A Note on the Exstencie of Periodic Solutions of the Navier-Stokes Equations

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In an earlier paper [1] I established the stability of solutions of the Navier-Stokes equations provided that a certain Reynolds number was small enough. This result enables one to deduce from certain plausible hypotheses the existence of stable periodic solutions of the Navier-Stokes equations. In particular we shall prove the following result:

Let $\mathscr{V}=\mathscr{V}(t)$ be a bounded region in space, and let a flow velocity be prescribed at each point of the boundary of $\mathscr{V}$. Assume furthermore that both $\mathscr{V}$ and the assigned velocities depend periodically on the time t. Then, under conditions 1° and 2° stated below, there exists a unique, stable, periodic solution of the Navier-Stokes equations in $\mathscr{V}$ which takes on the prescribed on the boundary of $\mathscr{V}$.

In view of condition 2° (see below) it is necessary that the boundary conditions be compatible with a flow of Reynolds number less than 5.7. The theorem may therefore be paraphrased by the statement that corresponding to sufficiently low assigned periodic velocities there exists a periodic flow to which every other motion eventually subsides. A simple example would be a fluid enclosed in a fixed container and stirred by a low speed bladed rotor.

Another case of interest occurs when the assigned conditions are *steady*. Then the theorem asserts the existence of a unique, stable, time-independent solution of the Navier-Stokes equations taking on the prescribed velocities on the boundary of $\mathscr{V}$. In several long papers (*e.g.* [2]–[5]) the existence of steady flows has been proved independently of conditions 1° and 2°, though in these cases a considerable degree of smoothness has been required of the boundary data. Needless to say, the flows thus obtained may be neither stable nor unique, unless their Reynolds numbers are low enough.

We now state conditions 1° and 2° envisaged in the above theorem and remarks.

1°. *To every continuous initial distribution of velocities over $\mathscr{V}$ there corresponds a solution of the Navier-Stokes equations satisfying the prescribed boundary conditions.*

2°. *There is one solution whose Reynolds number $Re \equiv Vd/\nu$ is less than 5.7. (Here V is the maximum speed of the flow during the whole time interval $0 \leqq t < \infty$, d is the maximum diameter of $\mathscr{V}$, and ν is the kinematic viscosity.) This solution is equicontinuous in $\boldsymbol{x}=(x, y, z)$ for all t.*

Before proceeding to the proof of the theorem there are several remarks to be made. First, it is understood that the flows guaranteed by Condition 1° should be valid for all $t \geqq 0$. Though this is mathematically a very stringent requirement, it is nevertheless quite plausible on the grounds of physical intuition and our knowledge of the behavior of other parabolic systems; moreover, from the proof it will be clear that 1° need hold only for initial velocity distributions whose Reynolds numbers are less than 5.7. Second, the number 5.7 in Condition 2° may possibly be improved, as will be apparent from the proof. Finally, the author wishes to make it clear that he does not consider the above theorem to be equivalent to the usual kind of mathematical existence theorem: it is agreed that for certain types of boundary behavior 1° and 2° might not hold, and that (correspondingly) the limits of application of the theorem are not well-defined. On the other hand, in view of the plausibility of 1° and 2° there is every reason to expect that they will be proved in the future, subject to sufficiently smooth boundary conditions[1], and once such proofs are given the existence of periodic solutions indisputably follows.

Now let $\boldsymbol{v} = \boldsymbol{v}(\boldsymbol{x}, t)$ be the velocity vector of the flow guaranteed by Condition 2°, and suppose the assigned boundary conditions have period 1. We consider the sequence of vector fields

$$\boldsymbol{\varphi}_n(\boldsymbol{x}) = \boldsymbol{v}(\boldsymbol{x}, n), \qquad n = 0, 1, 2, \ldots.$$

This sequence is bounded and equicontinuous by hypothesis, hence by the theorem of Arzela contains a *subsequence* which converges uniformly to a vector $\boldsymbol{\varphi}(\boldsymbol{x})$. We assert that actually the whole sequence converges to $\boldsymbol{\varphi}(\boldsymbol{x})$.

Indeed, if this were not the case we could find another subsequence converging to a vector $\boldsymbol{\psi}(\boldsymbol{x})$ different from $\boldsymbol{\varphi}(\boldsymbol{x})$. But this is impossible, as we shall now show. For positive integers m and n, $(m > n)$, set

$$\boldsymbol{v}'(x, t) = \boldsymbol{v}(\boldsymbol{x}, t + m - n), \qquad t \geqq 0.$$

Then obviously $\boldsymbol{v}'$ is a solution of the Navier-Stokes equations, and, according to the periodicity assumption, it satisfies the assigned boundary conditions. Now for any vector $\boldsymbol{f} = \boldsymbol{f}(\boldsymbol{x}, t)$, let $\mathscr{K}(\boldsymbol{f})$ be defined by

$$\mathscr{K}(\boldsymbol{f}) \equiv \tfrac{1}{2} \int_{\mathscr{V}(t)} |\boldsymbol{f}(\boldsymbol{x}, t)|^2 \, dv.$$

Then by Theorem 1 of [1] we have

$$\mathscr{K}(\boldsymbol{v}' - \boldsymbol{v}) \leqq \mathscr{K}_0 e^{-\varepsilon t}, \tag{1}$$

where $\mathscr{K}_0$ denotes the value of $\mathscr{K}(\boldsymbol{v}' - \boldsymbol{v})$ at $t = 0$, and

$$\varepsilon = \frac{1}{\nu} (\alpha \nu^2 / d^2 - V^2) > 0$$

since $\mathrm{Re}[\boldsymbol{v}] \leqq 5.7$. Because $\boldsymbol{v}'$ as well as $\boldsymbol{v}$ satisfies $\mathrm{Re} \leqq 5.7$, it is seen that

$$\mathscr{K}_0 \leqq \tfrac{1}{2} \int_{\mathscr{V}_0} (2V)^2 \, dv \leqq 2 d^3 V^2 = \text{Const.}$$

Setting $t = n$ in (1) yields

$$\mathscr{K}(\boldsymbol{\varphi}_m - \boldsymbol{\varphi}_n) \leqq \text{Const.}\, e^{-\varepsilon n}, \tag{2}$$

[1] Similar theorems are already known, *cf.* the papers of LERAY and KISELEV & LADYSHENSKAYA cited in the references.

and it follows that
$$\lim_{m,n\to\infty} \mathscr{K}(\boldsymbol{\varphi}_m - \boldsymbol{\varphi}_n) = 0 \tag{3}$$
(the region of integration in (2) and (3) is, of course, $\mathscr{V}(n) \equiv \mathscr{V}_0$). Now letting m and n tend to infinity through sequences of integers such that $\boldsymbol{\varphi}_m \to \boldsymbol{\psi}$, $\boldsymbol{\varphi}_n \to \boldsymbol{\varphi}$ gives an immediate contradiction. This proves that $\lim \boldsymbol{\varphi}_n(\boldsymbol{x}) = \boldsymbol{\varphi}(\boldsymbol{x})$.

By condition 1° there exists a flow $\boldsymbol{v}^* = \boldsymbol{v}^*(\boldsymbol{x}, t)$ such that $\boldsymbol{v}^*(\boldsymbol{x}, 0) = \boldsymbol{\varphi}(\boldsymbol{x})$. We assert that $\boldsymbol{v}^*$ is a *periodic* solution of the Navier-Stokes equation. For set $\boldsymbol{v}''(\boldsymbol{x}, t) = \boldsymbol{v}(\boldsymbol{x}, t+n)$. Then we have, in the same way as inequality (1),
$$\mathscr{K}(\boldsymbol{v}^* - \boldsymbol{v}'') \leqq \mathscr{K}_0 e^{-\varepsilon t}, \tag{4}$$
where ε has the same meaning as before, and $\mathscr{K}_0$ is the value of $\mathscr{K}(\boldsymbol{v}^* - \boldsymbol{v}'')$ when $t = 0$, that is $\mathscr{K}_0 \equiv \mathscr{K}(\boldsymbol{\varphi} - \boldsymbol{\varphi}_n)$. Putting $t = 1$ in (4) yields
$$\mathscr{K}[\boldsymbol{v}^*(\boldsymbol{x}, 1) - \boldsymbol{\varphi}_{n+1}(\boldsymbol{x})] < \mathscr{K}(\boldsymbol{\varphi} - \boldsymbol{\varphi}_n)$$
and letting $n \to \infty$ gives, finally,
$$\mathscr{K}[\boldsymbol{v}^*(\boldsymbol{x}, 1) - \boldsymbol{\varphi}(\boldsymbol{x})] = 0.$$
It follows that $\boldsymbol{v}^*(\boldsymbol{x}, 1) = \boldsymbol{\varphi}(\boldsymbol{x}) = \boldsymbol{v}^*(\boldsymbol{x}, 0)$, that is, $\boldsymbol{v}^*$ is periodic.

To complete the proof of the theorem it is enough to show that $\mathrm{Re}[\boldsymbol{v}^*] \leqq 5.7$, for then by [*1*] $\boldsymbol{v}^*$ is stable and unique[2]. But we have
$$\mathscr{K}(\boldsymbol{v}^* - \boldsymbol{v}) \to 0 \quad \text{as} \quad t \to \infty, \tag{5}$$
by virtue of Theorem 1 of [*1*]. Since both $\boldsymbol{v}$ and $\boldsymbol{v}^*$ are equicontinuous, (5) implies $(\boldsymbol{v}^* - \boldsymbol{v}) \to 0$ as $t \to \infty$, and therefore
$$\operatorname*{Max}_{\boldsymbol{x}} \boldsymbol{v}^*(\boldsymbol{x}, t) \leqq V + o(t).$$
$\boldsymbol{v}^*$ being periodic, this in turn proves $V^* \leqq V$ and $\mathrm{Re}[\boldsymbol{v}^*] \leqq 5.7$.

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[2] That $\boldsymbol{v}^*$ is stable follows from Theorem 1; that it is unique can be shown by the same argument used to prove Theorem 2.

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