On the Stability of Viscous Fluid Motions

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This paper deals with one aspect of the classical problem of hydrodynamical stability, namely, the determination of sufficient conditions for a basic (usually laminar) flow of an incompressible fluid to be stable under arbitrary disturbances. The technique to be applied is the well known method of energy, originated by OSBOURNE REYNOLDS and WILLIAM McF. ORR and used since that time by many other writers[1]. In spite of this intensive study, it appears that a number of new results can be secured from the method, and it is to these that the paper is devoted.

Our main conclusion is a Reynolds number criterion for the stability of an *arbitrary* fluid motion in a bounded region. In particular, we show that a basic flow in a bounded region $\mathcal{V}$ is stable whenever its Reynolds number $\mathrm{Re} = Vd/\nu$ is less than 5.71; here V is the maximum speed of the basic flow, d is the diameter of $\mathcal{V}$, and ν is the kinematic viscosity of the fluid. The number 5.71 in itself is neither especially good nor especially bad as a criterion for stability, but what *is* interesting is the fact that it is absolutely rigorous and applies independently of the geometry of the flow region and the particular flow involved. For this reason the result may appropriately be called a Reynolds number for universal stability. We also obtain similar criteria for the stability of flows in unbounded regions, and applications are made to the problem of uniqueness of steady flow.

In the second part of the paper (§ 4) we state a general variational problem connected with the stability of an arbitrary motion. The Euler-Lagrange equations corresponding to this problem bear an interesting and remarkable resemblance to the Navier-Stokes equations, but they are in general too difficult to solve, except for special cases.

The last part of the paper treats a particular example, the laminar Couette flow between rotating coaxial cylinders. The methods of the earlier sections of the paper, when applied to this case, yield the stability criterion

$$\left|\frac{\Omega_2-\Omega_1}{\nu}\right| \leqq (R_2^2 - R_1^2)\left\{\frac{\pi}{R_1 R_2 \log(R_2/R_1)}\right\}^2, \tag{1}$$

in which the notation is practically self-explanatory. The author knows of no other formula of this type which applies to *arbitrary* disturbances of the basic laminar motion. (The criteria of G. I. TAYLOR and others (*cf.* [*14*], Chapter 2)

[1] *Cf.* references [*1*] — [*9*] at the end of the paper.

all refer to infinitesimal disturbances of a special type; thus, although they have indisputable importance if one is considering the breakdown of already established flow, and give quite sharp results in certain limited situations, they do not by any means apply to the whole realm of possible disturbances of the motion.) Finally, the stability of Couette motion is considered from the point of view of the variational method noted above. The results here do not have the finality of criterion (1), but they are nevertheless of interest.

The paper begins with a derivation of a fundamental identity for the rate of change of energy of a perturbation motion. This formula is the basis for all that follows.

1. The Reynolds-Orr energy equation

We consider a basic fluid motion occupying a region $\mathscr{V}=\mathscr{V}(t)$ of space and subject to a prescribed velocity distribution on the boundary $\mathscr{S}$ of $\mathscr{V}$. In the cases of greatest interest $\mathscr{V}$ is bounded by material walls and the boundary conditions arise from the motion of these walls, as, for example, in the case of Couette flow. Now suppose the velocity field of the basic flow is altered at some initial instant $t=0$; it is natural to ask whether the subsequent motion, subject to the same boundary conditions, will alter only slightly from what it was, or whether it will change radically in character. To investigate this problem, we shall consider the energy $\mathscr{K}$ of the perturbation (difference) motion. If $\mathscr{K}$ tends to zero as $t\to\infty$, then the basic motion is said to be stable, or, more precisely, stable in the mean.

To be specific, suppose the region $\mathscr{V}$ is bounded, and let $\boldsymbol{v}$ and $\boldsymbol{v}'$ denote, respectively, the velocity vectors of the basic and altered motions. The velocity $\boldsymbol{u}=\boldsymbol{v}'-\boldsymbol{v}$ of the perturbation motion obviously satisfies

$$\boldsymbol{u}=0 \qquad \text{on } \mathscr{S}, \tag{2}$$

and its kinetic energy is given by

$$\mathscr{K}=\tfrac{1}{2}\int u^2 \tag{3}$$

(in writing integrals, we shall consistently omit the conventional volume infinitesimal; moreover, all integrals are understood to be extended over the entire region $\mathscr{V}$). The rate of change of $\mathscr{K}$ is governed by the fundamental formula

$$\boxed{\frac{d\mathscr{K}}{dt}=-\int(\boldsymbol{u}\cdot\boldsymbol{D}\cdot\boldsymbol{u}+\nu\,\mathrm{grad}\,\boldsymbol{u}:\mathrm{grad}\,\boldsymbol{u})} \tag{4}$$

essentially due to REYNOLDS and ORR. In this equation $\mathrm{grad}\,\boldsymbol{u}$ denotes the matrix with components $(\mathrm{grad}\,\boldsymbol{u})_{ik}=u_{k,i}$ and $\boldsymbol{D}$ denotes the deformation matrix of the basic motion, $D_{ik}=\frac{1}{2}(v_{i,k}+v_{k,i})$.

To prove (4), we begin with the observation that both $\boldsymbol{v}$ and $\boldsymbol{v}'$ satisfy the Navier-Stokes equation

$$\varrho\left(\frac{\partial\boldsymbol{v}}{\partial t}+\boldsymbol{v}\cdot\mathrm{grad}\,\boldsymbol{v}\right)=\varrho\boldsymbol{f}-\mathrm{grad}\,p+\mu\nabla^2\boldsymbol{v}.$$

By subtraction there arises

$$\frac{\partial\boldsymbol{u}}{\partial t}+\boldsymbol{u}\cdot\mathrm{grad}\,\boldsymbol{v}+\boldsymbol{v}'\cdot\mathrm{grad}\,\boldsymbol{u}=\mathrm{grad}\,\frac{p-p'}{\varrho}+\nu\nabla^2\boldsymbol{u}.$$

Forming the scalar product of this equation with the vector $\boldsymbol{u}$, and using the incompressibility conditions $\operatorname{div} \boldsymbol{v} = \operatorname{div} \boldsymbol{v}' = 0$, then leads to

$$\frac{\partial}{\partial t}\left(\frac{1}{2} u^2\right) = -(\boldsymbol{u} \cdot \boldsymbol{D} \cdot \boldsymbol{u} + \nu \operatorname{grad} \boldsymbol{u} : \operatorname{grad} \boldsymbol{u}) + \operatorname{div} \boldsymbol{\Phi}, \tag{5}$$

where

$$\boldsymbol{\Phi} = \frac{p - p'}{\varrho} \boldsymbol{u} + \nu \operatorname{grad}\left(\frac{1}{2} u^2\right) - \frac{1}{2} u^2 \boldsymbol{v}'.$$

On the other hand, we have obviously

$$\frac{d\mathscr{K}}{dt} = \int \frac{\partial}{\partial t}\left(\frac{1}{2} u^2\right) + \oint \frac{1}{2} u^2 \boldsymbol{v} \cdot \boldsymbol{n}, \tag{6}$$

the latter integral being taken over the entire boundary of $\mathscr{V}$. The required formula (4) now follows easily from (5), (6), the divergence theorem, and the fact that $\boldsymbol{u} = \boldsymbol{\Phi} = 0$ on $\mathscr{S}$.[2]

This derivation does not hold when the region $\mathscr{V}$ is unbounded, since both formula (6) and the divergence theorem are applicable only to bounded regions. Under suitable conditions on the asymptotic behavior of $\boldsymbol{v}, \boldsymbol{v}'$, however, the above formal steps can still be justified (we shall omit the details). Another justification of (4) for infinite regions is available whenever the flow geometry is such that the disturbances can be assumed spatially periodic at each instant. This will be the case, in particular, for the important Poiseuille and Couette flows. The disturbance $\boldsymbol{u}$ being supposed periodic in the direction of the axis of symmetry (what LIN calls sustained oscillations), the region $\mathscr{V}$ can then be chosen to cover exactly one period. Then the boundary integrals at either end of $\mathscr{V}$, neither of which vanishes separately, just cancel one another. Formula (4) may therefore be assumed to hold in these two important situations.

2. Criteria for universal stability

In this section we shall use the method of energy to establish certain criteria for the stability of arbitrary fluid motions. This method is based on the observation that if $\mathscr{K}$ tends to zero, then $\boldsymbol{u}$ must likewise tend to zero almost everywhere. Thus a basic flow will be stable (stable in the mean) provided the energy of any disturbance tends to zero as t increases. To apply the method one considers the right-hand side of (4): if it is negative for arbitrary non-vanishing vectors $\boldsymbol{u}$ satisfying $\operatorname{div} \boldsymbol{u} = 0$, then obviously $d\mathscr{K}/dt < 0$ and there is stability. Since the second term on the right of (4) is always negative, it is seen that viscosity tends to damp out any disturbance. On the other hand, a high rate of shear in the basic flow can cause the first term to be highly positive, thus fostering the growth of a disturbance. The relative importance of these two terms, then, determines the stability of the flow (*cf.* also the discussion in [*14*], § 4.5).

Another criterion of the same sort arises when the right-hand side of (4) is written in slightly different form. Indeed, since $\operatorname{div} \boldsymbol{u} = 0$, we have

$$\boldsymbol{u} \cdot \boldsymbol{D} \cdot \boldsymbol{u} = \operatorname{div}\left[(\boldsymbol{u} \cdot \boldsymbol{v})\, \boldsymbol{u}\right] - \boldsymbol{u} \cdot \operatorname{grad} \boldsymbol{u} \cdot \boldsymbol{v},$$

[2] Several alternate and occasionally useful forms of (4) arise from the simple identity

$$\int \operatorname{grad} \boldsymbol{u} : \operatorname{grad} \boldsymbol{u} = \int (\operatorname{curl} \boldsymbol{u})^2 = 2 \int \boldsymbol{D}' : \boldsymbol{D}',$$

where $\boldsymbol{D}'$ denotes the deformation matrix of the perturbation motion.

whence, by application of the divergence theorem, (4) can be written in the form

$$\boxed{\frac{d\mathscr{K}}{dt} = \int (\boldsymbol{u} \cdot \operatorname{grad} \boldsymbol{u} \cdot \boldsymbol{v} - \nu \operatorname{grad} \boldsymbol{u} : \operatorname{grad} \boldsymbol{u}).} \tag{7}$$

If the first term on the right of (7) is less than the second for all admissible vectors $\boldsymbol{u}$, then clearly the basic flow will be stable. But the size of the first term is governed by the magnitude of $\boldsymbol{v}$, from which it follows that high speeds in the basic flow, as well as high rates of shear, tend to cause instability. The qualitative nature of these effects will be investigated in the next several paragraphs.

It is important to note that the energy method cannot provide accurate knowledge of the limits of stability, such as can be gained from the linearized perturbation theory ([*6*], Part II; [*14*], Chap. 1). The reason is that in the energy method one establishes stability relative to arbitrary disturbances, while in reality only those satisfying the hydrodynamical equations need be considered. Nevertheless, because the energy method gives insight into the physical situation, and because the results have the merit of simplicity and complete mathematical rigor, the investigations based on it are both interesting and valuable.

With these preliminary remarks aside, we may now turn to the main result of the paper.

Theorem 1. *Let $\mathscr{V} = \mathscr{V}(t)$ be a bounded region of space, which can be included in some cube of edge length d. Let $\boldsymbol{v}$ be the velocity vector of any motion in $\mathscr{V}$ satisfying prescribed conditions at the boundary of $\mathscr{V}$. Then the kinetic energy of an arbitrary disturbance motion $\boldsymbol{u} = \boldsymbol{v}' - \boldsymbol{v}$ satisfies the inequalities*

$$\mathscr{K} \leqq \mathscr{K}_0 \, e^{2(m - \alpha\nu/d^2)t}, \tag{8}$$

$$\mathscr{K} \leqq \mathscr{K}_0 \, e^{(V^2 - \alpha\nu^2/d^2)t/\nu}. \tag{9}$$

Here $\mathscr{K}_0$ is the initial energy of the disturbance, $-m$ is a lower bound for the characteristic values of the deformation matrix of the basic flow over the time interval 0 to t, V is the maximum speed of the basic flow in the same time interval, and α is a pure number,

$$\alpha = \frac{3 + \sqrt{13}}{2} \pi^2 \cong 32.6. \tag{10}$$

This theorem is a generalization and improvement of certain results of T. Y. THOMAS and EBERHARD HOPF[3]. Before giving the proof of the theorem, we note two important corollaries, which in a certain sense constitute the heart of the paper. It is to be emphasized that these corollaries are absolutely rigorous, and apply to *all possible motions* in $\mathscr{V}$.

Universal stability criterion I. *If the dimensionless "Reynolds number" md^2/ν of a flow is less than 32.6, then $\mathscr{K} \to 0$ as $t \to \infty$, and the motion is stable.*

[3] THOMAS [*8*] proved that an arbitrary fluid motion is stable if its deformation matrix is sufficiently small, but did not compute numerical values; the present analysis is an extension of THOMAS' work. Similarly, HOPF has shown (essentially) that a fluid motion is stable if its maximum speed is small enough, again with no discussion of numerical values.

Universal stability criterion II. *If the Reynolds number $V d/\nu$ of a flow is less than 5.71, then $\mathcal{K} \to 0$ as $t \to \infty$, and the motion is stable. (Here V is the maximum speed of the flow, and d is the maximum diameter of the flow region.)*

Proof of Theorem 1. We begin with an auxiliary computation whose goal is the inequality

$$\alpha d^{-2} \int u^2 \leqq \int \operatorname{grad} \boldsymbol{u} : \operatorname{grad} \boldsymbol{u}, \tag{11}$$

where α is given by (10). Let $\boldsymbol{h}$ be an arbitrary continuously differentiable vector field in $\mathcal{V}$. Then for any value of the constant ε,

$$\begin{aligned} 0 &\leqq (u_i h_k + u_{i,k} + \varepsilon u_{k,i})(u_i h_k + u_{i,k} + \varepsilon u_{k,i}) \\ &= (1+\varepsilon^2) u_{i,k} u_{i,k} + h^2 u^2 + h_i (u^2)_{,i} + 2\varepsilon (u_i h_k + u_{i,k}) u_{k,i}. \end{aligned} \tag{12}$$

(For convenience we use tensor rather than vector notation in carrying out this and several subsequent calculations.) Some of the terms on the right-hand side of (12) can be transformed as follows,

$$\begin{aligned} h_i (u^2)_{,i} &= (u^2 h_i)_{,i} - u^2 h_{i,i} \\ u_i h_k u_{k,i} &= (u_i h_k u_k)_{,i} - u_i h_{k,i} u_k \\ u_{i,k} u_{k,i} &= (u_i u_{k,i})_{,k}. \end{aligned}$$

Here we have made use of the incompressibility condition $u_{i,i} = 0$. Making the above changes in (12), integrating the result over $\mathcal{V}$, and using the divergence theorem, we obtain the inequality

$$\int \left[(h_{i,i} - h^2) u^2 + 2\varepsilon u_i h_{i,k} u_k\right] \leqq (1+\varepsilon^2) \int u_{i,k} u_{i,k}. \tag{13}$$

Now the particular vector field $h_i = C \tan C x_i$ is differentiable in a cube of edge length π/C centered at the origin. If we set $C = \pi/d$ and suitably locate the origin in $\mathcal{V}$, then this vector can be substituted into (13). An easy computation then yields the inequality

$$(1+\varepsilon^2)^{-1} (3 + 2\varepsilon) C^2 \int u^2 \leqq \int u_{i,k} u_{i,k}.$$

The left-hand side is maximized by choosing $\varepsilon = \frac{1}{2}(\sqrt{13} - 3)$, and (11) is thereby proved.

The term $\boldsymbol{u} \cdot \boldsymbol{D} \cdot \boldsymbol{u}$ in (4) satisfies the inequality

$$\boldsymbol{u} \cdot \boldsymbol{D} \cdot \boldsymbol{u} \geqq -m u^2 \tag{14}$$

during the entire time interval 0 to t. Therefore, from (4), (11) and (14)

$$\frac{d\mathcal{K}}{dt} \leqq (m - \alpha \nu d^{-2}) \int u^2 = 2(m - \alpha \nu d^{-2}) \mathcal{K}.$$

Writing this in the form

$$\frac{d}{dt}\left\{\mathcal{K} e^{-2(m - \alpha \nu d^{-2}) t}\right\} \leqq 0$$

and integrating from 0 to t leads at once to (8).

The proof of (9) is similar but requires in addition the inequality

$$\boldsymbol{u} \cdot \operatorname{grad} \boldsymbol{u} \cdot \boldsymbol{v} \leqq \frac{1}{2}\left(\nu \operatorname{grad} \boldsymbol{u} : \operatorname{grad} \boldsymbol{u} + \frac{u^2 v^2}{\nu}\right), \tag{15}$$

which follows at once from the identity

$$\boldsymbol{A}:\boldsymbol{A} - 2\boldsymbol{u}\cdot\boldsymbol{A}\cdot\boldsymbol{v} + u^2 v^2 \equiv (\boldsymbol{A} - \boldsymbol{u}\,\boldsymbol{v}):(\boldsymbol{A} - \boldsymbol{u}\,\boldsymbol{v}) \geqq 0$$

with $\boldsymbol{A} = \nu\,\mathrm{grad}\,\boldsymbol{u}$. From (7), (15), and (11) we obtain

$$\frac{d\mathscr{K}}{dt} \leqq \frac{1}{\nu}\left(V^2 - \alpha\,\nu^2 d^{-2}\right)\mathscr{K},$$

and (9) then follows by integration as in the proof of (8).

3. Extensions and applications of Theorem 1

The preceding method can be applied equally well to flows which take place in channels and pipes, and to plane flow problems, so long as the fundamental formula (4) applies. To illustrate the method, consider for example a straight pipe whose cross section has a maximum diameter d. If the pipe is directed along the z-axis, then the vector

$$\boldsymbol{h} = C(\boldsymbol{i}\tan Cx + \boldsymbol{j}\tan Cy), \qquad (C = \pi/d),$$

can be used in (13). Choosing $\varepsilon = 0$, we get an inequality of the form (11) with $\alpha = 2\pi^2$. By using this result, the reader can easily obtain estimates for the rate of change of $\mathscr{K}$, and thus formulate analogues of Theorem 1 and the universal stability criteria for flow in a pipe.

Other cases can be treated similarly and the results conveniently grouped in the following table.

A. Straight channel, maximum width d:	$\alpha = \pi^2$.
B. Straight pipe, maximum diameter d:	$\alpha = 2\pi^2$.
C. Plane flow in a bounded region, maximum diameter d,	
i) Three dimensional disturbances:	$\alpha = 2\pi^2$,
ii) Plane disturbances only:	$\alpha = (1 + \sqrt{2})\,\pi^2$.
D. Bounded region, maximum diameter d:	$\alpha = \frac{3+\sqrt{13}}{2}\,\pi^2$.

(Note: It is only in cases C, ii) and D that we have been able to make use of the incompressibility condition $\mathrm{div}\,\boldsymbol{u} = 0$.)

We conclude the section with two simple applications of Theorem 1. First, suppose the boundaries of $\mathscr{V}$ consist of *rigid fixed* walls, so that any motion initially present will presumably die out due to lack of an energy source. By choosing $\boldsymbol{v} \equiv 0$ for the basic motion, we see from (8) that the energy $\mathscr{K}$ of an arbitrary motion $\boldsymbol{v}'$ in $\mathscr{V}$ must in fact tend to zero according to the law

$$\mathscr{K} \leqq \mathscr{K}_0\, e^{-2\alpha\nu t/d^2}, \qquad \alpha \cong 32.6.$$

Similar estimates, but with smaller values for the coefficient α, have been obtained by LERAY (for plane flow) and KAMPÉ DE FERIET[4] and BERKER (for spatial

[4] KAMPÉ DE FÉRIET obtained the value $\alpha = 3\pi^2$, using a method somewhat similar to the one presented here.

flows), and RAYLEIGH in a much earlier paper [4] proved that $\mathcal{K} \to 0$ as $t \to \infty$, though without estimating the rate of convergence. It remains an open question whether the velocity itself must tend to zero as $t \to \infty$; certainly one would expect this, but a strict proof seems to be a matter of more than ordinary difficulty.

As a second application, we have the following uniqueness theorem concerning *steady* motion in a fixed bounded region $\mathcal{V}$.

Theorem 2. *Let $\boldsymbol{v}$ and $\boldsymbol{v}'$ be two steady flows in $\mathcal{V}$, subject to a prescribed velocity distribution on the boundary of $\mathcal{V}$. Let $-m$ be a lower bound for the characteristic values of the deformation matrix of the motion $\boldsymbol{v}$, let $V = \max v$, and suppose that either*

$$m\,d^2/\nu \leqq \alpha \quad \text{or} \quad V\,d/\nu \leqq \sqrt{\alpha}. \tag{16}$$

Then the two flows are identical.

Proof. The kinetic energy of the difference motion $\boldsymbol{v}' - \boldsymbol{v}$ must be constant, since the flows are steady. On the other hand, it must satisfy both (8) and (9). In view of (16) this can happen only if $\mathcal{K} = 0$, which in turn implies $\boldsymbol{v} = \boldsymbol{v}'$.

This theorem depends strongly on the condition (16), but without some such assumption it is extremely unlikely that the conclusion is true.

4. Variational techniques

The key step in the proof of Theorem 1 lay in establishing inequality (11). We are concerned, however, not only with the validity of (11), but also with the determination of the largest possible coefficient α, for the size of α evidently determines the numerical values in the various stability criteria. Now the method of proof in Theorem 1 clearly gives no guarantee of providing the "best possible" value for α; moreover, it is quite crude in its estimate of the term $\boldsymbol{u} \cdot \boldsymbol{D} \cdot \boldsymbol{u}$. For these reasons, it is of great interest to consider an alternate approach which not only supplies a theoretical procedure for determining the best possible α, but in addition gives a way to avoid the estimate (14).

Stated in precise terms, our problem is twofold. First, we must determine the greatest coefficient α such that the inequality

$$\alpha\, d^{-2} \int u^2 \leqq \int \operatorname{grad} \boldsymbol{u} : \operatorname{grad} \boldsymbol{u}$$

holds for arbitrary vector fields $\boldsymbol{u}$ satisfying

$$\operatorname{div} \boldsymbol{u} = 0, \qquad \boldsymbol{u} = 0 \ \text{on}\ \mathcal{S}. \tag{17}$$

Second, we must determine the least coefficient $\tilde{\nu}$ such that the inequality

$$\int (\boldsymbol{u} \cdot \boldsymbol{D} \cdot \boldsymbol{u} + \tilde{\nu} \operatorname{grad} \boldsymbol{u} : \operatorname{grad} \boldsymbol{u}) \geqq 0$$

holds, again for arbitrary $\boldsymbol{u}$ satisfying (17). In this case it is clear that the basic flow will be stable provided simply that $\nu > \tilde{\nu}$. The two problems above can be consolidated into the single variational problem:

$$-\int \boldsymbol{u} \cdot \boldsymbol{D} \cdot \boldsymbol{u} = \text{Maximum}, \tag{18}$$

where the vector field $\boldsymbol{u}$ must satisfy the side conditions

$$\int \operatorname{grad} \boldsymbol{u} : \operatorname{grad} \boldsymbol{u} = 1, \quad \operatorname{div} \boldsymbol{u} = 0, \quad \boldsymbol{u} = 0 \text{ on } \mathscr{S}. \tag{19}$$

The first problem occurs when $\boldsymbol{D}$ is the negative of the identity matrix.

The variational problem (18)—(19) can be reformulated as a partial differential equation for the extremal function $\boldsymbol{u}$, according to well known procedures of the calculus of variations. Thus, through introduction of Lagrange multipliers ν^* and $\lambda = \lambda(x, y, z, t)$, the problem becomes

$$\delta \int (\boldsymbol{u} \cdot \boldsymbol{D} \cdot \boldsymbol{u} + \nu^* \operatorname{grad} \boldsymbol{u} : \operatorname{grad} \boldsymbol{u} - 2\lambda \operatorname{div} \boldsymbol{u}) = 0. \tag{20}$$

The Euler-Lagrange equation corresponding to (20) is easily found to be

$$\boldsymbol{u} \cdot \boldsymbol{D} = - \operatorname{grad} \lambda + \nu^* \nabla^2 \boldsymbol{u}, \tag{21}$$

and this is to be solved subject to the side conditions (19). The reader may observe the remarkable similarity between the eigenvalue equation (21) and the equations of hydrodynamics. The equations corresponding to the case $\boldsymbol{D} = -\boldsymbol{I}$ are of sufficient importance to be noted separately, namely

$$\nu^* \nabla^2 \boldsymbol{u} + \boldsymbol{u} = \operatorname{grad} \lambda, \quad \operatorname{div} \boldsymbol{u} = 0. \tag{22}$$

In spite of the relative simplicity of (22), we have been unable to determine its solution, and in what follows can only offer some general remarks concerning the system (19), (21).

Now for any solution of equation (21) we have

$$-\int \boldsymbol{u} \cdot \boldsymbol{D} \cdot \boldsymbol{u} = \int (\boldsymbol{u} \cdot \operatorname{grad} \lambda - \nu^* \boldsymbol{u} \cdot \nabla^2 \boldsymbol{u}) = \nu^* \int \operatorname{grad} \boldsymbol{u} : \operatorname{grad} \boldsymbol{u} = \nu^*,$$

where use has been made of the divergence theorem and the conditions (19). On the other hand, any vector $\boldsymbol{u}$ which provides the integral (18) with its maximum value must be a solution of (21). It follows that the eigenvalue $\tilde{\nu}$ associated with a maximizing vector is precisely the maximum value of the integral (18). Moreover, no eigenvalue ν^* can be larger than $\tilde{\nu}$, for if this were the case then the corresponding eigenvector $\boldsymbol{u}^*$ would give to the integral (18) a value larger than $\tilde{\nu}$. This proves the following theorem.

Theorem 3. *Suppose there exists a vector* $\boldsymbol{u}$ *which solves the variational problem* (18)—(19). *Then the eigenvalue problem* (19), (21) *has a greatest eigenvalue* $\tilde{\nu}$, *and the basic motion will be stable provided that* $\nu > \tilde{\nu}$.

The difficulty with Theorem 3 is, of course, that there is no direct way of verifying its hypothesis. In circumstances such as these the best that can be hoped for is to determine all the eigenvalues (by Theorem 1 they constitute a bounded set), and to assume that the least upper bound of these eigenvalues is the required maximum of the integral (18). If the eigenvectors are complete in the set of admissible vectors, then this gives another method for proving that the least upper bound of the eigenvalues is the maximum of (18); unfortunately, this appears at least as difficult to verify as the hypothesis of Theorem 3.

5. Example. Couette flow

We consider the well known circulatory flow between rotating concentric cylinders. If the inner and outer cylinders, respectively, have radii R_1 and R_2, and rotate with angular speeds Ω_1 and Ω_2 ($\Omega_1 > 0$), then the velocity field is given by

$$v_\theta = A r + B r^{-1}, \qquad v_r = v_z = 0,$$

where

$$A = \frac{R_2^2 \Omega_2 - R_1^2 \Omega_1}{R_2^2 - R_1^2}, \qquad B = -\frac{(R_1 R_2)^2 (\Omega_2 - \Omega_1)}{R_2^2 - R_1^2}.$$

One finds easily that the scalar vorticity ω has the constant value $2A$, while in polar coordinates

$$\boldsymbol{D} = -\frac{B}{r^2}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The characteristic values of $\boldsymbol{D}$ are $\pm B/r^2$, whence

$$\boldsymbol{u}\cdot\boldsymbol{D}\cdot\boldsymbol{u} \geqq -\frac{|B|}{r^2} u^2. \tag{23}$$

The special form of inequality (23) suggests that the stability estimates of § 2 can be considerably improved. In particular, in place of the inequality (11) we shall look for an estimate of the form

$$\beta \int \frac{u^2}{r^2} \leqq \int \operatorname{grad} \boldsymbol{u} : \operatorname{grad} \boldsymbol{u}. \tag{24}$$

To this end, let us seek a vector field $\boldsymbol{h}$ such that

$$\operatorname{div} \boldsymbol{h} - h^2 = C^2/r^2, \qquad R_1 < r < R_2. \tag{25}$$

A radially symmetric solution of (25) is readily found, namely

$$\boldsymbol{h} = \frac{C}{r} \tan(C \log r + D)\, \boldsymbol{i}_r, \tag{26}$$

where the constants C and D are given by

$$C = \frac{\pi}{\log(R_2/R_1)}, \qquad D = \frac{\pi}{2}\, \frac{\log(R_1 R_2)}{\log(R_2/R_1)}.$$

Substituting (26) into (13) and setting $\varepsilon = 0$ then yields an inequality of the form (24), with

$$\beta = C^2 = \left\{\frac{\pi}{\log(R_2/R_1)}\right\}^2.$$

Combining (23) and (24) with the fundamental energy equation (4) yields

$$\frac{d\mathscr{K}}{dt} \leqq (|B| - \beta\nu) \int \frac{u^2}{r^2},$$

and from this it follows that *Couette flow is stable relative to arbitrary disturbances whenever*

$$\left|\frac{\Omega_2 - \Omega_1}{\nu}\right| < (R_2^2 - R_1^2)\left\{\frac{\pi}{R_1 R_2 \log(R_2/R_1)}\right\}^2. \tag{27}$$

The reader should notice that this stability is in the "strong" sense, $\mathscr{K} \to 0$ as $t \to \infty$.

Previously, SYNGE [*13*] has proved stability relative to infinitesimal disturbances whenever A has the same sign as Ω_1, which in the present case ($\Omega_1 > 0$) gives the stability conditions $\omega = 2A \geqq 0$, or equivalently, $\Omega_2 \geqq (R_2/R_1)^2 \Omega_1$ [5]. The relative zones of stability are shown in Fig. 1. For the celebrated pair of radii $R_1 = 3.55$, $R_2 = 4.03$, inequality (27) reduces to

$$\left| \frac{\Omega_2 - \Omega_1}{\nu} \right| < 10.92. \tag{28}$$

In Fig. 2 we have indicated the stability zones based on (28), on SYNGE's criterion, and on the calculations and experiments of G. I. TAYLOR. If we suppose TAYLOR's

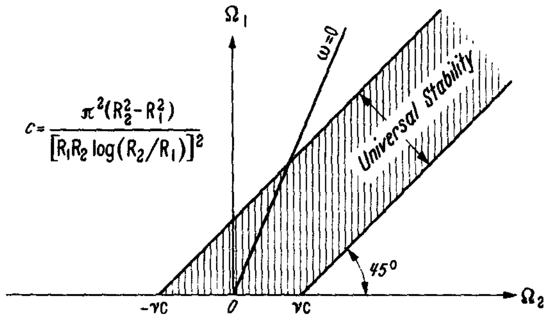

Fig. 1. Stability zones for Couette flow. The zone $\omega \geqq 0$ lies *below* the line $\omega = 0$

experimental data to be an accurate representation of the mathematical situation, it is seen that the right-hand side of (28) cannot possibly be greater than about 50, for otherwise there is an observed secondary motion. Our value is certainly not

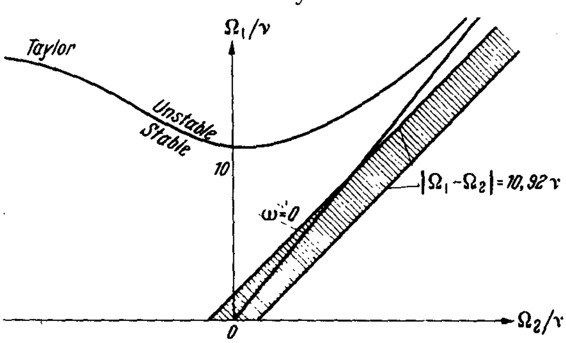

Fig. 2. Stability zones for Couette flow, $R_1 = 3.55$ and $R_2 = 4.03$

very near 50, but considering the difficulty of the problem and the fact that (28) applies to arbitrary disturbances, one does feel that 10.92 is a quite respectable value.

6. Couette flow (continued)

In the previous section we studied the stability of Couette flow using the methods of § 2. In this section the same problem will be treated by the variational techniques of § 4. The end result will be seen to have a somewhat tentative

[5] In this connection we may recall some earlier work of RAYLEIGH and SYNGE, in which the same criterion is established for inviscid fluids; *cf.* reference [*14*], § 4.2.

character, in line with the remarks at the close of § 4, though it still has considerable interest. For Couette flow the differential equation (21) and the side condition div $\boldsymbol{u} = 0$ take the following form (in polar coordinates)

$$\begin{aligned}
-\frac{Bv}{r^2} &= -\frac{\partial\lambda}{\partial r} + \nu^*\left\{\Delta u - \frac{2}{r^2}\frac{\partial v}{\partial\vartheta} - \frac{u}{r^2}\right\},\\
-\frac{Bu}{r^2} &= -\frac{1}{r}\frac{\partial\lambda}{\partial\vartheta} + \nu^*\left\{\Delta v + \frac{2}{r^2}\frac{\partial u}{\partial\vartheta} - \frac{v}{r^2}\right\},\\
0 &= -\frac{\partial\lambda}{\partial z} + \nu^*\Delta w,\\
&\frac{1}{r}\frac{\partial}{\partial r}(ru) + \frac{1}{r}\frac{\partial v}{\partial\vartheta} + \frac{\partial w}{\partial z} = 0,
\end{aligned} \tag{29}$$

where

$$\Delta \equiv \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial\vartheta^2} + \frac{\partial^2}{\partial z^2},$$

and u, v, w denote, respectively, the r, ϑ and z components of the perturbation velocity $\boldsymbol{u}$. Equations (29), as they stand, are too difficult to solve in full generality. We shall therefore seek a particular solution of the form

$$u = \hat{u}(r)\cos kz, \qquad v = \hat{v}(r)\cos kz, \qquad w = \hat{w}(r)\sin kz. \tag{30}$$

Eliminating λ and $\hat{w}$ from the system (29), we get

$$\begin{aligned}
(L-k^2)^2\hat{u} &= \frac{Bk^2}{\nu^*}\frac{\hat{v}}{r^2},\\
(L-k^2)\hat{v} &= -\frac{B}{\nu^*}\frac{\hat{u}}{r^2},
\end{aligned} \tag{31}$$

where L denotes the differential operator

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{d}{dr}\right) - \frac{1}{r^2}.$$

(The derivation of (31) is most simply carried out as follows. From the last equation of (29) we have

$$\frac{1}{r}\frac{d}{dr}(r\hat{u}) + k\hat{w} = 0,$$

whence $\boldsymbol{u}$ can be written in the form $\boldsymbol{u} = \operatorname{curl}\boldsymbol{\Psi}$, where

$$\boldsymbol{\Psi} = \frac{1}{k}(\hat{v}\,\boldsymbol{i}_r - \hat{u}\,\boldsymbol{i}_\theta)\sin kz.$$

Substituting $\boldsymbol{u} = \operatorname{curl}\boldsymbol{\Psi}$ directly into (21) and taking the curl of that equation yields

$$B\operatorname{curl}(v/r^2, u/r^2, 0) = \nu^*\operatorname{curl}^4\boldsymbol{\Psi}, \tag{32}$$

where the identity curl curl = grad div $-\nabla^2$ has been used. The first equation of (31) is now obtained as the ϑ-component of equation (32). Finally, the unkown λ must necessarily be independent of ϑ, so that the second equation of (29) reduces immediately to the second equation of (31).)

Equations (31) are formally similar to the classical first order equations for small disturbance in Couette flow. The solution of (31) is a difficult task unless the cylinders are close together, in which case we can suppose (31) to be sufficiently well approximated by the simpler equations

$$\begin{aligned}(D^2-k^2)^2\hat{u} &= \Lambda k^2\hat{v}\\ (D^2-k^2)\hat{v} &= -\Lambda\hat{u},\end{aligned} \tag{33}$$

where $D=d/dr$, $\Lambda=B/\nu^*\bar{R}^2$, and $\bar{R}$ is some appropriate mean radius, say $\bar{R}=\sqrt{R_1R_2}$. The boundary conditions associated with (33) are

$$\hat{u}=\hat{v}=\frac{d\hat{u}}{dr}=0 \quad \text{at} \quad r=R_1, R_2. \tag{34}$$

According to the calculations of JEFFREYS and others (*cf.* [14], § 2.3) the critical value of Λ is $41.2\,(R_2-R_1)^{-2}$ (that is, this is the smallest value of Λ for which a non-trivial solution of (33), (34) exists). We are thus led to the following stability criterion,

$$\left|\frac{\Omega_2-\Omega_1}{\nu}\right| \leqq 41.2\,\frac{R_1+R_2}{R_1R_2(R_2-R_1)}. \tag{35}$$

For the pair of radii 3.55, 4.03 this gives, in particular,

$$\left|\frac{\Omega_2-\Omega_1}{\nu}\right| \leqq 45.5. \tag{36}$$

(36) is certainly a real improvement over the earlier estimate (28), and, moreover, the right-hand side comes very close to the "best value" 50. It must be borne in mind, however, that (35) and (36) are not yet rigorously established, since it remains to be shown that we have really found the greatest eigenvalue of the system (29). There is some reason to believe that this is so, since hydrodynamical experiments have shown indisputably that the secondary motion occurring in Couette flow is approximately of the form (30), but barring a proof of this fact, we must accept (35)—(36) as only tentatively established, in contrast with the absolute certainty of (27)—(28).

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References

[1] REYNOLDS, O.: On the dynamical theory of incompressible viscous fluids and the determination of the criterion. Phil. Trans. Roy. Soc. London (A) **186**, 123—164 (1895); Scientific Papers II, pp. 535—577.

[2] ORR, W. McF.: The stability or instability of the steady motions of a liquid. Part II: A viscous liquid. Proc. Roy. Irish Acad. (A) **27**, 69—138 (1907).

[3] HAMEL, G.: Zum Turbulenzproblem. Nachr. Ges. Wiss. Göttingen. **1911**, 261—270.

[4] LORD RAYLEIGH: On the motion of a viscous fluid. Phil. Mag. (6) **26**, 776—786 (1913); Papers VI, pp. 187—197.

[5] KÁRMÁN, TH. VON: Über die Stabilität der Laminarströmung und die Theorie der Turbulenz. Proc. 1st. Int. Congress Appl. Mech. Delft, 1924, pp. 97—112.

[6] SYNGE, J. L.: Hydrodynamical stability. Semi-centennial publications of the Amer. Math. Soc. **2**, 227—269 (1938). Especially Part III.

[7] THOMAS, T. Y.: On the stability of viscous fluids. (1941). Univ. Calif. Publ. Math. New Series **2**, 13—43 (1944).

[8] THOMAS, T. Y.: On the uniform convergence of the solutions of the Navier-Stokes equations. Proc. Nat. Acad. Sci. U.S.A. **29**, 243—246 (1943).

[9] HOPF, E.: On non-linear partial differential equations. Lecture series of the Symposium on partial differential equations. Univ. of Calif. 1955, pp. 7—11.

[10] LERAY, J.: Essai sur les mouvements plan d'un liquide visqueux que limitent des parois. J. Math. Pures. Appl. (9) **13**, 331—418 (1934).

[11] KAMPÉ DE FÉRIET, J.: Sur la decroissance de l'énergie cinétique d'un fluide visqueux incompressible occupant un domaine borné ayant pour frontière des parois solides fixes. Ann. Soc. Sci. Bruxelles **63**, 35—46 (1949).

[12] BERKER, R.: Inégalité vérifiée par l'énergie cinétique d'un fluide visqueux incompressible occupant un domaine spatial borné. Bull. Tech. Univ. Istanbul **2**, 41—50 (1949).

[13] SYNGE, J. L.: On the stability of a viscous liquid between two rotating coaxial cylinders. Proc. Roy. Soc. London (A) **167**, 250—256 (1938).

[14] LIN, C. C.: The theory of hydrodynamic stability. Cambridge 1955. Especially Chaps. 1, 2, 4.

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