The Mechanics of Non-Linear Materials with Memory

Part II

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1. Introduction

In a previous paper (GREEN & RIVLIN 1957), subsequently referred to as Part I, the form of the constitutive equations governing the deformation of a class of materials possessing memory was discussed. It was assumed that the stress in an element of the material depends not only on the deformation gradients in the element at the instant of time considered, but also on those at previous instants of time. The limitations imposed on the constitutive equation by the fact that it must be form-invariant under a rotation of the physical system considered (consisting of the body and applied forces) were examined.

This was done by first considering that the stress depends on the deformation gradients at a number of discrete times up to the instant of measurement. Then, the number of instants of time was considered to increase indefinitely, so that the expression for the stress became a functional of the deformation gradients. In this analysis it was found that the form-invariance of the constitutive equation under a rotation of the physical system leads naturally to a particular form of dependence of the stress on the deformation gradients at the instant of measurement. It was assumed that, apart from this, the expression for the stress as a functional of the deformation gradients at times up to and including the instant of measurement is continuous.

In the present paper, we do not make this assumption, but allow that the stress may have arbitrary polynomial dependence on the deformation gradients at the instant of measurement, while its functional dependence on the deformation gradients at times preceding the instant of measurement is continuous. Under these conditions, the limitations imposed by isotropy of the material in its undeformed state on the form of the constitutive equation is considered.

2. Special Dependence of the Stress on the Displacement Gradients

We consider a three-dimensional body to undergo deformation described in a fixed rectangular Cartesian coordinate system x by

$$\begin{aligned} x_i(\tau) &= x_i(X_j, \tau) \qquad (\tau > 0), \\ x_i(\tau) &= X_i \qquad (\tau \leqq 0), \end{aligned} \tag{2.1}$$

where X_i and x_i are the coordinates in the system x of a generic particle of the body at zero time and time τ respectively. We assume that $x_i(\tau)$ are single-valued functions of the arguments, possessing continuous spatial derivatives up to any required order except possibly at singular points, lines and surfaces. If the deformation is to be possible in a real material we must have

$$|\partial x_i(\tau)/\partial X_j| > 0.$$

We assume the stress components $\sigma_{ij}(t)$ at time t in the system x to be polynomial functions of the deformation gradients $\partial x_i(\tau_\alpha)/\partial X_j$ ($\alpha = 0, 1, 2, \ldots, N$) at the $N+1$ instants of time $\tau_1, \tau_2, \ldots, \tau_N, \tau_0(=t)$ between $\tau = 0$ and $\tau = t$. It has been shown (GREEN & RIVLIN 1957), that σ_{ij} may then be expressed in the form

$$\sigma_{ij} = \frac{1}{\sqrt{g}}\left[f\,\delta_{ij} + \frac{\partial x_i}{\partial X_r}\frac{\partial x_j}{\partial X_s} f_{rs}\right], \tag{2.2}$$

where the notation

$$\begin{gathered}\sigma_{ij} = \sigma_{ij}(t), \qquad \frac{\partial x_i}{\partial X_r} = \frac{\partial x_i(t)}{\partial X_r},\\ g_{pq}(\tau) = \frac{\partial x_k(\tau)}{\partial X_p}\frac{\partial x_k(\tau)}{\partial X_q}, \qquad g_{pq} = g_{pq}(t),\\ g(\tau) = |g_{pq}(\tau)|, \qquad g = g(t),\end{gathered} \tag{2.3}$$

is used, δ_{ij} denotes the Kronecker delta and f and f_{rs} are polynomial functions of $g_{pq}(\tau_\alpha)$ and $\sqrt{g(\tau_\alpha)}$ ($\alpha = 0, 1, 2, \ldots, N$).

We may re-write (2.2) in a somewhat more succinct form by observing that

$$2g(t)\,\delta_{ij} = \frac{\partial x_i}{\partial X_r}\frac{\partial x_j}{\partial X_s} e_{rmn}\, e_{suv}\, g_{mu}\, g_{nv}, \tag{2.4}$$

so that

$$\sigma_{ij} = \frac{1}{2g^{\frac{3}{2}}}\frac{\partial x_i}{\partial X_r}\frac{\partial x_j}{\partial X_s} F_{rs}, \tag{2.5}$$

where

$$F_{rs} = f\, e_{rmn}\, e_{suv}\, g_{mu}\, g_{nv} + 2g\, f_{rs}. \tag{2.6}$$

Hence F_{rs} is a symmetric polynomial function of $g_{pq}(\tau_\alpha)$ and $\sqrt{g(\tau_\alpha)}$ ($\alpha = 0, 1, 2, \ldots, N$).

3. Isotropic Materials

In Part I the restrictions required for a material which is isotropic when $\tau \leqq 0$ were obtained using (2.2) and the corresponding results for (2.5) can then be deduced*. Here, however, we proceed directly from (2.5). Let $\bar{x}$ be a fixed rectangular Cartesian coordinate system related to x by

$$\bar{x}_i = A_{ij}\, x_j, \tag{3.1}$$

where A_{ij} are constants satisfying the orthogonality conditions

$$A_{ir}\, A_{jr} = A_{ri}\, A_{rj} = \delta_{ij}, \qquad |A_{ij}| = 1. \tag{3.2}$$

* A slight gap in the argument in Part I was completed later (GREEN & RIVLIN 1958).

If $\bar{x}_i(\tau)$ denotes the coordinates of $x_i(\tau)$ in the system $\bar{x}$ and $\bar{X}_i = \bar{x}_i(0)$, we have, from (3.1),

$$\bar{x}_i(\tau) = A_{ij} x_j(\tau), \qquad \bar{X}_i = A_{ij} X_j,$$
$$\bar{g}_{ij}(\tau) = \frac{\partial \bar{x}_r(\tau)}{\partial \bar{X}_i} \frac{\partial \bar{x}_r(\tau)}{\partial \bar{X}_j} = A_{ir} A_{js} g_{rs}(\tau), \tag{3.3}$$
$$\bar{g}(\tau) = |\bar{g}_{ij}(\tau)| = g(\tau).$$

Denoting the stress in the system $\bar{x}$ by $\bar{\sigma}_{ij}$ we obtain

$$\bar{\sigma}_{ij} = A_{im} A_{jn} \sigma_{mn}. \tag{3.4}$$

Hence, from (2.5), (3.1) and (3.4),

$$\bar{\sigma}_{ij} = \frac{1}{2\bar{g}^{\frac{1}{2}}} \frac{\partial \bar{x}_i}{\partial \bar{X}_r} \frac{\partial \bar{x}_j}{\partial \bar{X}_s} A_{rm} A_{sn} F_{mn}. \tag{3.5}$$

If the material is isotropic at $\tau = 0$ then

$$\bar{\sigma}_{ij} = \frac{1}{2\bar{g}^{\frac{1}{2}}} \frac{\partial \bar{x}_i}{\partial \bar{X}_r} \frac{\partial \bar{x}_j}{\partial \bar{X}_s} \left[F_{rs}\, \bar{g}_{pq}(\tau_\alpha), \sqrt{\bar{g}(\tau_\alpha)}\right], \tag{3.6}$$

so that, using (3.5) and (3.6), we have

$$\frac{\partial \bar{x}_i}{\partial \bar{X}_r} \frac{\partial \bar{x}_j}{\partial \bar{X}_s} \left\{F_{rs}\left[\bar{g}_{pq}(\tau_\alpha), \sqrt{\bar{g}(\tau_\alpha)}\right] - A_{rm} A_{sn} F_{mn}\left[g_{pq}(\tau_\alpha), \sqrt{g(\tau_\alpha)}\right]\right\} = 0. \tag{3.7}$$

Multiplying this equation by the non-zero expressions

$$\frac{\partial \bar{X}_k}{\partial \bar{x}_i} \frac{\partial \bar{X}_l}{\partial \bar{x}_j},$$

we obtain

$$F_{rs}\left[\bar{g}_{pq}(\tau_\alpha), \sqrt{\bar{g}(\tau_\alpha)}\right] = A_{rm} A_{sn} F_{mn}\left[g_{pq}(\tau_\alpha), \sqrt{g(\tau_\alpha)}\right]. \tag{3.8}$$

Using the notation

$$\boldsymbol{g}(\tau) = \|g_{ij}(\tau)\|, \qquad \bar{\boldsymbol{g}}(\tau) = \|\bar{g}_{ij}(\tau)\|, \tag{3.9}$$

it follows that F_{rs} are the components of a symmetric matrix polynomial in the matrices $\boldsymbol{g}(\tau_\alpha)$ in which the coefficients are scalar polynomials in $\sqrt{g(\tau_\alpha)}$ ($\alpha = 0, 1, 2, \ldots, N$) and traces of products formed from the matrices $\boldsymbol{g}(\tau_\alpha)$ ($\alpha = 0, 1, 2, \ldots, N$). Since $[g(\tau)]^{\frac{1}{2}}$ is a continuous single-valued function of $g_{pq}(\tau)$ and $1/g^{\frac{1}{2}}$ has no singularities, we can omit the factor $1/(2g^{\frac{1}{2}})$ and the arguments $[g(\tau_\alpha)]^{\frac{1}{2}}$ in (2.5) and write, to any desired degree of approximation,

$$\boldsymbol{\sigma} = \boldsymbol{c} \boldsymbol{F} \boldsymbol{c}', \tag{3.10}$$

where

$$\boldsymbol{\sigma} = \|\sigma_{ij}\|, \qquad \boldsymbol{c} = \|\partial x_i / \partial X_r\|. \tag{3.11}$$

In (3.10) $\boldsymbol{c}'$ is the transpose of $\boldsymbol{c}$ and $\boldsymbol{F}$ is a symmetric matrix polynomial in the matrices $\boldsymbol{g}(\tau_\alpha)$ in which the coefficients are scalar polynomial functions in the traces of products formed from the matrices $\boldsymbol{g}(\tau_\alpha)$ ($\alpha = 0, 1, 2, \ldots, N$).

The expression (3.10), valid for materials which are isotropic at time $\tau \leqq 0$, can be written in a convenient alternative form. Let*

$$h_{pq}(\tau) = \frac{\partial X_r}{\partial x_p} \frac{\partial X_s}{\partial x_q} g_{rs}(\tau) = \frac{\partial x_m(\tau)}{\partial x_p} \frac{\partial x_m(\tau)}{\partial x_q}, \qquad \boldsymbol{h}(\tau) = \|h_{pq}(\tau)\|, \tag{3.12}$$

and

$$\boldsymbol{C} = \|C_{ij}\|, \qquad C_{ij} = \frac{\partial x_i}{\partial X_m} \frac{\partial x_j}{\partial X_m}. \tag{3.13}$$

Then

$$\boldsymbol{C} = \boldsymbol{c}\,\boldsymbol{c}', \quad \boldsymbol{g}(\tau) = \boldsymbol{c}'\,\boldsymbol{h}(\tau)\,\boldsymbol{c}, \quad \boldsymbol{h}(\tau) = (\boldsymbol{c}')^{-1}\boldsymbol{g}(\tau)\,\boldsymbol{c}^{-1}. \tag{3.14}$$

The trace of products of matrices formed from $\boldsymbol{g}(\tau_\alpha)$ $(\alpha = 0, 1, 2, \ldots, N)$ can be expressed as the trace of products of matrices formed from $\boldsymbol{C}$ and $\boldsymbol{h}(\tau_\alpha)$ $(\alpha = 0, 1, 2, \ldots, N)$. Also a symmetric matrix polynomial formed from the matrices $\boldsymbol{g}(\tau_\alpha)$ can be expressed in the form $\boldsymbol{c}'\boldsymbol{L}\boldsymbol{c}$ where $\boldsymbol{L}$ is a symmetric matrix polynomial formed from the matrices $\boldsymbol{C}$ and $\boldsymbol{h}(\tau_\alpha)$. It follows that we may write (3.10) in the alternative form

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}[\boldsymbol{C}, \boldsymbol{h}(\tau_\alpha)], \tag{3.15}$$

where $\boldsymbol{\sigma}$ is a symmetric matrix polynomial in the matrices $\boldsymbol{C}, \boldsymbol{h}(\tau_\alpha)$ $(\alpha = 1, 2, \ldots, N)$, with coefficients which are polynomial functions of the traces of products formed from the matrices $\boldsymbol{C}, \boldsymbol{h}(\tau_\alpha)$ $(\alpha = 1, 2, \ldots, N)$.

Conversely, we can show that any constitutive equation of the form (3.15) can be expressed with any desired accuracy in the form (3.10), where $\boldsymbol{F}$ is a symmetric matrix polynomial in the matrices $\boldsymbol{g}(\tau_\alpha)$ with coefficients which are scalar polynomial functions in the traces of products formed from these matrices.

4. Isotropic Tensor Functional

When the material is isotropic initially we may make the transition from tensor functions to tensor functionals using either (3.10) or (3.15) as a starting point. In order to keep in line with the work of Part I we use (3.10) and suppose that

$$\sigma_{ij} = \frac{\partial x_i}{\partial X_r} \frac{\partial x_j}{\partial X_s} F_{rs}, \tag{4.1}$$

where we assume that F_{rs} is a symmetric matrix polynomial in g_{pq} with coefficients which are functionals of $g_{pq}(\tau)$ in the range $0 \leqq \tau \leqq t$. We assume that these functionals are continuous functionals of $g_{pq}(\tau)$ over the compact aggregate of functions which are continuous in the range $0 \leqq \tau \leqq t$. We denote the Fourier half-range cosine coefficients of $g_{pq}(\tau)$ by $G_{pq}^{(\alpha)}$, where

$$G_{pq}^{(\alpha)} = \frac{2}{t} \int_0^t g_{pq}(\tau) \cos \frac{\alpha \pi \tau}{t} \, d\tau \quad (\alpha > 0), \qquad G_{pq}^{(0)} = \frac{1}{t} \int_0^t g_{pq}(\tau) \, d\tau. \tag{4.2}$$

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* From (2.1) $x_m(\tau)$ may be expressed as a function of $x_m(t)$, τ, t by eliminating X_j, and then $\partial x_m(\tau)/\partial x_p$ can be evaluated. Alternatively, we can use the first expression in (3.12) to find $h_{pq}(\tau)$.

Then each continuous functional in the expression for F_{rs} may be approximated by a polynomial $P^{(\alpha)}(G_{pq}^{(0)}, G_{pq}^{(1)}, \ldots, G_{pq}^{(\alpha)})$ which tends uniformly to the functional as $\alpha \to \infty$. It follows that F_{rs} can be expressed with any desired approximation by a polynomial in g_{pq} and $G_{pq}^{(0)}, G_{pq}^{(1)}, \ldots, G_{pq}^{(N)}$ with a sufficiently large choice of N and we write

$$\sigma_{ij} = \frac{\partial x_i}{\partial X_r} \frac{\partial x_j}{\partial X_s} F_{rs}[g_{pq}, G_{pq}^{(0)}, G_{pq}^{(1)}, \ldots, G_{pq}^{(N)}], \tag{4.3}$$

where F_{rs} is a polynomial.

We next consider a change of rectangular Cartesian axes of the form (3.1) and (3.2). If $\bar{\sigma}_{ij}$ denotes the stress components at time t in the system $\bar{x}$, then $\bar{\sigma}_{ij}$ is given by (3.4) and (4.3). Also, if the material is isotropic for $\tau \leqq 0$, then

$$\bar{\sigma}_{ij} = \frac{\partial \bar{x}_i}{\partial \bar{X}_r} \frac{\partial \bar{x}_j}{\partial \bar{X}_s} F_{rs}[\bar{g}_{pq}, \bar{G}_{pq}^{(0)}, \bar{G}_{pq}^{(1)}, \ldots, \bar{G}_{pq}^{(N)}], \tag{4.4}$$

where

$$\begin{aligned} \bar{G}_{pq}^{(\alpha)} &= \frac{2}{t} \int_0^t \bar{g}_{pq}(\tau) \cos \frac{\alpha \pi \tau}{t} \, d\tau \qquad (\alpha > 0), \\ \bar{G}_{pq}^{(0)} &= \frac{1}{t} \int_0^t \bar{g}_{pq}(\tau) \, d\tau, \end{aligned} \tag{4.5}$$

and $\bar{g}_{pq}(\tau)$ is given by (3.3). From (3.3), (4.2) and (4.5), we have

$$\begin{aligned} \bar{g}_{pq} &= A_{pm} A_{qn} g_{mn}, \\ \bar{G}_{pq}^{(\alpha)} &= A_{pm} A_{qn} G_{mn}^{(\alpha)} \qquad (\alpha \geqq 0). \end{aligned} \tag{4.6}$$

Also, from (3.1), (3.3), (3.4), (4.3) and (4.4), we obtain

$$\frac{\partial \bar{x}_i}{\partial \bar{X}_r} \frac{\partial \bar{x}_j}{\partial \bar{X}_s} \{F_{rs}[\bar{g}_{pq}, \bar{G}_{pq}^{(\alpha)}] - A_{rm} A_{sn} F_{mn}[g_{pq}, G_{pq}^{(\alpha)}]\} = 0$$

and hence

$$F_{rs}[\bar{g}_{pq}, \bar{G}_{pq}^{(\alpha)}] = A_{rm} A_{sn} F_{mn}[g_{pq}, G_{pq}^{(\alpha)}]. \tag{4.7}$$

Using the notation

$$\boldsymbol{G}_\alpha = \|G_{pq}^{(\alpha)}\|, \tag{4.8}$$

it follows from (4.7) and (4.6) that $\boldsymbol{F}$ is a symmetric matrix polynomial in the symmetric matrices $\boldsymbol{g}, \boldsymbol{G}_\alpha$ $(\alpha = 0, 1, \ldots, N)$ and equation (4.3) may be written in the matrix form

$$\boldsymbol{\sigma} = \boldsymbol{c} \boldsymbol{F} \boldsymbol{c}', \tag{4.9}$$

where $\boldsymbol{F}$ is a symmetric matrix polynomial in the matrices $\boldsymbol{g}, \boldsymbol{G}_\alpha$ $(\alpha = 0, 1, \ldots, N)$.

Since $\boldsymbol{g}(\tau) = \boldsymbol{c}' \boldsymbol{h}(\tau) \boldsymbol{c}$, it follows that $\boldsymbol{G}_\alpha = \boldsymbol{c}' \boldsymbol{H}_\alpha \boldsymbol{c}$, where

$$\begin{aligned} \boldsymbol{H}_\alpha &= \|H_{pq}^{(\alpha)}\|, \\ H_{pq}^{(\alpha)} &= \frac{2}{t} \int_0^t h_{pq}(\tau) \cos \frac{\alpha \pi \tau}{t} \, d\tau \qquad (\alpha > 0), \\ H_{pq}^{(0)} &= \frac{1}{t} \int_0^t h_{pq}(\tau) \, d\tau. \end{aligned} \tag{4.10}$$

Hence, by an argument similar to that used in § 3, we may reduce (4.9) to the alternative form

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}[\boldsymbol{C}, \boldsymbol{H}_0, \boldsymbol{H}_1, \ldots, \boldsymbol{H}_N], \tag{4.11}$$

where $\boldsymbol{\sigma}$ is a symmetric matrix polynomial in the arguments stated and $\boldsymbol{C}$ is given by (3.13).

5. Further Development of the Equations for Isotropic Materials

Since $\boldsymbol{F}$ in (4.9) is a symmetric matrix polynomial in the matrices $\boldsymbol{g}, \boldsymbol{G}_\alpha$ $(\alpha = 0, 1, \ldots, N)$ it is apparent that it may be expressed as the sum of a number of terms of the form

$$\begin{aligned} f(t) \int_0^t \int_0^t \cdots \int_0^t \cos\frac{\alpha_1 \pi \tau_1}{t} \cos\frac{\alpha_2 \pi \tau_2}{t} \cdots \cos\frac{\alpha_P \pi \tau_P}{t} \times \\ \times (\operatorname{tr}\boldsymbol{\Pi}_1 \operatorname{tr}\boldsymbol{\Pi}_2 \ldots \operatorname{tr}\boldsymbol{\Pi}_\mu)(\boldsymbol{\Pi} + \boldsymbol{\Pi}')\, d\tau_1\, d\tau_2 \ldots d\tau_P, \end{aligned} \tag{5.1}$$

where $\boldsymbol{\Pi}_1, \boldsymbol{\Pi}_2, \ldots, \boldsymbol{\Pi}_\mu$ and $\boldsymbol{\Pi}$ are matrix products formed from $\boldsymbol{g}, \boldsymbol{g}(\tau_1), \boldsymbol{g}(\tau_2), \ldots, \boldsymbol{g}(\tau_P)$ and are such that $\boldsymbol{\Pi}_1 \boldsymbol{\Pi}_2 \ldots \boldsymbol{\Pi}_\mu \boldsymbol{\Pi}$ is linear in each of the matrices $\boldsymbol{g}(\tau_1), \boldsymbol{g}(\tau_2), \ldots, \boldsymbol{g}(\tau_P)$, and $\boldsymbol{\Pi}'$ denotes the transpose of $\boldsymbol{\Pi}$. Let $\boldsymbol{\Pi}_1$ be a matrix product formed from $\boldsymbol{g}(\tau_1), \boldsymbol{g}(\tau_2), \ldots, \boldsymbol{g}(\tau_{\beta_1})$ and possibly $\boldsymbol{g}$; let $\boldsymbol{\Pi}_2$ be a matrix product formed from $\boldsymbol{g}(\tau_{\beta_1+1}), \boldsymbol{g}(\tau_{\beta_1+2}), \ldots, \boldsymbol{g}(\tau_{\beta_2})$ and possibly $\boldsymbol{g}$ and so on, so that $\boldsymbol{\Pi}_\mu$ is a matrix product formed from $\boldsymbol{g}(\tau_{\beta_{\mu-1}+1}), \boldsymbol{g}(\tau_{\beta_{\mu-1}+2}), \ldots, \boldsymbol{g}(\tau_{\beta_\mu})$ and possibly $\boldsymbol{g}$, and let $\boldsymbol{\Pi}$ be a matrix product formed from $\boldsymbol{g}(\tau_{\beta_\mu+1}), \boldsymbol{g}(\tau_{\beta_\mu+2}), \ldots, \boldsymbol{g}(\tau_P)$ and possibly $\boldsymbol{g}$. Then, the term (5.1) may be re-written as

$$\begin{aligned} & f(t) \int_0^t \int_0^t \cdots \int_0^t \cos\frac{\alpha_1 \pi \tau_1}{t} \cos\frac{\alpha_2 \pi \tau_2}{t} \cdots \cos\frac{\alpha_{\beta_1} \pi \tau_{\beta_1}}{t} \operatorname{tr}\boldsymbol{\Pi}_1\, d\tau_1 \ldots d\tau_{\beta_1} \\ & \int_0^t \int_0^t \cdots \int_0^t \cos\frac{\alpha_{\beta_1+1} \pi \tau_{\beta_1+1}}{t} \cdots \cos\frac{\alpha_{\beta_2} \pi \tau_{\beta_2}}{t} \operatorname{tr}\boldsymbol{\Pi}_2\, d\tau_{\beta_1+1} \ldots d\tau_{\beta_2} \\ & \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ & \int_0^t \int_0^t \cdots \int_0^t \cos\frac{\alpha_{\beta_{\mu-1}+1} \pi \tau_{\beta_{\mu-1}+1}}{t} \cdots \cos\frac{\alpha_{\beta_\mu} \pi \tau_{\beta_\mu}}{t} \operatorname{tr}\boldsymbol{\Pi}_\mu\, d\tau_{\beta_{\mu-1}+1} \ldots d\tau_{\beta_\mu} \\ & \int_0^t \int_0^t \cdots \int_0^t \cos\frac{\alpha_{\beta_\mu+1} \pi \tau_{\beta_\mu+1}}{t} \cdots \cos\frac{\alpha_P \pi \tau_P}{t} (\boldsymbol{\Pi} + \boldsymbol{\Pi}')\, d\tau_{\beta_\mu+1} \ldots d\tau_P. \end{aligned} \tag{5.2}$$

We now consider the factor

$$\int_0^t \int_0^t \cdots \int_0^t \cos\frac{\alpha_1 \pi \tau_1}{t} \cos\frac{\alpha_2 \pi \tau_2}{t} \cdots \cos\frac{\alpha_{\beta_1} \pi \tau_{\beta_1}}{t} \operatorname{tr}\boldsymbol{\Pi}_1\, d\tau_1 \ldots d\tau_{\beta_1}. \tag{5.3}$$

It has been shown (SPENCER & RIVLIN 1959) that the trace of a matrix product formed from the R symmetric 3×3 matrices $\boldsymbol{a}_P$ $(P = 1, 2, \ldots, R)$ may be expressed as a polynomial in the traces of matrix products formed from those listed in Table 1 by replacing $K_1, K_2, \ldots, K_7$ by all permutations of $1, 2, \ldots, R$ seven at a time.

Since $\mathbf{\Pi}_1$ is linear in each of the matrices $\boldsymbol{g}(\tau_1), \boldsymbol{g}(\tau_2), \ldots, \boldsymbol{g}(\tau_{\beta_1})$, $\operatorname{tr} \mathbf{\Pi}_1$ must be expressible as a polynomial in the invariants listed in Table 2 and the invariants formed from these by replacing $\tau_1, \tau_2, \ldots, \tau_7$ by all possible permutations of $\tau_1, \tau_2, \ldots, \tau_{\beta_1}$ seven at a time.

Table 1

$\operatorname{tr} \boldsymbol{a}_{K_1}$, $\quad \operatorname{tr} \boldsymbol{a}_{K_1}^2$, $\quad \operatorname{tr} \boldsymbol{a}_{K_1}^3$;

$\operatorname{tr} \boldsymbol{a}_{K_1} \boldsymbol{a}_{K_2}$, $\quad \operatorname{tr} \boldsymbol{a}_{K_1} \boldsymbol{a}_{K_2}^2$, $\quad \operatorname{tr} \boldsymbol{a}_{K_1}^2 \boldsymbol{a}_{K_2}^2$;

$\operatorname{tr} \boldsymbol{a}_{K_1} \boldsymbol{a}_{K_2} \boldsymbol{a}_{K_3}$, $\quad \operatorname{tr} \boldsymbol{a}_{K_1} \boldsymbol{a}_{K_2} \boldsymbol{a}_{K_3}^2$, $\quad \operatorname{tr} \boldsymbol{a}_{K_1} \boldsymbol{a}_{K_2}^2 \boldsymbol{a}_{K_3}^2$;

$\operatorname{tr} \boldsymbol{a}_{K_1} \boldsymbol{a}_{K_2} \boldsymbol{a}_{K_3} \boldsymbol{a}_{K_4}$, $\quad \operatorname{tr} \boldsymbol{a}_{K_1} \boldsymbol{a}_{K_2} \boldsymbol{a}_{K_3} \boldsymbol{a}_{K_4}^2$, $\quad \operatorname{tr} \boldsymbol{a}_{K_1} \boldsymbol{a}_{K_2} \boldsymbol{a}_{K_3}^2 \boldsymbol{a}_{K_4}^2$,
$\operatorname{tr} \boldsymbol{a}_{K_1} \boldsymbol{a}_{K_2} \boldsymbol{a}_{K_4} \boldsymbol{a}_{K_3} \boldsymbol{a}_{K_4}^2$;

$\operatorname{tr} \boldsymbol{a}_{K_1} \boldsymbol{a}_{K_2} \boldsymbol{a}_{K_3} \boldsymbol{a}_{K_4} \boldsymbol{a}_{K_5}$, $\quad \operatorname{tr} \boldsymbol{a}_{K_1} \boldsymbol{a}_{K_2} \boldsymbol{a}_{K_3} \boldsymbol{a}_{K_4} \boldsymbol{a}_{K_5}^2$,
$\operatorname{tr} \boldsymbol{a}_{K_1} \boldsymbol{a}_{K_5} \boldsymbol{a}_{K_2} \boldsymbol{a}_{K_3} \boldsymbol{a}_{K_4} \boldsymbol{a}_{K_5}^2$, $\quad \operatorname{tr} \boldsymbol{a}_{K_1} \boldsymbol{a}_{K_2} \boldsymbol{a}_{K_5} \boldsymbol{a}_{K_3} \boldsymbol{a}_{K_4} \boldsymbol{a}_{K_5}^2$,
$\operatorname{tr} \boldsymbol{a}_{K_1} \boldsymbol{a}_{K_2} \boldsymbol{a}_{K_3} \boldsymbol{a}_{K_5} \boldsymbol{a}_{K_4} \boldsymbol{a}_{K_5}^2$, $\quad \operatorname{tr} \boldsymbol{a}_{K_1} \boldsymbol{a}_{K_2} \boldsymbol{a}_{K_3} \boldsymbol{a}_{K_4}^2 \boldsymbol{a}_{K_5}^2$;

$\operatorname{tr} \boldsymbol{a}_{K_1} \boldsymbol{a}_{K_2} \boldsymbol{a}_{K_3} \boldsymbol{a}_{K_4} \boldsymbol{a}_{K_5} \boldsymbol{a}_{K_6}$, $\quad \operatorname{tr} \boldsymbol{a}_{K_1} \boldsymbol{a}_{K_2} \boldsymbol{a}_{K_3} \boldsymbol{a}_{K_4} \boldsymbol{a}_{K_5} \boldsymbol{a}_{K_6}^2$;

$\operatorname{tr} \boldsymbol{a}_{K_1} \boldsymbol{a}_{K_2} \boldsymbol{a}_{K_3} \boldsymbol{a}_{K_4} \boldsymbol{a}_{K_5} \boldsymbol{a}_{K_6} \boldsymbol{a}_{K_7}$.

Denoting the invariants in Table 2 which involve none of the matrices $\boldsymbol{g}(\tau_1), \boldsymbol{g}(\tau_2), \ldots, \boldsymbol{g}(\tau_7)$ by $\operatorname{tr} \tilde{\boldsymbol{\omega}}_\alpha^{(0)}$ $(\alpha = 1, 2, 3)$, those which involve $\boldsymbol{g}(\tau_1)$ only or $\boldsymbol{g}$ and $\boldsymbol{g}(\tau_1)$ only by $\operatorname{tr} \tilde{\boldsymbol{\omega}}_\alpha^{(1)}$ $(\alpha = 1, 2, 3)$, those which involve $\boldsymbol{g}(\tau_1)$ and $\boldsymbol{g}(\tau_2)$

Table 2

$\operatorname{tr} \boldsymbol{g}$, $\quad \operatorname{tr} \boldsymbol{g}^2$, $\quad \operatorname{tr} \boldsymbol{g}^3$;

$\operatorname{tr} \boldsymbol{g}\, \boldsymbol{g}(\tau_1)$, $\quad \operatorname{tr} \boldsymbol{g}^2 \boldsymbol{g}(\tau_1)$;

$\operatorname{tr} \boldsymbol{g}\, \boldsymbol{g}(\tau_1)\, \boldsymbol{g}(\tau_2)$, $\quad \operatorname{tr} \boldsymbol{g}^2 \boldsymbol{g}(\tau_1)\, \boldsymbol{g}(\tau_2)$;

$\operatorname{tr} \boldsymbol{g}\, \boldsymbol{g}(\tau_1)\, \boldsymbol{g}(\tau_2)\, \boldsymbol{g}(\tau_3)$, $\quad \operatorname{tr} \boldsymbol{g}^2 \boldsymbol{g}(\tau_1)\, \boldsymbol{g}(\tau_2)\, \boldsymbol{g}(\tau_3)$,
$\operatorname{tr} \boldsymbol{g}(\tau_1)\, \boldsymbol{g}(\tau_2)\, \boldsymbol{g}\, \boldsymbol{g}(\tau_3)\, \boldsymbol{g}^2$;

$\operatorname{tr} \boldsymbol{g}\, \boldsymbol{g}(\tau_1)\, \boldsymbol{g}(\tau_2)\, \boldsymbol{g}(\tau_3)\, \boldsymbol{g}(\tau_4)$, $\quad \operatorname{tr} \boldsymbol{g}^2 \boldsymbol{g}(\tau_1)\, \boldsymbol{g}(\tau_2)\, \boldsymbol{g}(\tau_3)\, \boldsymbol{g}(\tau_4)$,
$\operatorname{tr} \boldsymbol{g}(\tau_1)\, \boldsymbol{g}\, \boldsymbol{g}(\tau_2)\, \boldsymbol{g}(\tau_3)\, \boldsymbol{g}(\tau_4)\, \boldsymbol{g}^2$, $\quad \operatorname{tr} \boldsymbol{g}(\tau_1)\, \boldsymbol{g}(\tau_2)\, \boldsymbol{g}\, \boldsymbol{g}(\tau_3)\, \boldsymbol{g}(\tau_4)\, \boldsymbol{g}^2$,
$\operatorname{tr} \boldsymbol{g}(\tau_1)\, \boldsymbol{g}(\tau_2)\, \boldsymbol{g}(\tau_3)\, \boldsymbol{g}\, \boldsymbol{g}(\tau_4)\, \boldsymbol{g}^2$;

$\operatorname{tr} \boldsymbol{g}\, \boldsymbol{g}(\tau_1)\, \boldsymbol{g}(\tau_2)\, \boldsymbol{g}(\tau_3)\, \boldsymbol{g}(\tau_4)\, \boldsymbol{g}(\tau_5)$, $\quad \operatorname{tr} \boldsymbol{g}^2 \boldsymbol{g}(\tau_1)\, \boldsymbol{g}(\tau_2)\, \boldsymbol{g}(\tau_3)\, \boldsymbol{g}(\tau_4)\, \boldsymbol{g}(\tau_5)$;

$\operatorname{tr} \boldsymbol{g}\, \boldsymbol{g}(\tau_1)\, \boldsymbol{g}(\tau_2)\, \boldsymbol{g}(\tau_3)\, \boldsymbol{g}(\tau_4)\, \boldsymbol{g}(\tau_5)\, \boldsymbol{g}(\tau_6)$;

$\operatorname{tr} \boldsymbol{g}(\tau_1)$, $\quad \operatorname{tr} \boldsymbol{g}(\tau_1)\, \boldsymbol{g}(\tau_2)$, $\quad \operatorname{tr} \boldsymbol{g}(\tau_1)\, \boldsymbol{g}(\tau_2)\, \boldsymbol{g}(\tau_3)$,
$\operatorname{tr} \boldsymbol{g}(\tau_1)\, \boldsymbol{g}(\tau_2)\, \boldsymbol{g}(\tau_3)\, \boldsymbol{g}(\tau_4)$, $\quad \operatorname{tr} \boldsymbol{g}(\tau_1)\, \boldsymbol{g}(\tau_2)\, \boldsymbol{g}(\tau_3)\, \boldsymbol{g}(\tau_4)\, \boldsymbol{g}(\tau_5)$,
$\operatorname{tr} \boldsymbol{g}(\tau_1)\, \boldsymbol{g}(\tau_2)\, \boldsymbol{g}(\tau_3)\, \boldsymbol{g}(\tau_4)\, \boldsymbol{g}(\tau_5)\, \boldsymbol{g}(\tau_6)$,
$\operatorname{tr} \boldsymbol{g}(\tau_1)\, \boldsymbol{g}(\tau_2)\, \boldsymbol{g}(\tau_3)\, \boldsymbol{g}(\tau_4)\, \boldsymbol{g}(\tau_5)\, \boldsymbol{g}(\tau_6)\, \boldsymbol{g}(\tau_7)$.

or $\boldsymbol{g}, \boldsymbol{g}(\tau_1)$ and $\boldsymbol{g}(\tau_2)$ only by $\operatorname{tr} \tilde{\boldsymbol{\omega}}_\alpha^{(2)}$ $(\alpha = 1, 2, 3)$, and so on, we see that the term (5.3) may be expressed as a polynomial in expressions of the forms

$$\begin{gathered} \operatorname{tr} \tilde{\boldsymbol{\omega}}_\alpha^{(0)}, \quad \int_0^t \varphi(\tau_1) \operatorname{tr} \tilde{\boldsymbol{\omega}}_\alpha^{(1)}\, d\tau_1, \quad \int_0^t \int_0^t \varphi(\tau_1, \tau_2) \operatorname{tr} \tilde{\boldsymbol{\omega}}_\alpha^{(2)}\, d\tau_1\, d\tau_2, \\ \ldots \int_0^t \cdots \int_0^t \varphi(\tau_1, \tau_2, \ldots, \tau_7) \operatorname{tr} \tilde{\boldsymbol{\omega}}_\alpha^{(7)}\, d\tau_1\, d\tau_2 \ldots d\tau_7, \end{gathered} \tag{5.4}$$

where the functions $\varphi(\tau_1), \varphi(\tau_1, \tau_2), \ldots, \varphi(\tau_1, \tau_2, \ldots, \tau_7)$ are analytic functions of their arguments. Each of the other factors in (5.2) except the last may be similarly expressed.

We now consider the last factor in (5.2), *i.e.*

$$\int_0^t \int_0^t \cdots \int_0^t \cos \frac{\alpha_{\beta_\mu+1} \pi \tau_{\beta_\mu+1}}{t} \cdots \cos \frac{\alpha_P \pi \tau_P}{t} (\boldsymbol{\Pi} + \boldsymbol{\Pi}') \, d\tau_{\beta_\mu+1} \ldots d\tau_P. \quad (5.5)$$

Since $\boldsymbol{\Pi} + \boldsymbol{\Pi}'$ is a symmetric isotropic matrix polynomial in the matrices $\boldsymbol{g}(\tau_{\beta_\mu+1}), \boldsymbol{g}(\tau_{\beta_\mu+2}), \ldots, \boldsymbol{g}(\tau_P)$ and $\boldsymbol{g}$, linear in each of them except $\boldsymbol{g}$, it follows (SPENCER & RIVLIN 1958) that it may be expressed as an isotropic matrix polynomial of the form

$$\boldsymbol{\Pi} + \boldsymbol{\Pi}' = \sum \varphi_\nu (\boldsymbol{\chi}_\nu + \boldsymbol{\chi}_\nu'), \quad (5.6)$$

where $\boldsymbol{\chi}_\nu$ denote the matrix products formed from those listed in Table 3 by replacing $\tau_1, \tau_2, \ldots, \tau_6$ by all possible permutations of $\tau_{\beta_\mu+1}, \tau_{\beta_\mu+2}, \ldots, \tau_P$ six at a time, while φ_ν are polynomials in the invariants obtained from those listed in Table 2, by replacing $\tau_1, \tau_2, \ldots, \tau_7$ by all possible permutations of $\tau_{\beta_\mu+1}, \tau_{\beta_\mu+2}, \ldots, \tau_P$ seven at a time. $\boldsymbol{\chi}'$ denotes the transpose of $\boldsymbol{\chi}$. Since $\boldsymbol{\Pi} + \boldsymbol{\Pi}'$ is linear in each of the matrices $\boldsymbol{g}(\tau_{\beta_\mu+1}), \boldsymbol{g}(\tau_{\beta_\mu+2}), \ldots, \boldsymbol{g}(\tau_P)$, φ_ν and $\boldsymbol{\chi}_\nu + \boldsymbol{\chi}_\nu'$ cannot involve any of these matrices in common. We may therefore express the factor (5.5) as an isotropic matrix polynomial in which the matrix terms are of the forms

$$\begin{aligned} &\boldsymbol{\chi}_\nu^{(0)}, \int_0^t \psi_\nu(\tau_1) (\boldsymbol{\chi}_\nu^{(1)} + \boldsymbol{\chi}_\nu^{(1)\prime}) \, d\tau_1, \\ &\int_0^t \int_0^t \psi_\nu(\tau_1, \tau_2) (\boldsymbol{\chi}_\nu^{(2)} + \boldsymbol{\chi}_\nu^{(2)\prime}) \, d\tau_1 \, d\tau_2, \ldots, \\ &\int_0^t \int_0^t \cdots \int_0^t \psi_\nu(\tau_1, \tau_2, \ldots, \tau_6) (\boldsymbol{\chi}_\nu^{(6)} + \boldsymbol{\chi}_\nu^{(6)\prime}) \, d\tau_1 \, d\tau_2 \ldots d\tau_6, \end{aligned} \quad (5.7)$$

where $\boldsymbol{\chi}_\nu^{(0)}$ are the matrix products listed in Table 3 which do not involve any of the matrices $\boldsymbol{g}(\tau_1), \boldsymbol{g}(\tau_2), \ldots, \boldsymbol{g}(\tau_6)$, $\boldsymbol{\chi}_\nu^{(1)}$ are those which involve only $\boldsymbol{g}(\tau_1)$

Table 3

$\boldsymbol{I}$;

$\boldsymbol{g}$, $\boldsymbol{g}^2$;

$\boldsymbol{g}\,\boldsymbol{g}(\tau_1)$, $\boldsymbol{g}^2\,\boldsymbol{g}(\tau_1)$;

$\boldsymbol{g}\,\boldsymbol{g}(\tau_1)\,\boldsymbol{g}(\tau_2)$, $\boldsymbol{g}(\tau_1)\,\boldsymbol{g}\,\boldsymbol{g}(\tau_2)$, $\boldsymbol{g}^2\,\boldsymbol{g}(\tau_1)\,\boldsymbol{g}(\tau_2)$, $\boldsymbol{g}(\tau_1)\,\boldsymbol{g}^2\,\boldsymbol{g}(\tau_2)$,
$\boldsymbol{g}\,\boldsymbol{g}(\tau_1)\,\boldsymbol{g}^2\,\boldsymbol{g}(\tau_2)$, $\boldsymbol{g}\,\boldsymbol{g}(\tau_1)\,\boldsymbol{g}(\tau_2)\,\boldsymbol{g}^2$;

$\boldsymbol{g}\,\boldsymbol{g}(\tau_1)\,\boldsymbol{g}(\tau_2)\,\boldsymbol{g}(\tau_3)$, $\boldsymbol{g}(\tau_1)\,\boldsymbol{g}\,\boldsymbol{g}(\tau_2)\,\boldsymbol{g}(\tau_3)$,
$\boldsymbol{g}^2\,\boldsymbol{g}(\tau_1)\,\boldsymbol{g}(\tau_2)\,\boldsymbol{g}(\tau_3)$, $\boldsymbol{g}(\tau_1)\,\boldsymbol{g}^2\,\boldsymbol{g}(\tau_2)\,\boldsymbol{g}(\tau_3)$,
$\boldsymbol{g}\,\boldsymbol{g}(\tau_1)\,\boldsymbol{g}(\tau_2)\,\boldsymbol{g}(\tau_3)\,\boldsymbol{g}^2$, $\boldsymbol{g}(\tau_1)\,\boldsymbol{g}\,\boldsymbol{g}(\tau_2)\,\boldsymbol{g}(\tau_3)\,\boldsymbol{g}^2$,
$\boldsymbol{g}(\tau_1)\,\boldsymbol{g}(\tau_2)\,\boldsymbol{g}\,\boldsymbol{g}(\tau_3)\,\boldsymbol{g}^2$, $\boldsymbol{g}(\tau_1)\,\boldsymbol{g}\,\boldsymbol{g}(\tau_2)\,\boldsymbol{g}^2\,\boldsymbol{g}(\tau_3)$;

$\boldsymbol{g}\,\boldsymbol{g}(\tau_1)\,\boldsymbol{g}(\tau_2)\,\boldsymbol{g}(\tau_3)\,\boldsymbol{g}(\tau_4)$, $\boldsymbol{g}(\tau_1)\,\boldsymbol{g}\,\boldsymbol{g}(\tau_2)\,\boldsymbol{g}(\tau_3)\,\boldsymbol{g}(\tau_4)$,
$\boldsymbol{g}(\tau_1)\,\boldsymbol{g}(\tau_2)\,\boldsymbol{g}\,\boldsymbol{g}(\tau_3)\,\boldsymbol{g}(\tau_4)$, $\boldsymbol{g}^2\,\boldsymbol{g}(\tau_1)\,\boldsymbol{g}(\tau_2)\,\boldsymbol{g}(\tau_3)\,\boldsymbol{g}(\tau_4)$,
$\boldsymbol{g}(\tau_1)\,\boldsymbol{g}^2\,\boldsymbol{g}(\tau_2)\,\boldsymbol{g}(\tau_3)\,\boldsymbol{g}(\tau_4)$, $\boldsymbol{g}(\tau_1)\,\boldsymbol{g}(\tau_2)\,\boldsymbol{g}^2\,\boldsymbol{g}(\tau_3)\,\boldsymbol{g}(\tau_4)$;

$\boldsymbol{g}\,\boldsymbol{g}(\tau_1)\,\boldsymbol{g}(\tau_2)\,\boldsymbol{g}(\tau_3)\,\boldsymbol{g}(\tau_4)\,\boldsymbol{g}(\tau_5)$, $\boldsymbol{g}(\tau_1)\,\boldsymbol{g}\,\boldsymbol{g}(\tau_2)\,\boldsymbol{g}(\tau_3)\,\boldsymbol{g}(\tau_4)\,\boldsymbol{g}(\tau_5)$,
$\boldsymbol{g}(\tau_1)\,\boldsymbol{g}(\tau_2)\,\boldsymbol{g}\,\boldsymbol{g}(\tau_3)\,\boldsymbol{g}(\tau_4)\,\boldsymbol{g}(\tau_5)$;

$\boldsymbol{g}(\tau_1)$, $\boldsymbol{g}(\tau_1)\,\boldsymbol{g}(\tau_2)$, $\boldsymbol{g}(\tau_1)\,\boldsymbol{g}(\tau_2)\,\boldsymbol{g}(\tau_3)$, $\boldsymbol{g}(\tau_1)\,\boldsymbol{g}(\tau_2)\,\boldsymbol{g}(\tau_3)\,\boldsymbol{g}(\tau_4)$,
$\boldsymbol{g}(\tau_1)\,\boldsymbol{g}(\tau_2)\,\boldsymbol{g}(\tau_3)\,\boldsymbol{g}(\tau_4)\,\boldsymbol{g}(\tau_5)$, $\boldsymbol{g}(\tau_1)\,\boldsymbol{g}(\tau_2)\,\boldsymbol{g}(\tau_3)\,\boldsymbol{g}(\tau_4)\,\boldsymbol{g}(\tau_5)\,\boldsymbol{g}(\tau_6)$.

or $\boldsymbol{g}$ and $\boldsymbol{g}(\tau_1)$, $\boldsymbol{\chi}_\nu^{(2)}$ are those which involve only $\boldsymbol{g}(\tau_1)$ and $\boldsymbol{g}(\tau_2)$ or $\boldsymbol{g}, \boldsymbol{g}(\tau_1)$ and $\boldsymbol{g}(\tau_2)$ and so on, and the functions ψ are analytic functions of their arguments.

It follows that (5.2) and hence $\boldsymbol{F}$ may be expressed as an isotropic matrix polynomial in which the matrix terms take the forms (5.7) and the coefficients are polynomials in the invariants (5.4) and functions of t. In general, the matrix polynomial will contain more than one term of each of the forms (5.7), except the first. Alternatively, we can bring the coefficients under the integration signs in the terms of the forms (5.7) to derive the result that $\boldsymbol{F}$ may be expressed in the form

$$\begin{aligned} \boldsymbol{F} = & \sum_{\nu} \vartheta_{\nu} \boldsymbol{\chi}_{\nu}^{(0)} + \sum_{\nu} \int_0^t \vartheta_{\nu}(\tau_1) (\boldsymbol{\chi}_{\nu}^{(1)} + \boldsymbol{\chi}_{\nu}^{(1)\prime}) \, d\tau_1 + \\ & + \sum_{\nu} \int_0^t \int_0^t \vartheta_{\nu}(\tau_1, \tau_2) (\boldsymbol{\chi}_{\nu}^{(2)} + \boldsymbol{\chi}_{\nu}^{(2)\prime}) \, d\tau_1 \, d\tau_2 + \cdots + \\ & + \sum_{\nu} \int_0^t \int_0^t \cdots \int_0^t \vartheta_{\nu}(\tau_1, \tau_2, \ldots, \tau_6) (\boldsymbol{\chi}_{\nu}^{(6)} + \boldsymbol{\chi}_{\nu}^{(6)\prime}) \, d\tau_1 \, d\tau_2 \ldots d\tau_6, \end{aligned} \tag{5.8}$$

where the ϑ's are functions of t and of their indicated arguments and polynomials in the invariants (5.4).

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