On Circulation in Relativistic Hydrodynamics

A. H. TAUB

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Abstract. In this paper it is shown that the relativistic analogue of the circulation in classical hydrodynamics is the integral

$$C = \int \left(1 + \frac{i}{c^2}\right) U_\alpha \, d x^\alpha ,$$

where i is the specific enthalpy of the fluid, U_α are the covariant components of the four velocity of the fluid and the integral is taken around a closed curve traveling with the fluid. This integral is similar to an integral given by LICHNEROWICZ, but differs in the factor multiplying U_α in the integrand.

The necessary and sufficient conditions for C to vanish are shown to be that the flow be irrotational and isentropic. The necessary and sufficient condition for C to be a constant of the motion is that the pressure be a function of the density alone at every point in space-time occupied by the fluid. This condition is shown to be violated when a shock is present. Results for similar integrals taken around vortex lines are also obtained. The relations between these results and Bernoulli's theorems are discussed.

1. Introduction

It is the purpose of this paper to discuss an integral formed from quantities describing the relativistic behavior of a perfect fluid. This integral is analogous to the integral defining circulation in classical hydrodynamics. The results obtained below are, in part, similar to results given by SYNGE [*1*] and reported by LICHNEROWICZ [*2*] but are obtained by using a different formulation for the stress-energy tensor of a perfect fluid. Our results hold for both special and general relativity and depend only on the assumed nature of the stress-energy tensor and the four conservation laws it obeys and the equation of conservation of matter.

A perfect fluid is described by the stress energy tensor

$$T^{\mu\nu} = \sigma U^\mu U^\nu - \frac{p}{c^2} g^{\mu\nu}, \tag{1.1}$$

where

$$U^\mu U_\mu = 1 ; \tag{1.2}$$

U^μ represents the four dimensional velocity vector of the fluid, $g_{\mu\nu}$, the metric tensor of space-time, p, the pressure, and

$$\sigma c^2 = \varrho \left(c^2 + \varepsilon + \frac{p}{\varrho}\right). \tag{1.3}$$

In this equation ϱ is the proper density, and

$$\varepsilon = \varepsilon(p, \varrho) \tag{1.4}$$

is the proper, specific internal energy of the fluid. Equation (1.4) is sometimes referred to as the caloric equation of state of the fluid.

The equations governing the motion of the fluid are the formulation of five conservation laws, namely, the conservation of mass, energy and momentum. Where there are no singularities, these equations take the form of partial differential equations which are

$$\begin{aligned} T^{\mu\nu}{}_{;\nu} &= 0, \\ (\varrho U^\mu)_{;\mu} &= 0, \end{aligned} \tag{1.5}$$

where the semicolon denotes the covariant derivative with respect to the metric tensor $g_{\mu\nu}$. This paper is not concerned with the determination of this tensor. It is assumed to be known.

Across a singular hyper-surface [3], the conservation laws are

$$\begin{aligned} [T^\mu{}_\nu \lambda_\mu] &= 0, \\ [\varrho U^\mu \lambda_\mu] &= 0, \end{aligned} \tag{1.6}$$

where λ_μ are the components of the unit normal vector to the singular hypersurface, and we have used the notation

$$[f] = \lim_{\varepsilon \to 0} [f(x^\mu - \varepsilon \xi^\mu) - f(x^\mu + \varepsilon \xi^\mu)] = f_- - f_+,$$

where the x^μ are the coordinates of a point on the singular surface and the ξ^μ are arbitrary.

If equations (1.1) are substituted into the first four of equations (1.5), they may be written as (*cf.* [3])

$$U^\mu{}_{;\nu} U^\nu = \frac{p_{,\nu}}{\sigma c^2} (g^{\mu\nu} - U^\mu U^\nu) \tag{1.7}$$

and

$$T S_{;\mu} U^\mu = 0, \tag{1.8}$$

where T is the proper temperature, and S is the specific proper entropy. The quantities T and S are defined as in classical hydrodynamics by the requirement that

$$T\,dS = d\varepsilon + p\,d\left(\frac{1}{\varrho}\right). \tag{1.9}$$

We define the specific enthalpy as

$$i = \varepsilon + \frac{p}{\varrho}. \tag{1.10}$$

Then

$$di = T\,dS + \frac{dp}{\varrho}, \tag{1.11}$$

and

$$\sigma c^2 = \varrho(c^2 + i). \tag{1.12}$$

We further define

$$-\varphi = \int_{p_0}^{p} \frac{dp}{\sigma c^2},$$

where the integrand is considered a function of p and S, and the integration is carried out with $S = \text{constant}$. Then

$$-\varphi = \int_{p_0}^{p} \frac{di}{c^2 + i} = \log \frac{c^2 + i(p, S)}{c^2 + i(p_0, S)}. \tag{1.13}$$

We shall treat p_0 as independent of the coordinates of points of space-time in our subsequent discussion. Then

$$-\frac{\partial \varphi}{\partial x^\mu} = + \frac{1}{\sigma c^2} \frac{\partial p}{\partial x^\mu} + \frac{\partial \varphi}{\partial S} \frac{\partial S}{\partial x^\mu}, \tag{1.14}$$

where p is kept constant in the partial differentiation of φ with respect to S.

For some functions $\varepsilon(p, \varrho)$ there exists a p_0 such that

$$i(p_0, S) = 0.$$

With this value of p_0, equation (1.13) becomes

$$e^{-\varphi} = 1 + \frac{i}{c^2} \tag{1.15}$$

and

$$\frac{\partial e^{-\varphi}}{\partial S} = -e^{-\varphi} \frac{\partial \varphi}{\partial S} = \frac{T}{c^2}.$$

2. The Tensor $\Omega_{\mu\nu}$

We define the antisymmetric tensor

$$\Omega_{\mu\nu} = V_{\mu;\nu} - V_{\nu;\mu}, \tag{2.1}$$

where

$$V_\mu = e^{-\varphi} U_\mu, \tag{2.2}$$

and φ is given by equation (1.13). This tensor will play an important role in our subsequent discussion. We, therefore, discuss some of its properties. It follows from equation (2.1) and (2.2) that

$$\Omega_{\mu\nu} = e^{-\varphi} [W_{\mu\nu} - (\varphi_{,\nu} U_\mu - \varphi_{,\mu} U_\nu)],$$

where

$$W_{\mu\nu} = U_{\mu;\nu} - U_{\nu;\mu}, \tag{2.3}$$

and hence

$$W_{\mu\nu} U^\nu = U_{\mu;\nu} U^\nu \tag{2.4}$$

in view of equation (1.2).

We further define the vorticity vector v^μ by the equation

$$v^\mu = \tfrac{1}{2} E^{\mu\nu\sigma\tau} W_{\sigma\tau} U_\nu, \tag{2.5}$$

where

$$E^{\mu\nu\sigma\tau} = \frac{1}{\sqrt{-g}} \varepsilon^{\mu\nu\sigma\tau} \tag{2.6}$$

and

$$E_{\mu\nu\sigma\tau} = \sqrt{-g}\, \varepsilon_{\mu\nu\sigma\tau}. \tag{2.7}$$

The relative tensors $\varepsilon_{\mu\nu\sigma\tau} = \varepsilon^{\mu\nu\sigma\tau}$ are zero unless all indices are different. In that case, they are plus or minus one, depending on whether the values taken by the indices are an even or an odd permutation of 1 2 3 4.

It is a consequence of the definition of these quantities that

$$E_{\mu_1\ldots\mu_k\nu_1\ldots\nu_r}E^{\mu_1\ldots\mu_k\lambda_1\ldots\lambda_r}=k!\,\delta^{\lambda_1\ldots\lambda_r}_{\nu_1\ldots\nu_r},\tag{2.8}$$

where the tensor $\delta^{\lambda_1\ldots\lambda_r}_{\nu_1\ldots\nu_r}$ is zero unless the indices $\lambda_1\ldots\lambda_r$ are an even or an odd permutation of the indices $\nu_1\ldots\nu_r$. In the former case it has the value $+1$, and in the latter case, the value -1.

Multiplying equation (2.5) by $E_{\mu\alpha\beta\gamma}$ and summing, we obtain

$$v^\mu E_{\mu\alpha\beta\gamma}=\tfrac{1}{2}\,\delta^{\nu\sigma\tau}_{\alpha\beta\gamma}W_{\sigma\tau}U_\nu=U_\alpha W_{\beta\gamma}+U_\beta W_{\gamma\alpha}+U_\gamma W_{\alpha\beta}.\tag{2.9}$$

It then follows that

$$W_{\beta\gamma}=v^\mu U^\alpha E_{\mu\alpha\beta\gamma}-U_\beta U_{\gamma;\alpha}U^\alpha+U_\gamma U_{\beta;\alpha}U^\alpha.\tag{2.10}$$

Substituting equation (1.7) into equation (2.10) and the resulting equation into the expression for $\Omega_{\mu\nu}$ in terms of $W_{\mu\nu}$, we obtain

$$\Omega_{\beta\gamma}=e^{-\varphi}\left[v^\mu U^\alpha E_{\mu\alpha\beta\gamma}+\frac{\partial\varphi}{\partial S}(S_{;\gamma}U_\beta-S_{;\beta}U_\gamma)\right]\tag{2.11}$$

after having made use of equation (1.14). When equation (1.15) holds, this may be written as

$$\Omega_{\beta\gamma}=v^\mu V^\alpha E_{\mu\alpha\beta\gamma}-\frac{T}{c^2}(S_{;\gamma}U_\beta-S_{;\beta}U_\gamma).\tag{2.12}$$

It is immediately evident that if

$$v^\mu=0,\quad\text{and}\quad S=\text{constant},\tag{2.13}$$

then $\Omega_{\beta\gamma}=0$. That is, if the flow is irrotational and isentropic, then $\Omega_{\beta\gamma}=0$. The converse also follows; for, if the right-hand side of equation (2.11) vanishes, we may multiply by U^γ and sum to obtain

$$\left(\frac{\partial e^{-\varphi}}{\partial S}\right)S_{;\beta}=0$$

in view of equation (1.8). Hence,

$$v^\mu U^\alpha E_{\mu\alpha\beta\gamma}=0.$$

This implies that

$$v^\mu=A\,U^\mu.$$

However, it is a consequence of the definition of v^μ (*cf.* equation (2.5)) that

$$v^\mu U_\mu=0.\tag{2.14}$$

Hence, the first of equations (2.13) also holds. Thus, equations (2.13) are the necessary and sufficient conditions for

$$\Omega_{\mu\nu}=0.$$

3. Algebraic Properties of $\Omega_{\mu\nu}$

Associated with the tensor $\Omega_{\mu\nu}$ there is the dual tensor defined by the equation

$$\breve{\Omega}_{\mu\nu}=\tfrac{1}{2}E^{\mu\nu\lambda\varrho}\Omega_{\lambda\varrho}.\tag{3.1}$$

It is a consequence of equations (2.8) and (2.12) that

$$\breve{\Omega}^{\mu\nu}=-\left(V^\mu v^\nu-v^\mu V^\nu+\frac{T}{c^2}S_{,\varrho}U_\lambda E^{\mu\nu\lambda\varrho}\right).\tag{3.2}$$

Hence
$$\Omega_1 = \Omega_{\mu\nu} \breve{\Omega}^{\mu\nu} = 4 \frac{T}{c^2} e^{-\varphi} v^\mu S_{,\mu}. \tag{3.3}$$

Thus, the tensor $\Omega_{\mu\nu}$ will be singular if, and only if, $v^\mu T S_{,\mu} = 0$. If it is singular, it can only be of rank zero or two.

Another invariant associated with the tensor $\Omega_{\mu\nu}$ is
$$\Omega_2 = \Omega_{\mu\nu} \Omega^{\mu\nu} = 2 \left[e^{-2\varphi} v^\mu v_\mu + \frac{T^2}{c^4} S_{,\mu} S_{,\nu} g^{\mu\nu} \right]. \tag{3.4}$$

Since both v^μ and $TS_{,\mu}$ are space-like vectors (*cf.* equations (2.15) and (1.8)), the quantity Ω_2 satisfies
$$\Omega_2 \leqq 0. \tag{3.5}$$
The equality holds if, and only if,
$$v_\mu = 0, \quad \text{and} \quad S_{,\mu} = 0,$$
that is, if, and only if,
$$\Omega_{\mu\nu} = 0.$$

It is a consequence of equations (2.11), (1.2), and (1.8) that
$$\begin{aligned} \Omega_{\mu\nu} U^\nu &= \frac{T}{c^2} S_{,\mu}, \\ \Omega_{\mu\nu} v^\nu &= -\frac{T}{c^2} S_{,\nu} v^\nu U_\mu, \\ \Omega_{\mu\nu} \frac{T}{c^2} S_{,\alpha} g^{\alpha\nu} &= w_\mu - \left(\frac{T^2}{c^4} g^{\alpha\beta} S_{,\alpha} S_{,\beta} \right) U_\mu, \end{aligned} \tag{3.6}$$
where
$$w_\mu = E_{\mu\nu\sigma\tau} v^\nu V^\sigma \frac{T}{c^2} g^{\tau\alpha} S_{,\alpha} \tag{3.7}$$
and is a vector orthogonal to u^μ, v^μ and $TS_{,\mu}$. It further follows that
$$\Omega_{\mu\nu} w^\nu = e^{-2\varphi} \left[\left(v^\sigma \frac{T}{c^2} S_{,\sigma} \right) v_\mu - (v_\alpha v^\alpha) \frac{T}{c^2} S_{,\mu} \right] \tag{3.8}$$
and that
$$w_\nu w^\nu = e^{-2\varphi} \left[(v^\alpha v_\alpha) \left(\frac{T^2}{c^4} S_{,\beta} g^{\beta\gamma} S_{,\gamma} \right) - \left(\frac{T}{c^2} v^\gamma S_{,\gamma} \right)^2 \right]. \tag{3.9}$$

In case $\Omega_1 = 0$, that is, $\Omega_{\mu\nu}$ is singular but not all of its components vanish (v^μ and $TS_{,\mu}$ are not both zero), it follows from the above that
$$\Omega_{\mu\nu} v^\nu = 0$$
and
$$\Omega_{\mu\nu} y^\nu = 0,$$
where
$$y^\mu = \left(U^\mu + \frac{e^{2\varphi}}{v^\alpha v_\alpha} w^\mu \right). \tag{3.10}$$

That is, the vectors v^ν and y^ν are annulled by the matrix of coefficients of the tensor $\Omega_{\mu\nu}$. The vector y^μ reduces to U^μ whenever w^μ vanishes, that is, whenever
$$v^\alpha = A \frac{TS_{,\gamma}}{c^2} g^{\gamma\alpha},$$
where A is an arbitrary scalar.

Since

$$y^\mu y_\mu = 1 + \frac{e^{4\varphi}}{(v^\alpha v_\alpha)^2} w_\mu w^\mu$$

and $w_\mu w^\mu \geqq 0$ as follows from equation (3.9) and the application of SCHWARZ's inequality to the space-like vectors involved, the vector y^μ is always time-like.

4. Tubes in Space-Time

If $z^\mu(x)$ is a vector field, the solutions of the ordinary differential equations

$$\frac{d x^\mu}{dW} = z^\mu(x) \tag{4.1}$$

define a three parameter family of curves passing through a hypersurface Σ in space-time. We shall assume that the hypersurface may be represented by the equation

$$x^{i_4} = f(x^{i_1}, x^{i_2}, x^{i_3}) \tag{4.2}$$

where i_1, i_2, i_3 and i_4 are some permutation of the numbers 1 2 3 4.

We may then write the solutions of equations (4.1) as

$$x^\mu = x^\mu(\xi^i, W) \qquad (i = i_1, i_2, i_3), \tag{4.3}$$

where the ξ^i are the initial values of the x^i, that is

$$x^i(\xi^i, 0) = \xi^i$$

and

$$x^{i_4}(\xi^i, 0) = f[x^{i_1}(\xi^i, 0), x^{i_2}(\xi^i, 0), x^{i_3}(\xi^i, 0)].$$

If in the surface Σ we have a closed curve described by the equations

$$\xi^i = \xi^i(\tau), \qquad \tau_1 \leqq \tau \leqq \tau_2$$

with

$$\xi^i(\tau_1) \equiv \xi^i(\tau_2),$$

then the equations

$$x^\mu = x^\mu[\xi^i(\tau), W] = x^\mu(\tau, W) \tag{4.4}$$

define a two-dimensional surface in space-time which we shall call a tube. The closed curves obtained by setting constant values for W into equation (4.4) are images of the closed curve on the initial surface. When in equations (4.4) τ is constant, the equations define a curve which is a solution of equation (4.1).

The tangent vector to the curves of parameter τ is given by

$$\lambda^\mu = \frac{\partial x^\mu}{\partial \tau} \tag{4.5}$$

where W is kept fixed in the differentiation. We have

$$\frac{\partial \lambda^\mu}{\partial W} = \frac{\partial^2 x^\mu}{\partial W \, \partial \tau} = \frac{\partial z^\mu}{\partial \tau}. \tag{4.6}$$

When the vector field $z^\mu(x)$ is set equal to the four-velocity $U^\mu(x)$, W is the proper-time and the curves of parameter W are called the world-lines of particles making up the fluid. The four variables ξ^i, W may be interpreted as a new set of coordinates in space-time, co-moving coordinates. They are the analogues of Lagrange coordinates in classical hydrodynamics. The curves of parameter τ (W fixed) are then said to "travel with the fluid".

Another example of a tube in space-time is obtained by setting $z^\mu(x)$ equal to the vorticity vector $v^\mu(x)$. Then the curves of parameter W given by equation (4.3) are said to be vortex-lines, and the tube is called a vortex tube.

5. Invariant Integrals

Consider the integral

$$\Gamma(W) = \int_{\tau_1}^{\tau_2} V_\alpha[x(\tau, W)]\, \lambda^\alpha\, d\tau, \tag{5.1}$$

where V_α is given by equation (2.2) and is considered as a function of τ and W in view of equation (4.4) and λ^α is the tangent vector to a curve of parameter W given by equation (4.5). The integral $\Gamma(W)$ represents the integral of the vector V_α which is proportional to the velocity vector, around a closed curve on the tube defined by the solutions of equation (4.1).

We now compute

$$\frac{d\Gamma}{dW} = \int_{\tau_1}^{\tau_2} \left(\frac{\partial V_\alpha}{\partial x^\sigma} z^\sigma \lambda^\alpha + V_\alpha \frac{\partial \lambda^\alpha}{\partial W}\right) d\tau.$$

This may be written as

$$\frac{d\Gamma}{dW} = \int_{\tau_1}^{\tau_2} \left(\Omega_{\alpha\sigma} z^\sigma \lambda^\alpha + \frac{\partial V_\sigma}{\partial x^\alpha} \lambda^\alpha z^\sigma + V_\alpha \frac{\partial \lambda^\alpha}{\partial W}\right) d\tau$$

$$= \int_{\tau_1}^{\tau_2} \left(\Omega_{\alpha\sigma} z^\sigma \lambda^\alpha + \frac{\partial V_\sigma}{\partial \tau} z^\sigma + V_\alpha \frac{\partial z^\alpha}{\partial \tau}\right) d\tau$$

where we have made use of equation (4.6). Thus

$$\frac{d\Gamma}{dW} = \int_{\tau_1}^{\tau_2} \left[\Omega_{\alpha\sigma} z^\sigma \lambda^\alpha + \frac{\partial}{\partial \tau}(V_\sigma z^\sigma)\right] d\tau.$$

Since we are integrating around a closed curve, we finally obtain

$$\frac{d\Gamma}{dW} = \int_{\tau_1}^{\tau_2} \Omega_{\alpha\sigma} z^\sigma \lambda^\alpha\, d\tau. \tag{5.2}$$

The necessary and sufficient condition for $\Gamma(W) = 0$ for tubes defined by the vector field z^μ by means of arbitrary initial surfaces and arbitrary curves in this surface is that

$$\Omega_{\alpha\beta} = 0. \tag{5.3}$$

Similarly, the necessary and sufficient condition that $d\Gamma/dW = 0$ for tubes defined with the same amount of generality is that

$$Z_{\alpha,\beta} - Z_{\beta,\alpha} = 0$$

where

$$Z_\alpha = \Omega_{\alpha\sigma} z^\sigma. \tag{5.4}$$

6. The Circulation

In this section we set $z^\alpha = U^\alpha$, the four-velocity vector and $W = s$, the proper time. We shall then name the integral in equation (5.1) the circulation and denote it by $C(s)$. Equation (5.2) then becomes

$$\frac{dC}{ds} = -\int_{\tau_1}^{\tau_2} e^{-\varphi} \frac{\partial \varphi}{\partial S} S_{,\alpha} \lambda^\alpha \, d\tau \tag{6.1}$$

in view of the first of equations (3.6).

Thus, a necessary and sufficient condition for the circulation to be independent of proper-time is that

$$\left(e^{-\varphi} \frac{\partial \varphi}{\partial S} S_{;\alpha}\right)_{;\beta} = \left(e^{-\varphi} \frac{\partial \varphi}{\partial S} S_{;\beta}\right)_{;\alpha}. \tag{6.2}$$

It is clear that for isentropic flows (S = constant), $d\Gamma/ds = 0$.

When equation (1.15) holds, equation (6.2) may be written as

$$(T S_{;\alpha})_{;\beta} - (T S_{;\beta})_{;\alpha} = 0. \tag{6.3}$$

In virtue of equation (1.9) this may be written as

$$p_{;\alpha}\, \varrho_{;\beta} - p_{;\beta}\, \varrho_{;\alpha} = 0. \tag{6.4}$$

Hence, in this case the necessary and sufficient condition that $dC/ds = 0$ is that at every point of space-time

$$p = p(\varrho). \tag{6.5}$$

That is, the motion is such that a knowledge of p (or ϱ) at a point determines ϱ (or p).

7. Bernoulli's Theorem (Time-dependent Flows)

The definition of the circulation, namely,

$$C(S) = \int_{\tau_1}^{\tau_2} e^{-\varphi} U_\alpha \lambda^\alpha \, d\tau$$

involves an integrand which, aside from the exponential factor, is the tangential component of the four-velocity field U^α taken at a point of a curve which travels with the fluid. $C(s)$ is, thus, similar to the classical circulation. To show that it is a generalization of this classical quantity, we shall show that as in classical hydrodynamics, the condition

$$C(s) = 0 \tag{7.1}$$

and is related to Bernoulli's theorem for time-dependent irrotational and isentropic flows.

It follows from equation (5.3) that the condition (7.1) is equivalent to

$$\Omega_{\mu\nu} = 0,$$

that is, to the requirement that the flow be isentropic and irrotational (*cf.* Section 2). Under these conditions (*cf.* [3]) there exists a scalar function $R(x)$ such that

$$V_\mu = e^{-\varphi} U_\mu = \frac{\partial R}{\partial x^\mu}. \tag{7.2}$$

Define the quantity

$$\frac{q^2}{c^2} = - g_{\mu\nu} U^\mu U^\nu + \frac{U_4^2}{g_{44}} = - \left(g_{\mu\nu} - \frac{g_{4\mu} g_{4\nu}}{g_{44}}\right) U^\mu U^\nu.$$

This is the length of the vector 0^μ which is given by

$$0^\mu = U^\mu - \frac{\delta_4^\mu U_4}{g_{44}}$$

and which is the projection of U^μ onto the three-space orthogonal to the curves of parameter x^4.

In view of equation (1.2) we have

$$U_4 = \sqrt{g_{44}} \sqrt{1 + \frac{q^2}{c^2}}. \tag{7.3}$$

Equations (7.2) with $\mu = 4$ imply that

$$e^{-\varphi} U_4 = e^{-\varphi} \sqrt{g_{44}} \sqrt{1 + \frac{q^2}{c^2}} = \frac{\partial R}{\partial x^4}.$$

When equation (1.15) holds, this becomes

$$\left(1 + \frac{i}{c^2}\right) \sqrt{g_{44}} \sqrt{1 + \frac{q^2}{c^2}} = \frac{\partial R}{\partial x^4}.$$

If we now write

$$g_{44} = 1 + \frac{2V}{c^2} \tag{7.4}$$

and expand the left-hand side of the above equation neglecting powers of $1/c$ higher than the second, we obtain

$$\left(i + V + \frac{1}{2} q^2\right) = c^2 \left(\frac{\partial R}{\partial x^4} - 1\right). \tag{7.5}$$

This is the classical form of the Bernoulli equation for a non-steady irrotational, isentropic flow in the gravitational field with potential V. The quantity $c^2 R$ is to be interpreted in terms of the velocity potential for the flow.

Equation (7.4) is the usual relation between the classical Newtonian potential and the metric tensor of space-time which holds in the case of weak fields.

In case we use equation (1.13), equation (7.5) becomes

$$\left(i + V + \frac{1}{2} q^2\right) = [c^2 + i(p_0, S)] \left(\frac{\partial R}{\partial x^4} - 1\right). \tag{7.6}$$

8. Bernoulli's Theorem (Steady Flows)

In classical theory Bernoulli's theorem is usually formulated for stationary flows. LICHNEROWICZ [2] has indicated a method of describing steady flows of fluid in their own gravitational fields. It consists of assuming that the space-time and the velocity field are invariant under a time-like one-parameter group of motions. That is, there exists a vector field ξ^α which is such that Killing's equations

$$\xi_{\alpha;\beta} + \xi_{\beta;\alpha} = 0 \tag{8.1}$$

are satisfied and such that

$$\begin{aligned} U^\tau \xi^\sigma{}_{;\tau} - \xi^\tau U^\sigma{}_{;\tau} &= 0, \\ S_{;\alpha} \xi^\alpha = p_{;\alpha} \xi^\alpha &= 0. \end{aligned} \tag{8.2}$$

Equations (8.2) state that the velocity field U^σ is invariant under the group of motions as are the entropy and pressure.

Define the quantity

$$H = e^{-\varphi} U_\alpha \xi^\alpha = V_\alpha \xi^\alpha \tag{8.3}$$

and compute

$$H_{;\beta} U^\beta = (e^{-\varphi} U_\alpha \xi^\alpha)_{;\beta} U^\beta.$$

We have

$$\begin{aligned} H_{;\beta} U^\beta &= e^{-\varphi}(-\varphi_{;\beta} U^\beta U_\alpha \xi^\alpha + U_{\alpha;\beta} U^\beta \xi^\alpha + U_\alpha \xi^\alpha_{;\beta} U^\beta) \\ &= e^{-\varphi}\left[\frac{p_{,\alpha}}{\sigma c^2}\xi^\alpha + \frac{1}{2} U^\alpha U^\beta (\xi_{\alpha;\beta} + \xi_{\beta;\alpha})\right] = 0 \end{aligned}$$

as a consequence of equations (1.7), (1.8), (1.14), (8.1) and (8.2). Thus, along a world-line of the fluid we have

$$H = \text{constant}.$$

If the coordinate system is chosen so that

$$\xi^\alpha = \delta^\alpha_4$$

this equation becomes

$$e^{-\varphi} U_4 = \text{constant}.$$

Using the results of the preceding section, we may write this as

$$i + \frac{q^2}{2} + V = \text{constant} \tag{8.4}$$

along a world-line of the fluid. It may very from world-line to world-line. Equation (8.4) is the usual form of the Bernoulli equation.

It is instructive to compute the derivative of H in an arbitrary direction. Thus,

$$\begin{aligned} H_{;\beta} &= e^{-\varphi}[-\varphi_{,\beta} U_\alpha \xi^\alpha + U_{\alpha;\beta}\xi^\alpha + U_\alpha \xi^\alpha_{;\beta}] \\ &= e^{-\varphi}[e^{\varphi}\Omega_{\alpha\beta}\xi^\alpha - \varphi_{,\alpha}\xi^\alpha U_\beta + U_{\beta;\alpha}\xi^\alpha - U^\alpha \xi_{;\alpha} + U^\alpha(\xi_{\alpha;\beta} + \xi_{\beta;\alpha})] \end{aligned}$$

as follows from the definition of $\Omega_{\alpha\beta}$ (*cf.* equation (3.2)). In view of equations (8.2) we have

$$H_{;\beta} = \Omega_{\alpha\beta}\xi^\alpha.$$

Hence, if $\Omega_{\alpha\beta} = 0$,

$$H = \text{constant}$$

thoughout space-time. That is, if the motion is isentropic and irrotational, the Bernoulli constant is independent of the world-line of the fluid particle as is the case in classical theory.

9. Vortex Tubes

In this section we shall discuss the integral (5.1) when the tubes in question are vortex tubes, that is, when we set $z^\mu(x) = v^\mu(x)$, and the motion is rotational ($v^\mu \neq 0$). In this case, we cannot have $\Gamma(W) = 0$. However, it follows from equations (5.2) and (3.6) that

$$\frac{d\Gamma}{dW} = \int_{\tau_1}^{\tau_2} e^{-\varphi}\frac{\partial\varphi}{\partial S} S_{,\nu} v^\nu U_\alpha \lambda^\alpha d\tau. \tag{9.1}$$

Equation (9.1) may be written as

$$\frac{d\Gamma}{dW} = \int_{\tau_1}^{\tau_2} h V_\alpha \lambda^\alpha d\tau \tag{9.2}$$

with

$$h = \frac{\partial \varphi}{\partial S} S_{,\nu} v^\nu. \tag{9.3}$$

The necessary and sufficient condition for $d\Gamma/dW = 0$ for tubes defined by arbitrary curves in these surfaces is, then,

$$(h V_\alpha)_{;\beta} - (h V_\beta)_{;\alpha} = 0.$$

This may be written as

$$h \Omega_{\alpha\beta} + h_{,\beta} V_\alpha - h_{,\alpha} V_\beta = 0.$$

Thus,

$$h \Omega_{\alpha\beta} = h_{,\alpha} V_\beta - h_{,\beta} V_\alpha$$

and is of rank 2. Therefore, $h = 0$ or $\Omega_1 = 0$. However, $\Omega_1 = e^\varphi h$. Hence, the necessary and sufficient condition for $d\Gamma/dW = 0$ for vortex tubes defined as above is that $\Omega_1 = 0$, and $\Omega_{\alpha\beta}$ be of rank 2.

In this case we also have

$$\frac{d\Gamma}{dW} = 0$$

for world tubes defined by the vector field y^μ given by equation (3.10).

10. The Change in Circulation Across Shocks

All of the preceding discussion was based on the assumption that the derivatives of the flow variables existed everywhere. We now consider the possibility of singular hyper-surfaces in space-time, representing shocks across which equations (1.6) obtain. These equations may be written as

$$\frac{m}{c} = \varrho_+ U_+^\mu \lambda_\mu = \varrho_- U_-^\mu \lambda_\mu \tag{10.1}$$

and

$$\frac{m}{c} (\mu_+ U_+^\mu - \mu_- U_-^\mu) = \frac{\lambda^\mu}{c^2} (p_+ - p_-), \tag{10.2}$$

where

$$\mu = \frac{\sigma}{\varrho} = 1 + \frac{i}{c^2}. \tag{10.3}$$

In case equation (1.15) holds, equation (10.2) may be written as

$$\frac{m}{c} (V_+^\mu - V_-^\mu) = \frac{\lambda^\mu}{c^2} (p_+ - p_-), \tag{10.4}$$

where λ^μ is normalized so that

$$\lambda^\mu \lambda_\mu = -1 \tag{10.5}$$

and for a shock wave $m \neq 0$.

It then follows from these equations that the components of the vector field V_μ tangent to the hyper-surface are continuous across it. As has been shown earlier, the components normal to the hyper-surface are discontinuous. We may determine the discontinuity in this component in terms of the discontinuity in pressure (or density) across the shock as follows: multiply equations (10.2) by $U_{+\mu}$, $U_{-\mu}$ and λ_μ to obtain

$$(\mu_+ - \mu_- U^\mu_- U_{+\mu}) = \frac{1}{c^2} \cdot \frac{1}{\varrho_+} (p_+ - p_-), \tag{10.6}$$

$$(\mu_+ U^\mu_+ U_{-\mu} - \mu_-) = \frac{1}{c^2} \cdot \frac{1}{\varrho_-} (p_+ - p_-), \tag{10.7}$$

$$\frac{m^2}{c^2} \left(\frac{\mu_+}{\varrho_+} - \frac{\mu_-}{\varrho_-} \right) = - \frac{1}{c^2} (p_+ - p_-), \tag{10.8}$$

respectively. If we now eliminate $U^\mu_- U_{+\mu}$ from the first two of these equations we have the equation

$$\mu_+^2 - \mu_-^2 = \frac{1}{c^2} (p_+ - p_-) \left(\frac{\mu_+}{\varrho_+} + \frac{\mu_-}{\varrho_-} \right). \tag{10.9}$$

From equations (10.7) and (10.1) we have

$$\frac{m}{c} = \frac{1}{c} \left(\frac{p_+ - p_-}{\frac{\mu_-}{\varrho_-} - \frac{\mu_+}{\varrho_+}} \right)^{\frac{1}{2}} = \frac{\varrho_+}{\mu_+} (V^\mu_+ \lambda_\mu) = \frac{\varrho_-}{\mu_-} (V^\mu_- \lambda_\mu). \tag{10.10}$$

Equations (10.9) and (10.10) are the relativistic Rankine-Hugoniot equations previously derived [4]. When μ is known as a function of p and ϱ, equation (10.9) may be used to determine p_+ in terms of p_-, ϱ_+ and ϱ_-. Equations (10.10) may then be used to express ϱ_+ and hence p_+ as a function of $V^\mu_- \lambda_\mu$, p_- and ϱ_-. They may also be used to express $V^\mu_+ \lambda_\mu$ as a function of the latter quantities.

We shall write

$$M = \frac{\mu_+}{\mu_-}, \qquad \eta = \frac{\varrho_+}{\varrho_-}, \quad \text{and} \quad y = \frac{p_+}{p_-}. \tag{10.11}$$

Then equations (10.9) and (10.10) may be written as

$$\begin{aligned} M^2 - 1 &= \left(1 - \frac{M^2}{\eta^2}\right) (V^\mu_- \lambda_\mu) = \alpha^2 (y - 1) \left(\frac{M}{\eta} + 1 \right), \\ V^\mu_+ \lambda_\mu &= \frac{M}{\eta} (V^\mu_- \lambda_\mu), \\ y - 1 &= \left(1 - \frac{M}{\eta}\right) \left[\frac{(V^\mu_- \lambda_\mu)^2}{\alpha^2} \right], \\ \alpha^2 &= \frac{p_-}{c^2 \varrho_- \mu_-}. \end{aligned} \tag{10.12}$$

When μ is known as a function of p and ϱ, the first of these equations may be used to determine y in terms of η. We may then use the remaining equations to determine all quantities as functions of $V^\mu_- \lambda_\mu$. Since the entropy will not be conserved across shocks, care must be taken to choose solutions of these equations such that

$$S_+ > S_-$$

across the singular hyper-surface.

When a shock is such that λ_μ is a function of the points of space-time, we say that it is of varying strength. This variation may be due to the curvature of the shock at each time $x^4 =$ constant or to the dependence of λ_μ on x^4 alone. The world-lines of particles which cut the singular hyper-surface have discontinuous tangents at the hyper-surface. Along such a world-line the entropy suffers a discontinuity. The size of this discontinuity in entropy varies from world-line to world-line. As a result, it is not true that the pressure and density at all points of space-time are related as in equation (6.5). Thus, when shocks are present, we cannot have $dC/ds=0$ for arbitrary tubes.

This result may be seen in another manner. Suppose we have a shock of varying strength (λ^μ not constant) and suppose it is such that V_-^μ, p_-, and ϱ_- are constant. Then, we would have

$$\Omega^-_{\mu\nu} = V_{-\mu;\nu} - V_{-\nu;\mu} = 0 .$$

However if

$$\Omega^+_{\mu\nu} = 0 ,$$

then it follows from results of Section 2 that the flow behind the shock would be both isentropic and irrotational. However, we have seen that it is not isentropic. Thus,

$$\Omega^+_{\mu\nu} \neq 0 .$$

This means that we can construct tubes of world-lines which cross shocks and for which $C(S)=0$ on one portion of the tube, but $C(S)\neq 0$ on other portions.

The detailed behavior of the circulation across shocks depends on the explicit expression obtained when the system of equations (10.12) are solved as described above. These expressions depend in turn on the nature of μ as a function of p and ϱ or equivalently on the nature of ε as a function of p and ϱ. Once this function is prescribed, we may study the geometry of the world-lines in terms of the geometry of the shock surface by introducing a coordinate system consisting of the world-lines of the fluid particles as curves of parameter x^4 and the singular hyper-surface as the hyper-surface $x^4=$ constant. The methods used in studying stationary and pseudo-stationary flows [5] may then be applied in a straightforward manner.

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Digital Computer Laboratory
University of Illinois
Urbana, Illinois

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