On the Entropy Inequality

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Contents

1. Introduction

The Thermodynamics of Irreversible Processes as a phenomenological theory describing processes in continua was initiated by ECKART [*1*] in 1940. Independently of ECKART's work, MEIXNER proposed essentially the same theory in a series of papers between 1939 and 1943*. Both authors introduce an equation of balance of entropy with positive production density. An important feature of this balance equation is that the entropy flux is assumed to be equal to the heat flux divided by the temperature, although this relation does not result from the theory; one can suggest possibly meaningful generalizations of this assumption [*3*].

The motivation for this relation rests upon the definition of entropy in thermostatics and on an approximate calculation of the entropy flux based on the kinetic theory of gases.

In recent years, COLEMAN & NOLL [*4*] have developed an improved method for exploiting the entropy balance. This method was applied to simple materials with fading memory by COLEMAN [*5*]. Here again the postulate is made that entropy flux and heat flux over temperature are equal.

In the present paper this assumption is omitted. Instead, we introduce an independent entropy flux, subject to constitutive assumptions like those made for heat flux, internal energy, stress, and entropy. By evaluation of the entropy inequality and application of a natural invariance principle, we are then able to derive a relation between entropy flux and heat flux which, for simple materials with fading memory, reduces to that usually postulated, except if these materials have uncommon symmetries. Calculations for a dipolar fluid, however, seem to indicate that the generalization of the entropy flux leads to alterations in the theory of multipolar materials.

* See the survey by J. MEIXNER & H. G. REIK [*2*].

In a recent publication GURTIN & WILLIAMS [6] have generalized the entropy balance in a different way, but in their work also the entropy flux is proportional to heat flux. However the kinetic theory of gases gives a motivation for the assumption of a more general entropy flux [3].

2. Basic Concepts*

We consider a body $\mathscr{B}$, whose particles are characterised by the material coordinates X_A. We take the X_A as coordinates of positions occupied by the particles in a reference configuration.

The motion of the body is then described by the function $x_i(X_A, t)$, which gives the position of the particles at time t. We call the function $x_i(X_A, t)$ the deformation and suppose that the deformation gradient

$$F_{iA}(X_B, t) = \frac{\partial x_i}{\partial X_A} \tag{2.1}$$

is nonsingular, *i.e.*

$$J = \det\{F_{iA}\} \neq 0 . \tag{2.2}$$

Without loss of generality we may then assume: $J>0$. The mass density ρ is given by

$$\rho(X_A, t) = \frac{1}{J}\,\rho_0(X_A), \tag{2.3}$$

where $\rho_0(X_A)$ is the mass density in the reference configuration. The deformation gradient may be expressed as the product

$$F_{jB} = R_{jk}\,U_{kB}, \tag{2.4}$$

where R_{jk} and U_{kB} are components of a proper orthogonal tensor and a symmetric positive-definite tensor, respectively.

We suppose that it is always possible to assign a positive temperature $\vartheta(X_A, t)$ to each $X_A \in \mathscr{B}$.

For any deformation of the body, the equations of balance of linear momentum, moment of momentum, and internal energy hold. Hence

$$\rho\,\ddot{x}_i - \frac{\partial t_{ij}}{\partial x_j} - \rho\,b_i = 0, \tag{2.5}$$

$$t_{ij} = t_{ji}, \tag{2.6}$$

$$\rho\,\dot{\varepsilon} + \frac{\partial q_i}{\partial x_i} - t_{ij}\frac{\partial \dot{x}_i}{\partial x_j} - \rho\,r = 0, \tag{2.7}$$

where t_{ij} is the stress tensor, b_i the specific body force, ε the specific internal energy, q_i the heat flux vector, and r the specific energy supply from the external world, per unit time. The dot denotes the material time derivative.

* Throughout this paper we employ Cartesian tensor notation.

Let $\eta(X_A, t)$ be the specific entropy and $p_i(X_A, t)$ the components of the entropy flux. We postulate that the entropy production is nonnegative and hence write the entropy balance in the form

$$\rho\dot{\eta}+\frac{\partial p_i}{\partial x_i}-\rho\frac{r}{\vartheta}\geqq 0. \tag{2.8}$$

Here we have assumed that the entropy supply from the external world is equal to the energy supply divided by the temperature. In the earlier works cited above it is also assumed that the entropy flux is equal to heat flux divided by temperature. However, we here make no such assumption.

Let us introduce the specific free energy ψ and a vector k_i signifying the difference between entropy flux and heat flux over temperature

$$\psi\equiv\varepsilon-\vartheta\eta, \quad k_i\equiv p_i-\frac{q_i}{\vartheta}. \tag{2.9}$$

If we insert these quantities into (2.8) and make use of (2.7), we are led to

$$-\dot{\psi}-\dot{\vartheta}\eta+\frac{1}{\rho}t_{ij}\frac{\partial\dot{x}_i}{\partial x_j}+\frac{\vartheta}{\rho}\frac{\partial k_i}{\partial x_i}-\frac{1}{\rho\vartheta}q_i\frac{\partial\vartheta}{\partial x_i}\geqq 0. \tag{2.10}$$

It is assumed that the histories of deformation and temperature within the body determine ε, η, t_{ij}, q_i, and p_i or, equivalently, ψ, η, t_{ij}, q_i, and k_i as functions of X_A and t. The functional relations which connect these functions with the histories of deformation and temperature are called constitutive equations; their form characterizes a material.

We postulate that the constitutive equations and the balance equations (2.5) to (2.8) hold for every history of deformation and temperature in the body $\mathscr{B}$, provided of course $\det\{F_{iA}\}>0$ and $\vartheta>0$.

Then the possible constitutive functionals are subjected to the requirement that the entropy production be nonnegative, *i.e.* they are restricted by (2.8) or (2.10). The balance of momentum and of internal energy provide no further restrictions on the constitutive equations; an arbitrary choice of $x_i(X_A, \tau)$ and $\vartheta(X_A, \tau)$ $[X_A\in\mathscr{B}$ and $-\infty<\tau<t]$ merely determines the body force and energy supply. The balance of moment of momentum is to be satisfied by requiring any constitutive equation for t_{ij} to be symmetric in i and j.

We wish to emphasize the contrast in the basic concept of this paper and the usual theory of nonequilibrium thermodynamics. We lay down a general constitutive equation for the entropy flux, or equivalently, for k_i as well as for free energy, entropy, stress, and heat flux, whereas normally it is assumed that $p_i=q_i/\vartheta$. This amounts to postulating a very special constitutive equation for p_i.

To simplify later calculations, we introduce material components of heat flux, entropy flux, and of the vector k_i:

$$Q_A=J\frac{\partial X_A}{\partial x_i}q_i, \quad P_A=J\frac{\partial X_A}{\partial x_i}p_i, \quad K_A=J\frac{\partial X_A}{\partial x_i}k_i \tag{2.11}$$

where J is the determinant of the deformation gradient. It is easy to show that

$$\frac{\partial}{\partial x_i}\left(\frac{1}{J}\frac{\partial x_i}{\partial X_A}\right)=0. \tag{2.12}$$

We use (2.11) and (2.12) to transform (2.10) into a form which is appropriate for later use:

$$-\dot{\psi}-\dot{\vartheta}\eta+\frac{1}{\rho}t_{ij}\dot{F}_{jA}(F^{-1})_{Ai}+\frac{\vartheta}{\rho}\frac{1}{J}\frac{\partial K_A}{\partial X_A}-\frac{1}{\rho\vartheta}\frac{1}{J}Q_A\frac{\partial\vartheta}{\partial X_A}\geqq 0. \tag{2.13}$$

In (2.13) we have also replaced $\partial\dot{x}_j/\partial x_i$ by $\dot{F}_{jA}(F^{-1})_{Ai}$.

The possible constitutive functionals are restricted not only by the entropy inequality but also by the principle of invariance under superposed rigid motions. Let x_i^* and x_i be the positions of a particle in two motions which differ only by a superposed rigid motion. These positions are related by

$$x_i^*(X_A,t)=O_{ij}(t)\,x_j(X_A,t)+b_i(t), \tag{2.14}$$

where $O_{ij}(t)$ are the elements of any proper orthogonal matrix. For two such motions we assume the following:

i) The scalars ϑ, ε, η, and ψ are unaffected by this superposed rigid motion.

ii) The transformations of the components q_i, p_i, and k_i are

$$q_i^*=O_{ij}\,q_j;\qquad p_i^*=O_{ij}\,p_j;\qquad k_i^*=O_{ij}\,k_j;$$

hence the material components Q_A, P_A, and K_A are unaltered.

iii) The components t_{ij} for the two motions are related by

$$t_{ij}^*=O_{ik}\,O_{jl}\,t_{kl}.$$

This principle is closely related to the principle of material frame indifference*. However, in the latter principle the above transformation properties are valid for all orthogonal matrices O_{ij} instead of for all proper orthogonal ones.

3. Homogeneous Simple Materials with Fading Memory

Constitutive Equations

In a simple material, the quantities ψ, t_{ij}, η, Q_A, and K_A at the particle X_C and time t are determined by the histories

$$F_{iB}^t(s,X_C)=F_{iB}(t-s,X_C),\qquad \vartheta^t(s,X_C)=\vartheta(t-s,X_C)\qquad [0\leqq s<\infty]$$

of the deformation gradient and the temperature at X_C and by the present value of the temperature gradient at this particle. It will turn out to be convenient in later calculations to treat the present values $F_{iB}(t,X_C)$ and $\vartheta(t,X_C)$ of the deformation gradient and the temperature and their past values separately; let us therefore introduce the difference histories:

$$\begin{aligned}F_{iBd}^t(s,X_C)&=F_{iB}^t(s,X_C)-F_{iB}(t,X_C)\\ \vartheta_d^t(s,X_C)&=\vartheta^t(s,X_C)-\vartheta(t,X_C).\end{aligned} \tag{3.1}$$

* See [7] for a review of the history of this principle.

Then the constitutive functional relations for ψ, $t_{il}(=t_{li})$, η, Q_A and K_A are

$$\begin{aligned}
\psi(X_C,t) &= \underset{s=0}{\overset{\infty}{\mathfrak{p}}}\left[F^t_{iBd}(s,X_C), \vartheta^t_d(s,X_C); F_{iB}(t,X_C), \vartheta(t,X_C), \frac{\partial\vartheta}{\partial X_C}\right]\\
t_{il}(X_C,t) &= \underset{s=0}{\overset{\infty}{\mathfrak{t}_{il}}}\left[F^t_{iBd}(s,X_C), \vartheta^t_d(s,X_C); F_{iB}(t,X_C), \vartheta(t,X_C), \frac{\partial\vartheta}{\partial X_C}\right]\\
\eta(X_C,t) &= \underset{s=0}{\overset{\infty}{\mathfrak{h}}}\left[F^t_{iBd}(s,X_C), \vartheta^t_d(s,X_C); F_{iB}(t,X_C), \vartheta(t,X_C), \frac{\partial\vartheta}{\partial X_C}\right]\\
Q_A(X_C,t) &= \underset{s=0}{\overset{\infty}{\mathfrak{Q}_A}}\left[F^t_{iBd}(s,X_C), \vartheta^t_d(s,X_C); F_{iB}(t,X_C), \vartheta(t,X_C), \frac{\partial\vartheta}{\partial X_C}\right]\\
K_A(X_C,t) &= \underset{s=0}{\overset{\infty}{\mathfrak{K}_A}}\left[F^t_{iBd}(s,X_C), \vartheta^t_d(s,X_C); F_{iB}(t,X_C), \vartheta(t,X_C), \frac{\partial\vartheta}{\partial X_C}\right].
\end{aligned} \tag{3.2}$$

We assume that the material is homogeneous. Then a reference configuration exists in which the functionals are independent of the particles; we may regard our coordinates X_C as the coordinates of the positions of the particles in this particular reference configuration. Then the functionals in (3.2) do not depend on X_C.

In formulating the constitutive equations (3.2) we have used the principle of equipresence [*8*], according to which the same independent variables should appear in all constitutive equations unless this contradicts the inequality (2.13), invariance under superposed rigid motions or some material symmetry.

More Compact Notation

Following COLEMAN [*5*], we introduce the ten-dimensional vectors

$$(F_{iA}, \vartheta).$$

If F_{iA} and ϑ were completely unrestricted quantities, the collection of all these vectors would form a normed linear vector space $\mathscr{A}$ under the following definitions:

$$\alpha(\overset{1}{F}_{iA}, \overset{1}{\vartheta}) + \beta(\overset{2}{F}_{iA}, \overset{2}{\vartheta}) = (\alpha \overset{1}{F}_{iA} + \beta \overset{2}{F}_{iA}, \alpha\overset{1}{\vartheta} + \beta\overset{2}{\vartheta}), \tag{3.3}$$

$$\|(F_{iA}, \vartheta)\| = \sqrt{F_{iA}F_{iA} + \vartheta^2}. \tag{3.4}$$

However, as is discussed by COLEMAN & MIZEL [*9*], the restrictions $\det\{F_{iA}\} > 0$ and $\vartheta > 0$ lead to the conclusion that

$$A_\alpha = (F_{iA}\,\vartheta) \qquad (\alpha = 1, 2, \dots, 10) \tag{3.5}$$

form a cone $C \subset \mathscr{A}$. We define

$$\Sigma_\alpha = \left(\frac{1}{\rho}\, t_{il}(F^{-1})_{Al}, -\eta\right) \tag{3.6}$$

and correspondingly the functional

$$\mathfrak{S}_\alpha = \left(\frac{1}{\rho}\, \mathfrak{t}_{il}(F^{-1})_{Al}, -\mathfrak{h}\right). \tag{3.7}$$

Let us introduce also

$$I_\alpha=(0,1), \tag{3.8}$$

Σ_α and I_α are vectors $\in\mathscr{A}$.

Using this notation, we can write the constitutive equations (3.2) in the form

$$\begin{aligned}
\psi(X_C,t) &= \underset{s=0}{\overset{\infty}{\mathfrak{p}}}\left[\Lambda^t_{\beta d}(s,X_C);\Lambda_\mu(t,X_C),\frac{\partial\Lambda_\gamma}{\partial X_B}I_\gamma\right]\\
\Sigma_\alpha(X_C,t) &= \underset{s=0}{\overset{\infty}{\mathfrak{S}_\alpha}}\left[\Lambda^t_{\beta d}(s,X_C);\Lambda_\mu(t,X_C),\frac{\partial\Lambda_\gamma}{\partial X_B}I_\gamma\right]\\
Q_A(X_C,t) &= \underset{s=0}{\overset{\infty}{\mathfrak{Q}_A}}\left[\Lambda^t_{\beta d}(s,X_C);\Lambda_\mu(t,X_C),\frac{\partial\Lambda_\gamma}{\partial X_B}I_\gamma\right]\\
K_A(X_C,t) &= \underset{s=0}{\overset{\infty}{\mathfrak{K}_A}}\left[\Lambda^t_{\beta d}(s,X_C);\Lambda_\mu(t,X_C),\frac{\partial\Lambda_\gamma}{\partial X_B}I_\gamma\right].
\end{aligned}\tag{3.9}$$

These functionals must satisfy the inequality (2.13), which in the present compact notation has the form

$$-\dot\psi+\Sigma_\alpha\dot\Lambda_\alpha+\frac{1}{J}\frac{\vartheta}{\rho}\frac{\partial K_A}{\partial X_A}-\frac{1}{J}\frac{1}{\rho\vartheta}Q_A\frac{\partial\Lambda_\alpha}{\partial X_A}I_\alpha\geqq 0. \tag{3.10}$$

Fading Memory

Let $h(s)\,[0\leqq s<\infty]$ denote a positive, monotone decreasing, square-integrable, continuous function.

Let $\Gamma_\alpha(s)$ $[0\leqq s<\infty]$ with $\Gamma_\alpha(0)=0$ be a vector $\in\mathscr{A}$ such that its h-norm

$$\|\Gamma_\alpha\|_h\equiv\int\limits_0^\infty\|\Gamma_\alpha(s)\|^2\,h^2(s)\,ds \tag{3.11}$$

is finite. The collection of all such Γ_α's forms a Hilbert space $\mathscr{S}_h$. $\|\Gamma_\alpha\|_h$ is called a fading-memory norm, because the recent past of $\Gamma_\alpha(s)$ contributes more to $\|\Gamma_\alpha\|_h$ than does the distant past. Coleman & Mizel in a recent paper [9] thoroughly investigate the properties of the "influence function" $h(s)$ in norms of the type (3.11), subject to physically reasonable hypotheses on the space.

The principle of fading memory as laid down by Coleman in [5] states that the functionals (3.9) are Fréchet-differentiable throughout their domain in $\mathscr{S}_h$ with respect to the h-norm.

Recently Mizel & Wang [10] have re-examined the assumption of fading memory, emphasizing the fact that the domain of the functionals (3.9) is only the cone $C\subset\mathscr{A}$. They get the result that the chain rule is applicable for these functionals if the following conditions hold:

i) $\Lambda^t_{\beta d}(s,X_A)=\Lambda_{\beta d}(t-s,t,X_A)$ as a function of the argument $(t-s)$ is smooth, *i.e.* $\Lambda_{\beta d}(t-s,t,X_A)$ is absolutely continuous, $\Lambda_{\beta d}(0+,t,X_A)$ exists, and $\Lambda_{\beta d}(t-s,t,X_A)\in\mathscr{S}_h$.

ii) The functionals (3.9) are smooth for each $\Lambda^t_{\beta d}(s, X_A)$, *i.e.*, for all $\Gamma_\alpha(s, X_A) \in \mathscr{S}_h$ for which $\Lambda^t_{\beta d}(s, X_A) + \Gamma_\beta(s, X_A) \in C \cap \mathscr{S}_h$ the following relation holds:

$$\underset{s=0}{\overset{\infty}{\mathfrak{F}}} [\Lambda^t_{\beta d}(s, X_A) + \Gamma_\beta(s, X_A)] = \underset{s=0}{\overset{\infty}{\mathfrak{F}}} [\Lambda^t_{\beta d}(s, X_A)] + \delta \underset{s=0}{\overset{\infty}{\mathfrak{F}}} [\Lambda^t_{\beta d}(s, X_A) \,|\, \Gamma_\gamma(s, X_A)] + O(\|\Gamma_\beta\|_h) \quad (^*)$$

where $\delta \mathfrak{F}[\Lambda^t_{\beta d}(s, X_A) | \Gamma_\beta(s, X_A)]$ denotes a continuous functional in $\Lambda^t_{\beta d}(s, X_A)$ and a continuous linear functional in $\Gamma_\beta(s, X_A)$. Continuous in both cases means continuous with respect to the h-norm.

MIZEL & WANG justify the application of the chain rule only for the time differentiation of functionals, but we must differentiate the functionals also with respect to X_A. By following the proof of MIZEL in [*10*] we can easily see that the conditions i) and ii) of MIZEL & WANG allow the application of the chain rule also in X_A-differentiation, if we complete i) by requiring that smoothness of $\Lambda^t_{\beta d}(s, X_A)$ includes that

$$\frac{\partial \Lambda^t_{\beta d}(s, X_A)}{\partial X_B} \in S_h \qquad (B = 1, 2, 3).$$

In using the chain rule, we do not wish to emphasize every time the mathematical refinement of COLEMAN's assumption of fading memory by MIZEL & WANG. Therefore we shall call our functionals $\mathfrak{F}$ Fréchet-differentiable (and $\delta \mathfrak{F}$ the Fréchet-differential) if they and their argument functions satisfy the conditions i) and ii) which we assume. Furthermore we require that the functionals be continuous and differentiable in

$$\Lambda_\beta(t, X_A) \quad \text{and} \quad \frac{\partial \Lambda_\gamma(t, X_A)}{\partial X_C} I_\gamma.$$

Then we obtain

$$\begin{aligned} \dot{\psi} = \delta \mathfrak{p} \left[\Lambda^t_{\beta d}(s); \Lambda_\mu, \frac{\partial \Lambda_\gamma}{\partial X_C} I_\gamma \,\middle|\, \dot{\Lambda}^t_{\delta d}(s)\right] + \partial_{\Lambda_\delta} \mathfrak{p} \left[\Lambda^t_{\beta d}(s); \Lambda_\mu, \frac{\partial \Lambda_\gamma}{\partial X_C} I_\gamma\right] \dot{\Lambda}_\delta + \\ + \partial_{\frac{\partial \Lambda_\gamma}{\partial X_B} I_\gamma} \mathfrak{p} \left[\Lambda^t_{\beta d}(s); \Lambda_\mu, \frac{\partial \Lambda_\alpha}{\partial X_C} I_\alpha\right] \left(\frac{\partial \Lambda_\delta}{\partial X_B}\right)^{\!\cdot} I_\delta \end{aligned} \tag{3.12}$$

and

$$\begin{aligned} \frac{\partial K_A}{\partial X_A} = \delta \mathfrak{K}_A \left[\Lambda^t_{\beta d}(s); \Lambda_\mu, \frac{\partial \Lambda_\gamma}{\partial X_C} I_\gamma \,\middle|\, \frac{\partial \Lambda^t_{\delta d}(s)}{\partial X_A}\right] + \partial_{\Lambda_\delta} \mathfrak{K}_A \left[\Lambda^t_{\beta d}(s); \Lambda_\mu, \frac{\partial \Lambda_\gamma}{\partial X_C} I_\gamma\right] \frac{\partial \Lambda_\delta}{\partial X_A} + \\ + \partial_{\frac{\partial \Lambda_\gamma}{\partial X_B} I_\gamma} \mathfrak{K}_A \left[\Lambda^t_{\beta d}(s); \Lambda_\mu, \frac{\partial \Lambda_\alpha}{\partial X_C} I_\alpha\right] \frac{\partial^2 \Lambda_\delta}{\partial X_A \partial X_B} I_\delta. \end{aligned} \tag{3.13}$$

Here we have omitted the dependence of the argument functions on t and X_A. $\delta \mathfrak{p}$ and $\delta \mathfrak{K}_A$ are the Fréchet-differentials of the functionals $\mathfrak{p}$ and $\mathfrak{K}_A$.

* For simplicity in notation, we have omitted here the dependence of $\mathfrak{F}$ on $\Lambda_\beta(t, X_A)$ and $(\partial \Lambda_\gamma / \partial X_C) I_\gamma$.

4. Restrictions Imposed on the Constitutive Equation (3.9)$_4$ by the Inequality (3.10)

Let us introduce the constitutive equation (3.9) into the inequality (3.10). Then by (3.12) and (3.13) we obtain

$$
\begin{aligned}
&\left\{\mathfrak{S}_\delta\left[\Lambda^t_{\beta d}(s);\Lambda_\mu,\frac{\partial\Lambda_\gamma}{\partial X_C}I_\gamma\right]-\partial_{\Lambda_\delta}\mathfrak{p}\left[\Lambda^t_{\beta d}(s);\Lambda_\mu,\frac{\partial\Lambda_\gamma}{\partial X_C}I_\gamma\right]\right\}\dot{\Lambda}_\delta-\\
&\qquad -\delta\mathfrak{p}\left[\Lambda^t_{\beta d}(s);\Lambda_\mu,\frac{\partial\Lambda_\gamma}{\partial X_C}I_\gamma\,|\,\dot{\Lambda}^t_{\delta d}(s)\right]+\\
&+\frac{1}{J}\frac{\vartheta}{\rho}\left\{\partial_{\Lambda_\delta}\mathfrak{R}_A\left[\Lambda^t_{\beta d}(s);\Lambda_\mu,\frac{\partial\Lambda_\gamma}{\partial X_C}I_\gamma\right]-\frac{1}{\vartheta^2}\mathfrak{Q}_A\left[\Lambda^t_{\beta d}(s);\Lambda_\mu,\frac{\partial\Lambda_\gamma}{\partial X_C}I_\gamma\right]I_\delta\right\}\frac{\partial\Lambda_\delta}{\partial X_A}+\\
&+\frac{1}{J}\frac{\vartheta}{\rho}\delta\mathfrak{R}_A\left[\Lambda^t_{\beta d}(s);\Lambda_\mu,\frac{\partial\Lambda_\gamma}{\partial X_C}I_\gamma\,\middle|\,\frac{\partial\Lambda^t_{\delta d}(s)}{\partial X_A}\right]- \qquad\qquad (4.1)\\
&\qquad -\partial_{\frac{\partial\Lambda_\gamma}{\partial X_A}I_\gamma}\mathfrak{p}\left[\Lambda^t_{\beta d}(s);\Lambda_\mu,\frac{\partial\Lambda_\alpha}{\partial X_C}I_\alpha\right]\left(\frac{\partial\Lambda_\delta}{\partial X_A}\right)^{\bullet}I_\delta+\\
&+\frac{1}{J}\frac{\vartheta}{\rho}\partial_{\frac{\partial\Lambda_\gamma}{\partial X_B}I_\gamma}\mathfrak{R}_A\left[\Lambda^t_{\beta d}(s);\Lambda_\mu,\frac{\partial\Lambda_\alpha}{\partial X_C}I_\alpha\right]\frac{\partial^2\Lambda_\delta}{\partial X_A\partial X_B}I_\delta\geqq 0.
\end{aligned}
$$

This inequality must hold at every particle $X_C\in\mathscr{B}$ for any history $\Lambda^t_\beta(s, Y_C)$ with $Y_C\in\mathscr{B}$. Hence it must hold in particular for any choice of

$$
\begin{aligned}
\Lambda^t_\beta(s,X_C)&=\left(\frac{\partial x^t_i(s,X_C)}{\partial X_B},\,\vartheta^t(s,X_C)\right),\\
\frac{\partial\Lambda^t_\beta(s,X_C)}{\partial X_A}&=\left(\frac{\partial^2 x^t_i(s,X_A)}{\partial X_A\partial X_B},\,\frac{\partial\vartheta^t(s,X_C)}{\partial X_A}\right),\\
\frac{\partial^2\Lambda^t_\beta(s,X_C)}{\partial X_A\partial X_B}I_\beta&=\frac{\partial^2\vartheta^t(s,X_C)}{\partial X_A\partial X_B}
\end{aligned}
$$

that does not violate the symmetry in the last two expressions. Hence the six independent quantities

$$\frac{\partial^2\Lambda_\delta}{\partial X_A\partial X_B}I_\delta=\frac{\partial^2\vartheta}{\partial X_A\partial X_B},$$

which appear only in the last term of the inequality, may be chosen arbitrarily. From the inequality and our constitutive assumptions we then obtain (with $(\partial\Lambda_\gamma/\partial X_B)\,I_\gamma=\partial\vartheta/\partial X_B$)

$$\partial_{\frac{\partial\vartheta}{\partial X_B}}\mathfrak{R}_A+\partial_{\frac{\partial\vartheta}{\partial X_A}}\mathfrak{R}_B=0. \qquad (4.2)$$

Similarly if we assign arbitrary values to

$$\frac{\partial\Lambda_\delta}{\partial X_A}\qquad(\delta=1,2,\ldots,9),$$

i.e. to

$$\frac{\partial F_{iB}}{\partial X_A}=\frac{\partial^2 x_i}{\partial X_A \partial X_B},$$

we see that the inequality requires that

$$\partial_{F_{iB}} \mathfrak{K}_A+\partial_{F_{iA}} \mathfrak{K}_B=0. \tag{4.3}$$

Let us introduce

$$(A_B^1, A_B^2, A_B^3, A_B^4)=\left(F_{1B}, F_{2B}, F_{3B}, \frac{\partial \vartheta}{\partial X_B}\right). \tag{4.4}$$

Then (4.2) and (4.3) can be expressed as

$$\partial_{A_B^i} \mathfrak{K}_A+\partial_{A_A^i} \mathfrak{K}_B=0.$$

Consequently it can easily be shown that K_A must have the following form:

$$\begin{aligned} K_A= & \sum_{\substack{i,j,k,l=1 \\ i<j<k<l}}^{4} \Omega_{ABCDE}^{ijkl} A_B^i A_C^j A_D^k A_E^l+\sum_{\substack{i,j,k=1 \\ i<j<k}}^{4} \Omega_{ABCD}^{ijk} A_B^i A_C^j A_D^k+ \\ & +\sum_{\substack{i,j=1 \\ i<j}}^{4} \Omega_{ABC}^{ij} A_B^i A_C^j+\sum_{i=1}^{4} \Omega_{AB}^i A_B^i+\Omega_A . \end{aligned} \tag{4.5}$$

Here the Ω-tensors are functionals of $A_{\beta d}^t(s)$ and functions of ϑ. They are antisymmetric with respect to permutation of any of the lower indices, *e.g.*

$$\Omega_{AB}^i=-\Omega_{BA}^i. \tag{4.6}$$

The main implication of (4.5) is that K_A must depend linearly on the components of each vector A_B^i $(i=1,2,3,4)$.

Let us go back to the inequality (4.1). By use of the assumed continuity of $\delta \mathfrak{K}_A$ with respect to the h-norm, it can be shown possible to choose

$$\left(\frac{\partial A_\delta}{\partial X_A}\right)^{\cdot} I_\delta$$

arbitrarily yet change the term

$$\delta \mathfrak{K}_A\left[A_{\beta d}^t(s); A_\mu, \frac{\partial A_\alpha}{\partial X_C} I_\alpha \middle| \frac{\partial A_{\delta d}^t(s)}{\partial X_A}\right]$$

as little as desired. Hence we obtain

$$\partial_{\frac{\partial \vartheta}{\partial X_A}} \mathfrak{p}\left[A_{\beta d}^t(s); A_\mu, \frac{\partial A_\alpha}{\partial X_C} I_\alpha\right]=0. \tag{4.7}$$

Now the history $\frac{\partial A_{\delta d}^t(s)}{\partial X_A}$ can be assigned arbitrarily and independently of all the remaining terms in (4.1). Thus we have

$$\delta \mathfrak{K}_A\left[A_{\beta d}^t(s); A_\mu, \frac{\partial A_\gamma}{\partial X_C} I_\gamma \middle| \frac{\partial A_{\delta d}^t(s)}{\partial X_A}\right]=0 \quad \text{for any } \frac{\partial A_{\delta d}^t(s)}{\partial X_A}. \tag{4.8}$$

Equations (4.5), (4.6), and (4.8) represent restrictions imposed on the constitutive functional for K_A by the inequality.

The inequality itself is reduced to the form (note that $\mathfrak{p}$ is independent of

$$\frac{\partial \Lambda_\gamma}{\partial X_C} I_\gamma = \frac{\partial \vartheta}{\partial X_C}$$

according to (4.7))

$$\begin{aligned}\left\{\mathfrak{S}_\delta\left[\Lambda^t_{\beta\,d}(s);\Lambda_\mu,\frac{\partial \Lambda_\gamma}{\partial X_C} I_\gamma\right]-\partial_{\Lambda_\delta}\mathfrak{p}\left[\Lambda^t_{\beta\,d}(s);\Lambda_\mu\right]\right\}\dot{\Lambda}_\delta-\delta\mathfrak{p}\left[\Lambda^t_{\beta\,d}(s);\Lambda_\mu|\dot{\Lambda}^t_{\delta\,d}(s)\right]+&\\ +\frac{1}{J}\frac{\vartheta}{\rho}\left\{\partial_\vartheta\mathfrak{K}_A\left[\Lambda^t_{\beta\,d}(s);\Lambda_\mu,\frac{\partial \Lambda_\gamma}{\partial X_C} I_\gamma\right]-\right.&\\ \left.-\frac{1}{\vartheta^2}\mathfrak{Q}_A\left[\Lambda^t_{\beta\,d}(s);\Lambda_\mu,\frac{\partial \Lambda_\gamma}{\partial X_C} I_\gamma\right]\right\}\frac{\partial\vartheta}{\partial X_A}\geqq 0.&\end{aligned} \tag{4.9}$$

COLEMAN* obtains this same form short of the term

$$\partial_\vartheta\mathfrak{K}_A\left[\Lambda^t_{\beta\,d}(s);\Lambda_\mu,\frac{\partial \Lambda_\gamma}{\partial X_C} I_\gamma\right]\frac{\partial\vartheta}{\partial X_A}.$$

COLEMAN'S reasoning following his equation (6.17) is not affected by this difference, and we finally obtain the inequality

$$\sigma-\frac{1}{J}\left(\frac{1}{\rho\,\vartheta^2}Q_A-\frac{1}{\vartheta}\partial_\vartheta\mathfrak{K}_A\right)\frac{\partial\vartheta}{\partial X_A}\geqq 0 \tag{4.10}$$

in the same way as COLEMAN finds his inequality (6.29).

σ is defined as

$$\sigma=\frac{1}{\vartheta}\delta\mathfrak{p}\left[\Lambda^t_{\beta\,d}(s);\Lambda_\mu\left|\frac{d}{ds}\Lambda^t_\delta(s)\right.\right]. \tag{4.11}$$

Putting $\partial\vartheta/\partial X_A$ zero, we see that σ is nonnegative. COLEMAN calls it the *internal dissipation.*

Summarizing this section, we can say that the constitutive functional $\mathfrak{K}_A$ is subjected to the two restrictive requirements (4.5) and (4.8) and that the entropy inequality reduces to the form (4.10).

5. Further Restrictions Imposed on the Constitutive Functional $(3.9)_4$ by Invariance under Superposed Rigid Motion

According to the principle of invariance under superposed rigid motions, laid down at the end of Section 2, K_A is unaffected by this superposition:

$$K^*_A = K_A.$$

* See equations (6.17) and (6.5) in [5]. In comparing, note that COLEMAN has replaced $\delta\mathfrak{p}\left[\Lambda^t_{\beta\,d}(s),\Lambda_\mu|\dot{\Lambda}^t_{\delta d}(s)\right]$ using equation (5.17) in [5].

Hence the functional $\mathfrak{K}_A$ is restricted by the following condition:

$$\mathfrak{K}_A\left[O_{ij}(t-s)\,F^t_{jB}(s),\,\vartheta^t(s),\,\frac{\partial\vartheta}{\partial X_C}\right]=\mathfrak{K}_A\left[F^t_{iB}(s);\,\vartheta^t(s),\,\frac{\partial\vartheta}{\partial X_C}\right],\tag{5.1}$$

which has to hold for any proper orthogonal matrix $O_{ij}(t)$.

According to (2.4) we may write

$$F^t_{jB}(s)=R_{jk}(t-s)\,U^t_{kB}(s),$$

where R is a proper orthogonal tensor and U is a symmetric positive-definite tensor. In (5.1) we may choose $O_{ij}(t-s)=R_{ji}(t-s)$, and so we have

$$\mathfrak{K}_A\left[F^t_{iB}(s),\,\vartheta^t(s);\,\frac{\partial\vartheta}{\partial X_C}\right]=\mathfrak{K}_A\left[U^t_{iB}(s),\,\vartheta^t(s),\,\frac{\partial\vartheta}{\partial X_C}\right].\tag{5.2}$$

It is more convenient for us to use the tensor

$$C_{DE}=U^2_{DE}=F_{jD}F_{jE},\qquad C_{DE}=C_{ED}.\tag{5.3}$$

Hence we obtain

$$\begin{aligned}K_A(X_C,t)&=\mathfrak{K}_A\left[F^t_{iBd}(s),\,\vartheta^t_d(s);\,F_{iB},\,\vartheta,\,\frac{\partial\vartheta}{\partial X_C}\right]\\&=\hat{\mathfrak{K}}_A\left[C^t_{DEd}(s),\,\vartheta^t_d(s),\,C_{DE},\,\vartheta,\,\frac{\partial\vartheta}{\partial X_C}\right].\end{aligned}\tag{5.4}$$

According to (5.4), K_A must depend on $C_{DE}(t)=F_{iD}(t)\,F_{iE}(t)$, *i.e.* K_A must depend quadratically on F_{1B}, F_{2B}, and F_{3B}; on the other hand, (4.5) showed that K_A can depend only linearly on these quantities. Hence we conclude that K_A can not depend at all on F_{iA}, and (4.5) reduces to

$$K_A=\Omega_{AB}\frac{\partial\vartheta}{\partial X_B}+\Omega_A,\tag{5.5}$$

where we have set $\Omega^4_{AB}=\Omega_{AB}$. In (5.5) Ω_{AB} and Ω_A may still be functionals of $F^t_{iBd}(s)$ and $\vartheta^t_d(s)$ and functions of ϑ, and

$$\Omega_{AB}=-\Omega_{BA}.\tag{5.6}$$

Let us now determine explicitly the restriction placed upon $\mathfrak{K}_A$ by (4.8). For this calculation we abandon the summation convention.

Let us consider the Fréchet-differential

$$\begin{aligned}&\delta\,\mathfrak{K}_A\left[C^t_{DEd}(s),\,\vartheta^t_d(s);\,C_{DE},\,\vartheta,\,\frac{\partial\vartheta}{\partial X_C}\,\middle|\,\frac{\partial C^t_{DEd}(s)}{\partial X_A},\,\frac{\partial\vartheta^t_d(s)}{\partial X_A}\right]\\&=\delta\,\hat{\mathfrak{K}}_A\left[\sim\middle|\,\frac{\partial C^t_{DEd}(s)}{\partial X_A},\,\frac{\partial\vartheta^t_d(s)}{\partial X_A}\right];\end{aligned}\tag{5.7}$$

this is a linear functional in the ten functions

$$\frac{\partial C^t_{DEd}(s)}{\partial X_A},\;\frac{\partial\vartheta^t_d(s)}{\partial X_A},$$

of which only seven are independent because of (5.3). We can rewrite (5.7) in the form

$$\delta \hat{\mathfrak{R}}_A \left[\sim \left| \frac{\partial C^t_{DEd}(s)}{\partial X_A}, \frac{\partial \vartheta^t_d(s)}{\partial X_A} \right.\right] = \sum_C \sum_D \delta_{CD} \hat{\mathfrak{R}}_A \left[\sim \left| \frac{\partial C^t_{CDd}(s)}{\partial X_A} \right.\right] + + \delta_\vartheta \hat{\mathfrak{R}}_A \left[\sim \left| \frac{\partial \vartheta^t_d(s)}{\partial X_A} \right.\right]. \tag{5.8}$$

Here

$$\delta_{CD} \mathfrak{R}_A \left[\sim \left| \frac{\partial C^t_{CDd}(s)}{\partial X_A} \right.\right] \quad \text{and} \quad \delta_\vartheta \mathfrak{R}_A \left[\sim \left| \frac{\partial \vartheta^t_d(s)}{\partial X_A} \right.\right]$$

are those parts of the functional

$$\delta \hat{\mathfrak{R}}_A \left[\sim \left| \frac{\partial C^t_{CDd}(s)}{\partial X_A}, \frac{\partial \vartheta^t_d(s)}{\partial X_A} \right.\right]$$

that are linear in

$$\frac{\partial C^t_{CDd}(s)}{\partial X_A} \quad \text{and} \quad \frac{\partial \vartheta^t_d(s)}{\partial X_A},$$

respectively. Without loss of generality we may sasume

$$\delta_{CD} \hat{\mathfrak{R}}_A \left[\sim \left| \frac{\partial C^t_{CDd}(s)}{\partial X_A} \right.\right] = \delta_{DC} \hat{\mathfrak{R}}_A \left[\sim \left| \frac{\partial C^t_{CDd}(s)}{\partial X_A} \right.\right], \tag{5.9}$$

since only six out of nine functionals

$$\delta_{CD} \hat{\mathfrak{R}}_A \left[\sim \left| \frac{\partial C^t_{CDd}(s)}{\partial X_A} \right.\right]$$

for each A are defined by (5.8). The functionals

$$\delta_{CD} \hat{\mathfrak{R}}_A \left[\sim \left| \frac{\partial C^t_{DEd}(s)}{\partial X_A} \right.\right] \quad \text{and} \quad \delta_\vartheta \hat{\mathfrak{R}}_A \left[\sim \left| \frac{\partial \vartheta^t_d(s)}{\partial X_A} \right.\right]$$

are equal to the Fréchet-differentials (5.7) if $C^t_{CDd}(s)$ or $\vartheta^t_d(s)$ are the only argument functions that depend on X_A. If those functionals are zero for all

$$\frac{\partial C^t_{CDd}(s)}{\partial X_A} \quad \text{and} \quad \frac{\partial \vartheta^t_d(s)}{\partial X_A},$$

respectively, then $\hat{\mathfrak{R}}_A$ does not depend on $C^t_{CDd}(s)$ or $\vartheta^t_d(s)$.

Now according to (4.8), (5.4) and (5.8) we have the condition

$$0 = \sum_A \sum_C \sum_D \delta_{CD} \hat{\mathfrak{R}}_A \left[\sim \left| \frac{\partial C^t_{CDd}(s)}{\partial X_A} \right.\right] + \sum_A \delta_\vartheta \hat{\mathfrak{R}}_A \left[\sim \left| \frac{\partial \vartheta^t_d(s)}{\partial X_A} \right.\right],$$

which has to hold for all

$$\frac{\partial C^t_{CDd}(s)}{\partial X_A}, \frac{\partial \vartheta^t_d(s)}{\partial X_A}.$$

Hence

$$\delta_{CD} \hat{\mathfrak{R}}_A \left[\sim \left| \frac{\partial C^t_{CDd}(s)}{\partial X_A} \right.\right] = 0, \quad \delta_\vartheta \hat{\mathfrak{R}}_A \left[\sim \left| \frac{\partial \vartheta^t_d(s)}{\partial X_A} \right.\right] = 0, \tag{5.10}$$

which means that $\hat{\mathfrak{K}}_A$ does not depend upon $C^t_{CDd}(s)$ and $\vartheta^t_d(s)$, and consequently by (5.4) $\mathfrak{K}_A$ *is independent of* $F^t_{iBd}(s)$ *and* $\vartheta^t_d(s)$.

Summarizing this result and (5.5) and (5.6), we can say that K_A has the following form:

$$K_A = \Omega_{AB}(\vartheta)\frac{\partial\vartheta}{\partial X_B} + \Omega_A(\vartheta), \qquad \Omega_{AB} = -\Omega_{BA}. \tag{5.11}$$

Hence the material components of the entropy flux are (see (2.9))

$$P_A = \frac{Q_A}{\vartheta} + \Omega_{AB}(\vartheta)\frac{\partial\vartheta}{\partial X_B} + \Omega_A(\vartheta), \tag{5.12}$$

and by (2.11)

$$p_j = \frac{q_j}{\vartheta} + \Omega_{AC}(\vartheta)\frac{1}{J}F_{jA}F_{iC}\frac{\partial\vartheta}{\partial x_i} + \Omega_A(\vartheta)\frac{1}{J}F_{jA}. \tag{5.13}$$

The second term on the right-hand side of (5.13) is solenoidal; therefore this term does not contribute to the entropy inequality. Using (2.12), we find that

$$\frac{\partial p_j}{\partial x_j} = \frac{\partial}{\partial x_j}\left(\frac{q_j}{\vartheta}\right) + \frac{1}{J}F_{jA}\frac{\partial\Omega_A(\vartheta)}{\partial\vartheta}\frac{\partial\vartheta}{\partial x_j}. \tag{5.14}$$

Accordingly, if we insert K_A from (5.11) into our reduced inequality (4.10), we obtain

$$\sigma - \frac{1}{J}\left(\frac{1}{\rho\vartheta^2}Q_A - \frac{1}{\rho}\frac{\partial\Omega_A}{\partial\vartheta}\right)\frac{\partial\vartheta}{\partial X_A} \geqq 0, \tag{5.15}$$

or finally with (2.11)

$$\sigma - \left(\frac{1}{\rho\vartheta^2}q_i - \frac{1}{\rho J}F_{iA}\frac{\partial\Omega_A}{\partial\vartheta}\right)\frac{\partial\vartheta}{\partial x_i} \geqq 0. \tag{5.16}$$

6. Material Symmetry

It might seem at first sight that (5.12) or (5.13) considerably modify the usual result. However, the class of materials in which a material vector $\vec{\Omega}$ and a material antisymmetric tensor Ω can exist is rather restricted. We now find what properties those materials must have.

Let us change the reference configuration of the body. The unimodular transformation matrix

$$H_{CE} = \begin{Vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{Vmatrix}, \qquad |\det\{H_{CE}\}| = 1 \tag{6.1}$$

maps the coordinates X_E into the new coordinates X'_C. In the new coordinates (5.11) has the form

$$K'_M = H_{MA}H_{NB}\Omega_{AB}\frac{\partial\vartheta}{\partial X'_N} + H_{MA}\Omega_A. \tag{6.2}$$

All transformations for which

$$\begin{aligned} \Omega'_{MN} &\equiv H_{MA} H_{NB} \Omega_{AB} = \Omega_{MN}, \\ \Omega'_M &\equiv H_{MA} \Omega_A = \Omega_M, \end{aligned} \tag{6.3}$$

are said to belong to the *symmetry group* of the material.

If the symmetry group is formed by all transformations that map (say) one plane into itself, the material has a preferred plane; if only transformations H_{CE} with $\det\{H_{CE}\}=1$ belong to the symmetry group, the material does not possess central symmetry.

We now determine the general transformation for which (6.3) holds and from this infer properties a material must have in order for $\vec{\Omega}$ and Ω to exist.

The components Ω_M and Ω_{MN} in (6.3) are referred to a certain basis $\mathfrak{e}_1$, $\mathfrak{e}_2$, $\mathfrak{e}_3$. In general we may assume for instance that this basis has been chosen so that

$$\Omega_M = (\Omega_1, 0, 0),$$

$$\Omega_{MN} = \begin{pmatrix} 0 & 0 & -\omega_2 \\ 0 & 0 & 0 \\ \omega_2 & 0 & 0 \end{pmatrix}.$$

To give an intuitive idea, we note that this means that $\mathfrak{e}_1$ is in the direction of the polar vector $\vec{\Omega}$ and $\mathfrak{e}_2$ is in the direction of the axial vector $\vec{\omega}$, which can be associated with the antisymmetric tensor Ω. This is not possible, of course, if $\vec{\Omega}$ is parallel to $\vec{\omega}$; we treat this case later.

Then it is an easy problem to show that $(6.3)_1$ requires

$$H_{MA} = \begin{Vmatrix} a & b & c \\ 0 & 1 & 0 \\ d & e & f \end{Vmatrix}, \qquad af - dc = \pm 1, \tag{6.4}$$

while $(6.3)_2$ requires

$$H_{MA} = \begin{Vmatrix} 1 & g & h \\ 0 & i & j \\ 0 & k & l \end{Vmatrix}, \qquad il - kj = +1. \tag{6.5}$$

Thus (6.3) allows for transformations of the form

$$H_{MA} = \begin{Vmatrix} 1 & b & c \\ 0 & 1 & 0 \\ 0 & e & 1 \end{Vmatrix}. \tag{6.6}$$

(6.6) describes all those transformations that

i) leave the direction and length of $\vec{\Omega}$ unaltered,

ii) preserve a plane through $\vec{\Omega}$ and the distance of lines parallel to $\vec{\Omega}$ in this plane,

iii) preserve the distance of the preserved planes mentioned in ii).

Hence we conclude that a material in which a polar vector $\vec{\Omega}$ and an axial vector $\vec{\omega}$ which are not parallel exist, cannot have central symmetry and must possess all those preferred elements mentioned in i) – iii) as preserved elements.

There is, however, a case in which the material may have fewer preferred elements and still allow for the existence of $\vec{\Omega}$ and $\vec{\omega}$. If $\vec{\Omega}$ and $\vec{\omega}$ both point into the $\mathfrak{e}_1$ direction, then (6.3) requires that

$$H_{MA} = \left\| \begin{matrix} 1 & 0 & 0 \\ 0 & c & d \\ 0 & e & \frac{1+d\,e}{c} \end{matrix} \right\|. \tag{6.7}$$

These are the transformations that

i) leave the direction and length of $\vec{\Omega}$ unaffected,

ii) preserve a plane that does not contain $\vec{\Omega}$.

There are still two special cases which must be discussed, namely the cases in which either $\vec{\Omega}$ or Ω is equal to zero. The symmetry group is then formed by the transformations (6.4) and (6.5), respectively.

The transformations (6.4) preserve a plane; hence if $\vec{\Omega}=0$ and $\Omega\neq 0$, the material must have a preferred plane.

Similarly, the transformations (6.5) preserve a length on a line and the direction of this line. Hence if $\vec{\Omega}\neq 0$and $\Omega=0$, the material must possess a preferred direction and a preferred length in this direction.

In a material which does not prefer any of these elements, we have

$$\Omega_M=0 \quad \text{and} \quad \Omega_{MN}=0;$$

hence

$$p_i=\frac{q_i}{\vartheta}.$$

Thus we have proved that in such a material, the entropy flux has the form usually *assumed* as a postulate.

If the material symmetry forbids the existence of a vector $\vec{\Omega}$ but allows for Ω, we still have (see (5.14))

$$\frac{\partial p_i}{\partial x_i}=\frac{\partial}{\partial x_i}\left(\frac{q_i}{\vartheta}\right)$$

i.e. the divergence of the entropy flux has the form usually *assumed.*

7. Heat Conduction in a Homogeneous Thermoelastic Medium

In a thermoelastic material, the quantities ε, η, t_{ij}, q_i, and p_i at a particle X_A at the time t are determined by the instantaneous values of deformation gradient, temperature and temperature gradient at that particle. Hence a thermoelastic material is the special case of a simple material in which the constitutive functionals (3.2) reduce to ordinary functions.

The same argument which we applied in the theory of simple materials with memory can be used to show that the final inequality for a thermoelastic material is

$$-\left(\frac{1}{\rho\,\vartheta^2}\,q_i-\frac{1}{\rho\,J}\,F_{iA}\,\frac{\partial\Omega_A(\vartheta)}{\partial\vartheta}\right)\frac{\partial\vartheta}{\partial x_i}\geqq 0. \tag{7.1}$$

Comparison with (5.16) shows that the internal dissipation σ vanishes in this case.

Let us consider a thermoelastic material in which the vector $\vec{\Omega}$, as discussed in Section 6 does not vanish. According to the considerations in the last section, the symmetry group of such a material contains transformations of the form (6.5). These are transformations with preserve the direction and length of $\vec{\Omega}$. Hence a material which has a non-zero $\vec{\Omega}$ cannot possess central symmetry and must at least prefer one line and a length along this line. In this section we consider a material for which these are the only preferred elements; if there are other preferred elements as well proper regard must be given them in the calculations below. The necessary alterations can easily be made.

Equation (7.1) shows that the heat flux in the absence of a temperature gradient need not vanish; we have

$$q_i(F_{jB},\vartheta,0)=\frac{\vartheta^2}{J}\,F_{iA}\,\frac{\partial\Omega_A(\vartheta)}{\partial\vartheta}. \tag{7.2}$$

But this is not as remarkable as it sounds because the divergence of this heat flux vanishes, and hence this heat flux cannot give rise to a time rate of change of the internal energy.

Let us now consider the case when the temperature gradient does not vanish. The heat flux in a thermoelastic material has the general form

$$q_i=\Phi_A\left(\vartheta,C_{DE},\frac{\partial\vartheta}{\partial X_B}\right)F_{iA}. \tag{7.3}$$

From (7.2) we get

$$\Phi_A(\vartheta,C_{DE},0)=\frac{\vartheta^2}{J}\,\frac{\partial\Omega_A(\vartheta)}{\partial\vartheta}. \tag{7.4}$$

Hence if we introduce

$$\bar{\Phi}_A=\Phi_A-\frac{\vartheta^2}{J}\,\frac{\partial\Omega_A(\vartheta)}{\partial\vartheta}\quad\text{with}\quad\bar{\Phi}_A(\vartheta,C_{DE},0)=0, \tag{7.5}$$

we have

$$q_i=\left(\bar{\Phi}_A+\frac{\vartheta^2}{J}\,\frac{\partial\Omega_A(\vartheta)}{\partial\vartheta}\right)F_{iA}. \tag{7.6}$$

Hence we see that such restrictions as were obtained for the coefficient Φ_A in (7.3) by means of the usual inequality remain correct if we merely replace Φ_A by $\bar{\Phi}_A$. Thus the presence of the term

$$\frac{\vartheta^2}{J}\,\frac{\partial\Omega_A(\vartheta)}{\partial\vartheta}\,F_{iA}$$

may have physical implications because q_i now contains a term linear in F_{iA} with a factor which does not depend on the temperature gradient.

Let us give an example of what may occur with such a term present: we consider an undeformed body ($F_{iA}=\delta_{iA}$), so that

$$q_i=\overline{\Phi}_i+\vartheta^2\frac{\partial\Omega_i(\vartheta)}{\partial\vartheta}, \tag{7.7}$$

where $\overline{\Phi}_i$ now may depend on ϑ and $\partial\vartheta/\partial x_j$. Since we consider a material which has only one preferred direction, the vector

$$\frac{\partial\Omega_i(\vartheta)}{\partial\vartheta}$$

must be parallel to $\Omega_i(\vartheta)$:

$$\vartheta^2\frac{\partial\Omega_i(\vartheta)}{\partial\vartheta}=a(\vartheta)\,\Omega_i(\vartheta). \tag{7.8}$$

In (7.7) we now restrict attention to terms of order lower than the second in $\partial\vartheta/\partial x_j$. Then $\overline{\Phi}_i$ must be a vectorial combination of the two available vectors $\vec{\Omega}$ and grad ϑ, which is linear in grad ϑ and vanishes if grad ϑ vanishes; hence

$$\overline{\Phi}_i=-\kappa\frac{\partial\vartheta}{\partial x_i}+b(\vec{\Omega}\times\operatorname{grad}\vartheta)_i+c\left(\Omega_j\frac{\partial\vartheta}{\partial x_j}\right)\Omega_i.$$

If we insert this into (7.7), we find with (7.8)

$$q_i=-\kappa\frac{\partial\vartheta}{\partial x_i}+b(\vec{\Omega}\times\operatorname{grad}\vartheta)_i+c\left(\Omega_j\frac{\partial\vartheta}{\partial x_j}\right)\Omega_i+a(\vartheta)\,\Omega_i. \tag{7.9}$$

The inequality shows that the tensor $\{\kappa\delta_{ij}-c\,\Omega_i\Omega_j\}$ is nonnegative definite.

From (7.9) and (7.8) we obtain the following heat conduction equation when ϑ is time-independent:

$$(\kappa\,\delta_{ij}-c\,\Omega_i\Omega_j)\frac{\partial^2\vartheta}{\partial x_i\,\partial x_j}=\Lambda\,\Omega_i^l\,\frac{\partial\vartheta}{\partial x_i}, \tag{7.10}$$

where

$$\Lambda=\left(\frac{\partial a}{\partial\vartheta}+\frac{a^2}{\vartheta^2}\right)|\vec{\Omega}|,\qquad \Omega_i^l=\frac{\Omega_i}{|\vec{\Omega}|},$$

for simplicity we have assumed that κ, b and c are constant coefficients.

Hence we can decide whether or not the proposed generalization of the entropy flux is meaningful by measuring the steady state temperature distribution in a thermoelastic material of the kind considered.

Let us treat the case in which ϑ depends on x only and Ω_i^l points in the positive or negative x-direction. Then we get the differential equation

$$\vartheta''=\pm\frac{1}{\lambda}\,\vartheta', \tag{7.11}$$

with the solution

$$\vartheta = A\, e^{\pm \frac{1}{\lambda} x} + B \tag{7.12}$$

and with + or − according to whether $\vec{\Omega}^l$ points in the positive or negative x-direction. In (7.11) and (7.12) we have set

$$\frac{1}{\lambda} = \frac{A}{\kappa - c\,|\vec{\Omega}|^2}. \tag{7.13}$$

If $\vartheta(0) = T_0$ and $\vartheta(L) = T_L$, we get

$$\vartheta_{\pm} = \frac{1}{e^{\frac{L}{2\lambda}} - e^{-\frac{L}{2\lambda}}} \left[(T_L - T_0)\, e^{\pm \frac{1}{\lambda}\left(x - \frac{L}{2}\right)} + T_0\, e^{\pm \frac{L}{2\lambda}} - T_L\, e^{\mp \frac{L}{2\lambda}} \right], \tag{7.14}$$

where again the upper sign corresponds to the case when $\Omega_i^l = (+1, 0, 0)$ and the lower sign to the case when $\Omega_i^l = (-1, 0, 0)$. Fig. 1 shows the different temperature

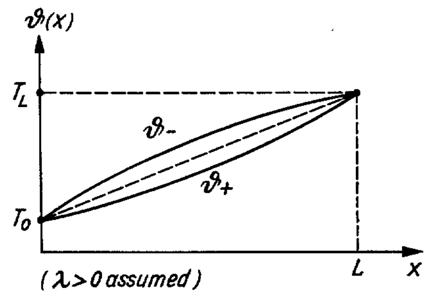

Fig. 1

distributions in these two cases, and the dotted line denotes the classical linear distribution

$$\vartheta^C = \frac{T_L - T_0}{L}\, x + T_0. \tag{7.15}$$

Thus if we measure the temperature at any point $0 < x < L$, we expect different results according as $\vec{\Omega}^l = (+1, 0, 0)$ or $\vec{\Omega}^l = (-1, 0, 0)$.

8. Dipolar Fluids

Constitutive Equations and Restrictions Imposed on Them by the Entropy Inequality

In the preceding sections we dealt with simple materials; it turned out that the generalizations of the entropy flux proposed here lead to a modification of the usual theory only if the material considered possesses rather uncommon symmetries. However, this is not necessarily so for non-simple materials. Let us briefly investigate the effect of this generalization on a non-simple material. We choose a dipolar fluid which is characterized by the following set of constitutive equations:

$$
\begin{aligned}
\eta &= \eta\left(\rho, d_{ij}, A_{ijk}, \vartheta, \frac{\partial \vartheta}{\partial x_i}\right)\\
\varepsilon &= \varepsilon\left(\rho, d_{ij}, A_{ijk}, \vartheta, \frac{\partial \vartheta}{\partial x_i}\right)\\
t_{ij} &= t_{ij}\left(\rho, d_{ij}, A_{ijk}, \vartheta, \frac{\partial \vartheta}{\partial x_i}\right)\\
q_i &= q_i\left(\rho, d_{ij}, A_{ijk}, \vartheta, \frac{\partial \vartheta}{\partial x_i}\right)\\
k_i &= k_i\left(\rho, d_{ij}, A_{ijk}, \vartheta, \frac{\partial \vartheta}{\partial x_i}\right).
\end{aligned}
\tag{8.1}
$$

The variables d_{ij} and A_{ijk} are defined by

$$
d_{ij} = \frac{1}{2}\left(\frac{\partial \dot{x}_i}{\partial x_j} + \frac{\partial \dot{x}_j}{\partial x_i}\right), \qquad A_{ijk} = \frac{\partial^2 \dot{x}_i}{\partial x_j \partial x_k}. \tag{8.2}
$$

Insertion of the constitutive equations (8.1) into the inequality (2.10) yields*

$$
\begin{aligned}
&-\left(\frac{\partial \psi}{\partial \vartheta} + \eta\right)\dot{\vartheta} - \frac{\partial \psi}{\partial\left(\frac{\partial \vartheta}{\partial x_i}\right)}\left(\frac{\partial \vartheta}{\partial x_i}\right)^{\cdot} - \frac{\partial \psi}{\partial d_{ij}}\dot{d}_{ij} - \frac{\partial \psi}{\partial A_{ijk}}\dot{A}_{ijk} +\\
&+\left(\frac{1}{\rho}t_{ij} + \rho\frac{\partial \psi}{\partial \rho}\delta_{ij}\right)d_{ij} - \frac{1}{\rho\vartheta}q_i\frac{\partial \vartheta}{\partial x_i} + \frac{\vartheta}{\rho}\left\{\frac{\partial k_i}{\partial \vartheta}\frac{\partial \vartheta}{\partial x_i} -\right.\\
&\left. - \frac{\partial k_i}{\partial\left(\frac{\partial \vartheta}{\partial x_j}\right)}\frac{\partial^2 \vartheta}{\partial x_i \partial x_j} + \frac{\partial k_i}{\partial d_{jk}}\frac{\partial d_{jk}}{\partial x_i} + \frac{\partial k_i}{\partial A_{jkl}}\frac{\partial A_{jkl}}{\partial x_i} + \frac{\partial k_i}{\partial \rho}\frac{\partial \rho}{\partial x_i}\right\} \geqq 0.
\end{aligned}
\tag{8.3}
$$

The inequality (8.3) must hold for any

$$
\dot{\vartheta}, \quad \left(\frac{\partial \vartheta}{\partial x_i}\right)^{\cdot}, \quad \dot{d}_{(ij)}, \quad \dot{A}_{i(jk)}, \quad \frac{\partial^2 \vartheta}{\partial x_{(i} \partial x_{j)}}, \quad \frac{\partial A_{j(kl}}{\partial x_{i)}}, \quad \frac{\partial \rho}{\partial x_i}, \tag{8.4}
$$

where the brackets ( ) indicate symmetry with respect to the bracketed indices. These quantities can be chosen arbitrarily and independently of other terms in the

* It is true that it is customary in the theory of multipolar materials — of which the dipolar fluid is a special case — to start with equations of balance more general than those used to derive (2.10) (*e.g.* see [*11*], [*12*]). But since we are interested in what modifications of the classical results are caused by the proposed generalization of the entropy flux, we do not consider generalization of the equations of balance as well.

inequality; hence we obtain the following statements:

$$\begin{aligned}
&\left(\frac{\partial\psi}{\partial\vartheta}+\eta\right)=0\\
&\frac{\partial\psi}{\partial\left(\frac{\partial\vartheta}{\partial x_i}\right)}=0\\
&\frac{\partial\psi}{\partial d_{ij}}=0\\
&\frac{\partial\psi}{\partial A_{ijk}}=0,
\end{aligned}\tag{8.5}$$

$$\frac{\partial k_i}{\partial\left(\frac{\partial\vartheta}{\partial x_j}\right)}+\frac{\partial k_j}{\partial\left(\frac{\partial\vartheta}{\partial x_i}\right)}=0,\tag{8.6}$$

$$\frac{\partial k_i}{\partial A_{jkl}}+\frac{\partial k_l}{\partial A_{jik}}+\frac{\partial k_k}{\partial A_{jli}}=0,\tag{8.7}$$

$$\frac{\partial k_i}{\partial\rho}=0.\tag{8.8}$$

Thus the inequality reduces to

$$\left(t_{ij}+\rho^2\frac{\partial\psi}{\partial\rho}\delta_{ij}\right)d_{ij}-\frac{1}{\vartheta}\left(q_i-\vartheta^2\frac{\partial k_i}{\partial\vartheta}\right)\frac{\partial\vartheta}{\partial x_i}+\vartheta\frac{\partial k_i}{\partial d_{jk}}A_{jki}\geqq 0.\tag{8.9}$$

The differential equations (8.6)–(8.8) can be solved. Omitting the rather tedious details, we obtain

$$\begin{aligned}
k_i=&\sum_{\substack{\alpha,\beta,\gamma=1\\ \alpha<\beta<\gamma}}^{3}\Omega^{\alpha\beta\gamma}_{i(ab)(lk)(rs)j}A_{\alpha ab}A_{\beta lk}A_{\gamma rs}\frac{\partial\vartheta}{\partial x_j}+\sum_{\substack{\alpha,\beta=1\\ \alpha<\beta}}^{3}\Omega^{\alpha\beta}_{i(ab)(lk)j}A_{\alpha ab}A_{\beta lk}\frac{\partial\vartheta}{\partial x_j}+\\
&+\sum_{\alpha=1}^{3}\Omega^{\alpha}_{i(ab)j}A_{\alpha(ab)}\frac{\partial\vartheta}{\partial x_j}+\Omega_{ij}\frac{\partial\vartheta}{\partial x_j}+\sum_{\substack{\alpha,\beta,\gamma=1\\ \alpha<\beta<\gamma}}^{3}\Omega^{\alpha\beta\gamma}_{i(ab)(lk)(rs)}A_{\alpha ab}A_{\beta lk}A_{\gamma rs}+\\
&+\sum_{\substack{\alpha,\beta=1\\ \alpha<\beta}}^{3}\Omega^{\alpha\beta}_{i(ab)(lk)}A_{\alpha ab}A_{\beta lk}+\sum_{\alpha=1}^{3}\Omega^{\alpha}_{i(ab)}A_{\alpha ab}+\Omega_i.
\end{aligned}\tag{8.10}$$

The Ω-tensors can depend on d_{kl} and ϑ, and the following symmetry relations must hold:

The Ω-tensors are symmetric with respect to the bracketed indices, *e.g.*

$$\Omega^{\alpha}_{i(ab)}=\Omega^{\alpha}_{i(ba)};\tag{8.11}$$

they are antisymmetric with respect to the index pair i, j, e.g.

$$\Omega_{ij} = -\Omega_{ji}; \tag{8.12}$$

and the sum of three components which result from cyclic permutation of i and two bracketed indices is zero, *e.g.*

$$\Omega^{\alpha}_{i\,(a\,b)} + \Omega^{\alpha}_{a\,(b\,i)} + \Omega^{\alpha}_{b\,(i\,a)} = 0\,. \tag{8.13}$$

Further Restrictions on the Constitutive Equations

From invariance under superposed rigid motion we know that the tensors Ω in (8.10) must be formed from d_{ij}, δ_{ij}, and ε_{ijk}. Hence it follows immediately that

$$\begin{aligned} \Omega_i &= 0 \\ \Omega_{ij} &= 0\,, \end{aligned} \tag{8.14}$$

and after some calculation using (8.13)

$$\begin{aligned} \Omega^{\alpha}_{inp} = {} & \omega_1\,\delta_{i\alpha}\,\delta_{np} - \frac{\omega_1}{2}\,(\delta_{in}\,\delta_{\alpha p} + \delta_{ip}\,\delta_{\alpha n}) + \omega_2\,\delta_{i\alpha}\,d_{np} - \omega_3\,\delta_{np}\,d_{i\alpha} - \\ & - \frac{\omega_2}{2}\,(\delta_{\alpha p}\,d_{in} + \delta_{\alpha n}\,d_{ip}) + \frac{\omega_3}{2}\,(\delta_{in}\,d_{\alpha p} + \delta_{ip}\,d_{\alpha n})\,. \end{aligned} \tag{8.15}$$

In (8.15) the coefficients ω_1, ω_2, and ω_3 may depend on ϑ and d_{nn}. It is also easy to find the form of the other tensors Ω, but we can omit this because from now on we want to restrict ourselves to the case linear in the sense that we neglect all terms in (8.9) which are of order higher than the second in the quantities

$$d_{ij}, \quad A_{ijk}, \quad \frac{\partial\vartheta}{\partial x_i}. \tag{8.16}$$

If we satisfy this requirement, the most general constitutive equations for t_{ij}, q_i and k_i are

$$t_{ij} = -p\,\delta_{ij} + (\zeta - \tfrac{2}{3}\mu)\,d_{nn}\,\delta_{ij} + 2\mu\,d_{ij}\,, \tag{8.17}$$

$$q_i = -\kappa_T\,\frac{\partial\vartheta}{\partial x_i} - \kappa_A\,A_{inn} - \kappa_B\,A_{nni}\,, \tag{8.18}$$

$$\begin{aligned} k_i = {} & \omega_1^0(A_{inn} - A_{nni}) + \omega_1^1\,d_{nn}(A_{inn} - A_{nni}) + \\ & + \omega_2(A_{ikl}\,d_{kl} - A_{llk}\,d_{ik}) + \omega_3(A_{kli}\,d_{kl} - A_{kll}\,d_{ik})\,. \end{aligned} \tag{8.19}$$

In (8.17)−8.19) the coefficients p, ζ, μ, κ_T, κ_A, and κ_B may depend on ϑ and ρ. From (8.8) we know that ω_1^0, ω_1^1, ω_2 and ω_3 depend only on ϑ.

Note that in (8.19) we must allow for those second-order terms in which d_{pq} occurs; otherwise second-order quantities in (8.9) arising from its last term would be neglected.

The Inequality, Final Form of the Constitutive Equations

Inserting (8.17)–(8.19) into the inequality (8.9), we get by a lengthy but straightforward calculation

$$
\begin{aligned}
&-\left(p+\rho^2\frac{\partial\psi}{\partial\rho}d_{nn}\right)+\zeta d_{nn}^2+\\
&+\begin{pmatrix}\frac{\kappa_T}{\vartheta} & -\frac{1}{2}\left(\frac{\kappa_A}{\vartheta}+\vartheta\frac{\partial\omega_1^0}{\partial\vartheta}\right) & \frac{1}{2}\left(\frac{\kappa_B}{\vartheta}-\vartheta\frac{\partial\omega_1^0}{\partial\vartheta}\right)\\ \times & -\vartheta(\frac{1}{5}\omega_2+\frac{7}{10}\omega_2) & \frac{1}{2}\vartheta(\omega_1^1+\frac{1}{5}\omega_2-\frac{4}{5}\omega_3)\\ \times & \times & -\vartheta(\omega_1^1+\frac{4}{5}\omega_2-\frac{7}{10}\omega_1)\end{pmatrix}\begin{pmatrix}\frac{\partial\vartheta}{\partial x_i}\\ A_{ill}\\ A_{lli}\end{pmatrix}\begin{pmatrix}\frac{\partial\vartheta}{\partial x_i}\\ A_{ill}\\ A_{lli}\end{pmatrix}+\\
&+2\mu\langle d_{ij}\rangle\langle d_{ij}\rangle+\vartheta(2\omega_2+\omega_3)\langle A_{jki}\rangle\langle A_{ijk}\rangle+\vartheta\omega_3\langle A_{jki}\rangle\langle A_{jki}\rangle+\\
&+\vartheta\sum_{i,s=1}^{3}A_{sss}(\omega_2A_{iis}+\omega_3A_{sii})-\vartheta\sum_{i,s=1}^{3}A_{ssi}(\omega_2A_{iss}+\omega_3A_{ssi})\geqq 0,
\end{aligned}
\tag{8.20}
$$

where the crosses indicate that the matrix in (8.20) is symmetric. Here we have introduced the traceless parts of d_{ij} and A_{ijk}.

$$
\begin{aligned}
\langle d_{ij}\rangle &= d_{ij}-\tfrac{1}{3}d_{nn}\delta_{ij},\\
\langle A_{ijk}\rangle &= A_{ijk}-\tfrac{1}{10}\{(4A_{ill}-2A_{lli})\delta_{jk}+(-A_{kll}+3A_{llk})\delta_{ij}+\\
&\quad+(-A_{jll}+3A_{llj})\delta_{ik}\}.
\end{aligned}
\tag{8.21}
$$

The inequality (8.20) is satisfied for all d_{nn}, $\partial\vartheta/\partial x_i$, A_{ill}, A_{lli}, $\langle d_{ij}\rangle$, and $\langle A_{ijk}\rangle$ if and only if the following relations hold:

$$
\begin{aligned}
&p=\rho^2\frac{\partial\psi}{\partial\rho}\\
&\zeta\geqq 0\\
&\mu\geqq 0\\
&\kappa_T\geqq 0\\
&\omega_2=0\\
&\omega_3=0\\
&\omega_1^1=0\\
&\kappa_A=\kappa_B=-\vartheta^2\frac{\partial\omega_1^0(\vartheta)}{\partial\vartheta}.
\end{aligned}
\tag{8.22}
$$

Thus the constitutive equations for ψ, t_{ij}, q_i, and

$$k_i=p_i-\frac{q_i}{\vartheta}$$

have been reduced to the following forms:

$$\begin{aligned}
&\psi=\psi(\rho,\vartheta) && \text{with } \frac{\partial\psi}{\partial\vartheta}=-\eta \\
&t_{ij}=-p(\rho,\vartheta)\,\delta_{ij}+[\zeta(\rho,\vartheta)-\tfrac{2}{3}\mu(\rho,\vartheta)]\,d_{nn}\delta_{ij}+ && \\
&\qquad+2\mu(\rho,\vartheta)\,d_{ij} && \text{with } p=\rho^2\frac{\partial\psi}{\partial\rho},\ \zeta\geqq 0,\ \mu\geqq 0 \qquad (8.23)\\
&q_i=-\kappa_T(\rho,\vartheta)\frac{\partial\vartheta}{\partial x_i}-\kappa_A(\vartheta)(A_{ill}-A_{lli}) && \text{with } \kappa_T\geqq 0 \\
&p_i=\frac{q_i}{\vartheta}+\omega_1^0(\vartheta)(A_{ill}-A_{lli}) && \text{with } \frac{\partial\omega_1^0}{\partial\vartheta}=-\frac{1}{\vartheta}\kappa_A .
\end{aligned}$$

Equation $(8.23)_4$ shows that in a dipolar fluid of the type considered, the entropy flux need not be equal to the heat flux divided by temperature. This result presumably indicates that the generalization of the entropy flux leads to alterations in the theory of multipolar materials which could well prove to be of less trivial character than they are for simple materials.

In the present case the main physical implication is that heat flux and temperature gradient are related by $(8.23)_3$ instead of being proportional. Although the heat flux does not vanish if the temperature gradient vanishes, its divergence does.

Simple Viscous Fluid

For the simple viscous fluid, there is no difference between the theory proposed here and the usual one. A simple viscous fluid is characterized by constitutive equations of the form

$$\begin{aligned}
\psi&=\psi\left(\rho, d_{ij}, \vartheta, \frac{\partial\vartheta}{\partial x_i}\right)\\
\eta&=\eta\left(\rho, d_{ij}, \vartheta, \frac{\partial\vartheta}{\partial x_i}\right)\\
t_{ij}&=t_{ij}\left(\rho, d_{ij}, \vartheta, \frac{\partial\vartheta}{\partial x_i}\right)\qquad (8.24)\\
q_i&=q_i\left(\rho, d_{ij}, \vartheta, \frac{\partial\vartheta}{\partial x_i}\right)\\
k_i&=k_i\left(\rho, d_{ij}, \vartheta, \frac{\partial\vartheta}{\partial x_i}\right).
\end{aligned}$$

Following the same procedure as in the case of the dipolar fluid, we obtain

$$k_i=\Omega_{ij}(\vartheta)\frac{\partial\vartheta}{\partial x_j}+\Omega_i(\vartheta),\qquad \Omega_{ij}(\vartheta)=-\Omega_{ji}(\vartheta);$$

hence by the same argument as led to (8.14),

$$k_i=0,\qquad p_i=\frac{q_i}{\vartheta}.$$

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