Galerkin's Procedure for Nonlinear Periodic Systems

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Contents

§0. Introduction

We shall consider a real nonlinear periodic system

$$\frac{dx}{dt}=X(x,t), \tag{0.1}$$

where x and $X(x, t)$ are vectors of the same dimension and $X(x, t)$ is periodic in t of period 2π. In this paper we discuss the question of existence and numerical approximation of periodic solutions of (0.1).

If

$$x_m(t)=a_0+\sqrt{2}\sum_{n=1}^{m}(a_n\cos nt+b_n\sin nt) \tag{0.2}$$

denotes a trigonometric polynomial of order m with undetermined coefficients $a_0, a_1, b_1, \ldots, a_m, b_m$, we may be able to determine these $2m+1$ coefficients so that $x_m(t)$ satisfies identically the reduced system

$$\begin{aligned}\frac{dx_m}{dt}=\frac{1}{2\pi}\int_0^{2\pi}X[x_m(s),s]\,ds+&\\ +\frac{1}{\pi}\sum_{n=1}^{m}\Big\{\cos nt\int_0^{2\pi}X[x_m(s),s]\cos ns\,ds+&\\ +\sin nt\int_0^{2\pi}X[x_m(s),s]\sin ns\,ds\Big\},&\end{aligned} \tag{0.3}$$

or equivalently

$$
(0.4)\quad \left.\begin{aligned}
F_0^{(m)}(\alpha) &= \frac{1}{2\pi}\int_0^{2\pi} X[x_m(s), s]\,ds = 0,\\
F_n^{(m)}(\alpha) &= \frac{1}{\sqrt{2\pi}}\int_0^{2\pi} X[x_m(s), s]\cos ns\,ds - n\,b_n = 0\\
G_n^{(m)}(\alpha) &= \frac{1}{\sqrt{2\pi}}\int_0^{2\pi} X[x_m(s), s]\sin ns\,ds + n\,a_n = 0,
\end{aligned}\right\}\qquad (n=1,2,\ldots,m),
$$

where α denotes the $(2m+1)$-vector $\alpha=(a_0, a_1, b_1, \ldots, a_m, b_m)$ and the $F_0^{(m)}$, $F_n^{(m)}$, $G_n^{(m)}$ are defined by the equalities above.

It is to be expected that, for m sufficiently large, a trigonometric polynomial $x_m(t)$ determined by these relations, may be a reasonable approximation to a periodic solution $\hat{x}(t)$ of (0.1). Actually, this process is exactly the Galerkin method [*3*] applied to the determination of periodic solutions of (0.1). A trigonometric polynomial $x_m(t)$ satisfying (0.3) or (0.4) (if any) will be denoted as a *Galerkin approximation of order m* and the system (0.4) will be called the *determining equation of Galerkin approximations.* In the present paper we discuss the question of the existence and error bounds of periodic solutions of (0.1) in association with the Galerkin approximations $x_m(t)$.

This problem was studied by Cesari [*2a*] under very mild conditions. Precisely, he proved that in association with a given Galerkin approximation $x_m(t)$, even of very low order (one or two), an algorithm is available which may answer the question as to whether there is an exact solution $\hat{x}(t)$ in some neighborhood of $x_m(t)$, and in the affirmative case may give a bound for $\hat{x}(t)-x_m(t)$. Cesari's process can be applied at a very low order m of approximation, and may even lead the way to a qualitative analysis, as shown by Knobloch [*4*]. Nevertheless, the process, when applied to a practical problem, actually requires a certain amount of discussion which may not be easy.

On the other hand, in a numerically given problem we may be able to obtain Galerkin's approximations of high order by an electronic computer.

Assuming that Galerkin's approximations of somewhat high order can be obtained, assuming some more conditions on the given system, and restricting somewhat the class of periodic solutions we deal with, we present here a theory which is more convenient to practical applications. Namely, we assume that $X(x, t)$ and its first order partial derivatives with respect to x are all continuously differentiable with respect to both x and t. We limit ourselves to those periodic solutions $\hat{x}(t)$ for which the multipliers of the equation of first variation are all different from one. Any such periodic solution will be called *isolated* in the present paper since there is no other periodic solution in some neighborhood of it. Both the smoothness condition on $X(x, t)$ and the restriction that the periodic solution $\hat{x}(t)$ be isolated are not severe limitations from the standpoint of practical applications. Under these hypotheses, we can summarize the conclusions of the present paper as follows:

1. The existence of an isolated periodic solution $\hat{x}(t)$ *of* (0.1) *lying in the interior of the region of definition of* $X(x, t)$ *always implies the existence of Galerkin approximations* $x_m(t)$ *of all orders m sufficiently high, as well as the boundedness of certain operators connected with the Jacobian matrix of* $X(x, t)$ *with respect to x.*

2. The existence of a Galerkin appromaximation $x_m(t)$ of a sufficiently high order m always implies the existence of an exact solution provided a boundedness condition as mentioned in 1. is satisfied.

Actually, from the first result, we prove the uniform convergence of the Galerkin approximations $x_m(t)$ toward $\hat{x}(t)$ as $m\to\infty$. From the second result we obtain simple criteria for the existence of the exact periodic solution $\hat{x}(t)$.

On the basis of the present results, practical numerical methods have been developed and applied to numerical problems. The methods and the numerical examples will be discussed elsewhere jointly with A. REITER.

If we apply our method to weakly nonlinear systems, then we obtain the first approximation of the averaging method [1], and moreover we can give an explicit bound for the magnitude of the parameter under which the averaging method is valid for the determination of periodic solutions. This will be shown in a later paper [5].

Lastly, we wish to acknowledge the comments made by Professor LAMBERTO CESARI.

§1. Preliminaries

In the present paper, we use *Euclidean norms* for vectors and matrices and denote them by the symbol $\|\ldots\|$. Let $f(t)$ be a continuous periodic vector-function of period 2π. In the present paper, for such a function, we use two kinds of norms $\|f\|_q$ and $\|f\|_n$, which are defined as follows:

$$\|f\|_q=\left[\frac{1}{2\pi}\int_0^{2\pi}\|f(t)\|^2\,dt\right]^{\frac{1}{2}},$$

$$\|f\|_n=\max_t\|f(t)\|.$$

The approach of the present paper is based on the following three propositions.

Proposition 1. *Let*

$$\frac{dx}{dt}=A(t)\,x+\varphi(t) \tag{1.1}$$

be a given linear periodic system where $A(t)$ is a continuous periodic matrix of period 2π and $\varphi(t)$ is a continuous periodic vector of the same period. If the multipliers of the corresponding homogeneous system

$$\frac{dy}{dt}=A(t)\,y \tag{1.2}$$

are all different from one, then (1.1) *has one and only one periodic solution of period 2π, which is given by*

$$x(t)=\int_0^{2\pi}H(t,s)\,\varphi(s)\,ds \tag{1.3}$$

where $H(t,s)$ is the piece-wise continuous periodic matrix

$$H(t,s)=\begin{cases}\Phi(t)\,[E-\Phi(2\pi)]^{-1}\,\Phi^{-1}(s) & \text{for}\quad 0\leqq s\leqq t\leqq 2\pi,\\ \Phi(t)\,[E-\Phi(2\pi)]^{-1}\,\Phi(2\pi)\,\Phi^{-1}(s) & \text{for}\quad 0\leqq t<s\leqq 2\pi,\end{cases} \tag{1.4}$$

$$H(t,s)=H(t+2m\pi,s+2n\pi)\qquad (m,n\ \text{integers}), \tag{1.5}$$

E is the unit matrix and $\Phi(t)$ the fundamental matrix of (1.2) *with $\Phi(0)=E$.*

Proof. Any solution of (1.1) is expressed as follows:

$$x(t)=\Phi(t)\,c+\Phi(t)\int_0^t \Phi^{-1}(s)\,\varphi(s)\,ds, \tag{1.6}$$

where c is a constant vector. The solution $x(t)$ given by (1.6) is periodic in t of period 2π if and only if

$$[E-\Phi(2\pi)]\,c=\Phi(2\pi)\int_0^{2\pi} \Phi^{-1}(s)\,\varphi(s)\,ds. \tag{1.7}$$

Since $\det[E-\Phi(2\pi)]\neq 0$ by the assumption, the equality (1.7) implies

$$c=[E-\Phi(2\pi)]^{-1}\,\Phi(2\pi)\int_0^{2\pi} \Phi^{-1}(s)\,\varphi(s)\,ds. \tag{1.8}$$

The desired formula (1.3) is now obtained by substituting (1.8) into (1.6). Q.E.D.

The formula (1.3) defines a linear mapping H in the space of continuous periodic functions of period 2π. Consequently, the norms of this linear mapping are defined corresponding to the norms of continuous periodic functions. We shall denote them by $\|H\|_q$ and $\|H\|_n$. By means of SCHWARZ' inequality, it is readily seen that

$$\|H\|_q \leqq \left[\int_0^{2\pi}\int_0^{2\pi} \sum_{k,l} H_{kl}^2(t,s)\,ds\,dt\right]^{\frac{1}{2}}, \tag{1.9}$$

$$\|H\|_n \leqq \left[2\pi\cdot\max_t \int_0^{2\pi} \sum_{k,l} H_{kl}^2(t,s)\,ds\right]^{\frac{1}{2}}, \tag{1.10}$$

where $H_{kl}(t,s)$ are the elements of the matrix $H(t,s)$.

In what follows, we shall call the linear mapping H defined by (1.3) *the H-mapping corresponding to a given matrix $A(t)$.*

Proposition 2. *Let*

$$F(\alpha)=0 \tag{1.11}$$

be a given real system of equations where α and $F(\alpha)$ are vectors of the same dimension and $F(\alpha)$ is a continuously differentiable function of α defined in some region Ω of α. Assume that (1.11) *has an approximate solution $\alpha=\hat{\alpha}$ for which the determinant of the Jacobian matrix $J(\alpha)$ of $F(\alpha)$ with respect to α does not vanish at $\alpha=\hat{\alpha}$ and there is a positive constant δ and a non-negative constant $\kappa<1$ such that*

$$\begin{aligned} &(i)\quad \Omega_\delta=\{\alpha \mid \|\alpha-\hat{\alpha}\|\leqq\delta\}\subset\Omega,\\ &(ii)\quad \|J(\alpha)-J(\hat{\alpha})\|\leqq\kappa/M' \quad \text{for any} \quad \alpha\in\Omega_\delta,\\ &(iii)\quad \frac{M'r}{1-\kappa}\leqq\delta, \end{aligned} \tag{1.12}$$

where r and M' (>0) are numbers such that

$$\|F(\hat{\alpha})\|\leqq r \quad\text{and}\quad \|J^{-1}(\hat{\alpha})\|\leqq M'. \tag{1.13}$$

Then the system (1.11) *has one and only one solution $\alpha=\bar{\alpha}$ in Ω_δ and*

$$\|\bar{\alpha}-\hat{\alpha}\|\leqq\frac{M'r}{1-\kappa}. \tag{1.14}$$

Proof. Put $A=J^{-1}(\hat{\alpha})$, and let us consider NEWTON's iterative process:

(1.15) $$\alpha_{n+1}=\alpha_n-AF(\alpha_n) \qquad (n=0,1,2,\ldots),$$

where $\alpha_0=\hat{\alpha}$.

First, we shall prove that this iterative process can be continued indefinitely and that

(1.16) $$\|\alpha_{n+1}-\alpha_n\|\leqq\kappa^n\|\alpha_1-\alpha_0\|,$$

(1.17) $$\alpha_{n+1}\in\Omega_\delta \qquad (n=0,1,2,\ldots).$$

For $n=0$, (1.16) is evident. For α_1, we have successively

(1.18) $$\begin{aligned}\|\alpha_1-\alpha_0\|=\|AF(\alpha_0)\|&\leqq M'r\\ &\leqq(1-\kappa)\delta<\delta,\end{aligned}$$

and consequently $\alpha_1\in\Omega_\delta$. This proves (1.17) for $n=0$.

Let us assume that (1.16) and (1.17) hold up to $n-1$. Then from (1.15) we have

$$\begin{aligned}\alpha_{n+1}-\alpha_n&=(\alpha_n-\alpha_{n-1})-A[F(\alpha_n)-F(\alpha_{n-1})]\\ &=A\int_0^1\{J(\alpha_0)-J[\alpha_{n-1}+\vartheta(\alpha_n-\alpha_{n-1})]\}\cdot(\alpha_n-\alpha_{n-1})\,d\vartheta.\end{aligned}$$

Here $\alpha_{n-1}+\vartheta(\alpha_n-\alpha_{n-1})\in\Omega_\delta$ $(0\leqq\vartheta\leqq1)$ since α_n, $\alpha_{n-1}\in\Omega_\delta$ by the assumption. Then, by (ii) of (1.12), we have

(1.19) $$\|\alpha_{n+1}-\alpha_n\|\leqq M'\cdot\frac{\kappa}{M'}\cdot\|\alpha_n-\alpha_{n-1}\|=\kappa\|\alpha_n-\alpha_{n-1}\|,$$

which proves (1.16) for n because

$$\|\alpha_n-\alpha_{n-1}\|\leqq\kappa^{n-1}\|\alpha_1-\alpha_0\|$$

by the assumption. Since

$$\|\alpha_{n+1}-\alpha_0\|\leqq\|\alpha_{n+1}-\alpha_n\|+\|\alpha_n-\alpha_{n-1}\|+\cdots+\|\alpha_1-\alpha_0\|,$$

it follows from (1.16) and (1.18) that

(1.20) $$\begin{aligned}\|\alpha_{n+1}-\alpha_0\|&\leqq(\kappa^n+\kappa^{n-1}+\cdots+\kappa+1)\|\alpha_1-\alpha_0\|\\ &\leqq\frac{M'r}{1-\kappa}\leqq\delta,\end{aligned}$$

which proves (1.17) for n.

By (1.16) and (1.17) it is evident that the iterative process (1.15) can be continued indefinitely in $\Omega_\delta\subset\Omega$.

Then, by the iterative process (1.15), we have an infinite sequence $\{\alpha_n\}$ in Ω_δ, which is convergent by (1.16) because $|\kappa|<1$. Let

$$\bar{\alpha}=\lim_{n\to\infty}\alpha_n.$$

This limit $\bar{\alpha}$ is evidently a solution of equation (1.11).

Inequality (1.14) readily follows from (1.20).

Lastly, let us prove the uniqueness of the solution. Let $\bar{\alpha}'$ be another solution of (1.11) lying in Ω_δ. Since

$$\bar{\alpha} = \bar{\alpha} - AF(\bar{\alpha}),$$
$$\bar{\alpha}' = \bar{\alpha}' - AF(\bar{\alpha}'),$$

analogously to (1.19) we have

$$\|\bar{\alpha} - \bar{\alpha}'\| \leqq \kappa \|\bar{\alpha} - \bar{\alpha}'\|,$$

which implies

(1.21) $$\|\bar{\alpha} - \bar{\alpha}'\| = 0$$

because $0 \leqq \kappa < 1$. Equality (1.21) proves the uniqueness of the solution. Q.E.D.

Proposition 3. *Let*

(1.22) $$\frac{dx}{dt} = X(x, t)$$

be a given real system of differential equations, where x and $X(x, t)$ are vectors of the same dimension, and $X(x, t)$ is periodic in t of period 2π and is continuously differentiable with respect to x in the region $D \times L$, where D is a given region of the x-space and L is the real line.

Assume that (1.22) *has a periodic approximate solution $x = \bar{x}(t)$ lying in D and there are a continuous periodic matrix $A(t)$, a positive constant δ, and a non-negative constant $\kappa < 1$ such that*

(i) *the multipliers of the linear homogeneous system*

(1.23) $$\frac{dy}{dt} = A(t)y$$

are all different from one, and

(ii) $D_\delta = \{x \mid \|x - \bar{x}(t)\| \leqq \delta \text{ for some } t \in L\} \subset D$,

(1.24) (iii) $\|\Psi(x, t) - A(t)\| \leqq k/M_1$ *for all x such that* $\|x - \bar{x}(t)\| \leqq \delta$ *and* $t \in L$,

(iv) $\dfrac{M_1 r}{1 - \kappa} \leqq \delta$.

Here

$\Psi(x, t)$ *is the Jacobian matrix of $X(x, t)$ with respect to x;*

M_1 *is a positive constant such that $\|H\|_n \leqq M_1$ where H is the H-mapping corresponding to $A(t)$;*

r *is a non-negative constant such that*

(1.25) $$\left\| \frac{d\bar{x}(t)}{dt} - X[\bar{x}(t), t] \right\| \leqq r.$$

Then the given system (1.22) *has one and only one periodic solution $x = \hat{x}(t)$ in D_δ, and this is an isolated periodic solution. Furthermore, for $x = \hat{x}(t)$, we have*

(1.26) $$\|\hat{x}(t) - \bar{x}(t)\| \leqq \frac{M_1 r}{1 - \kappa}.$$

Proof. Let us put

$$\frac{d\bar{x}(t)}{dt}=X[\bar{x}(t),t]+\eta(t). \tag{1.27}$$

Then this can be rewritten as follows:

$$\frac{d\bar{x}(t)}{dt}=A(t)\bar{x}(t)+\{X[\bar{x}(t),t]-A(t)\bar{x}(t)+\eta(t)\}.$$

Since $\bar{x}(t)$ is periodic of period 2π, by Proposition 1, $\bar{x}(t)$ can be expressed as follows:

$$\bar{x}(t)=\int_0^{2\pi} H(t,s)\{X[\bar{x}(s),s]-A(s)\bar{x}(s)+\eta(s)\}\,ds, \tag{1.28}$$

where $H(t,s)$ is the piecewise continuous periodic matrix defined by (1.4) and (1.5) in correspondence to $A(t)$.

As in Proposition 2, let us consider the iterative process

$$x_{n+1}(t)=\int_0^{2\pi} H(t,s)\{X[x_n(s),s]-A(s)x_n(s)\}\,ds \qquad (n=0,1,2,\dots), \tag{1.29}$$

where $x_0(t)=\bar{x}(t)$.

First we shall prove that this iterative process can be continued indefinitely and that

$$\|x_{n+1}-x_n\|_n\leqq\kappa^n\|x_1-x_0\|_n, \tag{1.30}$$

$$\|x_{n+1}-x_0\|_n\leqq\delta \qquad (n=0,1,2,\dots). \tag{1.31}$$

For $n=0$, (1.30) is evident. Since

$$x_1(t)-x_0(t)=-\int_0^{2\pi} H(t,s)\eta(s)\,ds, \tag{1.32}$$

we have

$$\|x_1-x_0\|_n\leqq M_1 r\leqq(1-\kappa)\delta<\delta. \tag{1.33}$$

This proves (1.31) for $n=0$.

Let us assume that (1.30) and (1.31) hold up to $n-1$. Then by (1.29) we have

$$x_{n+1}(t)-x_n(t)=\int_0^{2\pi} H(t,s)\{X[x_n(s),s]-X[x_{n-1}(s),s]-A(s)[x_n(s)-x_{n-1}(s)]\}\,ds,$$

and hence,

$$\|x_{n+1}-x_n\|_n\leqq M_1\cdot\|X[x_n(s),s]-X[x_{n-1}(s),s]-A(s)[x_n(s)-x_{n-1}(s)]\|_n. \tag{1.34}$$

However,

$$X[x_n(s),s]-X[x_{n-1}(s),s]-A(s)[x_n(s)-x_{n-1}(s)]$$
$$=\int_0^1\{\Psi[x_{n-1}(s)+\vartheta(x_n(s)-x_{n-1}(s)),s]-A(s)\}\{x_n(s)-x_{n-1}(s)\}\,d\vartheta,$$

and

$$\|[x_{n-1}(s)+\vartheta(x_n(s)-x_{n-1}(s))]-x_0(s)\|\leqq\delta \qquad (0\leqq\vartheta\leqq1)$$

since $\|x_{n-1}-x_0\|_n$, $\|x_n-x_0\|_n \leqq \delta$ by the assumption. Then, by (iii) of (1.24),

$$\|X[x_n(s), s]-X[x_{n-1}(s), s]-A(s)[x_n(s)-x_{n-1}(s)]\| \leqq \frac{\kappa}{M_1}\cdot\|x_n-x_{n-1}\|_n.$$

Therefore, from (1.34) we get

$$\|x_{n+1}-x_n\|_n \leqq \kappa\|x_n-x_{n-1}\|_n, \tag{1.35}$$

which proves (1.30) for n because

$$\|x_n-x_{n-1}\|_n \leqq \kappa^{n-1}\|x_1-x_0\|_n$$

by the assumption. Now since

$$\|x_{n+1}-x_0\|_n \leqq \|x_{n+1}-x_n\|_n+\|x_n-x_{n-1}\|_n+\cdots+\|x_1-x_0\|_n,$$

it follows from (1.30) and (1.33) that

$$\begin{aligned}\|x_{n+1}-x_0\|_n &\leqq (\kappa^n+\kappa^{n-1}+\cdots+\kappa+1)\|x_1-x_0\|_n \\ &\leqq \frac{M_1 r}{1-\kappa} \leqq \delta,\end{aligned} \tag{1.36}$$

which proves (1.31) for n.

By (1.30) and (1.31) it is evident that the iterative process (1.29) can be continued indefinitely in $D_\delta \subset D$.

Then, by the iterative process (1.29), we have a sequence $\{x_n(t)\}$ of continuous periodic functions lying in D_δ, and this sequence is uniformly convergent by (1.30) since $|\kappa|<1$. Therefore there exists a continuous periodic limit function

$$\hat{x}(t)=\lim_{n\to\infty} x_n(t)$$

lying in D_δ. For this limit function $\hat{x}(t)$, (1.29) yields

$$\begin{aligned}\hat{x}(t)-\int_0^{2\pi} H(t,s)\{X[\hat{x}(s), s]-A(s)\hat{x}(s)\}\,ds \\ =\{\hat{x}(t)-x_{n+1}(t)\}+\Big\{\int_0^{2\pi} H(t,s)[X(x_n(s), s)-A(s)x_n(s)]\,ds- \\ -\int_0^{2\pi} H(t,s)[X(\hat{x}(s), s)-A(s)\hat{x}(s)]\,ds\Big\}.\end{aligned}$$

Then analogously to (1.35) we have

$$\left\|\hat{x}(t)-\int_0^{2\pi} H(t,s)\{X[\hat{x}(s), s]-A(s)\hat{x}(s)\}\,ds\right\|_n \leqq \|\hat{x}-x_{n+1}\|_n+\kappa\|x_n-\hat{x}\|_n.$$

Letting $n\to\infty$, we see that

$$\hat{x}(t)=\int_0^{2\pi} H(t,s)\{X[\hat{x}(s), s]-A(s)\hat{x}(s)\}\,ds, \tag{1.37}$$

which implies that

$$\frac{d\hat{x}(t)}{dt}=A(t)\hat{x}(t)+\{X[\hat{x}(t), t]-A(t)\hat{x}(t)\},$$

that is,

$$\frac{d\hat{x}(t)}{dt} = X[\hat{x}(t), t].$$

This says that $x = \hat{x}(t)$ is a periodic solution of the given system (1.22).

Inequality (1.26) follows readily from (1.36).

Next, let us prove the uniqueness of the periodic solution. Let $\hat{x}'(t)$ be another periodic solution of (1.22) lying in D_δ. Then

$$\frac{d\hat{x}'(t)}{dt} = X[\hat{x}'(t), t] = A(t)\hat{x}'(t) + \{X[\hat{x}'(t), t] - A(t)\hat{x}'(t)\};$$

therefore $\hat{x}'(t)$ can be expressed as follows:

$$\hat{x}'(t) = \int_0^{2\pi} H(t, s)\{X[\hat{x}'(s), s] - A(s)\hat{x}'(s)\}\, ds.$$

Comparing this with (1.37), analogously to (1.35) we have

$$\|\hat{x} - \hat{x}'\|_n \leqq \kappa \|\hat{x} - \hat{x}'\|_n,$$

which implies that

$$\|\hat{x} - \hat{x}'\|_n = 0$$

because $0 \leqq \kappa < 1$. The above equality proves the uniqueness of the periodic solution.

Lastly, let us show that $x = \hat{x}(t)$ is an isolated periodic solution. Put

$$\hat{A}(t) = \Psi[\hat{x}(t), t],$$

and let us consider the linear homogeneous system

$$\text{(1.38)} \qquad \frac{dy}{dt} = \hat{A}(t)\, y.$$

This is the equation of first variation of (1.22) with respect to the periodic solution $x = \hat{x}(t)$. By (iii) of (1.24),

$$\text{(1.39)} \qquad \|\hat{A}(t) - A(t)\| \leqq \kappa / M_1.$$

Now any periodic solution $y = y(t)$ of (1.38) satisifes

$$\frac{dy}{dt} = A(t)\, y + [\hat{A}(t) - A(t)]\, y;$$

consequently, $y(t)$ is expressed as follows:

$$y(t) = \int_0^{2\pi} H(t, s)[\hat{A}(s) - A(s)]\, y(s)\, ds.$$

Then by (1.39) we see that

$$\|y\|_n \leqq M_1 \cdot \frac{\kappa}{M_1} \cdot \|y\|_n = \kappa \|y\|_n,$$

which implies that $\|y\|_n=0$ because $0 \leqq \kappa < 1$. This says there is no non-trivial periodic solution of (1.38). This implies that the multipliers of (1.38) are all different from one, namely $x=\hat{x}(t)$ is an isolated periodic solution. Q.E.D.

§2. The Existence of a Galerkin Approximation

2.1. A truncated trigonometric polynomial of a periodic solution. Let $f(t)$ be a continuous periodic vector-function of period 2π, and let its Fourier series be

$$f(t) \sim c_0 + \sqrt{2} \sum_{n=1}^{\infty} (c_n \cos n t + d_n \sin n t),$$

where $c_0, c_1, d_1, c_2, d_2, \ldots$ are vectors. Then the trigonometric polynomial

$$f_m(t) = c_0 + \sqrt{2} \sum_{n=1}^{m} (c_n \cos n t + d_n \sin n t)$$

is a truncated trigonometric polynomial of the given periodic function $f(t)$ (strictly speaking, a trigonometric polynomial obtained by truncating the Fourier series of the given periodic function). In the sequel we shall denote such a truncation of a periodic function (strictly speaking a truncation of the Fourier series of a periodic function) by P_m and write a truncated polynomial $f_m(t)$ of a periodic function $f(t)$ as follows:

$$f_m(t) = P_m f(t).$$

If we put $\gamma = (c_0, c_1, d_1, \ldots, c_m, d_m)$, then

$$\begin{aligned} \|f_m\|_q^2 &= \frac{1}{2\pi} \int_0^{2\pi} \|f_m(t)\|^2 \, dt \\ &= \|c_0\|^2 + \sum_{n=1}^{m} (\|c_n\|^2 + \|d_n\|^2) \\ &= \|\gamma\|^2; \end{aligned}$$

consequently

$$\|f_m\|_q = \|\gamma\|.$$

This property will often be used in the sequel.

We owe to Cesari [*2a*] the following lemma concerning continuously differentiable periodic functions.

Lemma 2.1. *Let $f(t)$ be a continuously differentiable periodic vector-function of period 2π. Then*

$$\|f - P_m f\|_n \leqq \sigma(m) \, \|\dot{f}\|_q \leqq \sigma(m) \, \|\dot{f}\|_n, \tag{2.1}$$

$$\|f - P_m f\|_q \leqq \sigma_1(m) \, \|\dot{f}\|_q, \tag{2.2}$$

where $\cdot = d/dt$ and

$$\begin{aligned} \sigma(m) &= \sqrt{2} \left[\frac{1}{(m+1)^2} + \frac{1}{(m+2)^2} + \cdots \right]^{\frac{1}{2}}, \\ \sigma_1(m) &= \frac{1}{m+1}. \end{aligned} \tag{2.3}$$

Also

$$\frac{\sqrt{2}}{m+1}<\sigma(m)<\frac{\sqrt{2}}{\sqrt{m}}. \tag{2.4}$$

Proof. Since $\dot{f}(t)$ is continuous and periodic and hence bounded, the Fourier series of $f(t)$ is uniformly convergent. Let the Fourier series of $f(t)$ and $\dot{f}(t)$ be respectively

$$f(t)=c_0+\sqrt{2}\sum_{n=1}^{\infty}(c_n\cos nt+d_n\sin nt)$$

and

$$\dot{f}(t)\sim c_0'+\sqrt{2}\sum_{n=1}^{\infty}(c_n'\cos nt+d_n'\sin nt),$$

with

$$c_0'=0,\quad c_n=-\frac{1}{n}d_n',\quad d_n=\frac{1}{n}c_n'\qquad(n=1,2,\ldots).$$

Hence,

$$f(t)-P_mf(t)=\sqrt{2}\sum_{n=m+1}^{\infty}\frac{1}{n}(-d_n'\cos nt+c_n'\sin nt). \tag{2.5}$$

By SCHWARZ' inequality it follows that

$$\|f(t)-P_mf(t)\|^2\leqq\sigma^2(m)\sum_{n=m+1}^{\infty}(\|c_n'\|^2+\|d_n'\|^2), \tag{2.6}$$

and by BESSEL's inequality,

$$\sum_{n=m+1}^{\infty}(\|c_n'\|^2+\|d_n'\|^2)\leqq\|\dot{f}\|_q^2.$$

Thus (2.1) follows from (2.6). Inequality (2.4) readily follows from the inequality

$$\frac{2}{(m+1)^2}<\sigma^2(m)<2\int_m^{\infty}\frac{du}{u^2}=\frac{2}{m}.$$

If we apply PARSEVAL's equality to (2.5), then we have

$$\begin{aligned}\|f-P_mf\|_q^2&=\sum_{n=m+1}^{\infty}\frac{1}{n^2}(\|c_n'\|^2+\|d_n'\|^2)\\&\leqq\frac{1}{(m+1)^2}\sum_{n=m+1}^{\infty}(\|c_n'\|^2+\|d_n'\|^2),\end{aligned}$$

from which (2.2) readily follows. Q.E.D.

If we apply Lemma 2.1 to a periodic solution of a differential equation, then we easily get the following lemma concerning its truncated trigonometric polynomials.

Lemma 2.2. *Let*

$$\frac{dx}{dt}=X(x,t) \tag{2.7}$$

be a given real nonlinear periodic system, where x and $X(x, t)$ are vectors of the same dimension, and $X(x, t)$ is periodic in t of period 2π. We assume that $X(x, t)$

and its first partial derivatives with respect to x are continuously differentiable with respect to x and t in the region $D \times L$, where D is a closed bounded region of the x-space and L is the real line.

Let K, K_1 and K_2 be non-negative constants such that

$$(2.8)\qquad \begin{aligned} &K=\max_{D\times L}\|X(x,t)\|, \qquad K_1=\max_{D\times L}\|\Psi(x,t)\|,\\ &K_2=\max_{D\times L}\left\|\frac{\partial X(x,t)}{\partial t}\right\|, \end{aligned}$$

where $\Psi(x,t)$ is the Jacobian matrix of $X(x,t)$ with respect to x.

If there exists a periodic solution $x=\hat{x}(t)$ of (2.7) lying in D, then

$$(2.9)\qquad \begin{aligned} &\text{(i)}\quad \|\hat{x}-\hat{x}_m\|_n \leqq K\,\sigma(m),\\ &\text{(ii)}\quad \|\hat{x}-\hat{x}_m\|_q \leqq K\,\sigma_1(m),\\ &\text{(iii)}\quad \|\dot{\hat{x}}-\dot{\hat{x}}_m\|_n \leqq (K K_1+K_2)\,\sigma(m), \end{aligned}$$

where $\hat{x}_m(t)=P_m\,\hat{x}(t)$.

This lemma yields the following corollary.

Corollary. *If $x=\hat{x}(t)$ is an isolated periodic solution of* (2.7) *lying inside D, then there exists a positive integer m_0 such that, for any $m \geqq m_0$,*

(i) $\hat{x}_m(t) \in D$;

(ii) *the multipliers of the linear homogeneous system*

$$(2.10)\qquad \frac{dy}{dt}=\Psi[\hat{x}_m(t),t]\,y$$

are all different from one and the H-mappings H_m corresponding to $\Psi[\hat{x}_m(t),t]$ are equibounded, that is, there exists a positive constant M such that

$$(2.11)\qquad \|H_m\|_q,\ \|H_m\|_n \leqq M;$$

(iii) $\dfrac{d}{dt}\Psi[\hat{x}_m(t),t]$ *is equibounded, that is, there exists a non-negative constant K_3 such that*

$$(2.12)\qquad \left\|\frac{d}{dt}\Psi[\hat{x}_m(t),t]\right\| \leqq K_3.$$

Proof. The conclusions (i) and (ii) are evident from (i) of (2.9), since (i) of (2.9) implies

$$\|\hat{x}_m-\hat{x}\|_n \to 0 \quad \text{as} \quad m\to\infty$$

and $\Psi(x,t)$ is, by the assumption, uniformly continuous with respect to x in $D\times L$.

Now for $\dfrac{d}{dt}\Psi[\hat{x}_m(t),t]$, we have

$$\frac{d}{dt}\Psi[\hat{x}_m(t),t]=\sum_k \frac{\partial\Psi}{\partial x_k}[\hat{x}_m(t),t]\cdot\frac{d\hat{x}_{mk}(t)}{dt}+\frac{\partial\Psi}{\partial t}[\hat{x}_m(t),t],$$

where x_k and $\hat{x}_{mk}(t)$ are respectively the components of the vector x and $\hat{x}_m(t)$. However

$$\left\|\frac{d\hat{x}_m(t)}{dt}\right\| \leqq K+(K K_1+K_2)\,\sigma(m)$$

by (iii) of (2.9), and all the elements of the matrices

$$\frac{\partial \Psi}{\partial x_k}[\hat{x}_m(t), t], \qquad \frac{\partial \Psi}{\partial t}[\hat{x}_m(t), t]$$

are equibounded so long as $\hat{x}_m(t) \in D$. Therefore, all the elements of the matrix

$$\frac{d}{dt}\Psi[\hat{x}_m(t), t]$$

are equibounded provided m is sufficiently large. This proves (iii). Q.E.D.

2.2 The Jacobian matrix of the determining equation of Galerkin approximations. Let $J_m(\alpha)$ be the Jacobian matrix of the left member of the determining equation (0.4) of Galerkin approximations. The elements of $J_m(\alpha)$ are of the following forms:

$$\begin{aligned}
&\frac{1}{2\pi}\int_0^{2\pi}\Psi[x_m(s), s]\,ds, \\
&\qquad\frac{1}{\sqrt{2}\pi}\int_0^{2\pi}\Psi[x_m(s), s]\cos ps\,ds, \\
&\qquad\qquad\frac{1}{\sqrt{2}\pi}\int_0^{2\pi}\Psi[x_m(s), s]\sin ps\,ds; \\
&\frac{1}{\sqrt{2}\pi}\int_0^{2\pi}\Psi[x_m(s), s]\cos ns\,ds, \\
&\qquad\frac{1}{\pi}\int_0^{2\pi}\Psi[x_m(s), s]\cos ns\cos ps\,ds, \\
&\qquad\qquad\frac{1}{\pi}\int_0^{2\pi}\Psi[x_m(s), s]\cos ns\sin ps\,ds - n\,\delta_{np}; \\
&\frac{1}{\sqrt{2}\pi}\int_0^{2\pi}\Psi[x_m(s), s]\sin ns\,ds, \\
&\qquad\frac{1}{\pi}\int_0^{2\pi}\Psi[x_m(s), s]\sin ns\cos ps\,ds + n\,\delta_{np}, \\
&\qquad\qquad\frac{1}{\pi}\int_0^{2\pi}\Psi[x_m(s), s]\sin ns\sin ps\,ds,
\end{aligned} \tag{2.13}$$

where $\alpha=(a_0, a_1, b_1, \ldots, a_m, b_m)$, $p, n=1, 2, \ldots, m$, and

$$x_m(t)=a_0+\sqrt{2}\sum_{n=1}^{m}(a_n\cos nt+b_n\sin nt). \tag{2.14}$$

To find the basic properties of $J_m(\alpha)$, let us consider the auxiliary linear system

$$J_m(\alpha)\,\xi+\gamma=0, \tag{2.15}$$

where $\xi=(u_0, u_1, v_1, \ldots, u_m, v_m)$ and $\gamma=(c_0, c_1, d_1, \ldots, c_m, d_m)$. If we put

$$\begin{aligned} y(t)&=u_0+\sqrt{2}\sum_{n=1}^{m}(u_n\cos nt+v_n\sin nt),\\ \varphi(t)&=c_0+\sqrt{2}\sum_{n=1}^{m}(c_n\cos nt+d_n\sin nt), \end{aligned} \tag{2.16}$$

then relations (2.13) and (2.15) show that $y(t)$ satisfies the differential system

$$\frac{dy(t)}{dt}=P_m\,\Psi[x_m(t),t]\,y(t)+\varphi(t), \tag{2.17}$$

where $x_m(t)$ is of the form (2.14).

First, we shall prove the following lemma.

Lemma 2.3. *Assume that the conditions of Lemma 2.2 are satisfied and that the system* (2.7) *has an isolated periodic solution* $x=\hat{x}(t)$ *lying inside* D.

Taking m_0 *sufficiently large, we consider the differential system*

$$\frac{dy}{dt}=P_m\,\Psi[\hat{x}_m(t),t]\,y+\varphi(t) \tag{2.18}$$

for $m\geqq m_0$, *where* $\hat{x}_m(t)=P_m\,\hat{x}(t)$ *and where* $\varphi(t)$ *is an arbitrary continuous periodic function of period* 2π.

Then, for any periodic solution $y=y(t)$ *of* (2.18) *(if any exists), we have*

$$\|y\|_q\leqq\frac{M[1+K_1\,\sigma_1(m)]}{1-M(K_3+K_1^2)\,\sigma_1(m)}\,\|\varphi\|_q. \tag{2.19}$$

Proof. For brevity let us put

$$\hat{A}_m(t)=\Psi[\hat{x}_m(t),t]. \tag{2.20}$$

Then for any periodic solution $y=y(t)$ of (2.18) we have

$$\frac{dy(t)}{dt}=\hat{A}_m(t)\,y(t)+\varphi(t)+\eta(t), \tag{2.21}$$

where

$$\eta(t)=-(I-P_m)\,\hat{A}_m(t)\,y(t). \tag{2.22}$$

Here I is the identity operator.

Put

$$u(t)=\hat{A}_m(t)\,y(t). \tag{2.23}$$

Then

$$\dot{u}(t)=\dot{\hat{A}}_m(t)\,y(t)+\hat{A}_m(t)\,[P_m\,\hat{A}_m(t)\,y(t)+\varphi(t)],$$

from which, by (2.8) and (2.12), it follows that

$$\|\dot{u}\|_q\leqq K_3\,\|y\|_q+K_1[\|P_m\,\hat{A}_m\,y\|_q+\|\varphi\|_q]. \tag{2.24}$$

But by Bessel's inequality,

$$\|P_m\,\hat{A}_m\,y\|_q\leqq\|\hat{A}_m\,y\|_q\leqq K_1\|y\|_q.$$

Therefore, from (2.24) we have

$$\|\dot{u}\|_q \leqq (K_3+K_1^2)\,\|y\|_q + K_1\|\varphi\|_q. \tag{2.25}$$

Since $\|\eta\|_q \leqq \sigma_1(m)\,\|\dot{u}\|_q$ by Lemma 2.1, we have then

$$\|\eta\|_q \leqq \sigma_1(m)\,[(K_3+K_1^2)\,\|y\|_q + K_1\|\varphi\|_q]. \tag{2.26}$$

On the other hand, $y(t)$ is a periodic solution of (2.21), so it can be expressed as follows:

$$y(t) = \int_0^{2\pi} H_m(t,s)\,[\varphi(s)+\eta(s)]\,ds, \tag{2.27}$$

where $H_m(t,s)$ is the matrix of the H-mapping corresponding to $\hat{A}_m(t)$. If m_0 is sufficiently large, by the Corollary of Lemma 2.2 we have

$$\|y\|_q \leqq M\,[\|\varphi\|_q + \|\eta\|_q] \tag{2.28}$$

for $m \geqq m_0$.

If we substitute (2.26) into (2.28), then we have

$$\|y\|_q \leqq M\,\|\varphi\|_q + M\,[(K_3+K_1^2)\,\|y\|_q + K_1\|\varphi\|_q]\,\sigma_1(m),$$

from which follows (2.19) since $1 - M(K_3+K_1^2)\,\sigma_1(m) > 0$ for sufficiently large m. Q.E.D.

Let

$$\hat{x}(t) = \hat{a}_0 + \sqrt{2}\sum_{n=1}^{\infty}(\hat{a}_n \cos nt + \hat{b}_n \sin nt)$$

be the Fourier series of an isolated periodic solution $x = \hat{x}(t)$ of (2.7) lying inside D, and let us consider the Jacobian matrix $J_m(\hat{\alpha})$ where $\hat{\alpha} = (\hat{a}_0, \hat{a}_1, \hat{b}_1, \dots, \hat{a}_m, \hat{b}_m)$. Then the lemma above yields the following corollaries.

Corollary 1. *There exists a positive integer m_0 such that*

$$\det J_m(\hat{\alpha}) \neq 0 \tag{2.29}$$

for any $m \geqq m_0$.

Proof. For $y(t)$ and $\varphi(t)$ of the form (2.16), the differential system (2.18) is equivalent to the linear system

$$J_m(\hat{\alpha})\,\xi + \gamma = 0 \tag{2.30}$$

as mentioned in the beginning of this section. Now put $\gamma = 0$. Then $\varphi(t) = 0$ by (2.16), and this implies $y(t) = 0$ by (2.19). Then $\xi = 0$ by (2.16). Thus, in (2.30), $\gamma = 0$ implies $\xi = 0$. This proves (2.29). Q.E.D.

Corollary 2. *There is a positive integer m_0 such that, for any $m \geqq m_0$, $J_m^{-1}(\hat{\alpha})$ exists and*

$$\|J_m^{-1}(\hat{\alpha})\| \leqq \frac{M[1+K_1\,\sigma_1(m)]}{1-M(K_3+K_1^2)\,\sigma_1(m)}. \tag{2.31}$$

Proof. By Corollary 1, $J_m^{-1}(\hat{\alpha})$ certainly exists for $m \geqq m_0$. Further, for $y(t)$ and $\varphi(t)$ of the form (2.16), the differential system (2.18) is equivalent to the linear system (2.30). Hence $\xi = -J_m^{-1}(\hat{\alpha})\gamma$. Since $\|y\|_q = \|\xi\|$ and $\|\varphi\|_q = \|\gamma\|$, (2.31) readily follows from (2.19). Q.E.D.

Lastly, for the difference $J_m(\alpha') - J_m(\alpha'')$, we shall prove the following lemma.

Lemma 2.4. *Under the conditions of Lemma 2.2, let K_4 be a positive constant such that*

$$K_4 \geqq \left[\max_{D\times L} \sum_{k,l,p} \left(\frac{\partial^2 X_k(x,t)}{\partial x_l \partial x_p}\right)^2\right]^{\frac{1}{2}}, \tag{2.32}$$

where $X_k(x,t)$ and x_l are respectively the components of the vectors $X(x,t)$ and x.

Then, if both

$$x'(t) = a_0' + \sqrt{2}\sum_{n=1}^{m}(a_n' \cos nt + b_n' \sin nt)$$

and

$$x''(t) = a_0'' + \sqrt{2}\sum_{n=1}^{m}(a_n'' \cos nt + b_n'' \sin nt)$$

belong to D together with $\vartheta x'(t) + (1-\vartheta)x''(t)$ $(0 \leqq \vartheta \leqq 1)$, then

$$\|J_m(\alpha') - J_m(\alpha'')\| \leqq K_4 \|x' - x''\|_n \leqq K_4 \sqrt{2m+1} \cdot \|\alpha' - \alpha''\|, \tag{2.33}$$

where

$$\alpha' = (a_0', a_1', b_1', \ldots, a_m', b_m') \quad \text{and} \quad \alpha'' = (a_0'', a_1'', b_1'', \ldots, a_m'', b_m'').$$

Proof. Take an arbitrary $\xi = (u_0, u_1, v_1, \ldots, u_m, v_m)$, and consider

$$y(t) = u_0 + \sqrt{2}\sum_{n=1}^{m}(u_n \cos nt + v_n \sin nt). \tag{2.34}$$

Put

$$\gamma' = -J_m(\alpha')\xi, \qquad \gamma'' = -J_m(\alpha'')\xi, \tag{2.35}$$

and let

$$\gamma' = (c_0', c_1', d_1', \ldots, c_m', d_m'), \qquad \gamma'' = (c_0'', c_1'', d_1'', \ldots, c_m'', d_m'').$$

If we put

$$\varphi'(t) = c_0' + \sqrt{2}\sum_{n=1}^{m}(c_n' \cos nt + d_n' \sin nt),$$

$$\varphi''(t) = c_0'' + \sqrt{2}\sum_{n=1}^{m}(c_n'' \cos nt + d_n'' \sin nt),$$

then by (2.17) and (2.35) we have

$$\frac{dy(t)}{dt} = P_m \Psi[x'(t), t]\, y(t) + \varphi'(t),$$

$$\frac{dy(t)}{dt} = P_m \Psi[x''(t), t]\, y(t) + \varphi''(t).$$

From this it readily follows that

$$\varphi'(t) - \varphi''(t) = -P_m\{\Psi[x'(t), t] - \Psi[x''(t), t]\}\, y(t). \tag{2.36}$$

Let us put

$$\varphi(t) = \varphi'(t) - \varphi''(t) \quad \text{and} \quad \gamma = \gamma' - \gamma''.$$

Then from (2.36) we have

(2.37) $$\varphi(t) = -P_m\{\Psi[x'(t), t] - \Psi[x''(t), t]\}\, y(t),$$

and from (2.35)

(2.38) $$\gamma = -[J_m(\alpha') - J_m(\alpha'')]\,\xi.$$

Now

$$\|\Psi[x'(t), t] - \Psi[x''(t), t]\|^2 \leqq \sum_{k,l} [\Psi_{kl}(x', t) - \Psi_{kl}(x'', t)]^2,$$

where $\Psi_{kl}(x', t)$ and $\Psi_{kl}(x'', t)$ are the elements of $\Psi[x'(t), t]$ and $\Psi[x''(t), t]$, respectively. Since $x''(t) + \vartheta[x'(t) - x''(t)] \in D\ (0 \leqq \vartheta \leqq 1)$ by the assumption, the quantity in the right member of the above inequality is estimated successively by means of SCHWARZ' inequality as follows:

$$\begin{aligned}
&\sum_{k,l} [\Psi_{kl}(x', t) - \Psi_{kl}(x'', t)]^2 \\
&\quad = \sum_{k,l} \left[\int_0^1 \left\{\sum_p \frac{\partial \Psi_{kl}}{\partial x_p}(x'' + \vartheta(x' - x''), t)(x'_p - x''_p)\right\} d\vartheta\right]^2 \\
&\quad = \sum_{k,l} \left[\sum_p \left\{\int_0^1 \frac{\partial \Psi_{kl}}{\partial x_p} d\vartheta \cdot (x'_p - x''_p)\right\}\right]^2 \\
&\quad \leqq \sum_{k,l} \left[\sum_p \left\{\int_0^1 \frac{\partial \Psi_{kl}}{\partial x_p} d\vartheta\right\}^2 \cdot \sum_p (x'_p - x''_p)^2\right] \\
&\quad \leqq \sum_{k,l} \left[\sum_p \int_0^1 \left(\frac{\partial \Psi_{kl}}{\partial x_p}\right)^2 d\vartheta\right] \cdot \|x'(t) - x''(t)\|^2 \\
&\quad \leqq \int_0^1 \left[\sum_{k,l,p} \left(\frac{\partial \Psi_{kl}}{\partial x_p}\right)^2\right] d\vartheta \cdot \|x' - x''\|_n^2 \\
&\quad \leqq K_4^2 \cdot \|x' - x''\|_n^2.
\end{aligned}$$

Hence

$$\|\{\Psi[x'(t), t] - \Psi[x''(t), t]\}\, y(t)\| \leqq K_4 \cdot \|x' - x''\|_n \cdot \|y(t)\|.$$

Then by BESSEL's inequality it follows from (2.37) that

$$\|\varphi\|_q \leqq K_4 \cdot \|x' - x''\|_n \cdot \|y\|_q.$$

Since $\|\gamma\| = \|\varphi\|_q$ and $\|\xi\| = \|y\|_q$, from (2.38) we have

$$\|[J_m(\alpha') - J_m(\alpha'')]\,\xi\| \leqq K_4 \cdot \|x' - x''\|_n \cdot \|\xi\|,$$

which implies that

(2.39) $$\|J_m(\alpha') - J_m(\alpha'')\| \leqq K_4 \cdot \|x' - x''\|_n.$$

Put $\alpha = \alpha' - \alpha''$, and suppose $\alpha = (a_0, a_1, b_1, \ldots, a_m, b_m)$. Then

$$x'(t) - x''(t) = a_0 + \sqrt{2} \sum_{n=1}^{m} (a_n \cos n t + b_n \sin n t),$$

and therefore,

$$\begin{aligned}
\|x'(t) - x''(t)\|^2 &= \sum_k \left[a_{0k} + \sqrt{2} \sum_{n=1}^{m} (a_{nk} \cos n t + b_{nk} \sin n t)\right]^2 \\
&\leqq \sum_k \left[|a_{0k}| + \sqrt{2} \sum_{n=1}^{m} \sqrt{a_{nk}^2 + b_{nk}^2}\right]^2,
\end{aligned}$$

where a_{0k}, a_{nk} and b_{nk} are respectively the components of the vectors a_0, a_n, and b_n. By means of SCHWARZ' inequality we have

$$\begin{aligned}\|x'-x''\|_n^2 &\leqq \sum_k (1+2m)\left[a_{0k}^2+\sum_{n=1}^m (a_{nk}^2+b_{nk}^2)\right] \\ &=(2m+1)\left[\|a_0\|^2+\sum_{n=1}^m (\|a_n\|^2+\|b_n\|^2)\right] \\ &=(2m+1)\cdot\|\alpha'-\alpha''\|^2.\end{aligned}$$

Then (2.33) follows readily from (2.39). Q.E.D.

2.3 The existence of a Galerkin approximation. The existence of a Galerkin approximation to an isolated periodic solution is proved by the following theorem.

Theorem 1. *Let*

$$\frac{dx}{dt}=X(x,t) \tag{2.40}$$

be a given real nonlinear periodic system, where x and $X(x,t)$ are real vectors of the same dimension and $X(x,t)$ is periodic in t of period 2π. We assume that $X(x,t)$ and its first partial derivatives with respect to x are continuously differentiable with respect to x and t in the region $D\times L$, where D is a closed bounded region of the x-space and L is the real line.

If there is an isolated periodic solution $x=\hat{x}(t)$ of (2.40) *lying inside D, then there exists a Galerkin approximation $x=\bar{x}_m(t)$ of any order $m\geqq m_0$ lying in D provided m_0 is sufficiently large.*

Proof. Setting $P_m\,\hat{x}(t)\doteq\hat{x}_m(t)$, we have

$$\frac{d\hat{x}_m(t)}{dt}=P_m\frac{d\hat{x}(t)}{dt}=P_m X[\hat{x}(t),t]. \tag{2.41}$$

Now let us take a small positive number δ_0 so that

$$U=\{x\,|\,\|x-\hat{x}(t)\|\leqq\delta_0 \text{ for some } t\in L\}\subset D.$$

This is possible because $x=\hat{x}(t)$ lies inside D by the assumption. Then, by Lemma 2.2, $\hat{x}_m(t)\in U\subset D$ for all $t\in L$ and for any $m\geqq m_0$ provided m_0 is sufficiently large.

For such m equation (2.41) can be rewritten as follows:

$$\frac{d\hat{x}_m(t)}{dt}=P_m X[\hat{x}_m(t),t]+R_m(t), \tag{2.42}$$

where

$$R_m(t)=P_m\{X[\hat{x}(t),t]-X[\hat{x}_m(t),t]\}. \tag{2.43}$$

Now

$$X[\hat{x}(t),t]-X[\hat{x}_m(t),t]=-\int_0^1 \Psi[\hat{x}(t)+\vartheta(\hat{x}_m(t)-\hat{x}(t)),t](\hat{x}_m(t)-\hat{x}(t))\,d\vartheta,$$

hence, by (2.8),

$$\|X[\hat{x}(t),t]-X[\hat{x}_m(t),t]\|\leqq K_1\cdot\|\hat{x}(t)-\hat{x}_m(t)\|.$$

Then by (ii) of (2.9) we have

$$\| X[\hat{x}(t), t] - X[\hat{x}_m(t), t] \|_q \leqq K_1 \cdot \| \hat{x} - \hat{x}_m \|_q \leqq K K_1 \sigma_1(m).$$

Hence, from (2.43) and BESSEL's inequality, we see that

$$\| R_m \|_q \leqq K K_1 \sigma_1(m). \tag{2.44}$$

Let us put

$$\hat{x}(t) = \hat{a}_0 + \sqrt{2} \sum_{n=1}^{\infty} (\hat{a}_n \cos n t + \hat{b}_n \sin n t) \tag{2.45}$$

and

$$R_m(t) = r_0^{(m)} + \sqrt{2} \sum_{n=1}^{m} (r_n^{(m)} \cos n t + s_n^{(m)} \sin n t). \tag{2.46}$$

Then (2.42) is equivalent to the following system:

$$\begin{aligned} F_0^{(m)}(\hat{\alpha}) &= \frac{1}{2\pi} \int_0^{2\pi} X[\hat{x}_m(t), t]\, dt &&= -r_0^{(m)}, \\ F_n^{(m)}(\hat{\alpha}) &= \frac{1}{\sqrt{2}\pi} \int_0^{2\pi} X[\hat{x}_m(t), t] \cos n t\, dt - n \hat{b}_n &&= -r_n^{(m)}, \\ G_n^{(m)}(\hat{\alpha}) &= \frac{1}{\sqrt{2}\pi} \int_0^{2\pi} X[\hat{x}_m(t), t] \sin n t\, dt + n \hat{a}_n &&= -s_n^{(m)}, \\ &(n = 1, 2, \ldots, m), \end{aligned} \tag{2.47}$$

where $F_0^{(m)}(\alpha)$, $F_n^{(m)}(\alpha)$, $G_n^{(m)}(\alpha)$ are defined by the expressions above, and where

$$\hat{x}_m(t) = \hat{a}_0 + \sqrt{2} \sum_{n=1}^{m} (\hat{a}_n \cos n t + \hat{b}_n \sin n t).$$

For brevity let us write equation (2.47) in vector form as follows:

$$F^{(m)}(\hat{\alpha}) = -\rho^{(m)}, \tag{2.48}$$

where

$$\hat{\alpha} = (\hat{a}_0, \hat{a}_1, \hat{b}_1, \ldots, \hat{a}_m, \hat{b}_m) \quad \text{and} \quad \rho^{(m)} = (r_0^{(m)}, r_1^{(m)}, s_1^{(m)}, \ldots, r_m^{(m)}, s_m^{(m)}).$$

Then, by the definition of $J_m(\alpha)$, $J_m(\alpha)$ is the Jacobian matrix of $F^{(m)}(\alpha)$ with respect to α, and by (2.44) and (2.46) we have

$$\| \rho^{(m)} \| \leqq K K_1 \sigma_1(m). \tag{2.49}$$

Now, for $m \geqq m_0$ and m_0 sufficiently large, let us consider the region

$$V_m = \{x \mid \| x - \hat{x}_m(t) \| \leqq \delta_0 - K \sigma(m) \text{ for some } t \in L\}.$$

Then, by (i) of (2.9),

$$V_m \subset U \subset D$$

for any $m \geqq m_0$. Consider

$$\Omega_m = \left\{ \alpha \mid \| \alpha - \hat{\alpha} \| \leqq \frac{\delta_0 - K \sigma(m)}{\sqrt{2m+1}} \right\},$$

where $\alpha=(a_0, a_1, b_1, \ldots, a_m, b_m)$. Then, as is shown in the proof of Lemma 2.4, for

$$x=x(t)=a_0+\sqrt{2}\sum_{n=1}^{m}(a_n\cos nt+b_n\sin nt)$$

with $(a_0, a_1, b_1, \ldots, a_m, b_m)=\alpha\in\Omega_m$, we have

$$\begin{aligned}\|x-\hat{x}_m\|_n &\leqq \sqrt{2m+1}\cdot\|\alpha-\hat{\alpha}\| \\ &\leqq \delta_0-K\sigma(m),\end{aligned}$$

and hence, $x(t)\in V_m\subset D$. Thus, it is proved that $F^{(m)}(\alpha)$ is well defined for any $\alpha\in\Omega_m$.

From (2.47) we note that a Galerkin approximation is a trigonometric polynomial whose Fourier coefficients satisfy the equation

$$F^{(m)}(\alpha)=0. \tag{2.50}$$

Since $\alpha=\hat{\alpha}$ is an approximate solution of the above equation, which follows from (2.48) and (2.49), we shall apply Proposition 2 to the above equation in order to prove the existence of an exact solution, namely the existence of a Galerkin approximation.

Let us take m_0 sufficiently large. Then by Corollary 2 of Lemma 2.3, for any $m\geqq m_0$, $J_m^{-1}(\hat{\alpha})$ exists and

$$\|J_m^{-1}(\hat{\alpha})\|\leqq\frac{M[1+K_1\sigma_1(m)]}{1-M(K_3+K_1^2)\sigma_1(m)}.$$

This implies that

$$\|J_m^{-1}(\hat{\alpha})\|\leqq M' \quad \text{for any} \quad m\geqq m_0, \tag{2.51}$$

where

$$M'=\frac{M[1+K_1\sigma_1(m_0)]}{1-M(K_3+K_1^2)\sigma_1(m_0)}. \tag{2.52}$$

Further by Lemma 2.4,

$$\|J_m(\alpha)-J_m(\hat{\alpha})\|\leqq K_4\sqrt{2m+1}\cdot\|\alpha-\hat{\alpha}\| \tag{2.53}$$

for any $\alpha\in\Omega_m$ provided $m\geqq m_0$.

Take an arbitrary number κ such that $0<\kappa<1$, and put

$$\min\left(\frac{\kappa}{K_4M'}, \delta_0-K\sigma(m_0)\right)=\delta_1. \tag{2.54}$$

Let us take $m_1\geqq m_0$ so that, for any $m\geqq m_1$,

$$\frac{M'KK_1}{1-\kappa}\sigma_1(m)=\frac{M'KK_1}{1-\kappa}\cdot\frac{1}{m+1}<\frac{\delta_1}{\sqrt{2m+1}}. \tag{2.55}$$

This is possible because

$$\frac{m+1}{\sqrt{2m+1}}\to\infty \quad \text{as} \quad m\to\infty.$$

By (2.55) we can take a positive number δ_m such that

$$\frac{M' K K_1}{1-\kappa}\sigma_1(m) \leqq \delta_m \leqq \frac{\delta_1}{\sqrt{2m+1}}. \tag{2.56}$$

Let us consider the set

$$\Omega_{\delta_m} = \{\alpha \,|\, \|\alpha - \hat{\alpha}\| \leqq \delta_m\}. \tag{2.57}$$

For any $\alpha \in \Omega_{\delta_m}$ we have

$$\begin{aligned} \|\alpha - \hat{\alpha}\| &\leqq \frac{\delta_1}{\sqrt{2m+1}} \\ &\leqq \frac{\delta_0 - K\sigma(m_0)}{\sqrt{2m+1}} \\ &\leqq \frac{\delta_0 - K\sigma(m)}{\sqrt{2m+1}} \qquad (m \geqq m_1 \geqq m_0), \end{aligned}$$

and consequently,

$$\Omega_{\delta_m} \subset \Omega_m. \tag{2.58}$$

Then, for any $\alpha \in \Omega_{\delta_m}$, by (2.53) we have

$$\|J_m(\alpha) - J_m(\hat{\alpha})\| \leqq K_4 \cdot \sqrt{2m+1} \cdot \delta_m,$$

and hence, by (2.56) and (2.54),

$$\|J_m(\alpha) - J_m(\hat{\alpha})\| \leqq \kappa / M'. \tag{2.59}$$

Further, from (2.49) and (2.56),

$$\frac{M' \|\rho^{(m)}\|}{1-\kappa} \leqq \frac{M' K K_1 \sigma_1(m)}{1-\kappa} \leqq \delta_m. \tag{2.60}$$

The expressions (2.57)–(2.60) show that the conditions of Proposition 2 are all fulfilled. Thus, by that proposition we see that equation (2.50) has one and only one solution $\alpha = \bar{\alpha}$ lying in Ω_{δ_m}. This proves the theorem. Q.E.D.

2.4 Error estimates and some properties of Galerkin approximations. An error estimate of a Galerkin approximation is given by the following theorem.

Theorem 2. *Assume that the conditions of Theorem 1 are satisfied. Let $x = \hat{x}(t)$ be an isolated periodic solution of* (2.40) *lying inside D and $x = \bar{x}_m(t)$ be its Galerkin approximation as stated in Theorem 1. If m_0 is sufficiently large, then for any positive integer $m \geqq m_0$,*

$$\|\bar{x}_m - \hat{x}\|_n \leqq \frac{M' K K_1}{1-\kappa} \cdot \frac{\sqrt{2m+1}}{m+1} + K\sigma(m), \tag{2.61}$$

$$\|\dot{\bar{x}}_m - \dot{\hat{x}}\|_n \leqq (K_2 + 2KK_1)\sigma(m) + \frac{M' K K_1^2}{1-\kappa} \cdot \frac{\sqrt{2m+1}}{m+1}, \tag{2.62}$$

where κ is an arbitrary fixed number such that $0 < \kappa < 1$, K, K_1 and K_2 are the numbers defined in (2.8), *$\sigma(m)$ the number defined in* (2.3), *and M' the number defined in* (2.52).

Proof. Put

(2.63) $$\bar{x}_m(t)=\bar{a}_0+\sqrt{2}\sum_{n=1}^{m}(\bar{a}_n\cos nt+\bar{b}_n\sin nt).$$

As shown in the proof of Theorem 1, $\bar{\alpha}=(\bar{a}_0, \bar{a}_1, \bar{b}_1, \ldots, \bar{a}_m, \bar{b}_m)$ is a solution of (2.50) lying in Ω_{δ_m}, and by Proposition 2 we have

(2.64) $$\|\bar{\alpha}-\hat{\alpha}\|\leqq\frac{M' K K_1}{1-\kappa}\cdot\sigma_1(m),$$

where $\hat{\alpha}=(\hat{a}_0, \hat{a}_1, \hat{b}_1, \ldots, \hat{a}_m, \hat{b}_m)$ is such that

$$\hat{a}_0+\sqrt{2}\sum_{n=1}^{m}(\hat{a}_n\cos nt+\hat{b}_n\sin nt)=\hat{x}_m(t)=P_m\hat{x}(t).$$

Inequality (2.64) is evidently equivalent to the inequality

$$\|\bar{x}_m-\hat{x}_m\|_q\leqq\frac{M' K K_1}{1-\kappa}\,\sigma_1(m).$$

As shown in the proof of Lemma 2.4, we have

(2.65) $$\|\bar{x}_m-\hat{x}_m\|_n\leqq\frac{M' K K_1}{1-\kappa}\,\sigma_1(m)\sqrt{2m+1}=\frac{M' K K_1}{1-\kappa}\cdot\frac{\sqrt{2m+1}}{m+1}.$$

On the other hand, by Lemma 2.2,

$$\|\hat{x}_m-\hat{x}\|_n\leqq K\,\sigma(m).$$

Thus, by combining this with (2.65), we obtain (2.61).

Since $\bar{\alpha}$ is a solution of (2.50), for $\bar{x}_m(t)$ of (2.63) we have

$$\frac{d\bar{x}_m(t)}{dt}=P_m X[\bar{x}_m(t), t].$$

This can be rewritten as follows:

(2.66) $$\frac{d\bar{x}_m(t)}{dt}=X[\bar{x}_m(t), t]+\eta_m(t),$$

where

(2.67) $$\eta_m(t)=-(I-P_m)X[\bar{x}_m(t), t],$$

and I is the identity operator. Since

$$\begin{aligned}\frac{d}{dt}X[\bar{x}_m(t), t]&=\Psi[\bar{x}_m(t), t]\cdot\frac{d\bar{x}_m(t)}{dt}+\frac{\partial X}{\partial t}[\bar{x}_m(t), t]\\&=\Psi[\bar{x}_m(t), t]\cdot P_m X[\bar{x}_m(t), t]+\frac{\partial X}{\partial t}[x_m(t), t],\end{aligned}$$

by (2.8) and BESSEL's inequality we have

$$\begin{aligned}\|\dot{X}_m\|_q&\leqq K_1\cdot\|P_m X_m\|_q+K_2\\&\leqq K_1 K+K_2,\end{aligned}$$

where $X_m = X[\bar{x}_m(t), t]$. Then, by Lemma 2.1 and (2.67), we have

$$\|\eta_m\|_n \leqq \sigma(m)(K K_1 + K_2). \tag{2.68}$$

Since

$$\frac{d\hat{x}(t)}{dt} = X[\hat{x}(t), t],$$

from (2.66) we have

$$\begin{aligned}\frac{d\bar{x}_m(t)}{dt} - \frac{d\hat{x}(t)}{dt} &= \{X[\bar{x}_m(t), t] - X[\hat{x}(t), t]\} + \eta_m(t) \\ &= \int_0^1 \Psi[\hat{x}(t) + \vartheta(\bar{x}_m(t) - \hat{x}(t)), t] \cdot [\bar{x}_m(t) - \hat{x}(t)]\, d\vartheta + \eta_m(t),\end{aligned}$$

and consequently, by (2.8),

$$\|\dot{\bar{x}}_m - \dot{\hat{x}}\|_n \leqq K_1 \cdot \|\bar{x}_m - \hat{x}\|_n + \|\eta_m\|_n.$$

If we substitute (2.61) and (2.68) into the right member of the above inequality, we get (2.62). Q.E.D.

Corollary. *Assume that the conditions of Theorem 1 are satisfied. If there is an isolated periodic solution $x = \hat{x}(t)$ of* (2.40) *lying inside D, then its Galerkin approximations $x = \bar{x}_m(t)$ stated in Theorem 1 converge uniformly to the original exact solution $x = \hat{x}(t)$ together with the first order derivatives as $m \to \infty$.*

Proof. This is evident from (2.61) and (2.62), because $\sqrt{2m+1}/(m+1)$ tends to zero as $m \to \infty$, and by (2.4), $\sigma(m)$ also tends to zero as $m \to \infty$. Q.E.D.

By Theorem 2, if we take m_0 sufficiently large, then for any $m \geqq m_0$ the conclusions of the Corollary of Lemma 2.2 are all valid for $\bar{x}_m(t)$ as well as for $\hat{x}_m(t)$. Thus, by Corollaries 1 and 2 of Lemma 2.3, we have the following theorem.

Theorem 3. *Assume that the conditions of Theorem 1 are satisfied, and suppose that* (2.40) *has an isolated periodic solution lying inside D. Let*

$$x = \bar{x}_m(t) = \bar{a}_0 + \sqrt{2} \sum_{n=1}^{m} (\bar{a}_n \cos nt + \bar{b}_n \sin nt)$$

be its Galerkin approximations as stated in Theorem 1, and suppose m_0 sufficiently large. Then, for any $m \geqq m_0$,

$$\det J_m(\bar{\alpha}) \neq 0, \tag{2.69}$$

and there exists a positive constant M' such that

$$\|J_m^{-1}(\bar{\alpha})\| \leqq M' \tag{2.70}$$

where $\bar{\alpha} = (\bar{a}_0, \bar{a}_1, \bar{b}_1, \ldots, \bar{a}_m, \bar{b}_m)$. Furthermore, the multipliers of the linear homogeneous system

$$\frac{dy}{dt} = \Psi[\bar{x}_m(t), t]\, y \tag{2.71}$$

are all different from one, and the H-mappings H_m corresponding to $\Psi[\bar{x}_m(t), t]$ are equibounded, namely, there is a positive constant M_1 such that

$$\|H_m\|_q, \ \|H_m\|_n \leqq M_1. \tag{2.72}$$

§3. The Existence of an Exact Isolated Periodic Solution

According to Theorem 3, let us assume the equiboundedness of the H-mappings H_m corresponding to $\Psi[\bar{x}_m(t), t]$. Then the following theorem holds.

Theorem 4. *Let us assume that the conditions of Theorem 1 are satisfied.*

Let m_0 be a positive integer, $\varepsilon > 0$ a given number, and Δ a region of points x whose ε-neighborhood is contained in D.

Then, if there are Galerkin approximations $x = \bar{x}_m(t)$ of all orders $m \geqq m_0$ lying in Δ such that the H-mappings H_m corresponding to $\Psi[\bar{x}_m(t), t]$ are equibounded in the norm $\| H_m \|_n$, then there exists an isolated periodic solution $x = \hat{x}(t)$ of (2.40) *lying in D, $\hat{x}(t)$ is unique in a neighborhood of $\bar{x}_m(t)$, and the following inequality holds:*

$$\| \hat{x} - \bar{x}_m \|_n \leqq \frac{M_1}{1-\kappa} (K K_1 + K_2) \sigma(m), \tag{3.1}$$

where

κ is an arbitrary fixed number such that $0 < \kappa < 1$;

K, K_1, and K_2 are the numbers defined in (2.8);

$\sigma(m)$ is the number defined in (2.3);

M_1 is a positive constant such that

$$\| H_m \|_n \leqq M_1. \tag{3.2}$$

Proof. By the definition of Galerkin approximations,

$$\frac{d\bar{x}_m(t)}{dt} = P_m X[\bar{x}_m(t), t]. \tag{3.3}$$

We have

$$\frac{d\bar{x}_m(t)}{dt} = X[\bar{x}_m(t), t] + \eta_m(t), \tag{3.4}$$

where

$$\eta_m(t) = -[I - P_m] X[\bar{x}_m(t), t]. \tag{3.5}$$

Equation (3.4) is of the same form as (2.66). Therefore, by (2.68) we have

$$\| \eta_m \|_n \leqq \sigma(m)(K K_1 + K_2). \tag{3.6}$$

Now let us take a positive integer $m_1 \geqq m_0$ so that

$$\frac{M_1}{1-\kappa} \cdot (K K_1 + K_2) \frac{\sqrt{2}}{\sqrt{m_1}} < \min\left(\varepsilon, \frac{\kappa}{M_1 K_4}\right),$$

where κ is an arbitrary fixed number such that $0 < \kappa < 1$ and K_4 is the number defined in Lemma 2.4. Take δ (>0) so that

$$\frac{M_1}{1-\kappa} \cdot (K K_1 + K_2) \frac{\sqrt{2}}{\sqrt{m_1}} \leqq \delta \leqq \min\left(\varepsilon, \frac{\kappa}{M_1 K_4}\right), \tag{3.7}$$

and let m be any positive integer such that $m \geqq m_1$.

By assumption there is a Galerkin approximation $x=\bar{x}_m(t)$ of the order m lying in $\varDelta$. Let D_δ be the set

$$D_\delta=\{x \mid \|x-\bar{x}_m(t)\|\leqq\delta \quad \text{for some} \quad t\in L\}.$$

Then

$$D_\delta\subset D \tag{3.8}$$

since $\delta\leqq\varepsilon$. By (3.4) and (3.6) we have

$$\begin{aligned}\left\|\frac{d\bar{x}_m(t)}{dt}-X[\bar{x}_m(t),t]\right\| &= \|\eta_m(t)\| \\ &\leqq (KK_1+K_2)\,\sigma(m) \\ &\leqq (KK_1+K_2)\,\sigma(m_1) \\ &\leqq (KK_1+K_2)\,\frac{\sqrt{2}}{\sqrt{m_1}} \quad (\text{cf. } (2.4)),\end{aligned}$$

and hence by (3.7)

$$\frac{M_1\|\eta_m\|_n}{1-\kappa}\leqq\frac{M_1}{1-\kappa}\cdot(KK_1+K_2)\,\frac{\sqrt{2}}{\sqrt{m_1}}\leqq\delta. \tag{3.9}$$

Furthermore, as shown in the proof of Lemma 2.4, for any $t\in L$ and any x such that $\|x-\bar{x}_m(t)\|\leqq\delta$ we have

$$\|\Psi(x,t)-\Psi[\bar{x}_m(t),t]\|\leqq K_4\,\|x-\bar{x}_m(t)\|\leqq K_4\,\delta,$$

from which, by (3.7), follows

$$\|\Psi(x,t)-\Psi[\bar{x}_m(t),t]\|\leqq\frac{\kappa}{M_1}. \tag{3.10}$$

Relations (3.8)–(3.10) show that conditions (ii)–(iv) of Proposition 3 are fulfilled for $A(t)=\Psi[\bar{x}_m(t),t]$. Condition (i) of Proposition 3 is naturally fulfilled by the assumption of equiboundedness of H_m. Thus, by Proposition 3 we see that there exists one and only one exact isolated periodic solution $x=\hat{x}(t)$ of (2.40) in D_δ. Inequality (3.1) readily follows from (3.6) and (1.26). Q.E.D.

Remark 1. In practical problems, after finding a Galerkin approximation $x=\bar{x}(t)$ of some order, we can compute (say by numerical integration) the approximate values of the matrix $H(t,s)$ of the H-mapping corresponding to the matrix $\Psi[\bar{x}(t),t]$. Then, by means of (1.10), we can estimate the value of $\|H\|_n$. Since

$$\left\|\frac{d\bar{x}(t)}{dt}-X[\bar{x}(t),t]\right\|_n$$

can also be estimated (say by numerical computation of the Fourier coefficients), we can check whether the conditions of Proposition 3 are all fulfilled or not. If the conditions are not fulfilled, then the order of the Galerkin approximation has to be raised. As seen from the proof of Theorem 4, such a procedure always ends at a certain finite order in case the given system has an isolated periodic solution lying inside D (*cf.* Theorems 1 and 3). In other words, in such a case we can always find a Galerkin approximation for which the conditions of Proposition 3 are all fulfilled, and thereby, we can affirm the existence of an exact isolated periodic solution. The error of such a Galerkin approximation is within the bound given by (3.1).

Remark 2. In Theorem 4 it is assumed that the multipliers of the linear homogeneous system

$$\frac{dy}{dt}=\Psi[\bar{x}_m(t),t]\,y$$

are all different from one. However, as is shown in the proposition of the Appendix, this can be proved if we assume the existence and the equiboundedness of $J_m^{-1}(\bar{\alpha})$ where $\bar{\alpha}$ is the vector corresponding to the Fourier coefficients of $\bar{x}_m(t)$.

However, the equiboundedness of the H-mapping corresponding to $\Psi[\bar{x}_m(t),t]$, which is also assumed in Theorem 4, does not seem to follow solely from the assumption of the existence and equiboundedness of $J_m^{-1}(\bar{\alpha})$.

§4. An Example

In this section we shall give an example of our method, using the equation

$$\ddot{x}+1.5^2\,x+(x-1.5\sin t)^3=2\sin t. \tag{4.1}$$

More realistic examples will be presented in the succeeding paper which will be mainly concerned with numerical techniques.

Equation (4.1) can be rewritten in the form of a first order system as follows:

$$\begin{aligned}\dot{x}&=y,\\ \dot{y}&=-1.5^2\,x-(x-1.5\sin t)^3+2\sin t.\end{aligned} \tag{4.2}$$

Now let $x=x(t)$ be any periodic solution of (4.1); then evidently $-x(-t)$ and $-x(t+\pi)$ are also periodic solutions. Therefore, if the periodic solution of (4.1) is unique, then it must be that

$$x(-t)=-x(t)\quad\text{and}\quad x(t+\pi)=-x(t).$$

Then the Fourier series of such a periodic solution must be of the form

$$x(t)\sim a_1\sin t+a_3\sin 3t+\cdots.$$

Taking this fact into consideration, first, for the system (4.2), let us calculate the 3rd order Galerkin approximation of the form

$$\begin{aligned}x&=\bar{x}(t)=a_1\sin t+a_3\sin 3t,\\ y&=\bar{y}(t)=a_1\cos t+3\,a_3\cos 3t.\end{aligned} \tag{4.3}$$

Substituting (4.3) into (4.2), we get the following determining equation:

$$\begin{aligned}&-2+1.25\,a_1+\tfrac{3}{4}(a_1-1.5)^3-\tfrac{3}{4}(a_1-1.5)^2\,a_3+\tfrac{3}{2}(a_1-1.5)\,a_3^2=0,\\ &-6.75\,a_3-\tfrac{1}{4}(a_1-1.5)^3+\tfrac{3}{2}(a_1-1.5)^2\,a_3+\tfrac{3}{4}\,a_3^3=0.\end{aligned} \tag{4.4}$$

Since $a_1=1.5$, $a_3=0$ is an approximate solution, let us put

$$a_1=1.5+u.$$

Then (4.4) is transformed to the following equation:

$$u=0.1-0.6u^3+0.6u^2a_3-1.2ua_3^2,$$
$$a_3=\tfrac{1}{27}(-u^3+6u^2a_3+3a_3^3).$$

This equation can be solved easily by the iterative method as follows:

$$u=0.09941, \quad a_3=-0.00004.$$

Thus, for (4.2), we get the following 3rd order Galerkin approximation:

$$\begin{aligned} x=\bar{x}(t)&=1.59941\sin t-0.00004\sin 3t,\\ y=\bar{y}(t)&=1.59941\cos t-0.00012\cos 3t. \end{aligned} \tag{4.5}$$

Next, putting

$$A(t)=\begin{pmatrix} 0 & 1\\ -\omega^2 & 0\end{pmatrix} \quad (\omega=1.5), \tag{4.6}$$

we calculate the matrix $H(t,s)$ of the H-mapping H corresponding to the above matrix. Let $\Phi(t)$ be the fundamental matrix of the linear system

$$\frac{dz}{dt}=A(t)z;$$

then evidently

$$\Phi(t)=\begin{pmatrix} \cos\omega t & \frac{1}{\omega}\sin\omega t\\ -\omega\sin\omega t & \cos\omega t\end{pmatrix}.$$

Hence, by (1.4) we have

$$H(t,s)=\frac{1}{2\sin\omega\pi}\begin{pmatrix} \sin\omega(\pi-t+s) & \frac{1}{\omega}\cos\omega(\pi-t+s)\\ -\omega\cos\omega(\pi-t+s) & \sin\omega(\pi-t+s)\end{pmatrix} \quad \text{for} \quad 0\leqq s\leqq t\leqq 2\pi,$$

$$H(t,s)=\frac{1}{2\sin\omega\pi}\begin{pmatrix} \sin\omega(-\pi-t+s) & \frac{1}{\omega}\cos\omega(-\pi-t+s)\\ -\omega\cos\omega(-\pi-t+s) & \sin\omega(-\pi-t+s)\end{pmatrix}$$
$$\text{for} \quad 0\leqq t<s\leqq 2\pi.$$

By (1.10) we see that

$$\|H\|_n\leqq\frac{13\sqrt{2}}{12}\pi=4.81312\ldots<4.8132. \tag{4.7}$$

Let $\Psi(x,y,t)$ be the Jacobian matrix of the right member of (4.2) with respect to x and y. Then, from (4.6),

$$\Psi(x,y,t)-A(t)=\begin{pmatrix} 0 & 0\\ -3(x-1.5\sin t)^2 & 0\end{pmatrix},$$

and hence,

(4.8) $$\| \Psi(x, y, t) - A(t) \| \leqq 3(x - 1.5 \sin t)^2.$$

Since

$$|\bar{x}(t) - 1.5 \sin t| \leqq 0.09945$$

by (4.5), inequality (4.8) implies that

(4.9) $$\| \Psi(x, y, t) - A(t) \| \leqq 3(\delta + 0.09945)^2$$

for x such that

(4.10) $$|x - \bar{x}(t)| \leqq \delta.$$

Lastly, from (4.5),

$$\frac{d\bar{x}(t)}{dt} - \bar{y}(t) = 0,$$

$$\frac{d\bar{y}(t)}{dt} + 1.5^2 \bar{x}(t) + [\bar{x}(t) - 1.5 \sin t]^3 - 2 \sin t$$
$$\approx -0.0000\,0040 \sin t + 0.0000\,2381 \sin 3t$$
$$+ 0.0000\,0030 \sin 5t.$$

Consequently

(4.11) $$\left[\left(\frac{d\bar{x}(t)}{dt} - \bar{y}(t)\right)^2 + \left(\frac{d\bar{y}(t)}{dt} + 1.5^2 \bar{x}(t) + (\bar{x}(t) - 1.5 \sin t)^3 - 2 \sin t\right)^2\right]^{\frac{1}{2}} < 0.0000\,25.$$

Now, according to Proposition 3, let us determine δ and κ (<1) so that

(4.12) $$3(\delta + 0.09945)^2 \leqq \frac{\kappa}{4.8132}, \qquad \frac{4.8132 \times 0.000025}{1 - \kappa} \leqq \delta.$$

The second inequality of (4.12) is

(4.13) $$\frac{0.00012033}{1 - \kappa} \leqq \delta.$$

Let us suppose

(4.14) $$\delta \leqq 0.0002.$$

Then the first inequality of (4.12) can be replaced by the stronger inequality

$$\delta + 0.09945 \leqq \frac{\kappa}{3 \times 0.09965 \times 4.8132} = \frac{\kappa}{1.43890614}$$

which can be further replaced by the stronger inequality

$$\delta \leqq \frac{\kappa}{1.439} - 0.09945.$$

Combining this with (4.13), we have

(4.15) $$\frac{0.00012033}{1 - \kappa} \leqq \delta \leqq \frac{\kappa}{1.439} - 0.09945.$$

Now let us consider the inequality

$$0.00012033 \times 1.439 \leqq (1-\kappa)(\kappa - 1.439 \times 0.09945),$$

i. e.,

$$0.00017315487 \leqq (1-\kappa)(\kappa - 0.14310855).$$

Then this is evidently satisfied by

$$\kappa = 0.144.$$

But for this value of κ,

$$\text{(4.16)} \qquad \begin{aligned} &\frac{0.00012033}{1-\kappa} = 0.00014057\ldots, \\ &\frac{\kappa}{1.439} - 0.09945 = 0.00061949\ldots. \end{aligned}$$

Consequently, by our restriction (4.14), it is seen that we have only to choose δ so that

$$\text{(4.17)} \qquad 0.000141 \leqq \delta \leqq 0.0002.$$

This shows that there are indeed positive constants δ and κ (<1) satisfying (4.12).

From this result, by Proposition 3, we see that the given equation which is equivalent to (4.2) has really one and only one exact isolated periodic solution $x = \hat{x}(t)$ in the region

$$\text{(4.18)} \qquad D_\delta = \{x \mid \| x - \bar{x}(t) \| \leqq \delta \text{ for some } t \in L\}.$$

Further, from (4.16) we see that

$$\text{(4.19)} \qquad |\bar{x}(t) - \hat{x}(t)| < 0.000141$$

for the Galerkin approximation $x = \bar{x}(t)$ of (4.5).

Remark. The region D_δ of (4.18) can be made broader by allowing a larger value to δ. For instance, instead of (4.14), let us suppose

$$\delta \leqq 0.1.$$

Then the first inequality of (4.12) can be replaced by the stronger inequality

$$\delta + 0.09945 \leqq \frac{\kappa}{3 \times 0.2 \times 5} = \frac{\kappa}{3}.$$

Then, by combining this with (4.13) we have

$$\frac{0.00012033}{1-\kappa} \leqq \delta \leqq \frac{\kappa}{3} - 0.09945.$$

Now let us consider the inequality

$$3 \times 0.00012033 \leqq (1-\kappa)(\kappa - 3 \times 0.09945),$$

i. e.

$$0.00036099 \leqq (1-\kappa)(\kappa - 0.29835).$$

This is evidently satisfied by

$$\kappa = 0.9.$$

But, for this value of κ,

$$\frac{0.00012033}{1-\kappa}=0.0012033,$$

$$\frac{\kappa}{3}-0.09945=0.20055.$$

Consequently, by our restriction it is seen that we have only to choose δ so that

$$0.0012033 \leqq \delta \leqq 0.1.$$

From this result we see that an isolated periodic solution of the given equation (4.1) is unique in the region

$$D'\{x \mid \|x-\bar{x}(t)\| \leqq 0.1 \text{ for some } t \in L\}.$$

For $\kappa=0.9$ we can obtain again an error estimate for the Galerkin approximation, but this error estimate is of course worse than that of (4.19).

Appendix

Proposition A. *Let us assume that the conditions of Theorem 1 are satisfied, and furthermore, let us assume that there is a positive integer m_0 such that, for any $m \geqq m_0$, there exists a Galerkin approximation*

$$x=\bar{x}_m(t)=\bar{a}_0+\sqrt{2}\sum_{n=1}^{m}(\bar{a}_n\cos n t+\bar{b}_n\sin n t)$$

of order m lying in D, for which

(A.1) $$\det J_m(\bar{\alpha}) \neq 0$$

and

(A.2) $$\|J_m^{-1}(\bar{\alpha})\| \leqq M'.$$

Here $\bar{\alpha}=(\bar{a}_0, \bar{a}_1, \bar{b}_1, \ldots, \bar{a}_m, \bar{b}_m)$, and M' is a given positive constant independent of m.

Then, if $m_1 \geqq m_0$ is sufficiently large, the multipliers of the linear homogeneous system

(A.3) $$\frac{dy}{dt}=\Psi[\bar{x}_m(t), t]\, y$$

are all different from one for any $m \geqq m_1$.

Proof. For brevity let us put

(A.4) $$\bar{A}_m(t)=\Psi[\bar{x}_m(t), t],$$

and let us consider the linear system

(A.5) $$\frac{dy}{dt}=P_m\bar{A}_m(t)\, y+\varphi(t),$$

where $\varphi(t)$ is an arbitrary trigonometric polynomial of the form

(A.6) $$\varphi(t)=c_0+\sqrt{2}\sum_{n=1}^{m}(c_n\cos n t+d_n\sin n t).$$

Because of the particular form of the right member of (A.5), a periodic solution $y=y(t)$ of (A.5), if it exists, is a trigonometric polynomial of the form

$$y(t)=u_0+\sqrt{2}\sum_{n=1}^{m}(u_n\cos n\,t+v_n\sin n\,t). \tag{A.7}$$

As has been mentioned at the beginning of Section 2.2, between $\xi=(u_0, u_1, \ldots, u_m, v_m)$ and $\gamma=(c_0, c_1, d_1, \ldots, c_m, d_m)$, the following relation holds:

$$J_m(\bar{\alpha})\,\xi+\gamma=0. \tag{A.8}$$

By assumption (A.1) this can be solved with respect to ξ as follows:

$$\xi=-J_m^{-1}(\bar{\alpha})\,\gamma. \tag{A.9}$$

This means that a periodic solution $y=y(t)$ of (A.5) always exists and is determined uniquely for any trigonometric polynomial $\varphi(t)$ of the form (A.6). Furthermore, from (A.2) it follows that

$$\|\xi\|\leqq M'\,\|\gamma\|,$$

which implies

$$\|y\|_q\leqq M'\,\|\varphi\|_q \tag{A.10}$$

since $\|y\|_q=\|\xi\|$ and $\|\varphi\|_q=\|\gamma\|$.

Now let us rewrite (A.5) as follows:

$$\frac{dy}{dt}=\bar{A}_m(t)\,y+\varphi(t)+\eta(t), \tag{A.11}$$

where

$$\eta(t)=-(I-P_m)\,\bar{A}_m(t)\,y(t). \tag{A.12}$$

As seen from (3.4) and (3.6), $\dot{\bar{x}}_m(t)$ is equibounded in the norm $\|\dot{\bar{x}}_m\|_n$; consequently, as in the proof of the Corollary of Lemma 2.2, there is a non-negative constant K_3' independent of m such that

$$\left\|\frac{d\bar{A}_m(t)}{dt}\right\|\leqq K_3' \quad \text{for} \quad m\geqq m_0.$$

Since (A.11) is of the same form as (2.21), from (2.26) we have

$$\|\eta\|_q\leqq\sigma_1(m)\,[(K_3'+K_1^2)\,\|y\|_q+K_1\,\|\varphi\|_q],$$

from which, by means of (A.10), we deduce that

$$\|\eta\|_q\leqq[(K_3'+K_1^2)\,M'+K_1]\,\sigma_1(m)\,\|\varphi\|_q. \tag{A.13}$$

On the other hand, $y(t)$ is a solution of (A.11), and consequently it can be written in the form

$$y(t)=\Phi(t)\,c+\Phi(t)\int_0^t\Phi^{-1}(s)\,[\varphi(s)+\eta(s)]\,ds,$$

where c is a constant vector and $\Phi(t)$ is a fundamental matrix of (A.3) such that $\Phi(0)=E$ (E is a unit matrix). Since $y(t)$ is periodic, we must have

(A.14) $$[\Phi(2\pi)-E]\,c+\Phi(2\pi)\int_0^{2\pi}\Phi^{-1}(s)\,[\varphi(s)+\eta(s)]\,ds=0.$$

Now let us suppose that at least one of the multipliers of (A.3) is one, and hence,

(A.15) $$\det[\Phi(2\pi)-E]=0.$$

Then there exists a non-trivial vector h such that

(A.16) $$h^*[\Phi(2\pi)-E]=0,$$

where h^* denotes the transpose of h. From (A.14) it follows that

(A.17) $$\int_0^{2\pi} v^*(s)\,[\varphi(s)+\eta(s)]\,ds=0,$$

where

$$v^*(s)=h^*\,\Phi(2\pi)\,\Phi^{-1}(s).$$

By (A.16), $v^*(s)$ can be written as

$$v^*(s)=h^*\,\Phi^{-1}(s),$$

or

(A.18) $$v(s)=\Phi^{*\,-1}(s)\,h.$$

Since $\Phi(s+2\pi)=\Phi(s)\,\Phi(2\pi)$, we have successively (*cf.* (A.16))

$$\begin{aligned} v(s+2\pi) &= \Phi^{*\,-1}(s)\,\Phi^{*\,-1}(2\pi)\,h \\ &= \Phi^{*\,-1}(s)\,h \\ &= v(s). \end{aligned}$$

This means that $v(s)$ is periodic in s of period 2π. Because h can be multiplied by any constant factor, we may suppose without loss of generality that

(A.19) $$\|v\|_q=1.$$

Since

$$\frac{dv(s)}{ds}=-\bar{A}_m^*(s)\,v(s),$$

we have also

(A.20) $$\|\dot{v}\|_q\leqq K_1',$$

where

$$K_1'=\max_{D\times L}\|\Psi^*(x,t)\|.$$

Since $\varphi(t)$ is an arbitrary trigonometric polynomial of the form (A.6), we can take $\varphi(t)$ so that

$$\varphi(t)=v_m(t)\overset{\text{def}}{=}P_m\,v(t).$$

By Lemma 2.1 it follows from (A.20) that

(A.21) $$\|\varphi-v\|_q=\|v_m-v\|_q\leqq K'_1\,\sigma_1(m).$$

Using (A.19), we have

$$\|\varphi\|_q\leqq 1+K'_1\,\sigma_1(m).$$

Then from (A.13) we have

(A.22) $$\|\eta\|_q\leqq[(K'_3+K_1^2)M'+K_1]\cdot[1+K'_1\,\sigma_1(m)]\,\sigma_1(m).$$

Now (A.17) can be rewritten as follows:

$$\frac{1}{2\pi}\int_0^{2\pi} v^*(s)\,v(s)\,ds=\frac{1}{2\pi}\int_0^{2\pi} v^*(s)\{[v(s)-\varphi(s)]-\eta(s)\}\,ds,$$

namely

$$\|v\|_q^2=\frac{1}{2\pi}\int_0^{2\pi} v^*(s)\{[v(s)-\varphi(s)]-\eta(s)\}\,ds.$$

Then by SCHWARZ' inequality we have

$$\|v\|_q^2\leqq\|v\|_q[\|v-\varphi\|_q+\|\eta\|_q],$$

from which, by (A.19), (A.21) and (A.22), it follows that

$$1\leqq K'_1\,\sigma_1(m)+[(K'_3+K_1^2)M'+K_1]\cdot[1+K'_1\,\sigma_1(m)]\,\sigma_1(m).$$

This is a contradiction for large m, since $\sigma_1(m)=1/(m+1)\to 0$ as $m\to\infty$. Thus, we see that (A.15) does not hold for any $m\geqq m_1$ provided $m_1\geqq m_0$ is sufficiently large. This proves the proposition. Q.E.D.

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