Material Symmetry Restrictions on Constitutive Equations

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Material symmetry imposes certain restrictions on the forms of the response functions or functionals which appear in constitutive equations. If a tensor-valued response function or functional satisfies the material symmetry requirements, then it must be representable in a certain canonical form. In Part I a canonical form is derived for response functions, and in Part II an analogous representation is derived for response functionals. The representations are complete and general, in the sense that no extraneous assumptions concerning continuity of the functions or functionals are used. The general representation theorems are valid for tensor-valued response functions or functionals of arbitrary rank, depending on an arbitrary number of tensors, or tensor-valued functions, respectively, of arbitrary ranks, and for any group of material symmetries which is a subgroup of the orthogonal group.

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Part I. Form-Invariant Response Functions

1. Introduction

A central problem in the formulation of constitutive equations is to find the effect of material symmetries in restricting the forms of response functions. Due largely to the efforts of RIVLIN and his co-workers, this problem may be regarded as solved, in the sense that standard techniques for deducing the implications of symmetry are known. However, the simplest and most widely used method involves the assumption that the response functions are polynomials. In the present paper, we show that the polynomial assumption is not essential to the technique motivated by it. Without any *a priori* assumption about the nature of the response functions, the restrictions imposed by material symmetries can be found by the polynomial technique.

The relevant theorem has been proved in an earlier work [*1*]*, but only for materials with a finite group of symmetries (*i.e.* crystals). In the present paper, we extend this theorem to include those continuous groups which characterize the symmetries of isotropic and transversely isotropic materials.

The general problem which we consider is the following. Some physical property of a given material is described by the dependence of a tensor $\boldsymbol{\chi}$ on a number of other tensors $\boldsymbol{\psi}^{(\nu)}$ $(\nu=1, 2, \ldots, N)$:

$$\boldsymbol{\chi}=\boldsymbol{f}(\boldsymbol{\psi}^{(1)}, \ldots, \boldsymbol{\psi}^{(N)}). \tag{1.1}$$

That is, the components $\chi_{i_1 \ldots i_r}$ of $\boldsymbol{\chi}$ in a Cartesian coordinate system x are assumed to be functions of the components $\psi^{(\nu)}_{i_1 \ldots i_{r(\nu)}}$ of the tensors $\boldsymbol{\psi}^{(\nu)}$ in that same system:

$$\chi_{i_1 \ldots i_r}=f_{i_1 \ldots i_r}(\psi^{(\nu)}_{p_1 \ldots p_{r(\nu)}}). \tag{1.2}$$

In a second Cartesian coordinate system $\bar{x}$, the components of $\boldsymbol{\psi}^{(\nu)}$ are given by

$$\bar{\psi}^{(\nu)}_{i_1 \ldots i_{r(\nu)}}=R_{i_1 j_1} \ldots R_{i_{r(\nu)} j_{r(\nu)}} \psi^{(\nu)}_{j_1 \ldots j_{r(\nu)}}, \tag{1.3}$$

where the matrix $\boldsymbol{R}=\|R_{ij}\|$ is orthogonal:

$$R_{ik}R_{jk}=R_{ki}R_{kj}=\delta_{ij}. \tag{1.4}$$

If the transformation with matrix $\boldsymbol{R}$ is a symmetry of the material, then the components $f_{i_1 \ldots i_r}$ of the tensor-valued response function $\boldsymbol{f}$ must satisfy the relations

$$f_{i_1 \ldots i_r}(\bar{\psi}^{(\nu)}_{p_1 \ldots p_{r(\nu)}})=R_{i_1 j_1} \ldots R_{i_r j_r} f_{j_1 \ldots j_r}(\psi^{(\nu)}_{p_1 \ldots p_{r(\nu)}}). \tag{1.5}$$

If $\boldsymbol{f}$ satisfies (1.5) identically in the tensors $\boldsymbol{\psi}^{(\nu)}$ $(\nu=1, 2, \ldots, N)$, for each matrix $\boldsymbol{R}$ belonging to the group of symmetries of the material, we shall call $\boldsymbol{f}$ *form-invariant* under the group considered. The property of form-invariance implies a restriction on the form of $\boldsymbol{f}$. The problem is to find the most general form which this property allows.

If attention is limited to solutions in which $\boldsymbol{f}$ is a polynomial (*i.e.* the components of $\boldsymbol{f}$ are polynomials), the general solution can be obtained in the following way ([*1*], [*2*]): First, introduce an auxiliary tensor $\boldsymbol{\varphi}$ of rank r, the rank

* Numbers in square brackets indicate references at the end of the paper.

of $\boldsymbol{f}$. (It may be convenient to let $\boldsymbol{\varphi}$ be an outer product of tensors $\boldsymbol{\varphi}^{(\mu)}$ ($\mu = 1, 2, \ldots, M$) of total rank r. In that case, the following description must be appropriately modified.) Second, calculate the elements of a finite integrity basis for polynomials in the components of $\boldsymbol{\varphi}$ and $\boldsymbol{\psi}^{(\nu)}$ ($\nu = 1, 2, \ldots, N$) which are invariant under the group considered. Third, single out those elements I_α ($\alpha = 1, 2, \ldots, A$) which are functions of $\boldsymbol{\psi}^{(\nu)}$ ($\nu = 1, 2, \ldots, N$) only, thus forming an integrity basis for invariants of these tensors. Fourth, select those invariants J_β ($\beta = 1, 2, \ldots, B$) which are linear in the components of $\boldsymbol{\varphi}$. Each such invariant is of the form

$$J_\beta = \varphi_{i_1 \ldots i_r} f^{(\beta)}_{i_1 \ldots i_r}(\boldsymbol{\psi}^{(1)}, \ldots, \boldsymbol{\psi}^{(N)}), \tag{1.6}$$

where the components $f^{(\beta)}_{i_1 \ldots i_r}$ of $\boldsymbol{f}^{(\beta)}$ are polynomials in the components of the indicated tensors. The tensor-valued function $\boldsymbol{f}^{(\beta)}$ is form-invariant, *i.e.* it satisfies (1.5). We call the functions $\boldsymbol{f}^{(\beta)}$ ($\beta = 1, 2, \ldots, B$) *basic form-invariant tensors.* Then, every form-invariant polynomial $\boldsymbol{f}$ is a linear combination of the basic form-invariant tensors $\boldsymbol{f}^{(\beta)}$, with coefficients p_β which are polynomials in the elements of the integrity basis I_α ($\alpha = 1, 2, \ldots, A$):

$$\boldsymbol{f} = \sum_{\beta=1}^{B} p_\beta(I_1, \ldots, I_A)\, \boldsymbol{f}^{(\beta)}(\boldsymbol{\psi}^{(1)}, \ldots, \boldsymbol{\psi}^{(N)}). \tag{1.7}$$

Conversely, for every choice of the coefficients p_β, the polynomial $\boldsymbol{f}$ defined by (1.7) is form-invariant. We call (1.7) the *polynomial canonical form.*

The technique just outlined has been used, with one variation or another, in applications to a variety of specific constitutive equations and material symmetry groups (see, *e.g.* [*3*]—[*9*]). The method is probably best understood by studying these specific examples. It is evident that the only possible difficulty involved in carrying out this procedure lies in the second step, where calculation of an integrity basis is required. This difficulty is alleviated by the fact that there exist standard tables of basic invariants for the cases most frequently encountered. The following list of such tables may be helpful. On this list, *tensor* means *symmetric second-rank tensor.*

Symmetry	*Variables*	*Source*
Crystal Classes	1 vector, 1 tensor	SMITH, SMITH & RIVLIN [*10*]
Crystal Classes	M vectors	SMITH & RIVLIN [*11*]
Orthotropy	N tensors	ADKINS [*12*]
Transverse Isotropy	N tensors	ADKINS [*12*]
Transverse Isotropy	M vectors, N tensors	ADKINS [*13*]
Isotropy	N tensors	SPENCER & RIVLIN [*14*], [*15*], SPENCER [*16*], RIVLIN [*17*]
Isotropy	M vectors, N tensors	SPENCER & RIVLIN [*18*].

Few form-invariance problems arising from specific constitutive equations and material symmetry groups have been solved without making use of the assumption that the response functions are polynomials. In the present paper we are concerned with the general solution of the form-invariance relations (1.5), with no *a priori* restriction of any kind on $\boldsymbol{f}$. We shall show that the general solution is obtained from the polynomial canonical form (1.7) merely by removing the stipulation that the coefficients p_β are polynomials. That is, with the definitions introduced earlier, *every form-invariant tensor-valued function* $\boldsymbol{f}$ *is a linear combination of the basic form-invariant tensors* $\boldsymbol{f}^{(\beta)}$ ($\beta = 1, 2, \ldots, B$), *with scalar coefficients* F_β *which are functions of the elements of an integrity basis:*

$$\boldsymbol{f} = \sum_{\beta=1}^{B} F_\beta(I_1, \ldots, I_A)\, \boldsymbol{f}^{(\beta)}(\boldsymbol{\psi}^{(1)}, \ldots, \boldsymbol{\psi}^{(N)}). \tag{1.8}$$

Conversely, for every choice of the functions F_β*, the function* $\boldsymbol{f}$ *so defined is form-invariant.* Thus, (1.8) furnishes a canonical representation for solutions of the material symmetry problem.

The proof of this canonical representation theorem is lengthy, although largely elementary and self-contained. Any facts about group theory which are required, excepting the most trivial, are mentioned in Section 2. The simplest relevant concepts in invariant theory are defined in Section 3. In Section 4 we prove two theorems characterizing functional bases.

In the special case in which $\boldsymbol{f}$ is a tensor of rank zero, the canonical representation theorem reduces to the statement that *an integrity basis is a functional basis.* This result, which is explained and proved in Sections 5 and 6, will be used as a lemma in proving the general theorem.

The proof of the general theorem is outlined in Section 7 and carried out in Sections 8 to 11.

2. *Notation and Preliminary Remarks*

Since we shall not be concerned with the number or ranks of the tensors $\boldsymbol{\psi}^{(\nu)}$ in (1.5), it is possible and convenient to use an abbreviated system of notation. Let the collection of components $\psi^{(\nu)}_{i_1 \ldots i_{r(\nu)}}$ ($\nu = 1, 2, \ldots, N$) be arranged in some sequence and renumbered with a single index as $\psi_1, \psi_2, \ldots, \psi_n$. We denote this sequence by the symbol $\boldsymbol{\psi}$. The collection of components $\bar{\psi}^{(\nu)}_{i_1 \ldots i_{r(\nu)}}$ ($\nu = 1, 2, \ldots, N$) in the system $\bar{x}$ can be arranged in a corresponding sequence, denoted by $\bar{\boldsymbol{\psi}}$. For brevity, we shall refer to $\boldsymbol{\psi}$ as a tensor, rather than a collection of components of various tensors. Alternatively, since it is convenient to think of the components ψ_i ($i = 1, 2, \ldots, n$) as coordinates in an n-dimensional space, we shall often refer to $\boldsymbol{\psi}$ as a point. Particular choices of $\boldsymbol{\psi}$ will be labeled with a superscript, as for example $\boldsymbol{\psi}^0, \boldsymbol{\psi}^1$, *etc.*

Similarly, the functions $f_{i_1 \ldots i_r}$ in (1.5) are renumbered with a single index as $f_1, f_2, \ldots, f_m$, and this sequence is denoted by $\boldsymbol{f}$. If we say that $\boldsymbol{f}$ (or some similar sequence of functions) is a polynomial, we mean that its components are polynomials.

The relation (1.3) between the components of $\bar{\boldsymbol{\psi}}$ and the components of $\boldsymbol{\psi}$ can be written as

$$\bar{\psi}_i = \sum_{j=1}^{n} S_{ij}\, \psi_j \qquad (i = 1, 2, \ldots, n). \tag{2.1}$$

Each element of the matrix $\boldsymbol{S} = \| S_{ij} \|$ is a product of elements of the matrix $\boldsymbol{R}$ which appears in (1.3). The relation (1.5) can be written as

$$f_i(\bar{\boldsymbol{\psi}}) = \sum_{j=1}^{m} T_{ij}\, f_j(\boldsymbol{\psi}) \qquad (i = 1, 2, \ldots, m), \tag{2.2}$$

where each element of $\boldsymbol{T}=\|T_{ij}\|$ is also a product of elements of $\boldsymbol{R}$. The two relations (2.1) and (2.2) can be abbreviated still further as

$$\bar{\boldsymbol{\psi}}=\boldsymbol{S}\boldsymbol{\psi} \quad \text{and} \quad \boldsymbol{f}(\bar{\boldsymbol{\psi}})=\boldsymbol{T}\boldsymbol{f}(\boldsymbol{\psi}), \tag{2.3}$$

respectively.

The symmetry groups which concern us are the 32 crystal point groups and the continuous groups which describe the symmetries of the various types of isotropic and transversely isotropic materials. There are two groups associated with isotropy, the full and proper orthogonal groups. Under the heading of transverse isotropy, there are four groups, differing from one another in their reflectional symmetries. Each of the four groups contains the group of rotations about a fixed axis as a subgroup. The four cases arise from the fact that the material may, or may not, be symmetric under reflections in planes perpendicular to the symmetry axis, and also may, or may not, have reflectional symmetry in planes containing the symmetry axis. Whenever we speak of a symmetry group, we shall mean one of the groups just mentioned.

In the case of a crystal symmetry group, there are a finite number of symmetries of the material. These symmetries can be numbered with a parameter λ which takes a finite number of values. In the case of the group of rotations about one axis, the group elements can be numbered with a continuous index λ representing the angle of rotation. For hemihedral isotropic materials, the group elements can be numbered with, say, the Euler angles of the rotation. Although there are three such angles, we denote the set of all three variables by the one symbol λ. In groups involving reflectional symmetries as well as a continuous subgroup of rotational symmetries, the group elements should be numbered with both a discrete and a continuous parameter. In such cases we shall continue to use the single symbol λ to denote the set of all such variables. In every case, $\boldsymbol{R}(\lambda)$, $\boldsymbol{S}(\lambda)$ and $\boldsymbol{T}(\lambda)$ will denote those matrices (appearing in (1.3) and (2.3)) which correspond to the symmetry numbered λ.

The set of matrices $\boldsymbol{S}(\lambda)$ is an n-dimensional representation of the symmetry group considered, and the matrices $\boldsymbol{T}(\lambda)$ form an m-dimensional representation. It follows from their definition that these matrices, like $\boldsymbol{R}(\lambda)$, are orthogonal, *i.e.*

$$\boldsymbol{S}(\lambda)\,\boldsymbol{S}^T(\lambda)=\boldsymbol{I} \quad \text{and} \quad \boldsymbol{T}(\lambda)\,\boldsymbol{T}^T(\lambda)=\boldsymbol{I}. \tag{2.4}$$

Here $\boldsymbol{S}^T$ and $\boldsymbol{T}^T$ are the transposes of $\boldsymbol{S}$ and $\boldsymbol{T}$, respectively, and $\boldsymbol{I}$ is the unit matrix in the appropriate number of dimensions.

The domain of each variable in the set λ can always be taken to be a finite number of closed and bounded intervals. (For example, $0\leqq\lambda\leqq 2\pi$ for the two-dimensional rotation group.) That is, the domain of λ is compact. When we speak of *all* values of λ, we shall mean all values in that compact set which is appropriate to the group considered.

The elements of the orthogonal matrix $\boldsymbol{R}(\lambda)$ are continuous functions of the variables denoted by λ. This is trivially true in those cases in which λ takes only a finite number of values. Since the elements of the matrices $\boldsymbol{S}(\lambda)$ and $\boldsymbol{T}(\lambda)$ are products of elements of $\boldsymbol{R}(\lambda)$, they too are continuous functions of λ. As continuous functions defined on a compact set, they are uniformly bounded.

In fact, it is easily seen from the orthogonality of these matrices that

$$|R_{ij}(\lambda)| \leqq 1, \qquad |S_{ij}(\lambda)| \leqq 1 \quad \text{and} \quad |T_{ij}(\lambda)| \leqq 1. \tag{2.5}$$

We shall have occasion to use the process of integration (or summation) over all group elements. This Hurwitz integration is discussed by WIGNER ([*19*], p. 98), for example. The Hurwitz integral of a function $g(\lambda)$ will be denoted by

$$\int g(\lambda)\, d\lambda. \tag{2.6}$$

We shall take the integral to be normalized in such a way that

$$\int d\lambda = 1. \tag{2.7}$$

When λ is a discrete parameter, the integral is to be interpreted as a sum over all values of λ, divided by the number of group elements in order to satisfy (2.7). In such cases, it is easy to see that for any λ_0, the following relation is satisfied:

$$\int g[\boldsymbol{S}(\lambda)\boldsymbol{S}(\lambda_0)\boldsymbol{\psi}]\, d\lambda = \int g[\boldsymbol{S}(\lambda)\boldsymbol{\psi}]\, d\lambda. \tag{2.8}$$

For, the set of matrices $\boldsymbol{S}(\lambda)\boldsymbol{S}(\lambda_0)$ obtained by letting λ take all values is the same as the set $\boldsymbol{S}(\lambda)$, and thus the argument of g runs over the same set of values in each of the sums in (2.8). For continuous groups, the integral is defined in such a way that (2.8) still holds. We note in passing that if the integrand $g[\boldsymbol{S}(\lambda)\boldsymbol{\psi}]$ is a polynomial in the components of $\boldsymbol{S}(\lambda)\boldsymbol{\psi}$, then the integral is a polynomial in the components of $\boldsymbol{\psi}$.

Nothing in the discussion to follow depends upon which finite or continuous material symmetry group we consider. We suppose that the group is fixed once and for all.

3. Invariants. Definitions

Given a tensor $\boldsymbol{\psi}$, the tensors $\boldsymbol{S}(\lambda)\boldsymbol{\psi}$ which can be obtained from it by transformations of the group will be called *equivalent* to $\boldsymbol{\psi}$. We observe that since the matrices $\boldsymbol{S}(\lambda)$ form a group, the tensors which are equivalent to $\boldsymbol{\psi}$ are equivalent to one another.

The set of tensors equivalent to $\boldsymbol{\psi}$ will be called the *orbit* of $\boldsymbol{\psi}$. The orbits of two tensors $\boldsymbol{\psi}^0$ and $\boldsymbol{\psi}^1$ either coincide or have no points in common, depending on whether $\boldsymbol{\psi}^0$ and $\boldsymbol{\psi}^1$ are equivalent or not. In the case of a continuous group, one can think of an orbit as one or more closed curves or surfaces in the space of points $\boldsymbol{\psi}$. In the case of a finite group, an orbit consists of a finite number of points. Since the group parameter λ ranges over a compact set, and the elements of $\boldsymbol{S}(\lambda)$ are continuous functions of λ, then the points $\boldsymbol{S}(\lambda)\,\boldsymbol{\psi}^0$ of the orbit of $\boldsymbol{\psi}^0$ form a compact set in the space of points $\boldsymbol{\psi}$.

An *invariant* of $\boldsymbol{\psi}$ is a scalar-valued function $F(\boldsymbol{\psi})$ such that

$$F(\boldsymbol{S}\boldsymbol{\psi}) = F(\boldsymbol{\psi}), \tag{3.1}$$

for every $\boldsymbol{\psi}$ and every $\boldsymbol{S}$ in the group considered. Thus, an invariant is a function which takes equal values at equivalent points. Invariants are functions which are constant on every orbit. In effect, an invariant is a function which assigns a number to each orbit.

A set of invariants $H_1(\psi), \ldots, H_K(\psi)$ is called a *functional basis* if every invariant $F(\psi)$ can be expressed as a function of the invariants $H_1, \ldots, H_K$. The notion of a functional basis will be examined in more detail in Section 4.

By a *polynomial in* ψ, we mean a polynomial function of the components of ψ. By a *polynomial invariant* we mean an invariant which is a polynomial. An *integrity basis* is a set of polynomial invariants $I_1(\psi), \ldots, I_A(\psi)$ such that every polynomial invariant can be expressed as a polynomial in the invariants $I_\alpha(\psi)$ $(\alpha = 1, 2, \ldots, A)$. In Sections 5 and 6 we shall prove that an integrity basis is a functional basis.

4. Criteria Characterizing Functional Bases

In the present section we prove two theorems characterizing functional bases. These results were stated in an earlier paper [*1*] without detailed proof. The theorems give criteria for determining whether or not a given set of invariants forms a functional basis. The first criterion will be used in Part II of the present paper. The second criterion will be applied in Section 5, in showing that an integrity basis is a functional basis.

In order for a set of invariants $H_1, \ldots, H_K$ to form a functional basis, it is necessary and sufficient that for every choice of ψ^0, the following system of equations is satisfied only by those tensors ψ which are equivalent to ψ^0:

$$H_\varkappa(\psi) = H_\varkappa(\psi^0), \qquad (\varkappa = 1, 2, \ldots, K). \tag{4.1}$$

To prove this, first suppose that the invariants $H_1, \ldots, H_K$ form a functional basis. Let ψ be a tensor which is not equivalent to ψ^0, so that the orbits of these tensors are distinct. Let Q be an invariant whose value on the orbit of ψ is different from its value on the orbit of ψ^0. By hypothesis, Q can be expressed as a function of the invariants $H_1, \ldots, H_K$. Hence, the equations (4.1) cannot all be satisfied. That is, the system (4.1) is not satisfied by tensors ψ which are not equivalent to ψ^0.

Conversely, suppose that the system (4.1) is satisfied only by those tensors ψ which are equivalent to ψ^0, regardless of how ψ^0 is chosen. Then one and only one orbit is determined by a set of values $H_\varkappa(\psi^0)$ $(\varkappa = 1, 2, \ldots, K)$. Any function which assigns values to orbits can be expressed as a function of those variables $H_1, \ldots, H_K$ which specify the orbit. Thus, any invariant can be expressed as a function of the invariants $H_1, \ldots, H_K$, which therefore form a functional basis.

A slightly different criterion can be obtained from the preceding one. *In order for a set of invariants $H_1, \ldots, H_K$ to form a functional basis, it is necessary and sufficient that for every choice of inequivalent tensors ψ^0 and ψ^1, there is a function $Q(H_1, \ldots, H_K; \psi^0, \psi^1)$ which takes different values on the orbits of ψ^0 and ψ^1:*

$$Q[H_1(\psi^0), \ldots, H_K(\psi^0); \psi^0, \psi^1] \neq Q[H_1(\psi^1), \ldots, H_K(\psi^1); \psi^0, \psi^1]. \tag{4.2}$$

We call such a function Q an *orbit-separator*.

If the invariants $H_1, \ldots, H_K$ form a functional basis, then by the first criterion above, at least one of these invariants can be used as an orbit-separator Q.

To prove the converse, suppose now that for every choice of inequivalent tensors $\boldsymbol{\psi}^0$ and $\boldsymbol{\psi}^1$, there is a function Q satisfying (4.2). From (4.2) it follows that there is some invariant $H_\varkappa$ which fails to satisfy (4.1) when $\boldsymbol{\psi}=\boldsymbol{\psi}^1$. Since $\boldsymbol{\psi}^1$ is any tensor inequivalent to $\boldsymbol{\psi}^0$, then the system (4.1) is not satisfied by tensors $\boldsymbol{\psi}$ which are inequivalent to $\boldsymbol{\psi}^0$. Hence, according to the first criterion, the invariants $H_1, \ldots, H_K$ form a functional basis.

5. *Integrity Bases as Functional Bases*

We now show that *an integrity basis is a functional basis.* Let I_α $(\alpha=1,2,\ldots,A)$ be an integrity basis. Let $\boldsymbol{\psi}^0$ and $\boldsymbol{\psi}^1$ be inequivalent tensors. We shall construct a polynomial Q in the invariants I_α $(\alpha=1,2,\ldots,A)$ which distinguishes the orbit of $\boldsymbol{\psi}^0$ from that of $\boldsymbol{\psi}^1$, *i.e.*

$$Q[I_1(\boldsymbol{\psi}^0),\ldots,I_A(\boldsymbol{\psi}^0);\boldsymbol{\psi}^0,\boldsymbol{\psi}^1] \neq Q[I_1(\boldsymbol{\psi}^1),\ldots,I_A(\boldsymbol{\psi}^1);\boldsymbol{\psi}^0,\boldsymbol{\psi}^1]. \tag{5.1}$$

According to the second criterion in Section 4, this is all that is required in order to prove that the invariants I_α $(\alpha=1,2,\ldots,A)$ form a functional basis.

The details of the construction of a polynomial orbit-separator Q will be carried out in Section 6, along the following lines. We first define the distance $D(\boldsymbol{\psi},\boldsymbol{\psi}^0)$ from $\boldsymbol{\psi}^0$ to $\boldsymbol{\psi}$. In terms of this distance, we define a measure $D^*(\boldsymbol{\psi};\boldsymbol{\psi}^0)$ of the distance from $\boldsymbol{\psi}^0$ to the orbit of $\boldsymbol{\psi}$. D^* is shown to be a continuous function of $\boldsymbol{\psi}$. It can accordingly be approximated as closely as desired, over a compact set containing the orbits of $\boldsymbol{\psi}^0$ and $\boldsymbol{\psi}^1$, by a polynomial in $\boldsymbol{\psi}$, $P(\boldsymbol{\psi};\boldsymbol{\psi}^0,\boldsymbol{\psi}^1)$. From P, a polynomial invariant $P^*(\boldsymbol{\psi};\boldsymbol{\psi}^0,\boldsymbol{\psi}^1)$ is constructed by Hurwitz integration. P^* is also a good approximation to D^*. Because P^* is a polynomial invariant, it can be expressed as a polynomial Q in the elements of an integrity basis. Since the distances from $\boldsymbol{\psi}^0$ to the orbits of $\boldsymbol{\psi}^0$ and $\boldsymbol{\psi}^1$ are different, and since Q is approximately the distance from $\boldsymbol{\psi}^0$ to the orbit of $\boldsymbol{\psi}$, it will satisfy (5.1). This will complete the proof that an integrity basis is a functional basis.

6. *Construction of a Polynomial Orbit-Separator*

We begin by defining the distance $D(\boldsymbol{\psi},\boldsymbol{\psi}^0)$ from $\boldsymbol{\psi}$ to $\boldsymbol{\psi}^0$:

$$\begin{aligned} D(\boldsymbol{\psi},\boldsymbol{\psi}^0) &= \Big[\sum_{i=1}^{n}(\psi_i-\psi_i^0)^2\Big]^{\frac{1}{2}} \\ &= [(\boldsymbol{\psi}-\boldsymbol{\psi}^0)\cdot(\boldsymbol{\psi}-\boldsymbol{\psi}^0)]^{\frac{1}{2}}. \end{aligned} \tag{6.1}$$

This distance is a continuous function of $\boldsymbol{\psi}$. Recalling from Section 3 that the set consisting of the points $\boldsymbol{S}(\lambda)\boldsymbol{\psi}$ (the orbit of $\boldsymbol{\psi}$) is compact, it follows that on this set D attains a minimum. Thus, we can define the distance from $\boldsymbol{\psi}^0$ to the orbit of $\boldsymbol{\psi}$ by

$$D^*(\boldsymbol{\psi};\boldsymbol{\psi}^0) = \min_\lambda D[\boldsymbol{S}(\lambda)\,\boldsymbol{\psi},\boldsymbol{\psi}^0]. \tag{6.2}$$

We note that D^* has the following properties:

1) $D^*(\boldsymbol{S}\boldsymbol{\psi};\boldsymbol{\psi}^0) = D^*(\boldsymbol{\psi};\boldsymbol{\psi}^0)$, (6.3)

2) $D^*(\boldsymbol{S}\boldsymbol{\psi}^0;\boldsymbol{\psi}^0) = 0$, (6.4)

3) $D^*(\boldsymbol{\psi};\boldsymbol{\psi}^0) > 0$ if $\boldsymbol{\psi}$ is not equivalent to $\boldsymbol{\psi}^0$. (6.5)

To prove property 1), that D^* is an invariant of ψ, we first note that since $S(\lambda)S$ varies over all transformations of the group when $S(\lambda)$ does, then $S(\lambda)S\psi$ varies over all the points of the orbit of ψ when $S(\lambda)\psi$ does. With this observation, (6.3) follows directly from (6.2). By the definition of D, the minimum value which D takes on any orbit must be non-negative. With this in mind, properties 2) and 3) follow from (6.2) and the fact that D vanishes only at $\psi=\psi^0$.

We now prove that $D^*(\psi;\psi^0)$ is a continuous function of ψ. To prove continuity at an arbitrary point $\psi=\psi^1$, we shall show that $D(\psi,\psi^1)\to 0$ implies that $D^*(\psi;\psi^0)\to D^*(\psi^1;\psi^0)$. First, the triangle inequality yields

$$D[S(\lambda)\psi,\psi^0]\leqq D[S(\lambda)\psi,S(\lambda)\psi^1]+D[S(\lambda)\psi^1,\psi^0]. \tag{6.6}$$

Since $S(\lambda)$ is an orthogonal matrix, it follows from (6.1) that

$$D[S(\lambda)\psi,S(\lambda)\psi^1]=D(\psi,\psi^1). \tag{6.7}$$

By using (6.7) in (6.6) and making use of the definition (6.2) of D^*, we obtain

$$D^*(\psi;\psi^0)\leqq D(\psi,\psi^1)+D^*(\psi^1;\psi^0). \tag{6.8}$$

Since the same inequality holds with ψ and ψ^1 interchanged, we have

$$|D^*(\psi;\psi^0)-D^*(\psi^1;\psi^0)|\leqq D(\psi,\psi^1). \tag{6.9}$$

This inequality implies that $D^*(\psi;\psi^0)$ is continuous at $\psi=\psi^1$.

To begin the next step in the construction of the polynomial orbit- separator, let ψ^1 be a tensor which is not equivalent to ψ^0. According to (6.5), we have

$$D^*(\psi^1;\psi^0)=D' \text{ (say)}, \qquad D'>0. \tag{6.10}$$

Let C be a compact domain containing the (compact) orbits of ψ^0 and ψ^1. Since $D^*(\psi;\psi^0)$ is a continuous function of ψ, then according to the Weierstrass theorem, we can approximate it arbitrarily well in the compact domain C by a polynomial in ψ. Thus, there is a polynomial in ψ, $P(\psi;\psi^0,\psi^1)$, such that for all ψ in C,

$$|P(\psi;\psi^0,\psi^1)-D^*(\psi;\psi^0)|<\tfrac{1}{4}D'. \tag{6.11}$$

From P we can construct a polynomial invariant in ψ by Hurwitz integration:

$$P^*(\psi;\psi^0,\psi^1)=\int P[S(\lambda)\psi;\psi^0,\psi^1]\,d\lambda. \tag{6.12}$$

That P^* is an invariant follows from property (2.8). P^*, like P, approximates D^* well enough to take different values at ψ^0 and ψ^1:

$$P^*(\psi^0;\psi^0,\psi^1)\neq P^*(\psi^1;\psi^0,\psi^1). \tag{6.13}$$

To prove this, we first note that since D^* is an invariant, we can write

$$D^*(\psi;\psi^0)=\int D^*[S(\lambda)\psi;\psi^0]\,d\lambda. \tag{6.14}$$

Then, from (6.12) and (6.14) we obtain

$$|P^*(\psi;\psi^0,\psi^1)-D^*(\psi;\psi^0)|\leqq\int|P[S(\lambda)\psi;\psi^0,\psi^1]-D^*[S(\lambda)\psi;\psi^0]|\,d\lambda. \tag{6.15}$$

Let $\boldsymbol{\psi}$ be a tensor whose orbit is in C, so that (6.11) is valid with $\boldsymbol{\psi}$ replaced by $\boldsymbol{S}(\lambda)\boldsymbol{\psi}$. Then, by using the bound (6.11) on the integrand in (6.15), and recalling that the integral is normalized according to (2.7), we obtain

$$|P^*(\boldsymbol{\psi};\boldsymbol{\psi}^0,\boldsymbol{\psi}^1)-D^*(\boldsymbol{\psi};\boldsymbol{\psi}^0)|<\tfrac{1}{4}D'. \tag{6.16}$$

Since the orbits of $\boldsymbol{\psi}^0$ and $\boldsymbol{\psi}^1$ are in C, (6.16) is valid when $\boldsymbol{\psi}$ takes either of these values. Since D^* vanishes at $\boldsymbol{\psi}=\boldsymbol{\psi}^0$ and takes the value D' at $\boldsymbol{\psi}=\boldsymbol{\psi}^1$, it follows from (6.16) that P^* takes different values at these two points.

Because P^* is a polynomial invariant, it can be expressed as a polynomial Q in the elements of an integrity basis $I_\alpha(\boldsymbol{\psi})$ ($\alpha=1, 2, \ldots, A$):

$$P^*(\boldsymbol{\psi};\boldsymbol{\psi}^0,\boldsymbol{\psi}^1)=Q\,[I_1(\boldsymbol{\psi}),\ldots,I_A(\boldsymbol{\psi});\boldsymbol{\psi}^0,\boldsymbol{\psi}^1]. \tag{6.17}$$

The inequality (6.13) guarantees that the values of Q at $\boldsymbol{\psi}=\boldsymbol{\psi}^0$ and $\boldsymbol{\psi}=\boldsymbol{\psi}^1$ are not equal. Hence, Q satisfies the condition (5.1) demanded of an orbit-separator.

7. *Canonical Representations for Form-Invariant Tensor-Valued Functions*

We return now to the main problem under consideration, which is to show that every form-invariant tensor-valued function $\boldsymbol{f}$ can be expressed in the canonical form (1.8). In the present notation, the form-invariance requirement (1.5) (or (2.3)) takes the form

$$\boldsymbol{f}[\boldsymbol{S}(\lambda)\boldsymbol{\psi}]=\boldsymbol{T}(\lambda)\,\boldsymbol{f}(\boldsymbol{\psi}). \tag{7.1}$$

It should be recalled that as λ varies over all values in its domain, $\boldsymbol{S}(\lambda)$ varies over all matrices in a certain representation of the group, and $\boldsymbol{T}(\lambda)$ varies over all matrices in another representation. For $\boldsymbol{f}$ to be form-invariant, (7.1) must be satisfied identically in $\boldsymbol{\psi}$ and λ.

The canonical form (1.8) involves the basic form-invariant tensors $\boldsymbol{f}^{(\beta)}$ ($\beta=1, 2, \ldots, B$), which are defined in Section 1. It is known ([1], [2]) that these functions are indeed form-invariant, *i.e.*

$$\boldsymbol{f}^{(\beta)}[\boldsymbol{S}(\lambda)\boldsymbol{\psi}]=\boldsymbol{T}(\lambda)\,\boldsymbol{f}^{(\beta)}(\boldsymbol{\psi}). \tag{7.2}$$

It is also known that every form-invariant polynomial $\boldsymbol{p}^*$ can be expressed in the polynomial canonical form (1.7):

$$\boldsymbol{p}^*(\boldsymbol{\psi})=\sum_{\beta=1}^{B}p_\beta(I_1,\ldots,I_A)\,\boldsymbol{f}^{(\beta)}(\boldsymbol{\psi}). \tag{7.3}$$

Here the coefficients p_β ($\beta=1, 2, \ldots, B$) are polynomials in the elements I_α ($\alpha=1, 2, \ldots, A$) of an integrity basis for invariants of $\boldsymbol{\psi}$.

The canonical torm (1.8) is, in the present notation,

$$\boldsymbol{f}(\boldsymbol{\psi})=\sum_{\beta=1}^{B}F_\beta(I_1,\ldots,I_A)\,\boldsymbol{f}^{(\beta)}(\boldsymbol{\psi}), \tag{7.4}$$

where the coefficients F_β are scalar-valued functions of the indicated invariants. A function of the form (7.4) is form-invariant, whatever the functions F_β may be. This can be verified immediately, by using (7.2) and the fact that the functions I_α ($\alpha=1, 2, \ldots, A$) are invariants.

In Sections 8 to 11 we will show that *every form-invariant function f can be expressed in the canonical form* (7.4).

The proof will proceed along the following lines. The main idea is that although we assume nothing about continuity of the components of f, the form-invariance condition (7.1) itself implies that f varies continuously along each orbit. That is, f can change discontinuously only as ψ moves from one orbit to another. It is then possible to find a polynomial p which approximates f on a given orbit, the orbit of ψ^0, say, with an error which is uniformly small. We shall make these statements more precise, and prove them, in Section 8.

From the polynomial p we can construct a form-invariant polynomial p^* which also approximates f on the orbit of ψ^0 (Section 9). Because p^* is a form-invariant *polynomial*, it can be expressed in the polynomial canonical form (7.3). Thus, f can be approximated as closely as desired, over the orbit of ψ^0, by a linear combination of the basic form-invariant tensors $f^{(\beta)}$. It follows that on the orbit of ψ^0, f is exactly a linear combination of the tensors $f^{(\beta)}$. This will be proved in Section 10.

Finally, we consider the manner in which f can vary as the orbit changes. By using the fact that dependence on the orbit amounts to dependence on those invariants I_α ($\alpha = 1, 2, \ldots, A$) which determine the orbit, in Section 11 we will complete the proof that f can be expressed in the canonical form (7.4).

8. Continuity Along an Orbit; Polynomial Approximation

Since $T(\lambda)$ is continuous with respect to λ, the right-hand side of (7.1) is continuous in λ, and therefore so is $f[S(\lambda)\psi]$. A stronger result is needed, however. We show that along a given orbit, $f(\psi)$ is a continuous function of ψ. That is, if a sequence of points $\{\psi^k\}$ ($k = 1, 2, \ldots$) lies on the orbit and approaches a limit ψ^0, then the sequence $\{f(\psi^k)\}$ ($k = 1, 2, \ldots$) approaches the limit $f(\psi^0)$.

Because each orbit is compact, the limit ψ^0 must lie on the same orbit as the points $\{\psi^k\}$. Then, for each point ψ^k there is at least one value λ_k such that

$$\psi^k = S(\lambda_k)\,\psi^0. \tag{8.1}$$

Convergence to the point ψ^0 can be expressed as

$$S(\lambda_k)\psi^0 \to \psi^0. \tag{8.2}$$

Since there may be more than one transformation which takes ψ^0 into itself, $S(\lambda_k)$ need not approach the identity, and the sequence $\{\lambda_k\}$ need not be convergent. However, the sequence $\{\lambda_k\}$ must have one or more limit points since the values λ_k belong to a compact set. Let λ_0 be one such limit point, and let $\{\lambda'_k\}$ be a subsequence converging to λ_0. Then by using (7.1) twice, and making use of the continuity of $T(\lambda)$, we find that

$$f[S(\lambda'_k)\psi^0] = T(\lambda'_k)\,f(\psi^0) \to T(\lambda_0)\,f(\psi^0) = f[S(\lambda_0)\psi^0]. \tag{8.3}$$

Since $S(\lambda)$ is continuous, we know that

$$S(\lambda'_k)\psi^0 \to S(\lambda_0)\psi^0. \tag{8.4}$$

Comparison of this result with (8.2) shows that

$$S(\lambda_0)\psi^0 = \psi^0. \tag{8.5}$$

By using this result in (8.3) we obtain

$$\boldsymbol{f}[\boldsymbol{S}(\lambda_k')\boldsymbol{\psi}^0]\to\boldsymbol{f}(\boldsymbol{\psi}^0). \tag{8.6}$$

Since this result is independent of the particular limit point λ_0 chosen, then

$$\boldsymbol{f}[\boldsymbol{S}(\lambda_k)\boldsymbol{\psi}^0]\to\boldsymbol{f}(\boldsymbol{\psi}^0). \tag{8.7}$$

The statement that (8.2) implies (8.7) is the required statement of continuity.

Since the orbit of $\boldsymbol{\psi}^0$ is compact and $\boldsymbol{f}(\boldsymbol{\psi})$ is continuous along the orbit, it follows from the Stone-Weierstrass theorem that $\boldsymbol{f}$ can be approximated arbitrarily well along this orbit by a polynomial in $\boldsymbol{\psi}$. That is, given $\boldsymbol{\psi}^0$ and $\varepsilon>0$, there is a polynomial in $\boldsymbol{\psi}$, denoted by $\boldsymbol{p}(\boldsymbol{\psi};\boldsymbol{\psi}^0,\varepsilon)$, such that for all λ, each component p_i $(i=1,2,\ldots,m)$ satisfies

$$|f_i[\boldsymbol{S}(\lambda)\boldsymbol{\psi}^0]-p_i[\boldsymbol{S}(\lambda)\boldsymbol{\psi}^0;\boldsymbol{\psi}^0,\varepsilon]|<\varepsilon. \tag{8.8}$$

The argument $\boldsymbol{\psi}^0$ is included in $\boldsymbol{p}(\boldsymbol{\psi};\boldsymbol{\psi}^0,\varepsilon)$ as a reminder that the form of $\boldsymbol{p}$ depends upon which orbit is considered. This polynomial will not necessarily be close to $\boldsymbol{f}$ on any other orbit. In order to approximate $\boldsymbol{f}$ on an orbit which is distinct from that of $\boldsymbol{\psi}^0$, a different polynomial must be used. With this clearly in mind, we will abbreviate $\boldsymbol{p}(\boldsymbol{\psi};\boldsymbol{\psi}^0,\varepsilon)$ as $\boldsymbol{p}(\boldsymbol{\psi};\varepsilon)$ in order to simplify the notation.

9. Form-Invariant Polynomial Approximation

We now construct a polynomial in $\boldsymbol{\psi}$, denoted by $\boldsymbol{p}^*(\boldsymbol{\psi};\varepsilon)$, which is form-invariant, *i.e.*

$$\boldsymbol{p}^*[\boldsymbol{S}(\lambda)\boldsymbol{\psi};\varepsilon]=\boldsymbol{T}(\lambda)\boldsymbol{p}^*(\boldsymbol{\psi};\varepsilon), \tag{9.1}$$

and which furnishes a close approximation to $\boldsymbol{f}$ over the orbit of $\boldsymbol{\psi}^0$. That is, for all λ and $i=1,2,\ldots,m$, the component p_i^* satisfies

$$|f_i[\boldsymbol{S}(\lambda)\boldsymbol{\psi}^0]-p_i^*[\boldsymbol{S}(\lambda)\boldsymbol{\psi}^0;\varepsilon]|<\varepsilon. \tag{9.2}$$

The polynomial $\boldsymbol{p}^*$ is defined in terms of the polynomial $\boldsymbol{p}$ in (8.8) by the Hurwitz integral

$$\boldsymbol{p}^*(\boldsymbol{\psi};\varepsilon)=\int \boldsymbol{T}^T(\lambda)\boldsymbol{p}[\boldsymbol{S}(\lambda)\boldsymbol{\psi};\varepsilon/m^2]\,d\lambda, \tag{9.3}$$

where $\boldsymbol{T}^T(\lambda)$ is the transpose of $\boldsymbol{T}(\lambda)$. We shall verify that if $\boldsymbol{p}^*$ is defined in this way, then (9.1) and (9.2) are satisfied.

First consider (9.1). From (9.3) we obtain

$$\begin{aligned}\boldsymbol{p}^*[\boldsymbol{S}(\lambda)\boldsymbol{\psi};\varepsilon]&=\int \boldsymbol{T}^T(\lambda')\,\boldsymbol{p}[\boldsymbol{S}(\lambda')\,\boldsymbol{S}(\lambda)\boldsymbol{\psi};\varepsilon/m^2]\,d\lambda'\\&=\boldsymbol{T}(\lambda)\int[\boldsymbol{T}(\lambda')\,\boldsymbol{T}(\lambda)]^T\boldsymbol{p}[\boldsymbol{S}(\lambda')\,\boldsymbol{S}(\lambda)\boldsymbol{\psi};\varepsilon/m^2]\,d\lambda',\end{aligned} \tag{9.4}$$

where we have used the orthogonality condition (2.4) for $\boldsymbol{T}(\lambda)$. From the defining property of Hurwitz integration, it follows that the integral in the final member of (9.4) is the same as the integral in (9.3). Thus, by using (9.3) in (9.4), we obtain (9.1) as desired.

To prove that the inequality (9.2) is satisfied, we first consider the difference between $\boldsymbol{f}$ and $\boldsymbol{p}^*$ at the point $\boldsymbol{\psi}=\boldsymbol{\psi}^0$. We note that

$$\boldsymbol{f}(\boldsymbol{\psi}^0)=\int \boldsymbol{f}(\boldsymbol{\psi}^0)\,d\lambda=\int \boldsymbol{T}^T(\lambda)\boldsymbol{f}[\boldsymbol{S}(\lambda)\boldsymbol{\psi}^0]\,d\lambda. \tag{9.5}$$

This follows from (2.4), (2.7), and the form-invariance of $\boldsymbol{f}$. Then from (9.3) and (9.5) we obtain

$$\boldsymbol{f}(\boldsymbol{\psi}^0) - \boldsymbol{p}^*(\boldsymbol{\psi}^0; \varepsilon) = \int \boldsymbol{T}^T(\lambda) \{\boldsymbol{f}[\boldsymbol{S}(\lambda)\boldsymbol{\psi}^0] - \boldsymbol{p}[\boldsymbol{S}(\lambda)\boldsymbol{\psi}^0; \varepsilon/m^2]\} \, d\lambda. \tag{9.6}$$

By using the inequality (8.8) with ε replaced by ε/m^2, and recalling that $|T_{ij}(\lambda)| \leqq 1$, we obtain bounds on the integrand in (9.6). It then follows that

$$\begin{aligned} &|f_i(\boldsymbol{\psi}^0) - p_i^*(\boldsymbol{\psi}^0; \varepsilon)| \\ &\quad \leqq \sum_{j=1}^{m} \int |T_{ji}(\lambda)| \, |f_j[\boldsymbol{S}(\lambda)\boldsymbol{\psi}^0] - p_j[\boldsymbol{S}(\lambda)\boldsymbol{\psi}^0; \varepsilon/m^2]| \, d\lambda < \varepsilon/m. \end{aligned} \tag{9.7}$$

Hence, (9.2) is satisfied at the point $\boldsymbol{\psi}^0$. Since $\boldsymbol{\psi}^0$ is a representative point on the orbit considered, it follows that (9.2) is satisfied at every point on the orbit. However, we shall also show this by direct manipulation. From the form-invariance of $\boldsymbol{f}$ and $\boldsymbol{p}^*$, we obtain

$$\boldsymbol{f}[\boldsymbol{S}(\lambda)\boldsymbol{\psi}^0] - \boldsymbol{p}^*[\boldsymbol{S}(\lambda)\boldsymbol{\psi}^0; \varepsilon] = \boldsymbol{T}^T(\lambda)\,[\boldsymbol{f}(\boldsymbol{\psi}^0) - \boldsymbol{p}^*(\boldsymbol{\psi}^0; \varepsilon)], \tag{9.8}$$

whence

$$|f_i[\boldsymbol{S}(\lambda)\boldsymbol{\psi}^0] - p_i^*[\boldsymbol{S}(\lambda)\boldsymbol{\psi}^0; \varepsilon]| \leqq \sum_{j=1}^{m} |T_{ji}(\lambda)| \, |f_j(\boldsymbol{\psi}^0) - p_j^*(\boldsymbol{\psi}^0; \varepsilon)|. \tag{9.9}$$

Then, by using (9.7) and the bound $|T_{ij}(\lambda)| \leqq 1$ we obtain (9.2), as desired.

10. Canonical Form on a Given Orbit

We have shown that there is a form-invariant polynomial $\boldsymbol{p}^*$ which is arbitrarily close to $\boldsymbol{f}$ everywhere on the orbit of $\boldsymbol{\psi}^0$. Because it is form-invariant, this polynomial $\boldsymbol{p}^*$ can be expressed in the polynomial canonical form (7.3). The forms of the coefficients p_β in the canonical form (7.3) will depend on the orbit considered and on ε, since $\boldsymbol{p}^*$ depends parametrically on these variables. We are interested in $\boldsymbol{p}^*$ only on the orbit of $\boldsymbol{\psi}^0$, where it furnishes a good approximation to $\boldsymbol{f}$. Because the coefficients p_β are invariants, they are constant over the orbit of $\boldsymbol{\psi}^0$. Thus, on the orbit of $\boldsymbol{\psi}^0$, $\boldsymbol{p}^*$ is of the form

$$\boldsymbol{p}^*[\boldsymbol{S}(\lambda)\boldsymbol{\psi}^0; \varepsilon] = \sum_{\beta=1}^{B} p_\beta(\varepsilon)\, \boldsymbol{f}^{(\beta)}[\boldsymbol{S}(\lambda)\boldsymbol{\psi}^0], \tag{10.1}$$

where $p_\beta(\varepsilon)$ is a constant depending on ε.

It then follows from (9.2) and (10.1) that the components of $\boldsymbol{f}$ satisfy the inequalities

$$|f_i[\boldsymbol{S}(\lambda)\boldsymbol{\psi}^0] - \sum_{\beta=1}^{B} p_\beta(\varepsilon)\, f_i^{(\beta)}[\boldsymbol{S}(\lambda)\boldsymbol{\psi}^0]| < \varepsilon \qquad (i = 1, 2, \ldots, m). \tag{10.2}$$

The functions $\boldsymbol{f}^{(\beta)}[\boldsymbol{S}(\lambda)\boldsymbol{\psi}^0]$, regarded as functions of λ, span a linear manifold of finite dimension ($\leqq B$). The functions $\boldsymbol{p}^*$ given by (10.1) lie in this manifold, for each value of ε. If we consider a sequence of these functions $\boldsymbol{p}^*$ with ε approaching zero, (10.2) states that this sequence converges to $\boldsymbol{f}$ in a certain norm. The manifold can be shown to be complete with respect to this norm, essentially because it is finite-dimensional. Hence, the limit $\boldsymbol{f}$ of the sequence of approximations $\boldsymbol{p}^*$ also belongs to the manifold. That is, on the orbit of $\boldsymbol{\psi}^0$, $\boldsymbol{f}$ can be expressed as a linear combination of the functions $\boldsymbol{f}^{(\beta)}$, with constant coefficients p_β:

$$\boldsymbol{f}[\boldsymbol{S}(\lambda)\boldsymbol{\psi}^0] = \sum_{\beta=1}^{B} p_\beta \boldsymbol{f}^{(\beta)}[\boldsymbol{S}(\lambda)\boldsymbol{\psi}^0]. \tag{10.3}$$

As a side issue, it is worth noting that if the functions $\boldsymbol{f}^{(\beta)}$ are not linearly independent on the orbit of $\boldsymbol{\psi}^0$, then the coefficients p_β in (10.3) are not unique.

11. Complete Canonical Form

Since $\boldsymbol{\psi}^0$ can be chosen arbitrarily, a relation of the form (10.3) is valid, whatever orbit may be considered. For tensors $\boldsymbol{\psi}$ which lie on a given orbit, $\boldsymbol{f}$ takes the form

$$\boldsymbol{f}(\boldsymbol{\psi}) = \sum_{\beta=1}^{B} p_\beta \boldsymbol{f}^{(\beta)}(\boldsymbol{\psi}). \tag{11.1}$$

The coefficients p_β depend upon which orbit is considered, but each coefficient is constant over any given orbit. Thus, these coefficients are invariants.

Since the coefficients are invariants, they can be expressed as functions of the elements of a functional basis. We have shown earlier that an integrity basis I_α $(\alpha=1, 2, \ldots, A)$ is a functional basis. Hence, each coefficient p_β can be expressed as a function F_β of these invariants:

$$p_\beta = F_\beta[I_1(\boldsymbol{\psi}), \ldots, I_A(\boldsymbol{\psi})] \qquad (\beta=1, 2, \ldots, B). \tag{11.2}$$

By using (11.2) in (11.1), we obtain the canonical representation (7.4), and the proof is complete.

Part II. Form-Invariant Response Functionals

1. Materials with Memory

Results analogous to those established in Part I can be obtained for constitutive equations involving functionals. Such constitutive equations arise when it is assumed that the value $\boldsymbol{\chi}$ at time t of some tensor depends upon the values of a number of other tensors $\boldsymbol{\psi}^{(\nu)}$ $(\nu=1, 2, \ldots, N)$ at times $t-\tau$ up to and including t. We will denote the value of $\boldsymbol{\psi}^{(\nu)}$ at time $t-\tau$ by $\boldsymbol{\psi}^{(\nu)}(\tau)$. This tensor-valued function of time will be called a *history*. Then, the assumption that $\boldsymbol{\chi}$ depends on the histories $\boldsymbol{\psi}^{(\nu)}(\tau)$ $(\nu=1, 2, \ldots, N)$ can be written as

$$\boldsymbol{\chi} = \boldsymbol{\mathfrak{F}}\{\boldsymbol{\psi}^{(1)}(\tau), \ldots, \boldsymbol{\psi}^{(N)}(\tau)\}, \tag{1.1}$$

where $\boldsymbol{\mathfrak{F}}$ is a tensor-valued functional of the indicated functions. That is, the components of $\boldsymbol{\chi}$ in a Cartesian system x are assumed to be functionals of the components of the histories $\boldsymbol{\psi}^{(\nu)}(\tau)$ $(\nu=1, 2, \ldots, N)$ in that same system:

$$\chi_{i_1\ldots i_r} = \mathfrak{F}_{i_1\ldots i_r}\{\psi^{(\nu)}_{p_1\ldots p_{r(\nu)}}(\tau)\}. \tag{1.2}$$

The range of τ is not important for our purpose, but for concreteness one may suppose that τ ranges over the interval $0 \leqq \tau < \infty$.

If the material described by the constitutive equation is symmetric under an orthogonal transformation with coordinate transformation matrix $\boldsymbol{R} = \|R_{ij}\|$, then the functionals $\mathfrak{F}_{i_1\ldots i_r}$ must satisfy the following relations:

$$\mathfrak{F}_{i_1\ldots i_r}\{R_{p_1q_1}\cdots R_{p_{r(\nu)}q_{r(\nu)}}\psi^{(\nu)}_{q_1\ldots q_{r(\nu)}}(\tau)\} = R_{i_1j_1}\cdots R_{i_rj_r}\mathfrak{F}_{j_1\ldots j_r}\{\psi^{(\nu)}_{p_1\ldots p_{r(\nu)}}(\tau)\}. \tag{1.3}$$

If (1.3) is satisfied identically in the histories $\psi^{(\nu)}(\tau)$ for each matrix $\boldsymbol{R}$ belonging to the group of symmetries of the material, we shall say that the tensor-valued functional $\mathfrak{F}$ is *form-invariant* under the group considered.

Our object is to derive canonical forms for form-invariant functionals, using only those properties of the functionals which are implied by (1.3). We shall not make any extraneous assumptions, either about the class of functions which are admissible as histories, or about the nature of the functionals. For our purpose, the functionals in (1.1) are to be regarded as any rules of correspondence which specify χ when the histories $\psi^{(\nu)}(\tau)$ are given.

The special case in which χ depends only on the values of the histories at the instant $\tau=0$ is treated in Part I of this paper. Another special case arises when χ is assumed to depend only on the values of the histories and their first n time-derivatives at the instant $\tau=0$. It should be noticed that in this case, only those functions which are sufficiently differentiable can be admitted as histories. However, for our present purpose, such restrictions on the class of admissible functions are irrelevant.

Relations which are special cases of (1.1) appear in a variety of forms in continuum physics. The stress-relaxation integral constitutive equation of linear viscoelasticity theory is an example in which χ is the stress, a second-rank tensor, and there is one history $\psi^{(1)}(\tau)$, the strain history, which is also of second rank. In this case the functional $\mathfrak{F}$ is defined in terms of integrals.

A second example in which χ is the stress and $\psi^{(1)}(\tau)$ is the strain history is given by the constitutive equations of plasticity theory. Here the functional $\mathfrak{F}$ is defined by a list of rules from which the stress can be determined when the strain history is known. The fact that no integral representation of $\mathfrak{F}$ is available in this case is not relevant to our purpose, since in deriving its canonical representation, we shall make use of no properties of $\mathfrak{F}$ other than its form-invariance.

The constitutive equation for magnetic hysteresis is a relation of the form (1.1) in which χ is the magnetization vector and $\psi^{(1)}(\tau)$ is the history of the field strength. In this case, with restriction to one-dimensional histories, the functional $\mathfrak{F}$ is defined by a hysteresis diagram. In Section 2 we shall show how form-invariance restricts the forms of the functionals in such relations between two vectors.

NOLL'S [*20*] definition of a simple fluid leads to a relation of the form (1.1) in which χ is the stress at time t and $\psi^{(1)}(\tau)$ is the strain at time $t-\tau$, measured relative to the configuration at time t. The definition of a simple fluid also specifies that the material is isotropic, *i.e.* that (1.3) must be satisfied for every orthogonal matrix $\boldsymbol{R}$. Nothing further is assumed about the dependence of the stress on the strain history. In Section 3, we shall show how isotropy limits the forms of the functionals in such a relation.

The first major work on the implications of form-invariance with respect to general non-linear functionals was GREEN & RIVLIN'S paper [*4*] on stress-deformation relations for materials with memory. They obtained a relation of the form (1.1) in which χ is the Piola-Kirchhoff stress and $\psi^{(1)}(\tau)$ is the strain history measured relative to the initial configuration of the body. By making assumptions about continuity of the histories and continuity of the functionals, they were able to obtain integral representations of the functionals. The form-invariance requirements could then be applied to the integrands of these integrals, and the problem was thus reduced to one involving functions rather than functionals. This method of procedure is illustrated in a less complicated case by RIVLIN'S work [*21*] on the dependence of one vector on the history of another; continuity assumptions are used to obtain integral representations of the functionals, and the form-invariance problem is reduced to the problem of finding canonical forms for the integrands. Although the results which we shall obtain are motivated by those of GREEN & RIVLIN, we avoid any assumption about continuity of the histories or of the functionals. We shall be concerned only with the restrictions which are imposed by material symmetry.

The explicit form of a canonical representation depends on the rank of the tensor-valued functional $\mathfrak{F}$, the number and ranks of the histories $\boldsymbol{\psi}^{(\nu)}(\tau)$, and the particular material symmetry group. However, the general method of construction and the general features of the representation can be expressed in terms which are independent of these details, just as in the case of functions, considered in Part I. The method is stated in Section 8 in the form of a general theorem covering all cases.

The main body of the paper is intentionally general and thus necessarily somewhat abstract. Furthermore, the type of result we obtain is not familiar from examples in the literature. For these reasons, in Sections 2 and 3 we will discuss a number of special cases in order to illustrate the method and to show the type of result which is obtained.

In Section 4 we shall introduce a general notation scheme similar to that used in Part I. Some relevant results pertaining to finite sets of tensors are summarized in Sections 5 and 6. In particular, in Section 5 we shall discuss the notion of a table of typical basic invariants, which plays an important role in the canonical representation. In Section 6, some results from Part I are rewritten in a form more suitable for our present purpose.

The canonical representation of scalar invariant functionals is derived in Section 7. It is shown that every such invariant functional can be represented as a functional of certain time-dependent invariant functions which are obtained directly from a table of typical basic invariants.

The main theorem on canonical representations is stated in Section 8. The proof is outlined in Section 9, and the remainder of the paper is devoted to carrying out the details of this proof.

2. Example. Dependence of One Vector on the History of Another

As a first example, let us suppose that the value $\boldsymbol{\chi}$ of some vector is dependent on the history $\boldsymbol{\psi}(\tau)$ of a second vector. Thus, in terms of the components of these vectors in a Cartesian coordinate system x, the relation we consider is the following special case of (1.2):

$$\chi_i = \mathfrak{F}_i\{\psi_p(\tau)\}. \tag{2.1}$$

Here $\mathfrak{F}_i$ is a functional of the three argument functions $\psi_1(\tau)$, $\psi_2(\tau)$ and $\psi_3(\tau)$. RIVLIN [*21*] has investigated the restrictions imposed on such relations by form-invariance under the orthogonal group, for cases in which the functionals can be represented in terms of integrals.

Let $\bar{x}$ be a second Cartesian system, related to the first by a transformation with matrix $\boldsymbol{R}$:

$$\bar{x}_i = R_{ij} x_j, \qquad R_{ik} R_{jk} = R_{ki} R_{kj} = \delta_{ij}. \tag{2.2}$$

If this transformation is a symmetry of the material, then the relation between the components $\bar{\chi}_i$ and $\bar{\psi}_p(\tau)$ of the vectors $\boldsymbol{\chi}$ and $\boldsymbol{\psi}(\tau)$, respectively, in the system $\bar{x}$ must be the same as the relation (2.1) between components in the system x. From this it can be shown that

$$\mathfrak{F}_i\{\bar{\psi}_p(\tau)\} = R_{ij} \mathfrak{F}_j\{\psi_p(\tau)\}, \tag{2.3}$$

where

$$\bar{\psi}_p(\tau) = R_{pq}\, \psi_q(\tau). \tag{2.4}$$

Equation (2.3), together with (2.4), is the special form which (1.3) takes in the present example.

Let $\boldsymbol{\varphi}$ be an arbitrary vector, whose components φ_i and $\bar{\varphi}_i$ in the systems x and $\bar{x}$, respectively, are related by

$$\bar{\varphi}_i = R_{ij}\, \varphi_j. \tag{2.5}$$

Then from (2.3) and (2.5), with the orthogonality conditions (2.2), we obtain

$$\bar{\varphi}_i \mathfrak{F}_i\{\bar{\psi}_p(\tau)\} = \varphi_i \mathfrak{F}_i\{\psi_p(\tau)\} = \mathfrak{F} \quad \text{(say)}. \tag{2.6}$$

Thus, $\mathfrak{F}$ is a scalar invariant of the vector $\boldsymbol{\varphi}$ and the history $\boldsymbol{\psi}(\tau)$ under each transformation $\boldsymbol{R}$ belonging to the group of symmetries of the material.

We shall regard the history $\boldsymbol{\psi}(\tau)$ as an infinite set of vectors, numbered with a parameter τ. From this point of view, $\mathfrak{F}$ is an invariant of an infinite number of vectors. If $\mathfrak{F}$ were dependent on only a finite number of vectors we could deduce its form, and thus the form of $\mathfrak{F}_i$, by the method of Part I. We would first find the elements I_α of an integrity basis for invariants of the vectors $\boldsymbol{\psi}(\tau)$. We would then find the elements J_β, linear in $\boldsymbol{\varphi}$, of an integrity basis for $\boldsymbol{\varphi}$ and the vectors $\boldsymbol{\psi}(\tau)$. Since an integrity basis is defined only for a finite number of tensors, we shall instead find a set of basic invariants which would be an integrity basis if τ could take only a finite number of values. This will be the first step in the procedure for constructing the canonical form for form-invariant functionals. We shall illustrate this step and the remaining steps of the procedure with a number of examples of particular symmetry groups.

2a. Holohedral Isotropic Materials. Let us first consider the case in which the functional $\mathfrak{F}$ defined by (2.6) is invariant under the full orthogonal group. An integrity basis for orthogonal invariants of a set of vectors is composed of inner products of these vectors. That is, there is only one type of basic invariant,

$$\boldsymbol{u} \cdot \boldsymbol{v} = u_i\, v_i. \tag{2.7}$$

An integrity basis for an arbitrary number of vectors can be obtained by substituting these vectors in all possible combinations, repeats allowed, for $\boldsymbol{u}$ and $\boldsymbol{v}$ in the typical basic invariant.

If we consider the set of vectors $\boldsymbol{\psi}(\tau)$, the set of basic invariants which are obtained in this way are given by

$$I[\boldsymbol{\psi}(\xi_1), \boldsymbol{\psi}(\xi_2)] = \boldsymbol{\psi}(\xi_1) \cdot \boldsymbol{\psi}(\xi_2), \tag{2.8}$$

where ξ_1 and ξ_2 range over all possible values of τ. The basic invariants of $\boldsymbol{\varphi}$ and $\boldsymbol{\psi}(\tau)$, linear in $\boldsymbol{\varphi}$, which are obtained from (2.7) are

$$J[\boldsymbol{\varphi}, \boldsymbol{\psi}(\xi)] = \boldsymbol{\varphi} \cdot \boldsymbol{\psi}(\xi), \tag{2.9}$$

where ξ ranges over all possible values of τ.

If there were only a finite number of the vectors $\boldsymbol{\psi}(\tau)$, we could now conclude that since $\mathfrak{F}$ in (2.6) is linear in $\boldsymbol{\varphi}$, it must be expressible as a linear combination of the invariants J, with coefficients which are functions of the invariants I. However, J and I take infinitely many values as their time arguments vary. Accordingly, we regard J as a function of ξ, and take $\mathfrak{F}$ to be a linear functional of the function J. We also regard the invariant I as a function of ξ_1 and ξ_2. Then, in analogy with coefficients depending on a finite number of invariants I, we suppose that the form of the linear functional depends on the function I. Thus, we arrive at the representation

$$\mathfrak{F} = \mathscr{L}\{J; I\}, \tag{2.10}$$

where $\mathscr{L}$ is a functional of J and I, linear with respect to J.

By using the definition (2.6), and making use of the linearity of $\mathscr{L}$ with respect to its first argument, from (2.10) we obtain

$$\varphi_i \mathfrak{F}_i\{\psi_p(\tau)\} = \mathscr{L}\{\varphi_i \psi_i(\xi); I\} = \varphi_i \mathscr{L}\{\psi_i(\xi); I\}. \tag{2.11}$$

Since $\boldsymbol{\varphi}$ is arbitrary, (2.11) yields

$$\mathfrak{F}_i\{\psi_p(\tau)\} = \mathscr{L}\{\psi_i(\xi); \psi_j(\xi_1)\, \psi_j(\xi_2)\}, \tag{2.12}$$

where we have also used (2.8). This is the canonical representation for the functionals in this example.

The constitutive equation (2.1) then takes the form

$$\chi_i = \mathscr{L}\{\psi_i(\xi); \psi_j(\xi_1)\, \psi_j(\xi_2)\}, \tag{2.13}$$

or, in vector notation,

$$\boldsymbol{\chi} = \mathscr{L}\{\boldsymbol{\psi}(\xi); \boldsymbol{\psi}(\xi_1) \cdot \boldsymbol{\psi}(\xi_2)\}. \tag{2.14}$$

It should be emphasized that $\mathscr{L}$ in (2.13) is the same functional for each choice of the index i.

The general theorem, to be stated in Section 8, justifies the assertion that if the vector-valued functional $\boldsymbol{\mathfrak{F}}$ is form-invariant under the orthogonal group, then the functionals $\mathfrak{F}_i$ must be of the form (2.12). The converse, that all functionals of this form are form-invariant under the orthogonal group, can be verified immediately. For, if $\boldsymbol{\mathfrak{F}}$ is of the form (2.12), then with (2.4) we obtain

$$\mathfrak{F}_i\{\overline{\psi}_p(\tau)\} = \mathfrak{F}_i\{R_{pq}\psi_q(\tau)\} = \mathscr{L}\{R_{ij}\psi_j(\xi); R_{kl}\psi_l(\xi_1)\, R_{km}\psi_m(\xi_2)\}. \tag{2.15}$$

By using the linearity of $\mathscr{L}$ with respect to its first argument and using the orthogonality conditions (2.2) to simplify the second argument, we obtain

$$\begin{aligned}\mathfrak{F}_i\{\overline{\psi}_p(\tau)\} &= R_{ij}\mathscr{L}\{\psi_j(\xi);\, \psi_k(\xi_1)\,\psi_k(\xi_2)\}\\ &= R_{ij}\mathfrak{F}_j\{\psi_p(\tau)\}.\end{aligned} \tag{2.16}$$

Here we have used (2.12). Thus, if $\mathfrak{F}_i$ has the form (2.12), the form-invariance condition (2.3) is satisfied for every history and every orthogonal matrix $\boldsymbol{R}$.

2b. Hemihedral Isotropic Materials. Let us now consider materials which are symmetric under the three-dimensional rotation group. In this case, (2.3) must be satisfied for every proper orthogonal matrix $\boldsymbol{R}$. The basic rotational invariants of vectors are of two types, inner products and scalar triple products. That is, an integrity basis for invariants of an arbitrary number of vectors can be obtained from the typical basic invariants

$$\boldsymbol{u}\cdot\boldsymbol{v} = u_i\, v_i \quad \text{and} \quad \boldsymbol{u}\cdot\boldsymbol{v}\times\boldsymbol{w} = e_{ijk} u_i v_j w_k, \tag{2.17}$$

by substituting the given vectors for $\boldsymbol{u}$, $\boldsymbol{v}$ and $\boldsymbol{w}$ in all possible combinations.

The invariants of the vectors $\boldsymbol{\psi}(\tau)$, as obtained from (2.17) are

$$\begin{aligned}&I_1[\boldsymbol{\psi}(\xi_1), \boldsymbol{\psi}(\xi_2)] = \boldsymbol{\psi}(\xi_1)\cdot\boldsymbol{\psi}(\xi_2)\\ \text{and}\\ &I_2[\boldsymbol{\psi}(\xi_1), \boldsymbol{\psi}(\xi_2), \boldsymbol{\psi}(\xi_3)] = \boldsymbol{\psi}(\xi_1)\cdot\boldsymbol{\psi}(\xi_2)\times\boldsymbol{\psi}(\xi_3),\end{aligned} \tag{2.18}$$

where ξ_1, ξ_2 and ξ_3 vary over all values in the range of τ. The invariants of $\boldsymbol{\varphi}$ and $\boldsymbol{\psi}(\tau)$, linear in $\boldsymbol{\varphi}$, which are obtained from (2.17) are

$$J_1[\boldsymbol{\varphi}, \boldsymbol{\psi}(\xi)] = \boldsymbol{\varphi}\cdot\boldsymbol{\psi}(\xi) \quad \text{and} \quad J_2[\boldsymbol{\varphi}, \boldsymbol{\psi}(\xi_1), \boldsymbol{\psi}(\xi_2)] = \boldsymbol{\varphi}\cdot\boldsymbol{\psi}(\xi_1)\times\boldsymbol{\psi}(\xi_2), \tag{2.19}$$

aside from redundant elements. The invariants (2.19) can be written as

$$J_\beta = \varphi_i\, f_i^{(\beta)} \qquad (\beta = 1, 2), \tag{2.20}$$

where

$$f_i^{(1)} = \psi_i(\xi) \quad \text{and} \quad f_i^{(2)} = e_{ijk}\,\psi_j(\xi_1)\,\psi_k(\xi_2). \tag{2.21}$$

Regarding I_α $(\alpha = 1, 2)$ and J_β $(\beta = 1, 2)$, now, as functions of the various time variables, the functional $\mathfrak{F}$ must be of the form

$$\mathfrak{F} = \mathscr{L}^{(1)}\{J_1;\, I_1, I_2\} + \mathscr{L}^{(2)}\{J_2;\, I_1, I_2\}. \tag{2.22}$$

Here $\mathscr{L}^{(\beta)}$ is a functional of J_β and the functions I_1 and I_2, linear with respect to J_β.

By using (2.22) together with (2.20) in (2.6) and taking into account the linearity of $\mathscr{L}^{(\beta)}$ with respect to its first argument, we obtain

$$\varphi_i\mathfrak{F}_i\{\psi_p(\tau)\} = \varphi_i\mathscr{L}^{(1)}\{f_i^{(1)};\, I_1, I_2\} + \varphi_i\mathscr{L}^{(2)}\{f_i^{(2)};\, I_1, I_2\}. \tag{2.23}$$

Since $\boldsymbol{\varphi}$ is arbitrary, (2.23) yields an expression for $\mathfrak{F}_i$.

Thus, in the case of hemihedral isotropic materials, the constitutive equation (2.1) must be expressible in the form

$$\chi_i = \mathscr{L}^{(1)}\{\psi_i(\xi);\, I_1, I_2\} + \mathscr{L}^{(2)}\{e_{ijk}\,\psi_j(\xi_1)\,\psi_k(\xi_2);\, I_1, I_2\} \tag{2.24}$$

or, in vector notation,

$$\boldsymbol{\chi}=\mathscr{L}^{(1)}\{\boldsymbol{\psi}(\xi)\,;I_1,I_2\}+\mathscr{L}^{(2)}\{\boldsymbol{\psi}(\xi_1)\times\boldsymbol{\psi}(\xi_2)\,;I_1,I_2\}, \tag{2.25}$$

where the functions I_1 and I_2 are defined in (2.18).

It is easy to verify that the functionals in (2.24) are form-invariant under the proper orthogonal group. The converse, that the relation (2.1) must reduce to the form (2.24) in the present case, follows from the general representation theorem to be proved later.

2c. Orthotropic Materials. For materials with three mutually orthogonal planes of reflectional symmetry, if the coordinate planes of the system x are taken to coincide with the symmetry planes, then a table of typical basic invariants for an arbitrary number of vectors consists of the invariants

$$u_1 v_1,\ u_2 v_2 \text{ and } u_3 v_3. \tag{2.26}$$

From these we obtain the following invariants of the vectors $\boldsymbol{\psi}(\tau)$:

$$I_1=\psi_1(\xi_1)\,\psi_1(\xi_2),\qquad I_2=\psi_2(\xi_1)\,\psi_2(\xi_2),\qquad I_3=\psi_3(\xi_1)\,\psi_3(\xi_2). \tag{2.27}$$

We also obtain invariants of $\boldsymbol{\varphi}$ and $\boldsymbol{\psi}(\tau)$, linear in $\boldsymbol{\varphi}$, of the form

$$J_\beta=\varphi_i\, f_i^{(\beta)}\qquad(\beta=1,2,3), \tag{2.28}$$

where

$$f_i^{(\beta)}=\delta_{\beta i}\,\psi_\beta(\xi)\quad(\text{no sum over } \beta). \tag{2.29}$$

Then, by following a procedure analogous to that used in the preceding examples, we find that the constitutive equation (2.1) must be of the form

$$\chi_i=\sum_{\beta=1}^{3}\mathscr{L}^{(\beta)}\{f_i^{(\beta)};I_1,I_2,I_3\}, \tag{2.30}$$

where $\mathscr{L}^{(\beta)}$ is a functional of the indicated functions, linear with respect to $f_i^{(\beta)}$. By using (2.29) and taking into account the linearity of $\mathscr{L}^{(\beta)}$ with respect to its first argument, from (2.30) we obtain

$$\chi_i=\mathscr{L}^{(i)}\{\psi_i(\xi)\,;I_1,I_2,I_3\}\quad(\text{no sum over } i), \tag{2.31}$$

where the functions I_α $(\alpha=1,2,3)$ are defined by (2.27).

3. Example. Dependence of One Symmetric Second-Rank Tensor on the History of Another

To consider a slightly different type of example, we now suppose that some symmetric second-rank tensor $\boldsymbol{\chi}$ is a functional of the history $\boldsymbol{\psi}(\tau)$ of another symmetric second-rank tensor:

$$\chi_{ij}=\mathfrak{F}_{ij}\{\psi_{pq}(\tau)\}. \tag{3.1}$$

In this case, form-invariance under the orthogonal transformation (2.2) requires that

$$\mathfrak{F}_{ij}\{\overline{\psi}_{pq}(\tau)\}=R_{ik}R_{jl}\,\mathfrak{F}_{kl}\{\psi_{pq}(\tau)\}, \tag{3.2}$$

where

$$\overline{\psi}_{pq}(\tau)=R_{pr}R_{qs}\,\psi_{rs}(\tau). \tag{3.3}$$

Form-invariance of a relation of the type (3.1) under the orthogonal group was investigated by Green & Rivlin [4] for cases in which the functionals $\mathfrak{F}_{ij}$ could be represented in terms of integrals.

Let φ_{ij} be components with respect to the system x of an arbitrary symmetric second-rank tensor $\boldsymbol{\varphi}$. The components of $\boldsymbol{\varphi}$ in the system $\bar{x}$ are then given by

$$\bar{\varphi}_{ij} = R_{ik} R_{jl} \varphi_{kl}. \tag{3.4}$$

From (3.2) and (3.4), with the orthogonality conditions (2.2), we obtain

$$\bar{\varphi}_{ij} \mathfrak{F}_{ij}\{\bar{\psi}_{pq}(\tau)\} = \varphi_{ij} \mathfrak{F}_{ij}\{\psi_{pq}(\tau)\} = \mathfrak{F} \quad \text{(say)}. \tag{3.5}$$

Thus, $\mathfrak{F}$ is an invariant of $\boldsymbol{\varphi}$ and the infinite set of tensors $\boldsymbol{\psi}(\tau)$.

3a. Isotropic Materials. In the case of the orthogonal group (full or proper), it is known from the work of Spencer & Rivlin [14] that a table of typical basic invariants for an arbitrary number of symmetric second-rank tensors is given in terms of six such tensors $\boldsymbol{M}^{(1)}, \ldots, \boldsymbol{M}^{(6)}$ by

$$\operatorname{tr} \boldsymbol{M}^{(1)}, \operatorname{tr} \boldsymbol{M}^{(1)} \boldsymbol{M}^{(2)}, \ldots, \operatorname{tr} \boldsymbol{M}^{(1)} \boldsymbol{M}^{(2)} \boldsymbol{M}^{(3)} \boldsymbol{M}^{(4)} \boldsymbol{M}^{(5)} \boldsymbol{M}^{(6)} \qquad (\operatorname{tr} \boldsymbol{M} = \operatorname{trace} \boldsymbol{M}). \tag{3.6}$$

That is, treating these tensors as matrices, an integrity basis is given by the traces of these matrices and the traces of their matrix products taken two at a time, three at a time, and so on up to six at a time.

From the typical invariants (3.6), we obtain the following invariants of the tensors $\boldsymbol{\psi}(\tau)$:

$$\begin{aligned} I_1 = \operatorname{tr} \boldsymbol{\psi}(\xi), \qquad & I_2 = \operatorname{tr} \boldsymbol{\psi}(\xi_1) \boldsymbol{\psi}(\xi_2), \ldots, \\ & I_6 = \operatorname{tr} \boldsymbol{\psi}(\xi_1) \boldsymbol{\psi}(\xi_2) \boldsymbol{\psi}(\xi_3) \boldsymbol{\psi}(\xi_4) \boldsymbol{\psi}(\xi_5) \boldsymbol{\psi}(\xi_6). \end{aligned} \tag{3.7}$$

Neglecting redundant elements, the invariants of $\boldsymbol{\varphi}$ and $\boldsymbol{\psi}(\tau)$, linear in $\boldsymbol{\varphi}$, obtained from (3.6) are

$$\begin{aligned} J_0 = \operatorname{tr} \boldsymbol{\varphi}, \quad J_1 = \operatorname{tr} \boldsymbol{\varphi} \boldsymbol{\psi}(\xi), \quad & J_2 = \operatorname{tr} \boldsymbol{\varphi} \boldsymbol{\psi}(\xi_1) \boldsymbol{\psi}(\xi_2), \ldots, \\ & J_5 = \operatorname{tr} \boldsymbol{\varphi} \boldsymbol{\psi}(\xi_1) \boldsymbol{\psi}(\xi_2) \boldsymbol{\psi}(\xi_3) \boldsymbol{\psi}(\xi_4) \boldsymbol{\psi}(\xi_5). \end{aligned} \tag{3.8}$$

These invariants J_β can be written as

$$J_\beta = \varphi_{ij} f_{ij}^{(\beta)} \qquad (\beta = 0, 1, 2, \ldots, 5), \tag{3.9}$$

where

$$f_{ij}^{(0)} = \delta_{ij} \quad \text{and} \quad 2 f_{ij}^{(\beta)} = [\boldsymbol{\psi}(\xi_1) \ldots \boldsymbol{\psi}(\xi_\beta) + \boldsymbol{\psi}(\xi_\beta) \ldots \boldsymbol{\psi}(\xi_1)]_{ij} \quad (\beta = 1, 2, \ldots, 5). \tag{3.10}$$

The invariant $\mathfrak{F}$ defined by (3.5), which is linear in $\boldsymbol{\varphi}$, can be expressed in terms of the invariants I_α and J_β in the form

$$\mathfrak{F} = \sum_{\beta=0}^{5} \mathscr{L}^{(\beta)} \{J_\beta; I_1, \ldots, I_6\}, \tag{3.11}$$

where $\mathscr{L}^{(\beta)}$ is a functional of J_β and I_α ($\alpha = 1, 2, \ldots, 6$), linear in J_β. By using (3.11) together with (3.9) in (3.5), taking into account the linearity of $\mathscr{L}^{(\beta)}$ with respect to its first argument, and making use of the arbitrariness of $\boldsymbol{\varphi}$, we obtain expressions for the functionals $\mathfrak{F}_{ij}$. The constitutive equation (3.1) then takes the form

$$\chi_{ij} = \sum_{\beta=0}^{5} \mathscr{L}^{(\beta)}\{f_{ij}^{(\beta)}; I_1, \ldots, I_6\}, \tag{3.12}$$

where the functions $f_{ij}^{(\beta)}$ and I_α are defined by (3.10) and (3.7), respectively.

The term involving $\mathscr{L}^{(0)}$ can be simplified slightly, by making use of the linearity of $\mathscr{L}^{(0)}$ with respect to its first argument δ_{ij}:

$$\mathscr{L}^{(0)}\{\delta_{ij}; I_1, \ldots, I_6\} = \delta_{ij}\mathscr{L}^{(0)}\{1; I_1, \ldots, I_6\}. \tag{3.13}$$

Thus, this term reduces to δ_{ij} times a scalar invariant functional.

3b. Orthotropic Materials. For the group of reflectional symmetries in the coordinate planes, ADKINS [*12*] has shown that a table of typical basic invariants for an arbitrary number of symmetric second-rank tensors is given by

$$M_{11},\ M_{22},\ M_{33},\ M_{23}N_{23},\ M_{31}N_{31},\ M_{12}N_{12},\ M_{23}N_{31}P_{12}. \tag{3.14}$$

From these we obtain the following invariants of the tensors $\boldsymbol{\psi}(\tau)$:

$$\begin{aligned} &I_1 = \psi_{11}(\xi), && I_2 = \psi_{22}(\xi), && I_3 = \psi_{33}(\xi), \\ &I_4 = \psi_{23}(\xi_1)\,\psi_{23}(\xi_2), && I_5 = \psi_{31}(\xi_1)\,\psi_{31}(\xi_2), && I_6 = \psi_{12}(\xi_1)\,\psi_{12}(\xi_2), \\ &I_7 = \psi_{23}(\xi_1)\,\psi_{31}(\xi_2)\,\psi_{12}(\xi_3). \end{aligned} \tag{3.15}$$

The invariants of $\boldsymbol{\varphi}$ and $\boldsymbol{\psi}(\tau)$, linear in $\boldsymbol{\varphi}$, which are obtained from (3.14) are

$$\begin{aligned} &J_1 = \varphi_{11}, && J_2 = \varphi_{22}, && J_3 = \varphi_{33}, \\ &J_4 = \varphi_{23}\psi_{23}(\xi), && J_5 = \varphi_{31}\psi_{31}(\xi), && J_6 = \varphi_{12}\psi_{12}(\xi), \\ &J_7 = \varphi_{23}\psi_{31}(\xi_1)\,\psi_{12}(\xi_2), && J_8 = \varphi_{31}\psi_{12}(\xi_1)\,\psi_{23}(\xi_2), && J_9 = \varphi_{12}\psi_{23}(\xi_1)\,\psi_{31}(\xi_2). \end{aligned} \tag{3.16}$$

The latter invariants can be written as

$$J_\beta = \varphi_{ij}\, f_{ij}^{(\beta)} \qquad (\beta = 1, 2, \ldots, 9), \tag{3.17}$$

where

$$f_{ij}^{(\beta)} = \delta_{\beta i}\,\delta_{\beta j} \qquad (\beta = 1, 2, 3;\ \text{no sum over } \beta), \tag{3.18}$$

$$2f_{ij}^{(4)} = (\delta_{2i}\,\delta_{3j} + \delta_{3i}\,\delta_{2j})\,\psi_{23}(\xi), \quad \text{etc.}, \tag{3.19}$$

and

$$2f_{ij}^{(7)} = (\delta_{2i}\,\delta_{3j} + \delta_{3i}\,\delta_{2j})\,\psi_{31}(\xi_1)\,\psi_{12}(\xi_2), \quad \text{etc.} \tag{3.20}$$

Then, for orthotropic materials, the constitutive equations (3.1) must be expressible in the form

$$\chi_{ij} = \sum_{\beta=1}^{9} \mathscr{L}^{(\beta)}\{f_{ij}^{(\beta)}; I_1, \ldots, I_7\}, \tag{3.21}$$

where $\mathscr{L}^{(\beta)}$ is a functional of the indicated functions, linear with respect to $f_{ij}^{(\beta)}$. Thus, for example, χ_{11} is of the form

$$\chi_{11} = \mathscr{L}^{(1)}\{1; I_\alpha\} \tag{3.22}$$

and χ_{23} is of the form

$$2\chi_{23} = \mathscr{L}^{(4)}\{\psi_{23}(\xi); I_\alpha\} + \mathscr{L}^{(7)}\{\psi_{31}(\xi_1)\,\psi_{12}(\xi_2); I_\alpha\}. \tag{3.23}$$

Here I_α stands for the set of seven functions defined by (3.15).

4. The General Problem. Notation

To treat the general problem posed by the form-invariance condition (1.3), we shall use a notation similar to that employed in Part I. The components $\psi^{(\nu)}_{p_1 \ldots p_{r(\nu)}}(\tau)$ $(\nu = 1, 2, \ldots, N)$ will be renumbered with a single subscript as $\psi_1(\tau), \ldots, \psi_n(\tau)$, and the ordered set will be denoted by $\boldsymbol{\psi}(\tau)$. We shall speak of $\boldsymbol{\psi}(\tau)$ as a history, or, if we mean the value at a particular time τ, as a tensor. The functionals $\mathfrak{F}_{i_1 \ldots i_r}$ will also be renumbered with a single subscript as $\mathfrak{F}_1, \ldots, \mathfrak{F}_m$, and the ordered set will be denoted by $\boldsymbol{\mathfrak{F}}$. This set will be called a tensor-valued functional. Matrices $\boldsymbol{S}(\lambda)$ and $\boldsymbol{T}(\lambda)$ are defined as in Section 2 of Part I. In this notation, the material symmetry requirement (1.3) takes the form

$$\boldsymbol{\mathfrak{F}}\{\boldsymbol{S}(\lambda)\boldsymbol{\psi}(\tau)\} = \boldsymbol{T}(\lambda)\boldsymbol{\mathfrak{F}}\{\boldsymbol{\psi}(\tau)\}. \tag{4.1}$$

The material symmetry groups which we shall consider are enumerated in Part I, Section 2. Since all of our remarks will be applicable to each of these groups individually, we regard the group under consideration as fixed once and for all. With the understanding that $\boldsymbol{S}(\lambda)$ and $\boldsymbol{T}(\lambda)$ vary over all transformations in certain representations of the group when the set of parameters denoted by λ varies over all values in its domain, then material symmetry requires that the relation (4.1) must be satisfied identically in $\boldsymbol{\psi}(\tau)$ and λ. The tensor-valued functional $\boldsymbol{\mathfrak{F}}$ will be called form-invariant (under the group considered) if (4.1) is so satisfied.

5. Finite Sets of Tensors

We restrict our attention in this and the following section to finite sets of tensors $\boldsymbol{\psi}^1, \boldsymbol{\psi}^2, \ldots, \boldsymbol{\psi}^L$, rather than histories. We are particularly interested in sets $\boldsymbol{\psi}(\tau_1), \ldots, \boldsymbol{\psi}(\tau_L)$ obtained by evaluating a history $\boldsymbol{\psi}(\tau)$ at a finite number of instants $\tau_1, \ldots, \tau_L$.

The definitions and results of Part I are stated in terms of a single symbol $\boldsymbol{\psi}$ which represents an ordered sequence of the components of some given finite number of tensors. In particular, the given tensors can be those denoted by $\boldsymbol{\psi}^1, \ldots, \boldsymbol{\psi}^L$. We now state some of these results explicitly in terms of sets $\boldsymbol{\psi}^1, \ldots, \boldsymbol{\psi}^L$. We shall also discuss the notion of a table of typical basic invariants for an arbitrary number of tensors.

By applying a common transformation $\boldsymbol{S}(\lambda)$ to each tensor of the set $\boldsymbol{\psi}^i$ $(i = 1, 2, \ldots, L)$, we can obtain a second set $\boldsymbol{S}(\lambda)\boldsymbol{\psi}^i$ $(i = 1, 2, \ldots, L)$. The latter set is called equivalent to the first (see Part I, Section 3). An invariant of the set of tensors is a function G which takes equal values on equivalent sets:

$$G[\boldsymbol{S}(\lambda)\boldsymbol{\psi}^1, \ldots, \boldsymbol{S}(\lambda)\boldsymbol{\psi}^L] = G[\boldsymbol{\psi}^1, \ldots, \boldsymbol{\psi}^L]. \tag{5.1}$$

The sets which are equivalent to a given set $\boldsymbol{\psi}^i$ $(i = 1, 2, \ldots, L)$ are said to form the orbit of that set. An invariant can be regarded as a function which assigns values to orbits.

A form-invariant tensor-valued function of the set $\boldsymbol{\psi}^i$ $(i = 1, 2, \ldots, L)$ is a function $\boldsymbol{f}$ (with components $f_1, \ldots, f_m$) such that

$$\boldsymbol{f}[\boldsymbol{S}(\lambda)\boldsymbol{\psi}^1, \ldots, \boldsymbol{S}(\lambda)\boldsymbol{\psi}^L] = \boldsymbol{T}(\lambda)\,\boldsymbol{f}(\boldsymbol{\psi}^1, \ldots, \boldsymbol{\psi}^L). \tag{5.2}$$

For each of the material symmetry groups, there exists (see WEYL [22], p. 44) a *finite table of typical basic invariants* for a tensor $\boldsymbol{\varphi}$ and an arbitrary number of tensors $\boldsymbol{\psi}^i$. By $\boldsymbol{\varphi}$ we mean a tensor whose transforms are of the form $\boldsymbol{T}(\lambda)\boldsymbol{\varphi}$. We shall be concerned only with those typical invariants which are either independent of $\boldsymbol{\varphi}$, or linear in $\boldsymbol{\varphi}$. There is a maximum number p of tensors $\boldsymbol{\psi}$ which appear as arguments in any of these typical invariants. Although some of these invariants may involve fewer than p tensors $\boldsymbol{\psi}$, for simplicity in notation we write each of them as a function of p such tensors. Let $I_\alpha(\boldsymbol{\psi}^1, \dots, \boldsymbol{\psi}^p)$ ($\alpha = 1, 2, \dots, A$) be those typical invariants which are independent of $\boldsymbol{\varphi}$, and let $J_\beta(\boldsymbol{\varphi}, \boldsymbol{\psi}^1, \dots, \boldsymbol{\psi}^p)$ ($\beta = 1, 2, \dots, B$) be those which are linear in $\boldsymbol{\varphi}$. Each invariant I_α or J_β is a polynomial, linear in each of the tensors on which it actually depends. This linearity greatly facilitates the calculation of a table of typical invariants.

The defining property of a table of typical basic invariants is that from it, one can obtain an integrity basis for any number of tensors, however large. This is done by letting the argument tensors in the typical invariants take on the values of the given tensors in all possible combinations, repeats allowed. Thus, for a given set of times $\tau_1, \dots, \tau_L$, a finite integrity basis for the tensors $\boldsymbol{\psi}(\tau_1), \dots, \boldsymbol{\psi}(\tau_L)$ is given in terms of the typical invariants I_α by

$$\begin{gathered} I_\alpha[\boldsymbol{\psi}(\tau_{i_1}), \dots, \boldsymbol{\psi}(\tau_{i_p})] \\ (\alpha = 1, 2, \dots, A;\ i_1 = 1, 2, \dots, L;\ \dots;\ i_p = 1, 2, \dots, L), \end{gathered} \tag{5.3}$$

no matter how large L may be. Similarly, any polynomial invariant in $\boldsymbol{\varphi}$ and $\boldsymbol{\psi}(\tau_1), \dots, \boldsymbol{\psi}(\tau_L)$, linear in $\boldsymbol{\varphi}$, can be expressed as a linear combination of the invariants

$$\begin{gathered} J_\beta[\boldsymbol{\varphi}, \boldsymbol{\psi}(\tau_{i_1}), \dots, \boldsymbol{\psi}(\tau_{i_p})] \\ (\beta = 1, 2, \dots, B; i_1 = 1, 2, \dots, L; \dots; i_p = 1, 2, \dots, L), \end{gathered} \tag{5.4}$$

with coefficients which are polynomials in the invariants (5.3).

Each typical invariant J_β is of the form

$$J_\beta = \sum_{i=1}^{m} \varphi_i f_i^{(\beta)}(\boldsymbol{\psi}^1, \dots, \boldsymbol{\psi}^p). \tag{5.5}$$

We recall from Part I that the functions $\boldsymbol{f}^{(\beta)}$ defined by relations of the form (5.5) are form-invariant. That is, in terms of the tensors $\boldsymbol{\psi}(\tau_{i_1}), \dots, \boldsymbol{\psi}(\tau_{i_p})$, $\boldsymbol{f}^{(\beta)}$ satisfies

$$\boldsymbol{f}^{(\beta)}[\boldsymbol{S}(\lambda)\boldsymbol{\psi}(\tau_{i_1}), \dots, \boldsymbol{S}(\lambda)\boldsymbol{\psi}(\tau_{i_p})] = \boldsymbol{T}(\lambda)\boldsymbol{f}^{(\beta)}[\boldsymbol{\psi}(\tau_{i_1}), \dots, \boldsymbol{\psi}(\tau_{i_p})]. \tag{5.6}$$

The functions

$$\boldsymbol{f}^{(\beta)}[\boldsymbol{\psi}(\tau_{i_1}), \dots, \boldsymbol{\psi}(\tau_{i_p})] \quad (\beta = 1, 2, \dots, B; i_1 = 1, 2, \dots, L; \dots; i_p = 1, 2, \dots, L) \tag{5.7}$$

are the basic form-invariant tensors for the set $\boldsymbol{\psi}(\tau_i)$ ($i = 1, 2, \dots, L$).

6. *Canonical Forms for Invariants and Form-Invariant Tensor-Valued Functions*

In Part I we have shown that an integrity basis is a functional basis. Since the invariants (5.3) form an integrity basis for the tensors $\boldsymbol{\psi}(\tau_i)$ ($i = 1, 2, \dots, L$), then they also form a functional basis for these tensors.

From this result and the first criterion for functional bases given in Section 4 of Part I, we immediately obtain the following theorem: *Given two sets of tensors* $\boldsymbol{\psi}^0(\tau_i)$ $(i=1, 2, \ldots, L)$ *and* $\boldsymbol{\psi}^1(\tau_i)$ $(i=1, 2, \ldots, L)$, *there exists a transformation* $\boldsymbol{S}(\lambda)$ *such that*

$$\boldsymbol{\psi}^1(\tau_i)=\boldsymbol{S}(\lambda)\boldsymbol{\psi}^0(\tau_i) \qquad (i=1, 2, \ldots, L), \tag{6.1}$$

if and only if the following equations are satisfied:

$$\begin{gathered} I_\alpha[\boldsymbol{\psi}^0(\tau_{i_1}), \ldots, \boldsymbol{\psi}^0(\tau_{i_p})]=I_\alpha[\boldsymbol{\psi}^1(\tau_{i_1}), \ldots, \boldsymbol{\psi}^1(\tau_{i_p})] \\ (\alpha=1, 2, \ldots, A; i_1=1, 2, \ldots, L; \ldots; i_p=1, 2, \ldots, L). \end{gathered} \tag{6.2}$$

Put differently, the values of the invariants (5.3) uniquely determine the orbit of the set $\boldsymbol{\psi}(\tau_i)$ $(i=1, 2, \ldots, L)$.

From Part I it is also known that any form-invariant function $\boldsymbol{f}$ of a set of tensors $\boldsymbol{\psi}(\tau_i)$ $(i=1, 2, \ldots, L)$ can be expressed as a linear combination of the basic form-invariant tensors (5.7) for this set, with invariant coefficients:

$$\boldsymbol{f}=\sum_{\beta=1}^{B} \sum_{i_1=1}^{L} \cdots \sum_{i_p=1}^{L} F_{\beta i_1 \ldots i_p} \boldsymbol{f}^{(\beta)}[\boldsymbol{\psi}(\tau_{i_1}), \ldots, \boldsymbol{\psi}(\tau_{i_p})]. \tag{6.3}$$

Here the coefficients $F_{\beta i_1 \ldots i_p}$ are functions of the invariants (5.3).

7. *Invariants of Histories*

Given a history $\boldsymbol{\psi}(\tau)$, any history $\boldsymbol{S}(\lambda)\boldsymbol{\psi}(\tau)$ which can be obtained from it by applying a time-independent transformation will be called *equivalent* to $\boldsymbol{\psi}(\tau)$. The set of histories equivalent to $\boldsymbol{\psi}(\tau)$ will be called the *orbit* of $\boldsymbol{\psi}(\tau)$. This set is also the orbit of any history equivalent to $\boldsymbol{\psi}(\tau)$. The orbits of two histories either coincide or have no common element.

A functional $\mathscr{I}\{\boldsymbol{\psi}(\tau)\}$ will be called an *invariant* of the history $\boldsymbol{\psi}(\tau)$ if it satisfies the relation

$$\mathscr{I}\{\boldsymbol{\psi}(\tau)\}=\mathscr{I}\{\boldsymbol{S}(\lambda)\boldsymbol{\psi}(\tau)\}, \tag{7.1}$$

identically in $\boldsymbol{\psi}(\tau)$ and λ. An invariant takes the same value for every history on a given orbit, and thus in effect assigns values to orbits.

We shall show how invariants can be characterized explicitly in terms of functions derived from the elements I_α of a table of typical invariants. We note first that for each choice of $\xi_1, \ldots, \xi_p$, the quantities $I_\alpha[\boldsymbol{\psi}(\xi_1), \ldots, \boldsymbol{\psi}(\xi_p)]$ $(\alpha=1, 2, \ldots, A)$ are invariants of the history $\boldsymbol{\psi}(\tau)$. Let us regard these invariants as functions of the variables $\xi_1, \ldots, \xi_p$. Then, we shall show that *every invariant functional can be expressed as a functional of the invariant functions* I_α $(\alpha=1, 2, \ldots, A)$. That is, if $\mathscr{I}$ satisfies (7.1), then

$$\mathscr{I}=\mathscr{I}\{I_1[\boldsymbol{\psi}(\xi_1), \ldots, \boldsymbol{\psi}(\xi_p)]; \ldots; I_A[\boldsymbol{\psi}(\xi_1), \ldots, \boldsymbol{\psi}(\xi_p)]\}. \tag{7.2}$$

This theorem is true provided that the values of the functions $I_\alpha[\boldsymbol{\psi}(\xi_1), \ldots, \boldsymbol{\psi}(\xi_p)]$ $(\alpha=1, 2, \ldots, A)$ uniquely determine the orbit of $\boldsymbol{\psi}(\tau)$. For, the value of an invariant is determined by the orbit on which the history $\boldsymbol{\psi}(\tau)$ lies and is thus determined by those values which specify the orbit. Hence, it is sufficient to prove that the functions I_α $(\alpha=1, 2, \ldots, A)$ determine the orbit: *Two* *histories* $\boldsymbol{\psi}^0(\tau)$ *and* $\boldsymbol{\psi}^1(\tau)$ *lie on the same orbit if and only if the following equations are satisfied for every choice of the variables* $\xi_1, \dots, \xi_p$:

$$I_\alpha[\boldsymbol{\psi}^0(\xi_1), \dots, \boldsymbol{\psi}^0(\xi_p)] = I_\alpha[\boldsymbol{\psi}^1(\xi_1), \dots, \boldsymbol{\psi}^1(\xi_p)] \qquad (\alpha = 1, 2, \dots, A). \tag{7.3}$$

To prove this theorem, we first note that if $\boldsymbol{\psi}^0(\tau)$ and $\boldsymbol{\psi}^1(\tau)$ lie on the same orbit, then, by the definition of an orbit, there is a transformation $\boldsymbol{S}$ such that for all τ,

$$\boldsymbol{\psi}^1(\tau) = \boldsymbol{S}\boldsymbol{\psi}^0(\tau). \tag{7.4}$$

In particular,

$$\boldsymbol{\psi}^1(\xi_i) = \boldsymbol{S}\boldsymbol{\psi}^0(\xi_i) \qquad (i = 1, 2, \dots, p) \tag{7.5}$$

for every choice of $\xi_1, \dots, \xi_p$. Since $I_\alpha[\boldsymbol{\psi}(\xi_1), \dots, \boldsymbol{\psi}(\xi_p)]$ is an invariant of the set of tensors $\boldsymbol{\psi}(\xi_i)$ $(i = 1, 2, \dots, p)$, then (7.3) is satisfied.

We now prove the converse, that satisfaction of (7.3) implies that there exists a matrix $\boldsymbol{S}$ such that (7.4) is satisfied. Since $\boldsymbol{\psi}^0(\tau)$ has a finite number of components $\psi_1^0(\tau), \dots, \psi_n^0(\tau)$, the values of $\boldsymbol{\psi}^0(\tau)$ for all τ span a linear manifold of finite dimension $L \leqq n$. Then, there exists a set of times $\tau_1, \dots, \tau_L$ such that the values $\boldsymbol{\psi}^0(\tau_1), \dots, \boldsymbol{\psi}^0(\tau_L)$ form a basis for this manifold. That is, every value $\boldsymbol{\psi}^0(\tau)$ can be expressed as a linear combination of the elements of this basis, with scalar coefficients $c_i(\tau)$:

$$\boldsymbol{\psi}^0(\tau) = \sum_{i=1}^{L} c_i(\tau)\, \boldsymbol{\psi}^0(\tau_i). \tag{7.6}$$

Assuming that (7.3) is satisfied for all choices of $\xi_1, \dots, \xi_p$, then in particular it is satisfied for all choices from the set $\tau_1, \dots, \tau_L$. Then (6.2) holds, and it follows from the theorem in Section 6 that there exists a matrix $\boldsymbol{S}$ such that

$$\boldsymbol{\psi}^1(\tau_i) = \boldsymbol{S}\boldsymbol{\psi}^0(\tau_i) \qquad (i = 1, 2, \dots, L). \tag{7.7}$$

In addition, (7.3) is satisfied, by hypothesis, for all choices of $\xi_1, \dots, \xi_p$ from the set of times consisting of $\tau_1, \dots, \tau_L$ and an arbitrary time τ. Then, for each choice of τ there is a matrix $\boldsymbol{S}_\tau$ such that

$$\boldsymbol{\psi}^1(\tau) = \boldsymbol{S}_\tau\boldsymbol{\psi}^0(\tau) \quad \text{and} \quad \boldsymbol{\psi}^1(\tau_i) = \boldsymbol{S}_\tau\boldsymbol{\psi}^0(\tau_i) \qquad (i = 1, 2, \dots, L). \tag{7.8}$$

We now find, from (7.8a), (7.6), and (7.8b), that the tensors $\boldsymbol{\psi}^1(\tau_i)$ $(i = 1, 2, \dots, L)$ form a basis for the manifold spanned by the values of $\boldsymbol{\psi}^1(\tau)$:

$$\boldsymbol{\psi}^1(\tau) = \sum_{i=1}^{L} c_i(\tau)\, \boldsymbol{\psi}^1(\tau_i). \tag{7.9}$$

Since $\boldsymbol{\psi}^1(\tau_i)$ is related to $\boldsymbol{\psi}^0(\tau_i)$ by the transformation $\boldsymbol{S}$, according to (7.7), then from (7.9) we obtain

$$\boldsymbol{\psi}^1(\tau) = \boldsymbol{S} \sum_{i=1}^{L} c_i(\tau)\, \boldsymbol{\psi}^0(\tau_i), \tag{7.10}$$

and thus with (7.6) we obtain (7.4) as desired. Hence, $\boldsymbol{\psi}^1(\tau)$ is related to $\boldsymbol{\psi}^0(\tau)$ by the same matrix $\boldsymbol{S}$ for every τ; the two histories are on the same orbit. This completes the proof.

Having proved that every invariant functional can be expressed as a functional of the functions $I_\alpha[\boldsymbol{\psi}(\xi_1), \dots, \boldsymbol{\psi}(\xi_p)]$ $(\alpha = 1, 2, \dots, A)$, we call these functions the *basic invariants* of the history $\boldsymbol{\psi}(\tau)$.

8. The Canonical Representation of a Form-Invariant Tensor-valued Functional

The canonical representation theorem will be stated in terms of quantities which have been defined in Sections 5 to 7. We shall summarize these definitions and then state the theorem. The summary of definitions is in effect an outline of the method to be employed in deriving a canonical representation.

Let $I_\alpha(\boldsymbol{\psi}^1, \ldots, \boldsymbol{\psi}^p)$ $(\alpha = 1, 2, \ldots, A)$ be a table of typical basic invariants for an arbitrary number of tensors $\boldsymbol{\psi}$. The basic invariants of the history $\boldsymbol{\psi}(\tau)$ are then the functions $I_\alpha[\boldsymbol{\psi}(\xi_1), \ldots, \boldsymbol{\psi}(\xi_p)]$ $(\alpha = 1, 2, \ldots, A)$ of the variables $\xi_1, \ldots, \xi_p$. Let $\boldsymbol{\varphi}$ be a tensor whose transforms are of the form $\boldsymbol{T}(\lambda)\boldsymbol{\varphi}$. Let $J_\beta(\boldsymbol{\varphi}, \boldsymbol{\psi}^1, \ldots, \boldsymbol{\psi}^p)$ $(\beta = 1, 2, \ldots, B)$ be the elements, linear in $\boldsymbol{\varphi}$, of a table of typical basic invariants for the tensor $\boldsymbol{\varphi}$ and an arbitrary number of tensors $\boldsymbol{\psi}$. Let $\boldsymbol{f}^{(\beta)}$ $(\beta = 1, 2, \ldots, B)$ be the basic form-invariant tensors, defined in terms of J_β by (5.5), and let the tensors $\boldsymbol{f}^{(\beta)}[\boldsymbol{\psi}(\xi_1), \ldots, \boldsymbol{\psi}(\xi_p)]$ be regarded as functions of the variables $\xi_1, \ldots, \xi_p$. A functional $\mathfrak{F}$ is said to be form-invariant if it satisfies the material symmetry requirements (4.1). We can now state the theorem.

Every form-invariant functional $\mathfrak{F}$ can be expressed in the form

$$\mathfrak{F}\{\boldsymbol{\psi}(\tau)\} = \sum_{\beta=1}^{B} \mathscr{L}^{(\beta)}\{\boldsymbol{f}^{(\beta)}; I_1, \ldots, I_A\}, \tag{8.1}$$

where $\mathscr{L}^{(\beta)}$ is a functional of the basic form-invariant tensor $\boldsymbol{f}^{(\beta)}$ and the basic invariants I_α $(\alpha = 1, 2, \ldots, A)$ of the history, and $\mathscr{L}^{(\beta)}$ is linear with respect to $\boldsymbol{f}^{(\beta)}$. Conversely, for every such choice of the functionals $\mathscr{L}^{(\beta)}$, a functional $\mathfrak{F}$ of the form (8.1) is form-invariant.

The fact that every form-invariant functional can be expressed in the form (8.1) will be proved in Sections 9 to 12. The converse, that the right-hand member of (8.1) is form-invariant, is easy to prove. We give the proof in order to promote a better understanding of the theorem.

By substituting $\boldsymbol{S}(\lambda)\boldsymbol{\psi}(\tau)$ for $\boldsymbol{\psi}(\tau)$ in (8.1), we obtain

$$\begin{aligned}&\mathfrak{F}\{\boldsymbol{S}(\lambda)\boldsymbol{\psi}(\tau)\}\\ &= \sum_{\beta=1}^{B} \mathscr{L}^{(\beta)}\{\boldsymbol{f}^{(\beta)}[\boldsymbol{S}(\lambda)\boldsymbol{\psi}(\xi_1), \ldots, \boldsymbol{S}(\lambda)\boldsymbol{\psi}(\xi_p)]; I_\alpha[\boldsymbol{S}(\lambda)\boldsymbol{\psi}(\xi_1), \ldots, \boldsymbol{S}(\lambda)\boldsymbol{\psi}(\xi_p)]\},\end{aligned} \tag{8.2}$$

where I_α denotes the set of functions $I_1, \ldots, I_A$. By now using the facts that $\boldsymbol{f}^{(\beta)}$ is form-invariant, according to (5.6), and that the functions I_α are invariants, from (8.2) we obtain

$$\mathfrak{F}\{\boldsymbol{S}(\lambda)\boldsymbol{\psi}(\tau)\} = \sum_{\beta=1}^{B} \mathscr{L}^{(\beta)}\{\boldsymbol{T}(\lambda)\boldsymbol{f}^{(\beta)}[\boldsymbol{\psi}(\xi_1), \ldots, \boldsymbol{\psi}(\xi_p)]; I_\alpha[\boldsymbol{\psi}(\xi_1), \ldots, \boldsymbol{\psi}(\xi_p)]\}. \tag{8.3}$$

Because $\mathscr{L}^{(\beta)}$ is linear with respect to its first argument, it follows from (8.3) that

$$\begin{aligned}\mathfrak{F}\{\boldsymbol{S}(\lambda)\boldsymbol{\psi}(\tau)\} &= \boldsymbol{T}(\lambda) \sum_{\beta=1}^{B} \mathscr{L}^{(\beta)}\{\boldsymbol{f}^{(\beta)}[\boldsymbol{\psi}(\xi_1), \ldots, \boldsymbol{\psi}(\xi_p)]; I_\alpha[\boldsymbol{\psi}(\xi_1), \ldots, \boldsymbol{\psi}(\xi_p)]\}\\ &= \boldsymbol{T}(\lambda)\,\mathfrak{F}\{\boldsymbol{\psi}(\tau)\}.\end{aligned} \tag{8.4}$$

Here we have used (8.1) again. Thus, if $\mathfrak{F}$ is of the form (8.1), it is form-invariant.

9. *Outline of Proof*

The proof that a form-invariant functional can be expressed in the canonical form (8.1) will proceed along the following lines. We first restrict our attention to histories on the orbit of a given history $\boldsymbol{\psi}^0(\tau)$. We shall show that histories on this orbit are distinguished, or characterized, by their values at a finite set of times $\tau_1, \ldots, \tau_L$. These times depend on the orbit, but not on the particular history considered. Then, with restriction to histories $\boldsymbol{\psi}(\tau)$ on the orbit of $\boldsymbol{\psi}^0(\tau)$, we shall show that $\mathfrak{F}$ is a function $\boldsymbol{f}$ of the finite set of values $\boldsymbol{\psi}(\tau_i)$ $(i=1, 2, \ldots, L)$ (Section 10).

The form-invariance of the functional $\mathfrak{F}$ implies that the function $\boldsymbol{f}$ is form-invariant. By using the canonical representation of a form-invariant function, which is known from Part I, we will show that $\mathfrak{F}$ is a linear combination of certain values of the basic form-invariant tensors $\boldsymbol{f}^{(\beta)}[\boldsymbol{\psi}(\xi_1), \ldots, \boldsymbol{\psi}(\xi_p)]$, and thus a linear functional of these functions (Section 11). The form of this linear functional depends upon which orbit is considered, since the form of $\boldsymbol{f}$ and the choice of times $\tau_1, \ldots, \tau_L$ depend on the orbit. Since dependence on the orbit amounts to dependence on the basic invariants $I_\alpha[\boldsymbol{\psi}(\xi_1), \ldots, \boldsymbol{\psi}(\xi_p)]$ $(\alpha=1, 2, \ldots, A)$, we arrive at (8.1) as desired (Section 12).

10. *Reduction of Functional to Function on a Given Orbit*

In this section we consider only those histories $\boldsymbol{\psi}(\tau)$ which are on the orbit of some arbitrarily selected history $\boldsymbol{\psi}^0(\tau)$, *i.e.* histories of the form $\boldsymbol{\psi}(\tau)=\boldsymbol{S}(\lambda)\boldsymbol{\psi}^0(\tau)$. We shall prove two results. First, there exists a finite number of fixed times $\tau_1, \ldots, \tau_L$ with the property that if two histories on this orbit are not identical, then there is at least one time τ_i at which they take different values. Second, on the orbit of $\boldsymbol{\psi}^0(\tau)$, $\mathfrak{F}\{\boldsymbol{\psi}(\tau)\}$ can be represented as a function of the values $\boldsymbol{\psi}(\tau_i)$ $(i=1, 2, \ldots, L)$. We now proceed with the proof of these statements.

It was shown in Section 7 that the values of $\boldsymbol{\psi}^0(\tau)$ for all τ span a manifold of some finite dimension L. Then, there are times $\tau_1, \ldots, \tau_L$ such that the values $\boldsymbol{\psi}^0(\tau_i)$ $(i=1, 2, \ldots, L)$ form a basis for the manifold. In other words, for every value $\boldsymbol{\psi}^0(\tau)$ there exists a set of numbers $c_i(\tau)$ $(i=1, 2, \ldots, L)$ such that

$$\boldsymbol{\psi}^0(\tau)=\sum_{i=1}^{L} c_i(\tau)\,\boldsymbol{\psi}^0(\tau_i). \tag{10.1}$$

Let $\boldsymbol{\psi}(\tau)$ be a history on the orbit of $\boldsymbol{\psi}^0(\tau)$. Then there is a fixed λ such that for all τ,

$$\boldsymbol{\psi}(\tau)=\boldsymbol{S}(\lambda)\boldsymbol{\psi}^0(\tau). \tag{10.2}$$

From (10.1) and (10.2) it follows that

$$\boldsymbol{\psi}(\tau)=\sum_{i=1}^{L} c_i(\tau)\,\boldsymbol{\psi}(\tau_i). \tag{10.3}$$

That is, the values $\boldsymbol{\psi}(\tau_i)$ $(i=1, 2, \ldots, L)$ form a basis for the linear manifold spanned by the values of $\boldsymbol{\psi}(\tau)$. We note that the scalar coefficients $c_i(\tau)$ $(i=1, 2, \ldots, L)$ are the same for all histories on the orbit of $\boldsymbol{\psi}^0(\tau)$.

If $\psi^1(\tau)$ and $\psi^2(\tau)$ are two histories on the orbit of $\psi^0(\tau)$, then

$$\psi^1(\tau)-\psi^2(\tau)=\sum_{i=1}^{L} c_i(\tau)\,[\psi^1(\tau_i)-\psi^2(\tau_i)]. \tag{10.4}$$

If at some time τ, $\psi^1(\tau)$ differs from $\psi^2(\tau)$, so that the left-hand member of (10.4) is not zero, then at least one of the terms in the right-hand member must also be different from zero. Thus, if two histories $\psi^1(\tau)$ and $\psi^2(\tau)$ are not identical, they take different values at one of the times $\tau_1, \ldots, \tau_L$.

This result means that if $\psi(\tau)$ is a history on the orbit of $\psi^0(\tau)$, then the values $\psi(\tau_i)$ $(i=1, 2, \ldots, L)$ serve to distinguish it from any other history on that orbit. If the orbit and the values $\psi(\tau_i)$ $(i=1, 2, \ldots, L)$ are given, $\psi(\tau)$ is completely determinate. Hence, a functional which assigns values to histories $\psi(\tau)$ on the orbit of $\psi^0(\tau)$ is a function of the finite set of tensors $\psi(\tau_i)$ $(i=1, 2, \ldots, L)$.

This can also be seen by direct manipulation. By substituting (10.3) into $\mathfrak{F}\{\psi(\tau)\}$ and making use of the fact that the scalar coefficients are the same for each history on the orbit of $\psi^0(\tau)$, we find that the functional $\mathfrak{F}$ becomes a function of the tensors $\psi(\tau_i)$:

$$\begin{aligned}\mathfrak{F}\{\psi(\tau)\}&=\mathfrak{F}\left\{\sum_{i=1}^{L} c_i(\tau)\psi(\tau_i)\right\}\\&=\boldsymbol{f}[\psi(\tau_1), \ldots, \psi(\tau_L); O(\psi^0)].\end{aligned} \tag{10.5}$$

The symbol $O(\psi^0)$, denoting the orbit considered, is attached as a reminder that (10.5) holds only for histories on that orbit. Indeed, $\boldsymbol{f}$ is not defined for histories which are not on the orbit of $\psi^0(\tau)$. Since it will be convenient to have $\boldsymbol{f}$ defined for all histories, we shall arbitrarily take $\boldsymbol{f}=0$ for histories which are not on the orbit of $\psi^0(\tau)$.

11. Canonical Form on a Given Orbit

If the functional $\mathfrak{F}$ is form-invariant, then the function $\boldsymbol{f}$ defined by (10.5) satisfies the form-invariance requirement. This follows from the fact that since $\boldsymbol{S}(\lambda)\psi(\tau)$ is on the orbit of $\psi^0(\tau)$ if $\psi(\tau)$ is, we may substitute (10.5) into (4.1) to obtain

$$\begin{aligned}\boldsymbol{f}[\boldsymbol{S}(\lambda)\psi(\tau_1), \ldots, \boldsymbol{S}(\lambda)\psi(\tau_L); O(\psi^0)]&=\mathfrak{F}\{\boldsymbol{S}(\lambda)\psi(\tau)\}\\=\boldsymbol{T}(\lambda)\,\mathfrak{F}\{\psi(\tau)\}&=\boldsymbol{T}(\lambda)\,\boldsymbol{f}[\psi(\tau_1), \ldots, \psi(\tau_L); O(\psi^0)].\end{aligned} \tag{11.1}$$

Since we have defined $\boldsymbol{f}$ to be zero for histories not on the orbit of $\psi^0(\tau)$, the form-invariance requirement is satisfied trivially for those histories.

Since $\boldsymbol{f}$ is a form-invariant function, it can be expressed in the canonical form (6.3). By using (6.3) in (10.5) we obtain the result that for histories $\psi(\tau)$ on the orbit of $\psi^0(\tau)$, $\mathfrak{F}$ is of the form

$$\mathfrak{F}\{\psi(\tau)\}=\sum_{\beta=1}^{B}\sum_{i_1=1}^{L}\cdots\sum_{i_p=1}^{L} F_{\beta i_1\ldots i_p}\boldsymbol{f}^{(\beta)}[\psi(\tau_{i_1}), \ldots, \psi(\tau_{i_p})], \tag{11.2}$$

where the coefficients $F_{\beta i_1\ldots i_p}$ are functions of the invariants (5.3). Since the functions $F_{\beta i_1\ldots i_p}$ are invariants, they are constant on the orbit of $\psi^0(\tau)$. Since

(11.2) is valid only on the orbit of $\psi^0(\tau)$, we may regard the coefficients $F_{\beta i_1 \dots i_p}$ as constants.

The right-hand member of (11.2) is a sum of certain functionals which are linear in the functions $f^{(\beta)}[\psi(\xi_1), \dots, \psi(\xi_p)]$. For, since the value of a function at a particular point is a linear functional of that function, then the values $f^{(\beta)}[\psi(\tau_{i_1}), \dots, \psi(\tau_{i_p})]$ $(i_1 = 1, 2, \dots, L; \dots; i_p = 1, 2, \dots, L)$ are linear functionals of the function $f^{(\beta)}[\psi(\xi_1), \dots, \psi(\xi_p)]$. Furthermore, since a linear combination of linear functionals of $f^{(\beta)}$ is itself a linear functional of $f^{(\beta)}$, then the sum of all terms involving $f^{(\beta)}$ in (11.2) is a linear functional of $f^{(\beta)}$. We shall denote this sum by $\mathscr{L}^{(\beta)}$:

$$\mathscr{L}^{(\beta)}\{f^{(\beta)}[\psi(\xi_1), \dots, \psi(\xi_p)]; O(\psi^0)\} = \sum_{i_1=1}^{L} \cdots \sum_{i_p=1}^{L} F_{\beta i_1 \dots i_p} f^{(\beta)}[\psi(\tau_{i_1}), \dots, \psi(\tau_{i_p})]. \quad (11.3)$$

By using (11.3) in (11.2), we obtain the desired expression for $\mathfrak{F}$ on the orbit of $\psi^0(\tau)$,

$$\mathfrak{F}\{\psi(\tau)\} = \sum_{\beta=1}^{B} \mathscr{L}^{(\beta)}\{f^{(\beta)}[\psi(\xi_1), \dots, \psi(\xi_p)]; O(\psi^0)\}. \quad (11.4)$$

The symbol $O(\psi^0)$ is attached to $\mathscr{L}^{(\beta)}$ as a reminder that the definition of $\mathscr{L}^{(\beta)}$ depends on the orbit considered, through the choice of times τ_i $(i = 1, 2, \dots, L)$ and the values of the coefficients $F_{\beta i_1 \dots i_p}$ in (11.3). The functional $\mathscr{L}^{(\beta)}$ is defined by (11.3) for all histories, but (11.4) is not necessarily valid unless $\psi(\tau)$ is on the orbit of $\psi^0(\tau)$.

12. General Canonical Form

The representation (11.4) is valid only for histories on the orbit of $\psi^0(\tau)$. However, since $\psi^0(\tau)$ was chosen arbitrarily, some representation of this general form is valid for each orbit. The linear functionals $\mathscr{L}^{(\beta)}$ in these representations depend upon which orbit is considered. Thus, for any history we can write

$$\mathfrak{F}\{\psi(\tau)\} = \sum_{\beta=1}^{B} \mathscr{L}^{(\beta)}\{f^{(\beta)}[\psi(\xi_1), \dots, \psi(\xi_p)]; O(\psi)\}, \quad (12.1)$$

where $O(\psi)$ indicates that the linear functionals depend upon the orbit on which $\psi(\tau)$ lies.

In Section 7 we have shown that dependence on the orbit of $\psi(\tau)$ is equivalent to dependence on all the values taken by the basic invariants $I_\alpha[\psi(\xi_1), \dots, \psi(\xi_p)]$ $(\alpha = 1, 2, \dots, A)$. Thus, the dependence of $\mathscr{L}^{(\beta)}$ on $O(\psi)$ in (12.1) can be expressed as dependence on the basic invariants of the history:

$$\begin{aligned} \mathfrak{F}\{\psi(\tau)\} = \sum_{\beta=1}^{B} \mathscr{L}^{(\beta)}\{&f^{(\beta)}[\psi(\xi_1), \dots, \psi(\xi_p)]; \\ &I_1[\psi(\xi_1), \dots, \psi(\xi_p)], \dots, I_A[\psi(\xi_1), \dots, \psi(\xi_p)]\}. \end{aligned} \quad (12.2)$$

If we regard $\mathscr{L}^{(\beta)}$ as a functional of the functions I_α $(\alpha = 1, 2, \dots, A)$, as well as a linear functional of $f^{(\beta)}$, then the representation (12.2) is valid for every history $\psi(\tau)$. This completes the proof that every form-invariant functional $\mathfrak{F}$ can be represented in the form (8.1).

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