $p > q$ is assumed hereafter. See Table A.1.

Next, the element of C_3 which lies between $[p_2, 1]$ and $[p_2+1, 1]$ takes either of the following two forms:

(a) $[2p_2+1, 2] + p_3[p_2, 1] = [(p_3+2)p_2+1,\ p_3+2]$,

(b) $[2p_2+1, 2] + p_3[p_2+1, 1]$
$= [(p_3+2)p_2 + (p_3+1),\ p_3+2]$,

Table A.1. Alternative expression of Table 4 between $C_1 = 2$ and $C_1 = 3$ by the use of prime pairs

C_1	C_2	C_3	C_1	C_2	C_3
2					
	...			[2, 1]	
	[3, 1]				...
		...			[7, 4]
		[11, 4]			[5, 3]
		[8, 3]			[3, 2]
		[5, 2]			[4, 3]
		[7, 3]			[5, 4]
		[9, 4]			...
		...		[1, 1]	
				[1, 2]	
				[1, 3]	
				...	
			3		

where $p_3 \geqq 0$, and it is easily seen that both of them are prime pairs.

Expression (a) above is rewritten as $[p_3p_2+1, p_3]$ $(p_3 \geqq 2)$, if p_3+2 is replaced by p_3. The element of C_4, which lies between $[p_3p_2+1, p_3]$ and $[(p_3+1)p_2+1, p_3+1]$, takes either of the following two forms:

(a) $[(2p_3+1)p_2+2,\ 2p_2+1] + p_4[p_3p_2+1, p_3]$
$= [((p_4+2)p_3+1)p_2 + (p_4+2),\ (p_4+2)p_3+1]$,

(b) $[(2p_3+1)p_2+2,\ 2p_2+1]$
$+ p_4[(p_3+1)p_2+1,\ p_3+1]$
$= [((p_4+2)p_3+p_4+1)p_2 + (p_4+2),$
$(p_4+2)p_3+p_4+1]$,

where $p_4 \geqq 0$, and both of them are prime pairs. The same result is obtained for the element of C_3 with expression (b).

Similar processes show that positive integers which appear between p_1 and p_1+1 are all prime pairs.

Our next task then is to show the fact that when a prime pair $[p, q]$ is given, the position it occupies between p_1 and p_1+1 in Table 4 is uniquely determined. If this is shown, it is known that every prime pair appears once and only once between p_1 and p_1+1 in Table 4.

Now, as described above,

$$C_2 = \{[p_2, 1],\ p_2 \geqq 1\}.$$

Next, the general form of the element of C_3, which lies between $[p_2, 1]$ and $[p_2+1, 1]$, is either of the following two forms:

(a) $[2p_2+1, 2] + p_3[p_2+1] = [(p_3+2)p_2+1, p_3+2]$,

(b) $[2p_2+1, 2] + p_3[p_2+1, 1]$
$= [(p_3+2)(p_2+1) - 1,\ p_3+2]$,

where $p_3 \geqq 0$. Thus the general form of C_3 can be expressed as $[p_3p_2' + \varepsilon_3, p_3]\,(p_3 \geqq 2)$ where $\varepsilon_3 = \pm 1$, and $p_2' = p_2$ if $\varepsilon_3 = +1$, $p_2' = p_2 + 1$ if $\varepsilon_3 = -1$.

Next, the general form of C_4, which lies between $[p_3p_2' + \varepsilon_3, p_3]$ and $[(p_3+1)p_2' + \varepsilon_3, p_3+1]$, takes either of the following two forms:

(a) $[(2p_3+1)p_2' + 2\varepsilon_3,\ 2p_3+1] + p_4[p_3p_2' + \varepsilon_3, p_3]$
$= [((p_4+2)p_3+1)p_2' + (p_4+2)\varepsilon_3,\ (p_4+2)p_3+1]$,

(b) $[(2p_3+1)p_2'+2\varepsilon_3, 2p_3+1]$

$$+p_4[(p_3+1)p_2'+\varepsilon_3, p_3+1]$$
$$=\left[\left((p_4+2)(p_3+1)-1\right)p_2'+(p_4+2)\varepsilon_3,\ (p_4+2)(p_3+1)-1\right],$$

where $p_4 \geqq 2$. Thus the general form of C_4 is given by

$$[(p_4p_3'+\varepsilon_4)p_2'+p_4\varepsilon_3, p_4p_3'+\varepsilon_4] \quad (p_4 \geqq 2),$$

where $\varepsilon_4 = \pm 1$, $p_3' = p_3$ if $\varepsilon_4 = +1$, $p_3' = p_3+1$ if $\varepsilon_4 = -1$.

Summing up, it is found that the element $[p, q]$ of C_3 is expressed as

$$p = p_2'q + \varepsilon_3,$$
$$q = p_3,$$

and the element of C_4 as

$$p = p_2'q + \varepsilon_3 q_3 \quad (q_3 < q/2),$$
$$q = p_3'q_3 + \varepsilon_4,$$
$$q_3 = p_4.$$

In like manner, the element $[p, q]$ of C_k can be expressed as follows:

$$\begin{aligned}
p &= p_2'q + \varepsilon_3 q_3 && (q_3 < q/2),\\
q &= p_3'q_3 + \varepsilon_4 q_4 && (q_4 < q_3/2),\\
q_3 &= p_4'q_4 + \varepsilon_5 q_5 && (q_5 < q_4/2),\\
&\cdots\cdots\cdots\cdots\cdots\cdots\\
q_{k-3} &= p_{k-2}'q_{k-2} + \varepsilon_{k-1}q_{k-1} && (q_{k-1} < q_{k-2}/2),\\
q_{k-2} &= p_{k-1}'q_{k-1} + \varepsilon_k,\\
q_{k-1} &= p_k,
\end{aligned}$$

where $\varepsilon_i = \pm 1$ for $i = 3, 4, \ldots, k$, and $p_{i-1}' = p_{i-1}$ if $\varepsilon_i = +1$, $p_{i-1}' = p_{i-1}+1$ if $\varepsilon_i = -1$; $p_2 \geqq 1$, $p_i \geqq 2$ for $i = 3, 4, \ldots, k$.

The above expression of the prime pair $[p, q]$ indicates the position it occupies between p_1 and p_1+1 in Table 4. An example is shown in Table A.2 where the prime pair $[143, 38]$ is found to belong to C_5.

Thus it is ascertained that the number of integer n $(n \geqq 5)$ which belongs to C_i's $(i \geqq 2)$ is equal to the number of ways of expressing n by the prime pair $[p, q]$; namely, the number of ways of expressing n as

$$\begin{aligned}
n &= pp_1 + q(p_1+1)\\
&= (p+q)p_1 + q\\
&= rp_1 + q \quad (r = p+q),
\end{aligned}$$

where $p_1 \geqq 2$; p and q are positive integers prime each other; r is prime to q and $r > q \geqq 1$; n is prime to r and $n > rp_1 \geqq 2r$. Hence $n/2 > r > 1$.

Given a positive integer n, let the positive integers which are less than and prime to n be

$$1 = r_1 < r_2 < r_3 < \cdots < r_s < r_{s+1} < \cdots < r_{\varphi-1} < r_\varphi = n-1,$$

where φ implies $\varphi(n)$. Since $r_1 + r_\varphi = n$, $r_2 + r_{\varphi-1} = n, \ldots, r_s + r_{s+1} = n$, it immediately follows that $s = \varphi(n)/2$. Since r is limited to $n/2 > r > 1$, the number of ways of expressing n in the form $n = rp_1 + q$ is equal to the number of r, such that

$$r = r_2, r_3, \ldots, r_s,$$

or $s-1$.

Table A.2. This table shows how one discovers the position occupied by prime pair [143, 38]. If $\varepsilon_i = \pm 1$, the position is located, as indicated by small arrows, in a half of the set C_i, which consists of smaller/larger prime pairs

$143 = 4\cdot 38 - 9$	$p_2' = 4$	$\varepsilon_3 = -1$	$p_2 = 3$	$q_3 = 9$
$38 = 4\cdot 9 + 2$	$p_3' = 4$	$\varepsilon_4 = +1$	$p_3 = 4(2)$	$q_4 = 2$
$9 = 4\cdot 2 + 1$	$p_4' = 4$	$\varepsilon_5 = +1$	$p_4 = 4(2)$	$(k = 5)$
			$p_5 = q_4 = 2(0)$	

C_2	C_3	C_4	C_5
$p_2 = 3$	$p_3 = 2$, $\varepsilon_3 = -1\uparrow$	$p_4 = 2$, $\varepsilon_4 = +1\downarrow$	$p_5 = 0$, $\varepsilon_5 = +1\uparrow$
	[19,5]		
	(2) [15,4]	(0) [34,9]	
	(1) [11,3]	(1) [49,13]	
[4,1]	(0) [7,2]	(2) [64,17]	
(0) [3,1]		[79,21]	(0) [143,38]

Besides, every positive integer not less than 2 appears once in C_1. Hence it is concluded that the number of n $(n \geqq 3)$ which appears in Table 4 is given by $(s-1)+1 = \varphi(n)/2$.

Appendix II

Let the series inductance and parallel capacitance per unit length of the delay line, assumed to be lossless, be L and C respectively. Then the relation between the voltage v and the current i of the line is given by

$$\frac{\partial v}{\partial x} = -L\frac{\partial i}{\partial t}, \qquad \frac{\partial i}{\partial x} = -C\frac{\partial v}{\partial t}. \tag{A.1}$$

As the line is shorted at $x = 0$ and connected with the negative resistance element at $x = l$, boundary conditions become

$$v(0, t) = 0, \tag{A.2}$$

$$i(l, t) = f\left(v(l, t) + E\right), \tag{A.3}$$

where E is the dc bias voltage, and $I = f(V)$ is the expression representing the characteristic curve of the negative resistance element (Fig. 10).

It is assumed that the voltage along the line at $t = 0$ is $\alpha(x)$ and the current is $\beta(x)$. Therefore, the initial condition may be written as

$$v(x, 0) = \alpha(x) \quad (0 < x < l), \tag{A.4}$$

$$i(x, 0) = \beta(x) \quad (0 < x < l), \tag{A.5}$$

where $\alpha(0) = 0$.

D'Alembert's solution of (A.1) is of the form

$$v(x, t) = \phi_1\left(t - \frac{x}{w}\right) + \phi_2\left(t + \frac{x}{w}\right), \tag{A.6}$$

$$i(x, t) = \frac{1}{Z}\left\{\phi_1\left(t - \frac{x}{w}\right) - \phi_2\left(t + \frac{x}{w}\right)\right\}, \tag{A.7}$$

where $w = (LC)^{-\frac{1}{2}}$ is the propagation velocity of waves in the line, and $Z = (L/C)^{\frac{1}{2}}$ is the characteristic impedance of the line.

The combination (A.6) and (A.2) gives

$$\phi_1(t) + \phi_2(t) = 0. \tag{A.8}$$

With the substitution of (A.8) into (A.6) and (A.7), the results are

$$v(x, t) = \phi_1\left(t - \frac{x}{w}\right) - \phi_1\left(t + \frac{x}{w}\right), \tag{A.9}$$

$$i(x, t) = \frac{1}{Z}\left\{\phi_1\left(t - \frac{x}{w}\right) + \phi_1\left(t + \frac{x}{w}\right)\right\}. \tag{A.10}$$

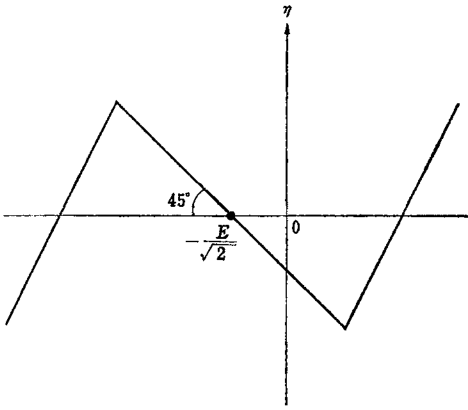

Fig. A.1. Graphic display of the relation between ξ and η in (A.19)

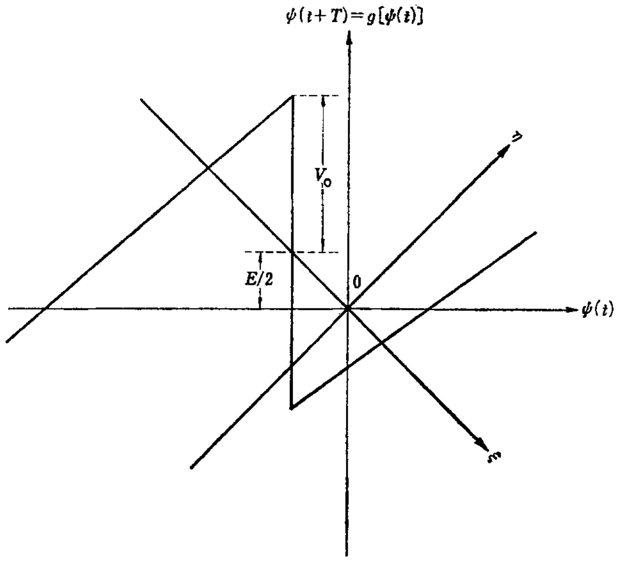

Fig. A.2. Comparison of this figure with Fig. 2 leads to the conclusion that the circuit in Fig. 9 is an analogue of the mathematical neuron model

From (A.3), (A.9) and (A.10)

$$\phi_1\left(t-\frac{T}{2}\right)+\phi_1\left(t+\frac{T}{2}\right) = Zf\left(\phi_1\left(t-\frac{T}{2}\right)-\phi_1\left(t+\frac{T}{2}\right)+E\right), \quad \textbf{(A.11)}$$

where $\frac{T}{2}=\frac{l}{w}$: the propagation time of waves in the line.

Equation (A.11) may be written as

$$\phi_1\left(t+\frac{T}{2}\right)=g\left[\phi_1\left(t-\frac{T}{2}\right)\right]. \quad \text{(A.12)}$$

This equation is a difference equation with the difference T if $\phi_1(t)$ is given for $-\frac{T}{2}<t<\frac{T}{2}$, $\phi_1(t)$ may be successively determined for

$$\frac{T}{2}<t<\frac{3T}{2}, \quad \frac{3T}{2}<t<\frac{5T}{2}, \ldots.$$

Now, from (A.4) and (A.9)

$$\phi_1\left(-\frac{x}{w}\right)-\phi_1\left(\frac{x}{w}\right)=\alpha(x) \quad (0<x<l), \quad \text{(A.13)}$$

and from (A.5) and (A.10)

$$\phi_1\left(-\frac{x}{w}\right)+\phi_1\left(\frac{x}{w}\right)=Z\beta(x) \quad (0<x<l). \quad \text{(A.14)}$$

Accordingly

$$\phi_1\left(-\frac{x}{w}\right)=\frac{1}{2}\{\alpha(x)+Z\beta(x)\} \quad (0<x<l), \quad \text{(A.15)}$$

$$\phi_1\left(\frac{x}{w}\right)=-\frac{1}{2}\{\alpha(x)-Z\beta(x)\} \quad (0<x<l). \quad \text{(A.16)}$$

Equation (A.15) gives $\phi_1(t)$ for $-\frac{T}{2}<t\leqq 0$, and (A.16) for $0\leqq t<\frac{T}{2}$. The value of $\phi_1(t)$ for $-\frac{T}{2}<t<\frac{T}{2}$, therefore, is determined by combining (A. 15) with (A.16). Thus our problem is reduced to difference equation (A.11) or (A.12).

By the introduction of a new variable $\psi(t)$:

$$\psi(t)=\phi_1\left(t-\frac{T}{2}\right),$$

(A.12) becomes

$$\psi(t+T)=g[\psi(t)]. \quad \text{(A.17)}$$

Furthermore, making use of new variables

$$\xi=\frac{1}{\sqrt{2}}\{\psi(t)-\psi(t+T)\}, \qquad \eta=\frac{1}{\sqrt{2}}\{\psi(t)+\psi(t+T)\}, \quad \text{(A.18)}$$

(A.17) is rewritten as

$$\eta=\frac{Z}{\sqrt{2}}f(\sqrt{2}\,\xi+E). \quad \text{(A.19)}$$

Obviously

$$\xi=\frac{1}{\sqrt{2}}v(l,t), \qquad \eta=\frac{Z}{\sqrt{2}}i(l,t). \quad \text{(A.20)}$$

From Fig. 10, (A.19) becomes as shown in Fig. A.1. Hence the relation between $\psi(t)$ and $\psi(t+T)$ is obtained by rotating the curve in Fig. A.1 45° to the right. This is shown in Fig. A.2. By comparing this figure with Fig. 2, (35) is obtained. Furthermore, a simple calculation yields (36), and (34) immediately follows from (A.20).

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References

Caianiello, E. R.: Outline of a theory of thought-processes and thinking machines. J. theor. Biol. 1, 204–235 (1961).
Caianiello, E. R., DeLuca, A.: Decision equation for binary systems. Application to neuronal behavior. Kybernetik 3, 33–40 (1966).
Hardy, G. H., Wright, E. M.: An introduction to the theory of numbers, p. 54. Oxford: Clarendon Press 1960.
Harmon, L. D.: Studies with artificial neurons, I: Properties and functions of an artificial neuron. Kybernetik 1, 89–101 (1961).
Nagumo, J., Shimura, M.: Self-oscillation in a transmission line with a tunnel diode. Proc. Inst. Radio Engineers 49, 1281–1291 (1961).
Titchmarsh, E. C.: The theory of functions, p. 366. Oxford: Clarendon Press 1968.

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