A Continuum Theory for Granular Materials

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1. Introduction

In 1776, COULOMB [1] presented the first yield criterion in mechanics, a yield criterion for soils, including granular materials. The Coulomb criterion states that slip is impending on a plane in the material when

$$S = bT + c, \tag{1.1}$$

where S and T are the shear stress and normal stress, respectively, acting on the plane, b is a coefficient of static friction, and c is a coefficient of cohesion. The Mohr-Coulomb theory of limiting equilibrium, based on the Coulomb criterion and the equations of static equilibrium, was formulated by KÖTTER [2] and extended and summarized by SOKOLOVSKII [3]. The Coulomb criterion also serves as the starting or focal point for most existing three-dimensional deformation theories for soils which, for the most part, are extensions or modifications of the theory of elastic-plastic bodies [4, 5, 6, 7, 8, 9]. Such theories, however, do not recognize the two (or three) phase nature of granular materials and, consequently, yield results which are independent of the magnitude and distribution of the void volume.

In this paper, we present a theory for granular materials formulated from formal arguments of continuum mechanics. The basic premise underlying the paper is that the concept of mass distribution must be extended to admit granular materials. In particular, the distribution of mass must be related to the volume distribution of granules. To achieve this, we introduce an independent kinematical variable, called the *volume distribution function.*

The following physically motivated assumptions associated with the volume distribution of granules in a granular material form the basis for the proposed theory of granular materials.

1. The volume of granules in a granular material is regarded as a measure on Euclidean space. The measure is equally valid for solid, porous materials (rock, cork, sponge, *etc.*) as well as granular materials (sand, grain, powder, *etc.*)

2. The mass measure is assumed absolutely continuous with respect to the volume distribution measure. This assumption is tantamount to neglecting the void mass and is consistent with one's intuitive notion of granular materials; *e.g.* consider dry sand. Since the void mass is neglected, only one type of material point need be considered for describing the motion of the body.

3. To account for energy flux and energy supply associated with the time rate of change of volume distribution, a higher order stress and body force are introduced. Such terms are expected since the volume distribution function and the motion are kinematically independent. They are also expected on physical grounds; see, for example [10, 11, 12, 13]. The introduction of these terms is not without precedent. Terms of this type are contained in the higher order elasticity theories developed by MINDLIN [14], TOUPIN [15], and GREEN & RIVLIN [16] and in the theory of liquid crystals as presented by ERICKSEN [17] and LESLIE [18]. For granular materials, GOODMAN [19] has shown that the higher order stress degenerates to an equilibrated stress related to a system of self-equilibrating forces resulting in either a center of compression or center of dilatation.

4. From a conceptual viewpoint, the flow behavior of granular materials is considered to be similar to fluid behavior. Specifically, the response of a granular material is unaltered by any change in reference configuration that does not change the density and, in addition, does not change the volume distribution. The condition on the volume distribution requires that granular materials have preferred reference configurations with respect to volume distribution. This is consistent with the experimental result reported by many investigators [20, 21, 22] that the bulk compressibility of granular materials is dependent on the initial porosity.

In Section 2, we introduce the concept of a distributed body which we propose as a continuum model for granular and porous bodies. The work of NOLL [23] is paralleled in formulating the concepts of mass distribution and volume distribution. Kinematics and thermodynamic processes for distributed bodies are considered in Sections 3 and 4. In the thermodynamic development, we leave the entropy flux arbitrary as proposed by MÜLLER [24]. In Sections 5 and 6, we consider constitutive equations for granular materials and restrictions imposed by the entropy inequality. We linearize the theory in Section 7 and show that a necessary condition for equilibrium is that the stress reduces to a generalized form of the Mohr-Coulomb stress state of limiting equilibrium. In Section 8, we present results for granular materials subject to the kinematical constraint of incompressible granules.

Cartesian tensor notation is employed throughout the paper.

2. The Distributed Body

The distribution of the solid constituent in porous and granular materials is a distinguishing characteristic of these materials. The distribution of solid volume must be known before one can determine the distribution of solid mass. To express this idea precisely, we introduce the concept of a distributed body.

A *distributed body* is a one-parameter family $\{B_t\}$, $-\infty < t < \infty$, of regions of Euclidean three space such that

(a) for any t and t', the region B_t is homeomorphic to the region $B_{t'}$, and

(b) for each t, the region B_t is endowed with a structure given by two real valued set functions $\mathcal{M}_t$ and $\mathcal{V}_t$ subject to the following axioms:

(b1) $\mathcal{M}_t$ and $\mathcal{V}_t$ are non-negative measures defined for all Borel subsets $P_t \subset B_t$, (b2) $\mathscr{V}_t(P_t) \leqq V(P_t)$*, for all $P_t \subset B_t$,

(b3) $\mathscr{M}_t$ is absolutely continuous with respect to $\mathscr{V}_t$.

In the above terminology, B_t is the *configuration* of the distributed body at time t and $\mathscr{V}_t$ and $\mathscr{M}_t$ are the *distributed volume* and *distributed mass*, respectively. If the homeomorphisms associated with $\{B_t\}$ are restricted to subsets of B_t, then $\{P_t\} \subset \{B_t\}$ is a *part* where it is understood that $P_t \subset B_t$ for each t. For any $P_t \subset B_t$, the quantities $\mathscr{V}_t(P_t)$ and $\mathscr{M}_t(P_t)$ are the distributed volume and distributed mass of the part $\{P_t\}$ at time t. Henceforth, it will be convenient to suppress the bracket notation and refer to P_t as a part of B_t.

Before proceeding, we wish to comment on the significance of axioms (b2) and (b3) in the above definition. Clearly, axiom (b2) is characteristic of all porous and granular bodies. The volume of solid is always less than or equal to the total volume of any part of the body. Axiom (b3), however, imposes certain features on the body. It provides a continuum aspect to an otherwise discrete medium. The idea of discrete pores and granules is no longer retained as volume continuity rules out the possibility of point, line, or surface concentrations of mass. Moreover, condition (b3) implies that the mass of a distributed body is only associated with the distributed volume $\mathscr{V}_t$ and not the void volume $V - \mathscr{V}_t$. We interpret this to mean that the void mass of a porous or granular material can be neglected.

From axiom (b2), it follows that the distributed volume measure $\mathscr{V}_t$ is absolutely continuous with respect to the Lebesgue volume measure V. Hence, by the Radon-Nikodym theorem, there exists a real valued Lebesgue integrable function $\nu(x_i, t)$ defined on B_t such that for any part $P_t \subset B_t$

$$\mathscr{V}_t(P_t) = \int_{P_t} \nu \, dV. \tag{2.1}$$

Moreover, the function ν, called the *volume distribution* function**, has the property that for almost all $x_i \in B_t$

$$0 \leqq \nu(x_i, t) \leqq 1. \tag{2.2}$$

Similarly, it follows from axiom (b3) that there exists an essentially bounded $\mathscr{V}_t$-integrable function $\gamma(x_i, t)$ defined on B_t such that for any part $P_t \subset B_t$

$$\mathscr{M}_t(P_t) = \int_{P_t} \gamma \, d\mathscr{V}_t. \tag{2.3}$$

The function γ is called the *distributed mass density* or simply the distributed density. From the absolute continuity of distributed volume with respect to Lebesgue volume, the mass $\mathscr{M}_t(P_t)$ can also be expressed as

$$\mathscr{M}_t(P_t) = \int_{P_t} \gamma \nu \, dV, \tag{2.4}$$

where the function $\rho (= \gamma \nu)$ is interpreted as the classical mass density function. For our purposes, this function is called the *bulk density* of the distributed body.

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* V is the Lebesgue volume measure.

** In soil mechanics terminology, the volume distribution ν is related to the porosity n or the void ratio e by $\nu = 1 - n = 1/(1+e)$.

For a granular material, the function γ would correspond to the mass density of the granules themselves and the function ν would represent the granular volume distribution. In this way, the mass density of the entire ensemble is uncoupled from the mass density of the individual granules, allowing a continuum interpretation of the idea of volume distribution as well as the concept of dilatancy introduced by REYNOLDS [25]. The total volume of the ensemble can increase or decrease as a result of a change in void volume induced by a change in the volume distribution.

3. Kinematics

Although the permeability and fluid flow characteristics of porous and granular materials are of general interest, here we are only concerned with the behavior of the bulk material. This is evident since we have neglected the void mass. Hence, we need only speak of one type of material point and describe its motion relative to a reference configuration.

For $t_0 \in (-\infty, \infty)$ the *motion* of a distributed body relative to the time t_0 is a mapping $\chi_i^{t_0}$ from the product set $B_{t_0} x(-\infty, \infty)$ into Euclidean three space having the following properties:

(a) For each time t, the function $\chi_i^{t_0}(\cdot, t)$, called the *deformation* function, is a smooth homeomorphism of B_{t_0} onto B_t.

(b) For each $X_A \in B_{t_0}$, the function $\chi_i^{t_0}(X_A, \cdot)$, called the *path* function, is defined from $(-\infty, \infty)$ into Euclidean three space and is twice differentiable for all $t \in (-\infty, \infty)$.

The traditional kinematical quantities are obtained in their familiar form. The *velocity* $v_i(X_A, t)$ and *acceleration* $a_i(X_A, t)$ of the material point X_A are given by

$$v_i(X_A, t) = \frac{\partial}{\partial t} \chi_i^{t_0}(X_A, t), \tag{3.1}$$

$$a_i(X_A, t) = \frac{\partial^2}{\partial t^2} \chi_i^{t_0}(X_A, t). \tag{3.2}$$

The *deformation gradient* $F_{iA}(X_B, t)$ is defined by

$$F_{iA}(X_B, t) = \frac{\partial}{\partial X_A} \chi_i^{t_0}(X_B, t), \tag{3.3}$$

and the *velocity gradient* $L_{ij}(x_k, t)$ has the form

$$L_{ij}(x_k, t) = \frac{\partial}{\partial x_j} v_i\left(\chi_B^{t_0^{-1}}(x_k, t), t\right). \tag{3.4}$$

The *rate of deformation* tensor $D_{ij}(x_k, t)$ and *spin* tensor $W_{ij}(x_k, t)$ are defined as the symmetric and skew-symmetric parts of the velocity gradient, respectively.

We now consider the kinematics of volume distribution and introduce the concept of dilatancy. Let dV_0 be an element of total or bulk volume in the reference configuration and let dV denote its image under the motion $\chi_i^{t_0}$. It is a well-known result from mechanics that

$$dV = J\, dV_0 \tag{3.5}$$

where $J = |\det F_{iA}|$. From (2.1) it follows that an element of distributed volume in the instantaneous configuration is related to an element of total or bulk volume by

$$d\mathscr{V}_t = \nu\, dV. \tag{3.6}$$

Similarly, in the reference configuration,

$$d\mathscr{V}_{t_0} = \nu_0\, dV_0. \tag{3.7}$$

Using (3.5)–(3.7), one can easily show that an element of distributed volume transforms according to

$$d\mathscr{V}_t = \frac{\nu}{\nu_0} J\, d\mathscr{V}_{t_0}. \tag{3.8}$$

If the distributed body is constrained so that the total or bulk volume is preserved under the motion, *i.e.* the motion is isochoric,

$$J = 1, \tag{3.9}$$

then the distributed body is said to be *non-dilatant*; otherwise, it is *dilatant*. We are interested in dilatancy and shall not employ the kinematical constraint (3.9).

The kinematical constraint that is more interesting for distributed bodies is the constraint of *incompressible distributed volume*. From (3.8) this kinematical constraint can be expressed by the equation

$$\frac{\nu}{\nu_0} J = 1, \tag{3.10}$$

or, equivalently,

$$\dot{\nu} + \nu v_{i,i} = 0. \tag{3.11}$$

Although (3.10) and (3.11) are similar in form to statements of mass conservation in continuum mechanics, they represent a kinematical constraint and are not statements of any conservation principle. A body whose distributed volume is incompressible can still exhibit dilatancy. The change in total or bulk volume will then correspond to the change in void volume.

In the development which follows, we consider the general unconstrained case of dilatant bodies with compressible distributed volume. The special case of dilatant bodies subject to the kinematical constraint of incompressible distributed volume is considered in Section 8.

4. Thermodynamic Processes

In this section, appropriate statements of the conservation of energy and the Clausius-Duhem inequality are assumed and some consequences of these statements are deduced from invariance principles (Noll [26] and Green & Rivlin [27]). A conservation law for the higher order forces associated with the volume distribution is postulated. The necessary thermal and mechanical field variables are introduced as primitive quantities. Specifically, there exists a *stress tensor* T_{ij}, *body force* b_i, *specific internal energy* ε, *heat flux vector* q_i, *heat supply* r, *specific entropy* η, *entropy flux vector* ϕ_i, and *temperature* θ. In addition, according to assumption three stated in the introduction, we introduce an *equilibrated inertia* k, *equilibrated stress vector* h_i, *external equilibrated body force* l, and *intrinsic equilibrated body force* g.

A *process* G for a distributed body $\{B_t\}$ with a motion $\chi_i^{t_0}$ is defined as the set

$$G=\{\chi_i^{t_0}, \gamma, \nu, T_{ij}, b_i, \varepsilon, q_i, r, \eta, \phi_i, \theta, k, h_i, l, g\}. \tag{4.1}$$

A process G is called a *thermodynamic process* if the elements of G satisfy the following balance relations:

Balance of Energy

$$\begin{aligned}\frac{d}{dt}\int_{P_t}\gamma\nu(\varepsilon+\tfrac{1}{2}v_i v_i+\tfrac{1}{2}k\dot{\nu}\dot{\nu})\,dV \\ =\int_{\partial P_t}(T_{ij}v_i+h_j\dot{\nu}-q_j)\,n_j\,dA+\int_{P_t}\gamma\nu(b_i v_i+l\dot{\nu}+r)\,dV,\end{aligned} \tag{4.2}$$

Entropy Inequality

$$\frac{d}{dt}\int_{P_t}\gamma\nu\eta\,dV \geqq -\int_{\partial P_t}\phi_i n_i\,dA+\int_{P_t}\gamma\nu\frac{r}{\theta}\,dV, \tag{4.3}$$

Balance of Equilibrated Force

$$\frac{d}{dt}\int_{P_t}\gamma\nu k\dot{\nu}\,dV=\int_{\partial P_t}h_i n_i\,dA+\int_{P_t}\gamma\nu(l+g)\,dV, \tag{4.4}$$

Balance of Equilibrated Inertia

$$\frac{d}{dt}\int_{P_t}\gamma\nu k\,dV=0. \tag{4.5}$$

The statements of the conservation of energy and entropy inequality given above differ from the traditional statements by the occurrence of the power terms associated with $\dot{\nu}$ in the energy equation and the fact that the entropy flux is not assumed to be the heat flux divided by temperature in the entropy inequality. In the less restrictive assumption concerning the entropy flux, we are following MÜLLER [24]. The introduction of the power terms associated with $\dot{\nu}$ are necessary because ν is kinematically independent of the motion $\chi_i^{t_0}$ and the temperature θ. That is to say, ν, $\chi_i^{t_0}$ and θ independently characterize energy storing or absorbing aspects of the model. It was noted in the introduction that similar terms appear in the theoretical developments of MINDLIN [14], TOUPIN [15], GREEN & RIVLIN [16], ERICKSEN [17], and LESLIE [18] and also in the soil mechanics literature [10, 11, 12, 13].

The balance equations (4.4) and (4.5) are analogous to the classical balance equations of linear momentum and mass. The balance of equilibrated force is motivated by a variational analysis [19]. A similar equation also arises in the theories of MINDLIN [14], TOUPIN [15], *etc.** The balance equation for equilibrated

* In a certain sense, the present theory may be regarded as a special case of the theories of microstructure (see MINDLIN [14], TOUPIN [15], and GREEN & RIVLIN [16]) where only the dilatation of the micromedium is considered.

inertia is necessary for a complete theory; we have assumed the simplest form. A more general expression would include on the right-hand side of (4.5) an inertia supply term to be described by a constitutive equation.

The classical laws of mechanics are deduced by invoking the principle of material objectivity and the following postulate: every process G' obtained from a thermodynamic process G by a change of frame must itself be a thermodynamic process. From arguments outlined by NOLL [26] and GREEN & RIVLIN [27], the familiar balance relations for mass, linear momentum, and angular momentum are obtained:

$$\dot{\overline{\gamma\nu}}+\gamma\nu v_{i,i}=0, \tag{4.6}$$

$$\gamma\nu\dot{v}_i=T_{ij,j}+\gamma\nu b_i, \tag{4.7}$$

$$T_{ij}=T_{ji}. \tag{4.8}$$

Employing (4.6)–(4.8) in the balance relations (4.2)–(4.5), we obtain the following field expressions after some manipulation:

$$\dot{k}=0, \tag{4.9}$$

$$\gamma\nu k\ddot{\nu}=h_{i,i}+\gamma\nu(l+g), \tag{4.10}$$

$$\gamma\nu\dot{\varepsilon}=T_{ij}D_{ij}+h_i(\dot{\nu})_{,i}-\gamma\nu g\dot{\nu}-q_{i,i}+\gamma\nu r, \tag{4.11}$$

$$\gamma\nu\dot{\eta}\geqq-\phi_{i,i}+\gamma\nu\frac{r}{\theta}. \tag{4.12}$$

Solving for r in (4.11), substituting the resulting expression into (4.12), and introducing the *free energy*

$$\psi=\varepsilon-\eta\theta \tag{4.13}$$

and *extra entropy flux*

$$k_i=\phi_i-\frac{q_i}{\theta}, \tag{4.14}$$

we have

$$-\gamma\nu(\dot{\psi}+\eta\dot{\theta})+T_{ij}D_{ij}+h_i(\dot{\nu})_{,i}-\gamma\nu g\dot{\nu}+\theta k_{i,i}-\frac{q_i\theta_{,i}}{\theta}\geqq 0. \tag{4.15}$$

This form of the entropy inequality will be used to investigate the constitutive postulates.

5. Constitutive Equations

The fundamental concepts and principles presented in the previous sections apply to a general class of materials, including porous materials as well as granular materials. Here we specialize the theory to granular materials. The basic distinction between porous materials and granular materials is characterized by the particular constitutive postulates.

Generally speaking, it is assumed that the histories of the motion, temperature, and volume distribution determine the thermodynamic response of porous and granular materials. In accordance with the fourth assumption presented in the introduction, the behavior of granular materials is similar to fluid behavior except that the response functionals for granular materials depend on the reference configuration B_{t_0} through the reference volume distribution ν_0. In particular, we consider in this paper granular materials whose response is characterized by constitutive functions of

$$\nu_0,\, \nu,\, \nu_{,i},\, \dot{\nu},\, \gamma,\, \theta,\, \theta_{,i},\, v_i,\, v_{i,j}. \tag{5.1}$$

Such materials will be called *granular materials* for the purposes of this paper. Invoking the principle of material objectivity which implies that the response of granular materials is independent of the velocity and skew-symmetric part of the velocity gradient, and assuming equipresence in the constitutive equations, we have

$$\begin{aligned}
\psi&=\psi(\nu_0, \nu, \nu_{,i}, \dot{\nu}, \gamma, \theta, \theta_{,i}, D_{ij}),\\
\eta&=\eta(\nu_0, \nu, \nu_{,i}, \dot{\nu}, \gamma, \theta, \theta_{,i}, D_{ij}),\\
T_{ij}&=T_{ij}(\nu_0, \nu, \nu_{,i}, \dot{\nu}, \gamma, \theta, \theta_{,i}, D_{ij}),\\
h_i&=h_i(\nu_0, \nu, \nu_{,i}, \dot{\nu}, \gamma, \theta, \theta_{,i}, D_{ij}),\\
g&=g(\nu_0, \nu, \nu_{,i}, \dot{\nu}, \gamma, \theta, \theta_{,i}, D_{ij}),\\
q_i&=q_i(\nu_0, \nu, \nu_{,i}, \dot{\nu}, \gamma, \theta, \theta_{,i}, D_{ij}),\\
k_i&=k_i(\nu_0, \nu, \nu_{,i}, \dot{\nu}, \gamma, \theta, \theta_{,i}, D_{ij}).
\end{aligned} \tag{5.2}$$

A further consequence of the principle of material objectivity is that the constitutive functionals for ψ, η, T_{ij}, h_i, g, q_i, and k_i must be isotropic functions of their tensor arguments $\nu_{,i}$, $\theta_{,i}$, and D_{ij}. Thus, the principle of material objectivity implies that granular materials are, in some sense, isotropic.

A thermodynamic process is said to be *admissible* for a granular material if the constitutive equations (5.2) are satisfied.

If the functional dependence of ψ and k_i expressed in (5.2) is incorporated in (4.15) by use of the chain rule, and if the relationship

$$\dot{\gamma}=-\gamma D_{ii}-\frac{\gamma}{\nu}\dot{\nu}, \tag{5.3}$$

which follows from (4.6), and the identity

$$\dot{\overline{\nu_{,i}}}=(\dot{\nu})_{,i}-\nu_{,j}L_{ji} \tag{5.4}$$

are employed, then (4.15) becomes

$$\begin{aligned}
&-\gamma\nu\left(\frac{\partial\psi}{\partial\theta}+\eta\right)\dot{\theta}-\left(\gamma\nu g-\frac{p-\hat{p}}{\nu}\right)\dot{\nu}\\
&\quad+\left(h_i-\gamma\nu\frac{\partial\psi}{\partial\nu_{,i}}+\theta\frac{\partial k_i}{\partial\dot{\nu}}\right)(\dot{\nu})_{,i}+\left(T_{ij}+p\delta_{ij}+\gamma\nu\frac{\partial\psi}{\partial\nu_{,i}}\nu_{,j}\right)D_{ij}\\
&\quad+\gamma\nu\frac{\partial\psi}{\partial\nu_{,i}}\nu_{,j}W_{ji}-\gamma\nu\frac{\partial\psi}{\partial\dot{\nu}}\ddot{\nu}-\gamma\nu\frac{\partial\psi}{\partial\theta_{,i}}\dot{\overline{\theta_{,i}}}-\gamma\nu\frac{\partial\psi}{\partial D_{ij}}\dot{D}_{ij}\\
&\quad+\theta\left[\frac{\partial k_i}{\partial\nu_0}\nu_{0,i}+\frac{\partial k_i}{\partial\nu}\nu_{,i}+\frac{\partial k_i}{\partial\nu_{,j}}\nu_{,ij}+\frac{\partial k_i}{\partial\gamma}\gamma_{,i}+\frac{\partial k_i}{\partial\theta_{,j}}\theta_{,ij}+\frac{\partial k_i}{\partial D_{jk}}D_{jk,i}\right]\\
&\quad+\left(\theta\frac{\partial k_i}{\partial\theta}-\frac{q_i}{\theta}\right)\theta_{,i}\geqq 0,
\end{aligned} \tag{5.5}$$

where we have introduced the definitions

$$p=\gamma^2 \nu \frac{\partial\psi}{\partial\gamma}, \quad \hat{p}=\gamma\nu^2 \frac{\partial\psi}{\partial\nu}. \tag{5.6}$$

If we employ arguments outlined by COLEMAN & NOLL [28] and COLEMAN & MIZEL [29], it follows that there exists at least one admissible thermodynamic process for a granular material in which the values of $\dot{\theta}$, $\ddot{\nu}$, $(\dot{\nu})_{,i}$, $\dot{\overline{\theta_{,i}}}$, $\dot{D}_{ij}$, $\nu_{0,i}$, $\nu_{,ij}$, $\gamma_{,i}$, $\theta_{,ij}$, and $D_{ij,k}$ can be specified independently of any other term in the inequality. The entropy inequality (5.5) then implies the following restrictions:

$$\eta=-\frac{\partial\psi}{\partial\theta}, \quad h_i=\gamma\nu\frac{\partial\psi}{\partial\nu_{,i}}-\theta\frac{\partial k_i}{\partial\dot{\nu}}, \tag{5.7}$$

$$\frac{\partial\psi}{\partial\dot{\nu}}=0, \quad \frac{\partial\psi}{\partial\theta_{,i}}=0, \quad \frac{\partial\psi}{\partial D_{ij}}=0, \tag{5.8}$$

$$\frac{\partial k_i}{\partial\nu_{,j}}+\frac{\partial k_j}{\partial\nu_{,i}}=0, \quad \frac{\partial k_i}{\partial\theta_{,j}}+\frac{\partial k_j}{\partial\theta_{,i}}=0, \tag{5.9}$$

$$\frac{\partial k_i}{\partial\nu_0}=0, \quad \frac{\partial k_i}{\partial\gamma}=0, \quad \frac{\partial k_i}{\partial D_{jk}}=0. \tag{5.10}$$

The result $(5.7)_1$ is a familiar result in thermostatics. We simply note that it is valid for granular materials in non-equilibrium.

The restrictions (5.8) imply

$$\psi=\psi(\nu_0, \nu, \nu_{,i}, \gamma, \theta) \tag{5.11}$$

or, from the isotropic dependence of ψ upon $\nu_{,i}$,

$$\psi=\psi(\nu_0, \nu, \nu_{,k}\nu_{,k}, \gamma, \theta). \tag{5.12}$$

The dependence of the free energy on $\nu_{,i}$ is an essential result. As we shall show shortly, the representation (5.12) allows the equilibrium stress to depend on $\nu_{,i}$ and, moreover, to include a shear stress component associated with the tensor product $\nu_{,i}\nu_{,j}$. Real granular materials can, of course, support shear in equilibrium as evidenced by the characteristic angle of repose of these materials.

It follows from (5.12) that

$$\frac{\partial\psi}{\partial\nu_{,i}}\nu_{,j}=\frac{\partial\psi}{\partial\nu_{,j}}\nu_{,i} \tag{5.13}$$

and, hence, the term in (5.5) involving W_{ij} vanishes identically.

The restrictions (5.9) and (5.10) on the extra entropy flux k_i can be investigated further using arguments presented by MÜLLER [24]. Integrating the differential equations (5.9) yields the general solution for k_i,

$$k_i=A_{ijk}\theta_{,j}\nu_{,k}+A_{ij}\theta_{,j}+B_{ij}\nu_{,j}+A_i, \tag{5.14}$$

where the coefficients are functions of only ν, $\dot{\nu}$, and θ, and A_{ijk}, A_{ij}, and B_{ij} are completely skew-symmetric tensors. Since k_i must be an isotropic tensor function of $\theta_{,i}$ and $v_{,i}$ and since there are no isotropic skew symmetric tensors of ranks two and three, it follows that (5.14) reduces to

$$k_i = 0. \tag{5.15}$$

Thus, by (4.14), the entropy flux takes on its traditional form. Combining (5.15) with $(5.7)_2$ and recalling the functional form (5.12) for ψ implies

$$h_i = \gamma v \frac{\partial \psi}{\partial v_{,i}} = 2\alpha v_{,i}, \tag{5.16}$$

where

$$\alpha = \alpha(v_0, v, v_{,k} v_{,k}, \gamma, \theta). \tag{5.17}$$

Returning now to the entropy inequality (5.5) and employing the restrictions (5.7)–(5.10), (5.13), (5.15), and (5.16), we obtain

$$(T_{ij} + p\delta_{ij} + 2\alpha v_{,i} v_{,j}) D_{ij} - \left(\gamma v g - \frac{p-\hat{p}}{v}\right) \dot{v} - \frac{q_i \theta_{,i}}{\theta} \geqq 0. \tag{5.18}$$

6. Equilibrium

Additional information can be extracted from the entropy inequality without postulating specific constitutive equations. This information concerns the appropriate equilibrium state for granular materials. We define an *equilibrium* process as one in which the independent variables

$$Y_A = (\dot{v}, \theta_{,i}, D_{ij}), \qquad A = 1, 2, \ldots, 10 \tag{6.1}$$

all vanish.

Denoting the left-hand side of (5.18) by σ, we see that σ has a minimum in equilibrium. A necessary condition for this minimum is that

$$\left.\frac{\partial \sigma}{\partial Y_A}\right|_0 = 0, \qquad A = 1, 2, \ldots, 10 \tag{6.2}$$

where the subscript 0 denotes the equilibrium values. This condition yields the equilibrium values of the stress, intrinsic equilibrated body force, and heat flux,

$$T_{ij}^0 = T_{ij}(v_0, v, v_{,i}, 0, \gamma, \theta, 0, 0) = -p\delta_{ij} - 2\alpha v_{,i} v_{,j}, \tag{6.3}$$

$$g^0 = g(v_0, v, v_{,i}, 0, \gamma, \theta, 0, 0) = \frac{p-\hat{p}}{\gamma v^2} \tag{6.4}$$

$$q_i^0 = q_i(v_0, v, v_{,i}, 0, \gamma, \theta, 0, 0) = 0. \tag{6.5}$$

Recalling that p and $\hat{p}$ are defined by (5.6), we find that T_{ij}^0 and g^0 are totally derivable from the free energy function and, moreover, are functions of only v_0, v, $v_{,i}$, γ, and θ. By virtue of (6.4), the equilibrium stress for granular materials with compressible granules has the dual representation

$$T_{ij}^0 = -\hat{p}\delta_{ij} - \gamma v^2 g^0 \delta_{ij} - 2\alpha v_{,i} v_{,j}. \tag{6.6}$$

The term $2\alpha v_{,i} v_{,j}$ in the representations (6.3) and (6.6) demonstrates the point we mentioned previously about the ability of granular materials to support shear in equilibrium*.

The pressures p and $\hat{p}$ associated with the distributed density γ and the volume distribution v, respectively, appear similar to the pressure in a compressible fluid. Indeed, if we consider ρ as an independent variable in the constitutive equations instead of either γ or v, we obtain

$$p=\rho^2 \frac{\partial \psi'}{\partial \rho}, \quad \hat{p}=\rho^2 \frac{\partial \hat{\psi}}{\partial \rho}, \tag{6.7}$$

where ψ' is a function of ρ instead of γ and $\hat{\psi}$ is a function of ρ instead of v. In the present development, the pressure p is interpreted as a material pressure related to the compressibility of granules whereas the pressure $\hat{p}$ is interpreted as a configuration pressure related to the volume distribution of granules.

7. Linear Theory

The restrictions on the constitutive equations established in the foregoing sections indicate that the specific entropy η, the equilibrated stress h_i, and the equilibrium parts of the stress T^0_{ij} and intrinsic equilibrated body force g^0 are all derivable from the free energy ψ. Consequently, in the theory of granular materials, we need only specify particular constitutive representations for the free energy ψ, heat flux q_i, and dissipative parts of the stress $T_{ij}-T^0_{ij}$ and intrinsic equilibrated body force $g-g^0$. In this section, we consider a linear theory in which the representations for q_i, $T_{ij}-T^0_{ij}$, and $g-g^0$ are linear in the variables Y_A defined by (6.1) and, in addition, h_i given by (5.16) is linear in the variable $v_{,i}$. The linearity condition on h_i implies

$$h_i=2\alpha v_{,i}, \quad \alpha=\alpha(v_0, v, \gamma, \theta). \tag{7.1}$$

Recalling the functional dependence expressed by (5.2) for q_i, T_{ij}, and g, we have in the linear theory

$$q_i=-\kappa\theta_{,i}, \tag{7.2}$$

$$T_{ij}-T^0_{ij}=\xi\dot{v}\delta_{ij}+\lambda D_{kk}\delta_{ij}+2\mu D_{ij}, \tag{7.3}$$

$$g-g^0=-\zeta\dot{v}-\delta D_{kk} \tag{7.4}$$

where the coefficients are, in general, scalar functions of v_0, v, $v_{,i}$, γ, and θ.

We first investigate the restrictions imposed by the entropy inequality on the coefficients in the constitutive equations (7.2)–(7.4). Substituting the representations (7.2)–(7.4) in the inequality (5.18) and employing the expressions (6.3) and (6.4) for T^0_{ij} and g^0, respectively, we obtain

$$(\gamma v\delta+\xi)\dot{v}D_{jj}+\lambda D_{ii}D_{jj}+2\mu D_{ij}D_{ij}+\zeta\gamma v\dot{v}\dot{v}+\kappa\frac{\theta_{,i}\theta_{,i}}{\theta}\geqq 0. \tag{7.5}$$

* The representation (6.3) for stress is a special case of a representation given by ERICKSEN [30, eqn. (3.8) with (3.5)] for transversely isotropic fluids; ERICKSEN's material vector n_i, the counterpart of our vector $v_{,i}$, is not necessarily the gradient of a scalar field.

The expression on the left-hand side of (7.5), which is just σ, is a positive quadratic form in the variables Y_A. Hence

$$\left\| \frac{\partial^2 \sigma}{\partial Y_A \partial Y_B} \right\| \text{ is a positive matrix}, \quad A, B = 1, 2, \ldots, 10. \tag{7.6}$$

Performing the operations indicated by (7.6) yields the following necessary and sufficient conditions that (7.5) hold:

$$\kappa \geqq 0, \tag{7.7}$$

$$\mu \geqq 0, \quad 3\lambda + 2\mu \geqq 0, \tag{7.8}$$

$$\zeta \geqq 0, \quad 4\gamma\nu\zeta(3\lambda + 2\mu) - 3(\xi + \gamma\nu\delta)^2 \geqq 0. \tag{7.9}$$

The restrictions (7.8) are familiar inequalities obtained for a linearly viscous fluid whose dissipative stress is precisely that given by (7.3) except for the pressure term associated with $\dot{\nu}$. From (7.2) it follows that the heat flux reduces to its familiar form. Inasmuch as the conductivity κ can depend on the volume distribution, a modified theory of heat conduction has resulted. Finally, the restrictions (7.9) arise due to the inclusion of terms involving $\dot{\nu}$ in the constitutive equations. Such terms are purely dissipative.

Turning now to the constitutive equation (7.1) for h_i and recalling the expression (5.16), we integrate to obtain

$$\gamma\nu\psi = \alpha_0(\nu_0, \nu, \gamma, \theta) + \alpha\nu_{,i}\nu_{,i}. \tag{7.10}$$

Requiring the free energy per unit volume to be positive for all values of its arguments implies

$$\alpha_0 \geqq 0, \quad \alpha \geqq 0. \tag{7.11}$$

The representation (7.10) use in conjunction with (5.6) yields the following expressions for p and $\hat{p}$:

$$p = \left(\gamma \frac{\partial \alpha_0}{\partial \gamma} - \alpha_0\right) + \left(\gamma \frac{\partial \alpha}{\partial \gamma} - \alpha\right) \nu_{,i}\nu_{,i}, \tag{7.12}$$

$$\hat{p} = \left(\nu \frac{\partial \alpha_0}{\partial \nu} - \alpha_0\right) + \left(\nu \frac{\partial \alpha}{\partial \nu} - \alpha\right) \nu_{,i}\nu_{,i}. \tag{7.13}$$

The representation (7.12) for p together with the equilibrium stress relation (6.3) require that the equilibrium normal stress and equilibrium shear stress acting on an arbitrary plane at an arbitrary point bear a special relationship to one another. A similar result occurs in fluid equilibrium in that the shear stress must always vanish. In granular material equilibrium, the shear stress has a specific non-zero value related to the magnitude of the normal stress. To develop this relationship, consider an arbitrary plane with normal n_i. From (6.3) the normal stress T acting across the plane is given by

$$T = T_{ij} n_i n_j = -p - 2\alpha(\nu_{,i} n_i)^2 \tag{7.14}$$

and is related to the shear stress S in the plane by

$$T^2 + S^2 = T_{ij} n_j T_{ik} n_k = p^2 + 4\alpha p(\nu_{,i} n_i)^2 + 4\alpha^2(\nu_{,i}\nu_{,i})(\nu_{,j} n_j)^2. \tag{7.15}$$

Employing (7.14) to eliminate the term $(\nu_{,i} n_i)^2$ in (7.15), completing the square in the resulting expression, and introducing the notation

$$s = \alpha \nu_{,i} \nu_{,i}, \qquad t = -p - \alpha \nu_{,i} \nu_{,i}, \tag{7.16}$$

we deduce the relationship

$$S^2 + (T - t)^2 = s^2. \tag{7.17}$$

Furthermore, recalling the representation (7.12) for p, we can write the expression $(7.16)_2$ in the form

$$s = b(-t + c), \tag{7.18}$$

where

$$c = \alpha_0 - \gamma \frac{\partial \alpha_0}{\partial \gamma}, \qquad \frac{1}{b} = \frac{\gamma}{\alpha} \frac{\partial \alpha}{\partial \gamma}. \tag{7.19}$$

Combining (7.17) with (7.18), we obtain the sought-after relationship between the shear stress S and normal stress T. Considering S and T as Cartesian coordinates, we find that (7.17) is the equation for a circle centered at $S=0$, $T=t$ with radius s. The relationship (7.18) requires the circle radius to be a function of displacement of the circle from the origin. This result is a generalization of the Coulomb stress condition (1.1) and shows that the linear theory considered here imbeds the Mohr-Coulomb theory [3] of limiting equilibrium. The angle of internal friction associated with b and the cohesion associated with c are, in general, functions of ν_0, ν, γ, and θ. If b and c are constants, the traditional Mohr-Coulomb criterion is obtained; *i.e.*, the Mohr circle (7.17) is tangent to the straight line (7.18).

8. Granular Materials with Incompressible Granules

The previous sections concerned the unconstrained case of dilatant granular materials with compressible granules. In this section, results are obtained for granular materials subject to the internal constraint of incompressible granules. This constraint is expressed by (3.11),

$$\frac{\dot{\nu}}{\nu} + v_{i,i} = 0, \tag{8.1}$$

which, when combined with the continuity equation (4.6) implies

$$\dot{\gamma} = 0. \tag{8.2}$$

Two versions of the constrained theory are developed here. First, we use the approach taken by Doria [31] for considering constraints. Then a more traditional approach, that of Lagrange multipliers, is employed. The difference between the two methods lies in the starting assumptions and in the generality of the resulting theory.

Paralleling the approach of Doria, we reformulate the general unconstrained theory using the thermodynamic pressure p as an independent variable instead of γ. The basic assumption is that, with ψ given by (5.12), the expression $(5.6)_1$ for $p(\nu_0, \nu, \nu_{,i}, \gamma, \theta)$ can be inverted to give a smooth function

$$\gamma = \gamma(\nu_0, \nu, \nu_{,i} \nu_{,i}, p, \theta). \tag{8.3}$$

This expression can then be used to eliminate γ as an independent variable in the constitutive equations. In the analysis which follows, we use an overbar on any function to denote that it has been rendered a function of p instead of γ. By use of the chain rule and the expression $(5.6)_1$ for p, we obtain results corresponding to $(5.6)_2$, $(5.7)_1$, and (5.16):

$$\bar{\bar{p}}=\gamma v^2\left(\frac{\partial\bar{\psi}}{\partial v}-\frac{p}{\gamma^2 v}\frac{\partial\gamma}{\partial v}\right), \tag{8.4}$$

$$\bar{\eta}=\frac{\partial\bar{\psi}}{\partial\theta}-\frac{p}{\gamma^2 v}\frac{\partial\gamma}{\partial\theta}, \tag{8.5}$$

$$\bar{h}_i=\gamma v\left(\frac{\partial\bar{\psi}}{\partial v_{,i}}-\frac{p}{\gamma^2 v}\frac{\partial\gamma}{\partial v_{,i}}\right)=2\bar{\alpha}v_{,i} \tag{8.6}$$

where the coefficient $\bar{\alpha}$ is a function of v_0, v, $v_{,i}v_{,i}$, p, and θ. The representation $(8.6)_2$ is possible because $\bar{\psi}$ and γ depend upon $v_{,i}$ only through $v_{,i}v_{,i}$. The entropy inequality (5.18) can be written in the form

$$(\bar{T}_{ij}+p\delta_{ij}+2\bar{\alpha}v_{,i}v_{,j})D_{ij}-\left(\gamma v\bar{g}-\frac{p-\bar{\bar{p}}}{v}\right)\dot{v}-\frac{\bar{q}_i\theta_{,i}}{\theta}\geqq 0. \tag{8.7}$$

The necessary conditions (6.2) for equilibrium then yield the results

$$\bar{T}^0_{ij}=-p\delta_{ij}-2\bar{\alpha}v_{,i}v_{,j}, \tag{8.8}$$

$$\bar{g}^0=\frac{p-\bar{\bar{p}}}{\gamma v^2}, \tag{8.9}$$

$$\bar{q}^0_i=0, \tag{8.10}$$

which differ from (6.3), (6.4), and (6.5) only by the change of independent variable from γ to p.

In order to constrain the theory so that the volume of granules is incompressible, we interpret the constraint (8.2) as an approximation whereby γ is insensitive to changes in the independent variables. Thus, by (8.2) and (8.3), we impose the restrictions

$$\frac{\partial\gamma}{\partial v_0}=\frac{\partial\gamma}{\partial v}=\frac{\partial\gamma}{\partial v_{,i}}=\frac{\partial\gamma}{\partial p}=\frac{\partial\gamma}{\partial\theta}=0 \tag{8.11}$$

upon the expressions (8.4)–(8.6) and (8.8)–(8.10). The resulting simplification can be read directly. The dependence upon p remains in all the constitutive equations. In particular, note that the coefficient $\bar{\alpha}$ depends on p.

In view of the constraint (8.1), the only dissipation in the constrained theory occurs through the independent variables D_{ij} and $\theta_{,i}$. For the linear case, $\bar{T}_{ij}-\bar{T}^0_{ij}$ and $\bar{q}_i$ take on their traditional representations

$$\bar{T}_{ij}-\bar{T}^0_{ij}=\bar{\lambda}D_{kk}\delta_{ij}+2\bar{\mu}D_{ij}, \tag{8.12}$$

$$\bar{q}_i=-\bar{\kappa}\theta_{,i} \tag{8.13}$$

and $\bar{g}-\bar{g}^0$ has the form

$$\bar{g}-\bar{g}^0=-\delta D_{kk}, \tag{8.14}$$

where the coefficients are, at most, scalar functions of ν_0, ν, $\nu_{,i}\nu_{,i}$, p, and θ. Substitution of (8.12)–(8.14) into the entropy inequality (8.7) and use of (8.1), (8.8), and (8.9) yields the following restrictions on the coefficients:

$$\bar{\kappa}\geqq 0, \tag{8.15}$$

$$\bar{\mu}\geqq 0, \quad 3(\bar{\lambda}-\gamma\nu^2\bar{\delta})+2\bar{\mu}\geqq 0. \tag{8.16}$$

Finally, the assumption that $\bar{h}_i$ is a linear function of $\nu_{,i}$ implies

$$\bar{h}_i=2\bar{\alpha}\nu_{,i}, \quad \bar{\alpha}=\bar{\alpha}(\nu_0, \nu, p, \theta), \tag{8.17}$$

which, when combined with (8.6), gives in the constrained theory

$$\gamma\nu\bar{\psi}=\bar{\alpha}_0+\bar{\alpha}\nu_{,i}\nu_{,i}, \quad \bar{\alpha}_0=\bar{\alpha}_0(\nu_0, \nu, p, \theta). \tag{8.18}$$

The representation for $\bar{\bar{p}}$ determined from (8.4) and (8.18) has the form

$$\bar{\bar{p}}=\left(\nu\frac{\partial\bar{\alpha}_0}{\partial\nu}-\bar{\alpha}_0\right)+\left(\nu\frac{\partial\bar{\alpha}}{\partial\nu}-\bar{\alpha}\right)\nu_{,i}\nu_{,i}. \tag{8.19}$$

The Mohr-Coulomb condition expressed by (7.17) and (7.18) is still valid. By the same analysis as before, the relation (7.17) follows from (8.8). To obtain equation (7.18), we use (8.19), together with (8.9), in (7.16). In this case, the scalar functions associated with the angle of internal friction and cohesion, respectively, are given by

$$\frac{1}{b}=\frac{\nu}{\bar{\alpha}}\frac{\partial\bar{\alpha}}{\partial\nu}, \quad c=-\gamma\nu^2\bar{g}_0+\bar{\alpha}_0-\nu\frac{\partial\bar{\alpha}_0}{\partial\nu}. \tag{8.20}$$

We present now a more traditional treatment of constraints based on the method of Lagrange multipliers. We return to the initial constitutive assumptions (5.2) and note that, in view of the restrictions (8.1) and (8.2), the list of variables appearing in (5.2) is no longer independent. We delete $\dot{\nu}$ and γ from the constitutive equations and repeat the analysis of Section 5. We find the same results $(5.6)_2$, $(5.7)_1$, (5.15), (5.16) and the entropy inequality (4.15) reduces to

$$(T_{ij}+2\alpha\nu_{,i}\nu_{,j})D_{ij}-\left(\gamma\nu g+\frac{\hat{p}}{\nu}\right)\dot{\nu}-\frac{q_i\theta_{,i}}{\theta}\geqq 0. \tag{8.21}$$

We note that a stress and intrinsic equilibrated body force of the form

$$T_{ij}=-p^*\delta_{ij}, \quad g=\frac{1}{\gamma\nu^2}p^*, \tag{8.22}$$

where p^* is an arbitrary scalar, do not contribute to the inequality (8.21) because of the condition (8.1). Thus we set

$$T_{ij}=-p^*\delta_{ij}+\tilde{T}_{ij}, \quad g=\tilde{g}+\frac{1}{\gamma\nu^2}p^*, \tag{8.23}$$

and (8.21) can be written in the form

$$(\tilde{T}_{ij}+2\alpha\nu_{,i}\nu_{,j})D_{ij}-\left(\gamma\nu\tilde{g}+\frac{\hat{p}}{\nu}\right)\dot{\nu}-\frac{q_i\theta_{,i}}{\theta}\geqq 0. \tag{8.24}$$

Equilibrium considerations then yield

$$T_{ij}^0=-p^*\delta_{ij}-2\alpha\nu_{,i}\nu_{,j}, \tag{8.25}$$

$$g^0=\frac{p^*-p}{\gamma\nu^2}, \tag{8.26}$$

$$q_i^0=0 \tag{8.27}$$

which correspond to (6.3)–(6.5) with p replaced by p^*. The arbitrary scalar p^* is the indeterminate pressure associated with the constraint (8.1). If p^* is eliminated from (8.25) by using (8.26), the result (6.6) is obtained with g^0 given by (8.26). The treatment of linear dissipation in the constrained case is the same as before except there is no longer a functional dependence upon p. The equations (8.12) through (8.20) are obtained but with all the superimposed bars removed, indicating the lack of functional dependence upon p.

The two methods of constructing the constrained theory here differ in basic assumptions and in the generality of the results. With the method of DORIA, the constraint is regarded as an approximation to be imposed on the general theory only after restrictions from the entropy inequality have been deduced. On the other hand, the Lagrange multiplier approach considers the constraint from the outset; the independent variables in the constitutive assumptions must be compatible with the constraint. The difference in generality of the two methods is seen in the constitutive equations. The approach of DORIA leads to greater generality in the sense that the scalar coefficients $\bar{\alpha}$, $\bar{\lambda}$, $\bar{\mu}$, $\bar{\delta}$, and $\bar{\kappa}$ all depend on p. The method of DORIA is, however, subject to the restriction of invertibility of the functional p.

9. Concluding Remarks

The theory presented here for granular materials demonstrates the significance of considering volume distribution as a kinematical variable independent of the motion. In the present theory the volume distribution serves to distinguish granular material behavior from ordinary fluid behavior. The equilibrated stress and body forces associated with the volume distribution are critical in the development. It is these forces which allow the material to support a density gradient in equilibrium (through the gradient of the volume distribution) as well as to sustain shear in equilibrium, properties uncommon to ordinary fluids. These forces are also responsible for the inclusion of a flow criterion which contains the generally accepted Mohr-Coulomb criterion. As a further consequence of the existence of these forces, the principal axes of stress and deformation rate do not coincide in the present theory. This result, which is in agreement with experimental evidence on the behavior of granular materials (see, for example, DE JONG [32]), has not been recognized in most existing theories for granular material behavior, particularly in the plasticity type theories. An exception to this is the theory of SPEN-

CER'S [8] that incorporates a dependence on stress rate to avoid this coincidence of the principal axes.

In a recent paper [33] we considered a special case of the linear theory of granular materials for the solution to two flow problems. The main assumptions underlying this special case are (1) the equilibrated momentum $k\ddot{\nu}$ and the external equilibrated body force l are negligible; (2) the distributed volume is incompressible in regions of non-equilibrium; (3) α_0 is analytic in its argument ν at $\nu=\nu_0$, where ν_0 is taken as the critical volume distribution at which no volume change occurs during shearing; and (4) the Mohr-Coulomb criterion is a sufficient, as well as necessary, condition for equilibrium. This last assumption implies that the equilibrium theory reduces to the traditional Mohr-Coulomb theory of limiting equilibrium.

The constitutive equation for the stress in this special case is expressed by

$$\begin{aligned} T_{ij} &= (\beta_0-\beta\nu^2+\alpha\nu_{,k}\nu_{,k}+2\alpha\nu\nu_{,kk})\delta_{ij} \\ &\quad -2\alpha\nu_{,i}\nu_{,j}+\lambda D_{kk}\delta_{ij}+2\mu D_{ij}, \qquad \text{if } D_{ij}\neq 0, \\ s &= b(-t+c), \qquad \text{if } D_{ij}=0, \end{aligned} \tag{9.1}$$

where the coefficients are material constants. A material described by (9.1) is referred to as a *Coulomb granular material.*

The theory of Coulomb granular materials is amenable to problem solution. Although the governing differential equations obtained by inserting (9.1) into the balance of linear momentum (4.7) are non-linear and coupled in the variables ν and v_i, the one-dimensional problems considered in [33] render these equations linear and uncoupled.

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