Singular Solutions for some Semilinear Elliptic Equations

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Dedicated to James Serrin on his sixtieth birthday

1. Introduction

Let $B_R = \{x \in \mathbb{R}^N;\ |x| < R\}$ with $N \geqq 2$. Consider a function u which satisfies

$$\begin{gathered} u \in C^2(B_R \setminus \{0\}), \quad u \geqq 0 \quad \text{on } B_R \setminus \{0\}, \\ -\Delta u + u^p = 0 \quad \text{on } B_R \setminus \{0\}. \end{gathered} \tag{1}$$

We are concerned with the behavior of u near $x = 0$. There are two distinct cases:

1) When $p \geqq N/(N-2)$ and $(N \geqq 3)$ it has been shown by BREZIS & VÉRON [9] that u must be smooth at 0 (See also BARAS & PIERRE [1] for a different proof). In other words, isolated singularities are *removable*.

2) When $1 < p < N/(N-2)$ there are solutions of (1) with a singularity at $x = 0$. Moreover all singular solutions have been classified by VÉRON [22]. We recall his result:

Theorem 1. *Assume that* $1 < p < N/(N-2)$ *and that* u *satisfies* (1). *Then one of the following holds:*

(i) *either* u *is smooth at* 0,

(ii) or $\lim_{x \to 0} u(x)/E(x) = c$ *where* c *is a constant which can take any value in the interval* $(0, \infty)$,

(iii) or $\lim_{x \to 0} |u(x) - l(p, N)|x|^{-2/(p-1)}| = 0$.

Here $E(x)$ denotes the fundamental solution of $-\Delta$ and $l = l(p, N)$ is the (unique) positive constant C such that $C|x|^{-2/(p-1)}$ satisfies (1)—more precisely

$$l = l(p, N) = \left[\frac{2}{(p-1)}\left(\frac{2p}{p-1} - N\right)\right]^{1/(p-1)}.$$

We shall first present a proof of Theorem 1 which is simpler than the original proof of Véron. In particular, it does not make use of FOWLER's results [10] for the Emden differential equation. Instead, it relies on some simple *scaling argument* (see the proof of Lemma 5) which is similar to the one used by KAMIN & PELETIER [12] for parabolic equations.

Next, we emphasize that a *singular behavior* such as (ii) or (iii) *can be prescribed* together with a boundary condition, and these determine uniquely the solution.

More precisely, let Ω be a smooth bounded domain in $\mathbb{R}^N$ with $0 \in \Omega$ and let $\varphi \geqq 0$ be a smooth function defined on $\partial\Omega$. We consider the problem

$$\begin{gathered} u \in C^2(\bar{\Omega} \setminus \{0\}), \quad u \geqq 0 \quad \text{on } \Omega \setminus \{0\}, \\ -\Delta u + u^p = 0 \quad \text{on } \Omega \\ u = \varphi \quad \text{on } \partial\Omega. \end{gathered} \tag{2}$$

Theorem 2. *Assume* $1 < p < N/(N-2)$. *Then*

(i) *There is a unique solution* u_0 *of* (2) *which belongs to* $C^2(\bar{\Omega})$.

(ii) *Given any constant* $c \in (0, +\infty)$ *there is a unique solution* u_c of (2) *which satisfies*

$$\lim_{x \to 0} u(x)/E(x) = c.$$

(iii) *There is a unique solution* u_∞ *of* (2) *which satisfies*

$$\lim_{x \to 0} |x|^{2/(p-1)} u(x) = l(p, N)$$

In addition, $\lim_{c \downarrow 0} u_c = u_0$ *and* $\lim_{c \uparrow \infty} u_c = u_\infty$.

Singular solutions of (1) occur in the THOMAS-FERMI theory with $N = 3$ and $p = 3/2$ (see *e.g.* [13] for a detailed exposition). Other results dealing with singular solutions of nonlinear elliptic equations have been obtained by a number of authors: J. SERRIN [20], [21], VERON and VAZQUEZ (See the exposition in [23]), P. L. LIONS [14], W. M. NI & J. SERRIN [16]. Semilinear parabolic equations with isolated singularities have been considered by BREZIS & FRIEDMAN [5], BREZIS & PELETIER & TERMAN [8], KAMIN & PELETIER [12], OSWALD [18].

2. Some preliminary facts

We recall some known results dealing with functions u satisfying (1). Set $\alpha = 2/(p-1)$ (for $1 < p < \infty$).

Lemma 1. *Assume* $u \in C^2(B_R)$ *satisfies* (1).
Then

$$u(0) \leqq C(p, N)/R^\alpha$$

where $C(p, N)$ *is defined by* $C(p, N) = \text{Max}\,\{2\alpha N, 4\alpha(\alpha + 1)\}^{1/(p-1)}$.

The proof of Lemma 1 uses a comparison function U of the same type as in OSSERMAN [17] (or LOEWNER & NIRENBERG [15]), namely set

$$U(x) = \frac{C(p, n)\, R^{\alpha}}{(R^2 - |x|^2)^{\alpha}} \text{ on } B_R\,.$$

A direct computation shows that

$$-\Delta U + U^p \geqq 0 \quad \text{on } B_R.$$

By the maximum principle we see that

$$u \leqq U \quad \text{on } B_R$$

and in particular $u(0) \leqq U(0)$.

Lemma 2. *Assume u satisfies* (1) *with* $1 < p < N/(N-2)$. *Then, for* $0 < |x| < R/2$,

$$u(x) \leqq \frac{l(p, N)}{|x|^{\alpha}} \left(1 + \frac{C(p, N)}{l(p, N)} \left(\frac{|x|}{R}\right)^{\beta}\right)$$

where $\beta = 2\alpha + 2 - N > \alpha$.

Lemma 2 is established in BREZIS & LIEB [6] (Proposition A.4) for the special case where $N = 3$ and $p = 3/2$. The proof in the general case is just the same.

Lemma 3. *Assume* $1 < p < N/(N-2)$ *and let* $c > 0$ *be a constant. Then there is a unique function u satisfying*

$$\begin{aligned} &u \in L^p(\mathbb{R}^N) \cap C^2(\mathbb{R}^N \setminus \{0\}), \\ &u \geqq 0 \quad \text{on } R^N \setminus \{0\}, \\ &-\Delta u + u^p = c\delta \quad \text{on } \mathbb{R}^N \end{aligned} \tag{3}$$

We set $u = W_c$.

Lemma 3, as well as Lemma 4 below, are due to BENILAN & BREZIS (unpublished); the ingredients for the proofs may be found in [2], [3], [4] (and also [1] and [11]).

Finally, we assume that Ω is a smooth bounded domain in $\mathbb{R}^N$ with $0 \in \Omega$ and that $\varphi \geqq 0$ is a smooth function defined as $\partial\Omega$.

Lemma 4. *Assume* $1 < p < N/(N-2)$ *and let* $c > 0$ *be a constant. Then, there is a unique function u satisfying*

$$\begin{aligned} &u \in L^p(\Omega) \cap C^2(\overline{\Omega} \setminus \{0\}) \\ &u \geqq 0 \quad \text{on } \Omega \setminus \{0\} \\ &-\Delta u + u^p = c\delta \quad \text{on } \Omega \\ &u = \varphi \quad \text{on } \partial\Omega. \end{aligned} \tag{4}$$

3. A Scaling Argument

An important step in the proof of Theorem 1 is the following

Lemma 5. *Assume* $1 < p < N/(N-2)$. *Then*

$$\lim_{c\uparrow\infty} W_c(x) = l\,|x|^{-\alpha} \equiv W_\infty(x).$$

Proof. It is clear (by comparison) that $W_c(x)$ is a nondecreasing function of c. Moreover we have

$$W_c(x) \leqq l\,|x|^{-\alpha}$$

(by letting $R \to \infty$ in Lemma 2). Therefore $\lim_{c\uparrow\infty} W_c(x) = W_\infty(x)$ exists pointwise (for $x \neq 0$) and $W_\infty(x) \leqq l\,|x|^{-\alpha}$. The uniqueness of the solution of (3) implies that $W_c(x)$ is radial and so is $W_\infty(x)$. Next, we observe that the function

$$u(x) = k^\alpha W_c(kx) \qquad (k > 0)$$

satisfies

$$-\Delta u(x) + u^p(x) = k^{\alpha p} c\delta(kx) = k^{\alpha p - N} c\delta(x).$$

It follows, again by uniqueness, that

$$k^\alpha W_c(kx) = W_{ck^{\alpha p - N}}(x).$$

As $c \uparrow \infty$ we see that

$$k^\alpha W_\infty(kx) = W_\infty(x).$$

Choosing $k = 1/|x|$ we obtain

$$W_\infty(x) = W_\infty\left(\frac{x}{|x|}\right)|x|^{-\alpha} = C\,|x|^{-\alpha}$$

where $C > 0$ is some constant. Finally we note that since

$$-\Delta W_c + W_c^p = 0 \quad \text{in } \mathscr{D}'(\mathbb{R}^N \setminus \{0\})$$

and

$$W_c \to W_\infty \quad \text{in } L^p_{\mathrm{loc}}(\mathbb{R}^N \setminus \{0\}),$$

it follows that

$$-\Delta W_\infty + W_\infty^p = 0 \quad \text{in } \mathscr{D}'(\mathbb{R}^N \setminus \{0\}).$$

This determines the value of the constant C to be $C = l$.

There is a similar result in balls: Set $u = V_c$ to be the unique solution of problem (4) with $\Omega = B_R$.

Lemma 6. *Assume* $1 < p < N/(N-2)$. *Then* $V_\infty(x) \equiv \lim_{c \uparrow \infty} V_c(x)$ *exists pointwise on* $B_R \setminus \{0\}$ *and moreover*

$$W_\infty(x) - lR^{-\alpha} \leqq V_\infty(x) \leqq W_\infty(x) \quad \text{on } B_R.$$

Proof. It is again clear (by comparison) that $V_c(x)$ is a nondecreasing function of c. Also we have

$$0 \leqq V_c(x) \leqq W_c(x). \tag{5}$$

It follows from (4) and (5) that

$$-\Delta(W_c - V_c) \leqq 0 \quad \text{on } B_R,$$

and consequently $\operatorname{Sup}_{B_R} (W_c - V_c) \leqq \operatorname{Sup}_{\partial B_R} (W_c - V_c) \leqq \operatorname{Sup}_{\partial B_R} W_\infty = lR^{-\alpha}$. The conclusion follows by letting $c \to \infty$.

4. Proof of Theorem 1

Throughout this section we suppose $1 < p < N/(N-2)$. Assume u satisfies (1) and set

$$c = \limsup_{x \to 0} u(x)/E(x).$$

We distinguish three cases:

Case (i) $c = 0$

Case (ii) $0 < c < \infty$

Case (iii) $c = \infty$.

Cases (i) *and* (ii). Here the main ingredient is the following:

Lemma 7. *In cases* (i) *and* (ii) *the function u belongs to* $L^p_{\mathrm{loc}}(B_R)$ *and satisfies*

$$-\Delta u + u^p = c_0 \delta \quad \text{in } \mathscr{D}'(B_R)$$

for some constant c_0.

Proof. It is clear that $u \in L^p_{\mathrm{loc}}(B_R)$ since $E \in L^p_{\mathrm{loc}}(B_R)$ and $c < \infty$. We now use the same argument as in [7]: set

$$T = -\Delta u + u^p \in \mathscr{D}'(B_R).$$

Since the support of T is contained in $\{0\}$, it follows from a classical result about distributions (see [19]) that

$$T = \sum_{0 \leqq |\alpha| \leqq m} c_\alpha D^\alpha(\delta). \tag{6}$$

We claim $c_\alpha = 0$ when $|\alpha| \geqq 1$. Indeed let $\zeta \in \mathscr{D}(B_R)$ be any fixed function such that $(-1)^{|\alpha|} D^\alpha \zeta(0) = c_\alpha$ for every α with $|\alpha| \leqq m$. Multiplying (6) through by $\zeta_\varepsilon(x) = \zeta(x/\varepsilon)$ we obtain

$$-\int u \Delta \zeta_\varepsilon + \int u^p \zeta_\varepsilon = \sum_{0 \leq |\alpha| \leq m} c_\alpha^2 \varepsilon^{-|\alpha|}.$$

An easy computation—using the estimate $u \leqq CE$—shows that

$$\left|\int u \Delta \zeta_\varepsilon\right| \leqq C \quad \text{when } N \geqq 3$$

$$\left|\int u \Delta \zeta_\varepsilon\right| \leqq C\,|\log \varepsilon| + C \quad \text{when } N = 2.$$

Since $\int u^p \zeta_\varepsilon \to 0$ as $\varepsilon \to 0$, we conclude that $c_\alpha = 0$ for $|\alpha| \geqq 1$. Therefore we obtain

$$-\Delta u + u^p = c_0 \delta \quad \text{in } \mathscr{D}'(B_R).$$

We conclude the proof of Theorem 1 in cases (i) and (ii) with the help of the following:

Lemma 8. *Assume* $u \in C^2(B_R \setminus \{0\}) \cap L^p_{\text{loc}}(B_R)$ *satisfies*

$$\begin{cases} u \geqq 0 & \text{on } B_R, \\ -\Delta u + u^p = c_0 \delta & \text{in } \mathscr{D}'(B_R) \end{cases}$$

for some constant c_0. *Then*

(i) if $c_0 = 0$, *then* u *is smooth on* B_R,

(ii) if $c_0 \neq 0$, *then* $\lim_{x \to 0} u(x)/E(x) = c_0$.

Proof.
(i) Assume $c_0 = 0$. Since u is subharmonic it follows that $u \in L^\infty_{\text{loc}}(B_R)$ and thus $\Delta u \in L^\infty_{\text{loc}}(B_R)$. We deduce that $u \in C^1(B_R)$ and then $u \in C^2(B_R)$. In fact $u \in C^\infty(B_R)$ since, by the strong maximum principle, we have either $u \equiv 0$ or $u > 0$ or B_R.

(ii) Assume $c_0 \neq 0$. By the maximum principle we have

$$u \leqq c_0 E + C \quad \text{on } B_{R/2}$$

and therefore

$$\begin{aligned} -\Delta u &\geqq c_0 \delta - (c_0 E + C)^p \\ &\geqq c_0 \delta - C(E^p + 1) \quad \text{on } B_{R/2} \end{aligned}$$

An elementary computation leads to

$$u(x) \geqq c_0 E - o(E) \quad \text{as } x \to 0,$$

and we conclude that $\lim_{x \to 0} u(x)/E(x) = c_0$.

Remark 1. Assume $c_0 \neq 0$. The argument above provides in fact an estimate for $|u - c_0 E|$ as $x \to 0$. More precisely we have

a) If $N = 2$ and $1 < p < \infty$ or $N = 3$ and $1 < p < 2$, then

$$|u - c_0 E| \leqq C \quad \text{on } B_{R/2}$$

b) If $N = 3$ and $p = 2$, then

$$|u(x) - c_0 E(x)| \leqq C(|\log |x|| + 1) \quad \text{on } B_{R/2}$$

c) If $N = 3$ and $2 < p < 3$ or $N \geqq 4$ and $1 < p < N/(N-2)$ then

$$|u(x) - c_0 E(x)| \leqq C |x|^{2-(N-2)p} \quad \text{on } B_{R/2}$$

and consequently

$$\left| \frac{u(x)}{E(x)} - c_0 \right| \leqq C |x|^{\nu} \quad \text{on } B_{R/2}$$

with $\nu = N - (N-2)p > 0$.

Proof of Theorem 1 in Case (iii). We first recall a result of VÉRON [22] (Lemma 1.5):

Lemma 9. *Assume u satisfies* (1). *Then there is a constant V (depending only on p and N) such that*

$$\operatorname*{Sup}_{|x|=r} u(x) \leqq C \operatorname*{Inf}_{|x|=r} u(x) \quad \textit{for } 0 < r < R/2.$$

The conclusion of Lemma 9 is a simple consequence of Harnack's inequality and the estimate of Lemma 1, see [22] for the details.

We may now complete the proof of Theorem 1 with the help of the following:

Lemma 10. *Assume u satisfies* (1) and $\limsup_{x \to 0} u(x)/E(x) = \infty$. *Then*

$$|u(x) - l\,|x|^{-\alpha}| \leqq C\,|x|^{\gamma} \quad \text{on } B_{R/2}$$

for some constants $C = C(p, N, R)$ *and* $\gamma = \gamma(p, N) > 0$.

Proof. By Lemma 2 we already have the estimate

$$u(x) \leqq l\,|x|^{-\alpha} + C\,|x|^{\gamma} \quad \text{on } B_{R/2}$$

with

$$\gamma = \beta - \alpha = \alpha + 2 - N > 0.$$

We now establish an estimate from below. Let $x_n \to 0$ be such that $\lim u(x_n)/E(x_n) = \infty$. Set $r_n = |x_n|$, so that we obtain from Lemma 9

$$\operatorname*{Inf}_{|x|=r_n} u(x)/E(x) \underset{n\to\infty}{\to} \infty. \tag{7}$$

We recall that V_c is the unique solution of (4) when $\Omega = B_R$, so that $V_c \leqq cE$ on B_R.

Given any constant $c > 0$, we see (by (7)) that

$$u(x) \geqq cE(x) \quad \text{for } |x| = r_n \text{ and } n \text{ large enough}.$$

Therefore

$$u(x) \geqq V_c(x) \quad \text{for } |x| = r_n \text{ and } n \text{ large enough}.$$

Applying the maximum principle in the domain $\{x \in \mathbb{R}^N;\, r_n < |x| < R\}$ we find that

$$u(x) \geqq V_c(x) \quad \text{for } r_n < |x| < R \text{ and } n \text{ large enough}.$$

As $n \to \infty$ we conclude that

$$u(x) \geqq V_c(x) \quad \text{on } B_R \setminus \{0\}$$

and as $c \to \infty$ we see that

$$u(x) \geqq V_\infty(x) \quad \text{on } B_R \setminus \{0\}.$$

In Lemma 6 we had the estimate

$$V_\infty(x) \geqq l(|x|^{-\alpha} - R^{-\alpha}).$$

However it is not good enough to deduce conclusion (iii) of Theorem 1. We need a better estimate from below for $V_\infty(x)$; we claim that

$$V_\infty(x) \geqq l\,|x|^{-\alpha}\left(1 - \left(\frac{|x|}{R}\right)^{\beta}\right) \quad \text{on } B_R, \tag{8}$$

where β is defined in Lemma 2.

Clearly, it suffices to establish (8) for $R = 1$. The function V_∞ is radial and so we write $V_\infty(r)$. We define the function v on (0, 1) by the relation

$$v(r^\beta) = l^{-1} r^\alpha V_\infty(r)$$

so that $0 \leqq v \leqq 1$ on (0, 1), $v(1) = 0$ and $v(0) = 1$. Using the relation $-\Delta V_\infty + V_\infty^p = 0$ it is easy to deduce (as in the proof of Proposition A.4 [6]) that

$$-\beta^2 t^2 v''(t) + l^{p-1} v(t)\,(v^{p-1}(t) - 1) = 0 \quad \text{for } t \in (0, 1).$$

Consequently v is concave and thus we have

$$v(t) \geqq 1 - t \qquad t \in (0, 1),$$

which is (8).

Remark 2. VÉRON [22] obtains in case (iii) an estimate of the form $|u(x) - l\,|x|^{-\alpha}| \leqq C\,|x|^{\delta}$ with an exponent δ which is better than $\gamma = \beta - \alpha$.

5. Proof of Theorem 2

Case (i) is classical.

Case (ii). The existence of a solution follows from Lemma 4 and 8. Suppose now u satisfies (2) and $\lim_{x\to 0} u(x)/E(x) = c$. We deduce from Lemma 7 and 8 that $-\Delta u + u^p = c\delta$; uniqueness follows from Lemma 4.

Case (iii). We denote by u_c the unique solution of (4) given by Lemma 4. We claim that $u_\infty = \lim_{c\uparrow\infty} u_c$ has all the required properties.

Indeed $u_c(x)$ is a nondecreasing function of c. Fix $R > 0$ such that $2R <$ dist $(0, \partial\Omega)$. By Lemma 1 we have

$$u_c(x) \leqq C(p, N)\, R^{-\alpha} \quad \text{for } |x| = R.$$

The maximum principle applied in the region

$$\Omega_R = \{x \in \Omega;\ |x| > R\}$$

shows that, in Ω_R,

$$u_c(x) \leqq \operatorname{Max}\{\operatorname{Sup}_{\partial\Omega} \varphi,\ C(p, N)\, R^{-\alpha}\}.$$

Therefore $u_\infty(x) = \lim_{c\uparrow\infty} u_c(x)$ exists and u_∞ satisfies (2). By comparison on B_R we have

$$V_c \leqq u_c \quad \text{on } B_R$$

and as $c \to \infty$ we obtain $V_\infty \leqq u_\infty$ on B_R. It follows that $\lim_{x\to 0} |u_\infty(x) - l|x|^{-\alpha}| = 0$ (by Lemma 6 and Theorem 1).

We turn now to the question of uniqueness. Suppose u_1 and u_2 satisfy (2) and $\lim_{x\to 0} |x|^\alpha u_i(x) = l$ for $i = 1, 2$. Lemma 10 implies that

$$|u_1(x) - u_2(x)| \leqq C\,|x|^\gamma \quad \text{on } B_R.$$

On the other hand

$$-\Delta(u_1 - u_2) + u_1^p - u_2^p = 0 \quad \text{on } \Omega \setminus \{0\}.$$

Applying the maximum principle in Ω_R we obtain

$$\operatorname*{Max}_{\Omega_R} |u_1 - u_2| \leqq \operatorname*{Max}_{\partial B_R} |u_1 - u_2| \leqq CR^\gamma$$

and then we let $R \to 0$ to conclude that $u_1 = u_2$.

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