Dispersion of Low-Energy Waves for Two Conservative Equations

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1.

The equations proposed by ZAKHAROV & SHABAT [1] and by KORTEWEG & DE VRIES [2] both possess solitary waves with striking properties. The first of these equations has the form

$$\frac{\partial u}{\partial t}=i\left[\frac{\partial^2 u}{\partial x^2}+F(u)\right]. \tag{1}$$

In case $F(u)=k|u|^{p-1}u$, $p>1$, $k>0$, it possesses the four-parameter family of solitary waves

$$u(x,t)=f(x-ct;\lambda)\exp[ig(x-bt)],$$

where

$$g(x)=\frac{c}{2}x+\varphi, \qquad \lambda=\frac{c}{2}\left[\frac{c}{2}-b\right]>0,$$

and

$$f(x;\lambda)=\left\{\frac{(p+1)\lambda}{2k}\operatorname{sech}^2\left[\frac{p-1}{2}\sqrt{\lambda}(x-x_0)\right]\right\}^{1/(p-1)}.$$

For small λ these solutions have uniformly small amplitudes. In case $p>3$, however, they are not in every sense small, for their L^1 norms go to infinity as $\lambda\to 0$. On the other hand, as we shall show in Section 2, solutions of equation (1) which are initially small in terms of a certain norm stronger than the L^1 norm may remain small in absolute value. We show that, in case $|F(u)|=O(|u|^{4+\varepsilon})$ as $|u|\to 0$, all such solutions decay uniformly to zero as the time tends to infinity.

The second equation is

$$\frac{\partial u}{\partial t}=-\frac{\partial}{\partial x}\left[\frac{\partial^2 u}{\partial x^2}+F(u)\right], \tag{2}$$

with u real-valued. In case $F(u)=ku^p$, $p>1$, $k>0$, this equation possesses the two-parameter family of solitary waves

$$u(x,t)=f(x-ct;c),$$

where $f(x;c)$ is defined as above and $c>0$. For small c these solutions have small amplitudes. On the other hand, for $p>3$ their L^1 norms go to infinity as $c\to 0$, and for $p>5$ their L^2 norms do the same. In Section 3 we consider solutions of (2) whose first three conserved quantities are initially small. In case $|F(u)|=O(|u|^{5+\varepsilon})$ as $|u|\to 0$, we show that all such solutions decay uniformly to zero. Moreover, no sign condition is required on the nonlinear term.

The existence of waves which do not decay reflects the balance between nonlinearity and dispersion. In these two examples we have relaxed the nonlinear effects in such a way that the waves do decay and their asymptotic behavior can be obtained from the linearized equation.

In summary, if the KORTEWEG-DE VRIES or the ZAKHAROV-SHABAT equation is modified so that the nonlinear term is of high enough degree, then all disturbances which are initially small enough in a certain norm disperse and die out as $|t|\to\infty$.

We use a standard iteration procedure [3] which is applicable not only here but also for other equations (for example, n-dimensional versions of (1) and (2)).

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2.

We shall use subscripts to denote partial derivatives. The notation $|\ |_p$ refers to the usual Lebesgue L^p norms in the variable x: thus

$$|\varphi|_p^p=\int_{-\infty}^{\infty}|\varphi(x)|^p\,dx;\qquad |\varphi|_\infty=\operatorname*{ess\,sup}_{-\infty<x<\infty}|\varphi(x)|.$$

In addition, we define

$$\|\varphi\|=|\varphi|_1+|\varphi_x|_2.$$

We shall use the elementary inequalities $|\varphi|_\infty\leqq\|\varphi\|$ and

$$|\varphi|_\infty^2\leqq 2|\varphi\varphi_x|_1\leqq|\varphi|_2^2+|\varphi_x|_2^2.$$

Let $F(s)$ be a complex function of class C^1 (as a function of two real variables) having the properties $F(0)=F'(0)=0$ and $|F''(s)|=O(|s|^{2+\varepsilon})$ as $s\to 0$, for some $\varepsilon>0$. (The primes denote derivatives of $F(s)$ in any direction.)

Theorem 1. *There exists a positive number k_0, depending only on $F(s)$, with the following property. For any function $\varphi(x)$ with $\|\varphi\|\leqq k_0$, there exists a unique solution $u(x,t)$ of equation (1) with $u(x,0)=\varphi(x)$ which is uniformly bounded in time and has values in $L^2\cap L^\infty$. This solution decays to zero as $t\to\infty$, and in fact we have the estimate*

$$|u(x,t)|\leqq c(1+|t|)^{-1/2}$$

for all x and t.

Proof. We convert (1) into the integral equation

$$u(t)=R(t)*\varphi+\int_0^t R(t-\tau)*iF(u(\tau))\,d\tau, \tag{3}$$

where the space variable has been suppressed, $*$ denotes the convolution operation in space, and $R(x,t)$ is the solution of the equation (linearized about zero)

$$\frac{\partial v}{\partial t}=i\frac{\partial^2 v}{\partial x^2} \tag{4}$$

with the initial data $R(x,0)=\delta(x)$. Thus $R(t)*\varphi$ is the solution of (4) with data $\varphi(x)$, and we have $|R(t)*\varphi|_2=|\varphi|_2$. By Fourier transformation, $R(x,t)$ can easily be computed, thus:

$$R(x,t)=(4\pi i t)^{-1/2}\exp(x^2/4it).$$

To prove the uniqueness, let u and v be two solutions of (1) which agree initially. The difference $w=u-v$ satisfies

$$w(t)=i\int_0^t R(t-\tau)*[F(u(\tau))-F(v(\tau))]\,d\tau.$$

Forming the L^2 norm, we find

$$|w(t)|_2\leqq\int_0^t|F(u(\tau))-F(v(\tau))|_2\,d\tau\leqq c\int_0^t|w(\tau)|_2\,d\tau$$

since u and v are bounded. Thus w must vanish identically and $u=v$.

For brevity let us write the integral term in (3) as $\mathscr{I}u(t)$ and also define the "triple" norm

$$|||v|||^2=\operatorname*{ess\,sup}_t\{|v(t)|_2^2+|v_x(t)|_2^2+t|v(t)|_\infty^2\}.$$

For any pair of functions u and v of triple norm less than k, say, and any $t>0$, we have the following estimate (here we denote by c various constants which depend only on k and which tend to zero as $k\to 0$):

$$\begin{aligned}|\mathscr{I}u(t)-\mathscr{I}v(t)|_2&\leqq\int_0^t|F(u)-F(v)|_2\,d\tau\\&\leqq c\int_0^t(|u|_\infty+|v|_\infty)^{3+\varepsilon}|u-v|_2\,d\tau\leqq c\,|||u-v|||.\end{aligned}$$

Similarly, we can bound $|[\mathscr{I}u(t)-\mathscr{I}v(t)]_x|_2$ and therefore also $|\mathscr{I}u(t)-\mathscr{I}v(t)|_\infty$. To obtain the required decay as $t\to\infty$, we use the explicit expression for $R(x,t)$. Thus

$$|\mathscr{I}u(t)-\mathscr{I}v(t)|_\infty\leqq\int_0^t(4\pi(t-\tau))^{-1/2}|F(u)-F(v)|_1\,d\tau.$$

The norm in the integrand is bounded by

$$c(|u|_\infty+|v|_\infty)^{2+\varepsilon}(|u|_2+|v|_2)|u-v|_2.$$

Hence

$$|\mathscr{I}u(t)-\mathscr{I}v(t)|_\infty\leqq c\,|||u-v|||\int_0^t(t-\tau)^{-1/2}(1+\tau)^{-1+\varepsilon/2}\,d\tau,$$

the last integral being $O(t^{-1/2})$. Combining these estimates, we have

$$|||\mathscr{I}u-\mathscr{I}v|||\leqq c_0(k)\,|||u-v|||$$

where $c_0(k)\to 0$ as $k\to 0$.

Now fix k so small that $c_0(k) \leqq 1/2$. We define the approximating sequence

$$u_0 = R * \varphi, \quad u_n = u_0 + \mathscr{I} u_{n-1} \quad (n \geqq 1)$$

where $\|\varphi\| \leqq k/4 = k_0$. Then $|||u_0||| \leqq k/2$ and by induction

$$|||u_n||| \leqq k/2 + c_0(k)\, |||u_{n-1}||| \leqq k.$$

In addition, $|||u_{n+1} - u_n||| \leqq c_0^n |||u_1 - u_0|||$. Since $c_0 < 1$, $\{u_n\}$ converges in triple norm to some function u. It is easy to see that u is the required solution.

The asymptotic behavior can be described most accurately in terms of the linearized equation (4), as follows.

Theorem 2. *Let $u(x, t)$ be the solution of Theorem* 1. *There is a unique pair of solutions $u_\pm(x, t)$ of* (4) *such that $|u(t) - u_\pm(t)|_2 \to 0$ as $t \to \pm\infty$.*

Proof. It is enough to consider the case $t \to +\infty$. We define

$$u_+(t) = u(t) + \int_t^\infty R(t-\tau) * iF(u(\tau))\, d\tau.$$

The integral converges absolutely in L^2 norm because

$$|F(u)|_2 \leqq c\, |u|_\infty^{3+\varepsilon} |u|_2 = O(\tau^{-(3+\varepsilon)/2}).$$

Therefore

$$|u_+(t) - u(t)|_2 \leqq \int_t^\infty |F(u(\tau))|_2\, d\tau \to 0$$

as $t \to \infty$. Using (3), we may write u_+ in the form

$$u_+(t) = R(t) * \varphi + \int_0^\infty R(t-\tau) * iF(u(\tau))\, d\tau.$$

Since the right side is a linear combination of solutions of (4), u_+ also satisfies (4). The uniqueness of u_+ is immediate.

Remarks. It is clear, in addition, that as $t \to +\infty$ we have

$$t^{1/2} |u(t) - u_+(t)|_\infty + |[u(t) - u_+(t)]_x|_2 \to 0.$$

Moreover, the regularity of the solutions presents no problem if the data is smooth enough.

3.

Here we consider the generalized Korteweg-de Vries equation (2). We define $\|\varphi\|$ exactly as in Section 2. Let $F(s)$ be a real C^2 function such that $|F'(s)| = O(|s|^{4+\varepsilon})$ as $s \to 0$, for some $\varepsilon > 0$. Then the following result holds.

Theorem 3. *There exists a constant $k_0 > 0$ depending only on $F(s)$, with the following property. For any function $\varphi(x)$ with $\|\varphi\| \leqq k_0$, there is a solution $u(x, t)$ of equation* (2) *with $u(x, 0) = \varphi(x)$ which is uniformly bounded in time and has values in L^2. This solution moreover satisfies the uniform estimate*

$$|u(x, t)| \leqq c(1 + |t|)^{-1/3}.$$

Proof. We shall use the energy method (cf. for example TEMAM [4]) to prove the existence of a uniformly bounded solution, after which we shall consider its decay. Given $\delta>0$, we consider the parabolic regularization

$$u_t+u_{xxx}+F(u)_x=\delta u_{xx}-\delta u_{xxxx}, \qquad t>0 \tag{5}$$

(of course the solution of (5) with $u(x,o)=\varphi(x)$ depends on δ). By parabolic theory, this problem is well-posed. In particular, assuming that the data $\varphi(x)$ is everywhere less than 1, say, the solution does not exceed 1 in some time interval $0<t\leqq T_\delta$. We proceed to derive a priori bounds independent of δ which are valid in this time interval. Multiplying (5) by u and integrating, we obtain

$$\int[(\tfrac{1}{2}u^2)_t+\delta(u_x^2+u_{xx}^2)]\,dx=0.$$

Hence $|u(t)|_2\leqq|\varphi|_2$ and

$$\delta\int_0^\infty\int(u_x^2+u_{xx}^2)\,dx\,dt\leqq\tfrac{1}{2}|\varphi|_2^2.$$

Multiplying next by $u_{xx}+F(u)$, we get

$$\frac{d}{dt}\int[\tfrac{1}{2}u_x^2-G(u)]\,dx\leqq\delta\int[F'(u)(u_x^2+u_{xx}^2)+F''(u)u_x^2u_{xx}]\,dx$$

where G is a primitive of F. We may assume that $G(0)=F(0)=0$. In the time interval when $|u(x,t)|<1$, F', F'' and $u^{-2}G(u)$ are bounded. Thus

$$\begin{aligned}\int\tfrac{1}{2}u_x^2\,dx&\leqq\int[\tfrac{1}{2}u_x^2-G(u)+cu^2]\,dx\\&\leqq\int[\tfrac{1}{2}\varphi_x^2-G(\varphi)+c\varphi^2]\,dx+c\delta\int_0^t[u_x^2+u_{xx}^2+u_x^2|u_{xx}|]\,dx\,dt,\end{aligned}$$

where c denotes various constants depending only on F. The first two terms of the last integral are bounded by a constant times $|\varphi|_2^2$. To take care of the last term, $u_x^2|u_{xx}|$, we bound the first factor using the relation $|u_x|_\infty\leqq|u_x|_2+|u_{xx}|_2$. The last term thus is bounded by

$$c\delta\int_0^t(|u_x|_2+|u_{xx}|_2)\,|u_x|_2\,|u_{xx}|_2\,d\tau.$$

Therefore $|u_x|_2^2$ satisfies an inequality of the form

$$|u_x|_2^2\leqq\text{constant}+\int_0^t l\,|u_x|_2^2\,d\tau,$$

where $l=l(\tau)$ is integrable. Hence

$$|u_x|_2^2\leqq c_1(|\varphi_x|_2^2+|\varphi|_2^2)\exp(c_2|\varphi|_2^2).$$

Thus if $\|\varphi\|$ is small, so are $|u_x|_2$ and $|u|_\infty$. In particular, $T_\delta=+\infty$ and the estimates are valid for all positive times.

Call the preceding solutions $u^{(\delta)}(x,t)$. As δ varies, they are uniformly bounded together with $F(u^{(\delta)})$. Thus a passage to the limit is easily justified. This shows that if $|\varphi_x|_2+|\varphi|_2$ is sufficiently small, then there is a solution of (2) with initial data $\varphi(x)$ which satisfies

$$\sup_{x,t}|u(x,t)|<\infty, \qquad \sup_{t}\big(|u_x(t)|_2+|u(t)|_2\big)<\infty.$$

To obtain the required decay as $t\to+\infty$, we convert (2) into the integral equation

$$u(t)=R(t)*\varphi-\int_0^t R(t-\tau)*F\big(u(\tau)\big)_x\,d\tau \tag{6}$$

as in Section 2, where the source function $R(x,t)$ is the solution of the linearized equation

$$v_t+v_{xxx}=0 \tag{7}$$

with initial data equal to the delta function. Once again $|R(t)*\varphi|_2=|\varphi|_2$. The source function [5] is given by

$$R(x,t)=\frac{1}{\pi}\int_0^\infty \exp(ix\xi+it\xi^3)\,d\xi=(3t)^{-1/3}Ai\big(x(3t)^{-1/3}\big),$$

so that $t^{1/3}R(x,t)$ is uniformly bounded. Thus from (6) we have

$$|u(t)|_\infty\leqq c\,\|\varphi\|(1+t)^{-1/3}+\int_0^t c(t-\tau)^{-1/3}\,\big|F\big(u(\tau)\big)_x\big|_1\,d\tau.$$

By means of the L^2 bounds already established, we obtain

$$|F(u)_x|_1\leqq|u^{-1}F'(u)|_\infty\,|u|_2\,|u_x|_2\leqq c\,|u|_\infty^{3+\varepsilon}.$$

We define $m(t)$ as the supremum of $(1+\tau)^{1/3}|u(\tau)|_\infty$ over the time interval $0\leqq\tau\leqq t$. Then

$$m(t)\leqq c\,\|\varphi\|+c\,m(t)^{3+\varepsilon}.$$

If $\|\varphi\|$ is sufficiently small, $m(t)$ is bounded. This completes the proof for $t>0$. For $t<0$ we need only reverse the time.

Theorem 4. *Let $u(x,t)$ be the solution of Theorem* 3. *There is a unique pair of solutions $u_\pm(x,t)$ of the linearized equation* (7) *such that $|u(t)-u_\pm(t)|_2\to 0$ as $t\to\pm\infty$.*

Proof. We define

$$u_+(t)=u(t)-\int_t^\infty R(t-\tau)*F\big(u(\tau)\big)_x\,d\tau.$$

Since $|F(u)_x|_2\leqq c|u|_\infty^{4+\varepsilon}|u_x|_2$, the integral converges absolutely in the L^2 norm. The proof then proceeds exactly as in Theorem 2.

Remarks. As before, $t^{1/3}|u(t)-u_\pm(t)|_\infty\to 0$ as $t\to\pm\infty$. In addition, if $\varphi_{xx}\in L^2$, we can use higher-order estimates to prove the uniqueness of the solution of (2).

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