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On a consistent theory, and variational formulation of finitely stretched and rotated 3-D space-curved beams

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Abstract. This paper deals with finite rotations, and finite strains of three-dimensional space-curved elastic beams, under the action of conservative as well as nonconservative type external distributed forces and moments. The plausible deformation hypothesis of "plane sections remaining plane" is invoked. Exact expressions for the curvature, twist, and transverse shear strains are given; as is a consistent set of boundary conditions. General mixed variational principles, corresponding to the stationarity of a functional with respect to the displacement vector, rotation tensor, stress-resultants, stress-couples, and their conjugate strain-measures, are stated for the case when conservative-type external moments act on the beam. The momentum-balance conditions arising out of these functionals, either coincide exactly with, or are equivalent to, those from the "static method". The incremental variational functionals, governing both the Total and Updated Lagrangian incremental finite element formulations, are given. An example of the case of the buckling of a beam subject to axial compression and non-conservative type axial twisting couple, is presented and discussed.

1 Introduction

There exist three approaches that are commonly used for describing the large displacements and large rotations of space-curved beams. The first approach is based on a direct use of the 3-dimensional finite-elasticity theory. The second one is based on certain plausible hypotheses such as the Euler-Bernoulli hypotheses, while the last one, on the work of Reissner (1973). From a mathematical viewpoint, the first approach may lead to a consistent beam theory; however, it is not so easy to derive the kinematic relations. Therefore, asymptotic expansions have been used in the first approach (Parker 1979, Pleus and Sayir 1983). Each of the variables employed in such theories does not always carry a physical meaning, and their interpretation becomes difficult especially in highly nonlinear problems. The approach based on plausible deformation hypotheses, on the other hand, may not yield a beam theory consistent with 3-D finite-elasticity theory; however, as indicated by the theory of the "elastica", this approach is often found to be practically useful. Also in buckling problems, the second and the third approach enables one to easily take into account the prebuckling deformations, since each variable has a clear physical meaning.

There has been a limited number of earlier works concerning theories for beams undergoing large deformations, large rotations, and large strains. Notable among these is due to Reissner (1973, 1981), who developed a finite strain beam theory based on the differential equations of force and moment equilibrium for elements of the deformed curve. The exact definitions for kinematic relationships have been derived (Reissner 1973, 1981), while the expressions for boundary conditions consistent with the equilibrium equations have been obtained implicitly.

In this paper, using plausible and consistent kinematic hypotheses, a large deformation (and large rotation) beam theory is developed. The effects of stretching, bending, torsion and transverse shear, are taken into account, while the cross-sectional warping deformations are neglected. In the present formulation, we do not restrict the magnitude of strains, but assume that the material is linearly elastic. Using the principle of virtual work, we present a set of boundary conditions which are consistent with the presently developed finite strain beam theory.

As indicated by Argyris et al. (1979), the use of an arbitrary set of mathematical variables to describe rotations may lead to unsymmetric geometric stiffnesses of finite beam elements, even when the beam is subjected to a conservative system of external moments. One of the objectives of this paper is to present well-defined variational functionals, and associated 'principles' corresponding to the vanishing of the first variation of such functionals, when a conservative system of external forces and moments act on the finitely deformed beam. Using these functionals, one may construct a *symmetric* geometric stiffness of a beam element in *its current equilibrium state*. It is noted however that when a *nonconservative* system of external forces and moments act on the beam, the geometric stiffness of a beam element, in its current equilibrium state, will be unsymmetric. A systematic approach to solve such problems of nonconservative loading, has been discussed by Kondoh and Atluri (1987), based on a direct statement of the weak form of the associated balance laws.

The variational functionals, in the presence of a conservative system of external forces and moments, which are presented in this paper, form the bases of general mixed-hybrid finite element methods for finitely strained and rotated space-curved beams. The *modus operandi* for such finite element methods, involving finite rotation kinematics, has been discussed earlier by Atluri and Murakawa (1977), Murakawa and Atluri (1978).

The remainder of the paper is as follows. Section 2 deals with preliminaries; Sect. 3 with the geometry of the undeformed beam; and Sect. 4 with the geometry of the deformed beam. In Sect. 5 we deal with the principle of virtual work for the finitely strained beam; and discuss how this virtual work principle may be cast in the form of a condition of stationarity of well-defined functional, even when only a system of conservative external forces and *moments* act on the beam undergoing finite rotations. Depending on the form of virtual variations of the rotation parameters considered, (if $\mathbf{R}$ is the rotation tensor, one form of rotational variation corresponds to the vector $\delta\boldsymbol{\phi}$ such that $\delta\boldsymbol{\phi} \times \mathbf{I} = \delta\mathbf{R} \cdot \mathbf{R}^t$ and the other form corresponds directly to $\delta\alpha^j$ where α^j are the three parameters that describe the Lagrangian components of $\mathbf{R}$), the linear and angular momentum balance conditions take on different but equivalent forms; with only one of these forms coinciding with those derived a priori from the so-called "static-method". In Sect. 6 we deal with the constitutive equations; and Sect. 7 deals with the most general mixed variational principles under conservative loading, and their "incremental" counterparts. In the general variational principles, the variables are: the displacement vector, the rotation tensor, the stress-resultant vector acting on the beam cross-section, the stress-couple vector acting on the beam cross-section, and the appropriate strain and curvature measures that are conjugate to these mechanical variables.

To demonstrate the novel features of the presently developed theory, we consider, in Sect. 8, a problem of buckling of a beam subjected to an axial compression and a nonconservative twisting couple with the emphasis on the boundary conditions. The effects of prebuckling and shear deformations are manifested in the presently derived buckling load.

2 Preliminaries

The fundamental hypotheses for deriving the present finite-strain beam theory are itemized as follows:

(1) The plane cross-sections of the beam remain plane and do not undergo any shape-change during the deformation.

(2) The cross-sections are constant along the beam axis which remains a smooth space curve throughout the deformation.

Throughout this paper, the summation convention is adopted; and the Latain indices will have the range 1, 2, and 3, and the Greek indices the range 1 and 2.

3 The geometry of the undeformed beam

Consider a naturally curved and twisted beam in a fixed Cartesian coordinate system X^m, with base vectors $\boldsymbol{I}_m$, as shown in Fig. 1. An orthogonal curvilinear coordinate system Y^m, with base vectors $\overset{0}{\boldsymbol{E}}_m$, is introduced to describe the motion of the beam. The coordinates Y^α are taken in the cross sections, while the coordinate Y^3 is taken along the beam axis. The way to select the origin of the coordinates Y^m will be discussed in Sect. 6. The orientation of the present coordinate systems follows the familar "right hand rule".

The position vector of a point at the beam-axis is represented as

$$\boldsymbol{X} = X^m(Y^3)\,\boldsymbol{I}_m. \tag{1}$$

The tangent base vector $\overset{0}{\boldsymbol{E}}_3$ is a defined by:

$$\overset{0}{\boldsymbol{E}}_3 = \mathrm{d}\boldsymbol{X}/\mathrm{d}Y^3. \tag{2}$$

In general, the base vector $\overset{0}{\boldsymbol{E}}_3$ is not a unit vector, while the base vectors $\overset{0}{\boldsymbol{E}}_\alpha$ are chosen to be unit vectors without loss of generality. For latter convenience, we introduce the unit vectors $\boldsymbol{E}_m$ defined by

$$\boldsymbol{E}_\alpha = \overset{0}{\boldsymbol{E}}_\alpha, \qquad \boldsymbol{E}_3 = \overset{0}{\boldsymbol{E}}_3/|\overset{0}{\boldsymbol{E}}_3|. \tag{3a, b}$$

The well-known Frenet-Serret formulae lead to the relations:

$$\boldsymbol{E}_{m,3} = \boldsymbol{K} \times \boldsymbol{E}_m; \qquad \boldsymbol{K} = K_m\,\boldsymbol{E}_m, \tag{4a, b}$$

where $(\ )_{,3} = \mathrm{d}(\ )/\mathrm{d}L$ where $\mathrm{d}L = |\overset{0}{\boldsymbol{E}}_3|\,\mathrm{d}Y^3$; K_α are the components of initial curvature, and K_3 is the initial twist.

The position vector of an arbitrary point in a cross-section of the beam is given by:

$$\boldsymbol{R} = \boldsymbol{X} + Y^\alpha \boldsymbol{E}_\alpha. \tag{5}$$

Then, the base vectors at an arbitrary point in a cross-section of the beam are given by:

$$\boldsymbol{A}_\alpha = \boldsymbol{E}_\alpha, \qquad \boldsymbol{A}_3 = -\,Y^2 K_3\,\boldsymbol{E}_1 + Y^1 K_3\,\boldsymbol{E}_2 + g_0\,\boldsymbol{E}_3, \tag{6a, b}$$

where

$$g_0 = 1 - Y^1 K_2 + Y^2 K_1. \tag{7}$$

The contravariant base vectors $\boldsymbol{A}^m$ are defined through the relation:

$\boldsymbol{A}^m \cdot \boldsymbol{A}_n = \delta^m_n$ where δ^m_n is the Kronecker delta.

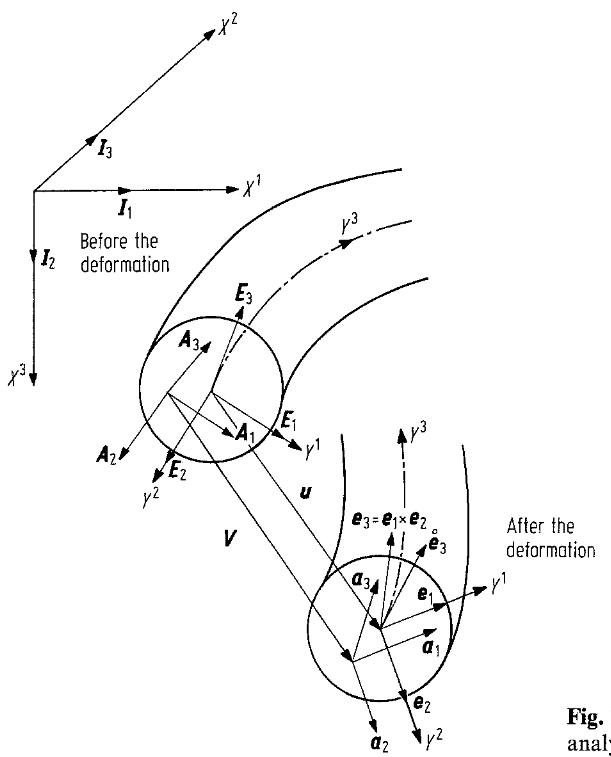


Fig. 1. Kinematic scheme for large deformation and large rotation analysis

4 The geometry of the deformed beam

Let $\overset{0}{e}_3$ be the unit vector tangential to the deformed beam axis. After deformation, the unit base vectors $\boldsymbol{E}_\alpha$ are transformed to the unit base vectors $\boldsymbol{e}_\alpha$, as shown in Fig. 1. Without loss of generality, the base vectors $\boldsymbol{e}_\alpha$ and $\boldsymbol{e}_3$ are assumed to be the maps of the base vectors $\boldsymbol{E}_\alpha$ and $\boldsymbol{E}_3$ after a purely rigid rotation, denoted by the tensor $\mathbf{R}$, alone.
Accordingly, we have

$$\boldsymbol{e}_\alpha \cdot \boldsymbol{e}_\beta = \delta_{\alpha\beta}; \quad \boldsymbol{e}_\alpha \cdot \overset{0}{\boldsymbol{e}}_3 \neq 0. \tag{8a, b}$$

Equation (8a) is consistent with assumption (1). The nonorthogonality condition given by Eq. (8b) is due to the transverse shear deformation which renders $\boldsymbol{e}_3 \neq \overset{0}{\boldsymbol{e}}_3$.

The relationship between the unit orthogonal vectors $\boldsymbol{e}_m$ and $\boldsymbol{E}_m$ is written, in terms of a *finite rotation tensor* $\mathbf{R}$, (Atluri 1984; Pietraszkiewicz and Badur 1983), as:

$$\boldsymbol{e}_m = \mathbf{R} \cdot \boldsymbol{E}_m; \quad \mathbf{R} = R_{ij}\, \boldsymbol{E}_i\, \boldsymbol{E}_j. \tag{9a, b}$$

Because of the condition that $\mathbf{R} \cdot \mathbf{R}^T = \mathbf{I}$, where $\mathbf{I}$ is the identity tensor and $(\;)^T$ a transpose, the number of independent components of $\mathbf{R}$ are three.

Finite rotation vectors are also used by Atluri (1984), Kane, Likins and Levinson (1983), Pietraszkiewicz and Badur (1983), Reissner (1973) and Simmonds and Danielson (1972) to represent the relationship between $\boldsymbol{e}_m$ and $\boldsymbol{E}_m$. Let $\boldsymbol{e}$ be a unit vector satisfying $\mathbf{R} \cdot \boldsymbol{e} = \boldsymbol{e}$ and ω a magnitude of rotation about the axis of rotation defined by $\boldsymbol{e}$. The alternate representations commonly used for finite rotation vectors are:

$$\boldsymbol{\Omega} = \sin\omega\, \boldsymbol{e}; \quad \boldsymbol{\theta} = 2\tan\frac{\omega}{2}\, \boldsymbol{e}; \quad \text{and} \quad \boldsymbol{\omega} = \omega\, \boldsymbol{e}. \tag{10a–c}$$

In terms of the finite rotation vectors $\boldsymbol{\Omega}$, $\boldsymbol{\theta}$, and $\boldsymbol{\omega}$, the relationship between $\boldsymbol{e}_m$ and $\boldsymbol{E}_m$ may be written as (Pietraszkiewicz and Badur 1983).

$$\begin{aligned}
\boldsymbol{e}_m &= \boldsymbol{E}_m + \boldsymbol{\Omega} \times \boldsymbol{E}_m + \frac{1}{2\cos^2\dfrac{\omega}{2}}\, \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{E}_m); \\
&= \boldsymbol{E}_m + \frac{1}{1 + \dfrac{1}{4}\theta^2}\, \boldsymbol{\theta} \times \left(\boldsymbol{E}_m + \frac{1}{2}\boldsymbol{\theta} \times \boldsymbol{E}_m\right); \\
&= \boldsymbol{E}_m + \frac{\sin\omega}{\omega}\, \boldsymbol{\omega} \times \boldsymbol{E}_m + \frac{1 - \cos\omega}{(\omega)^2}\, \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{E}_m).
\end{aligned} \tag{11a–c}$$

The finite rotation tensor $\mathbf{R}$ is often expressed very conveniently, in terms of $\boldsymbol{\omega}$, as:

$$\mathbf{R} = \exp(\boldsymbol{\omega} \times \mathbf{I}). \tag{12}$$

It is well known that no 3-parameter representation of $\mathbf{R}$ can be both global and nonsingular (Stuelpnagel 1964); for this reason the four quaternion or Euler parameters have been introduced to describe the finite rotations (Kane, Likins and Levinson 1983; Stuelpnagel 1964). In spite of this drawback, the concept of a finite rotation vector is useful for the study of finite rotations in structural members. For example, the three Rodrigues parameters, defined by $\boldsymbol{E}_m \cdot \boldsymbol{\theta}/2$, are not global, i.e., $\omega = \pi$ rad yields the Rodrigues vector of infinite magnitude. However, the finite rotation tensor $\mathbf{R}$ is determined uniquely even if $\omega = \pi$ rad (Kane, Likins and Levinson 1983). In this way, the finite rotation vectors have been frequently employed to describe the finite rotations in structural mechanics.

In the present case of a space-curved beam, we define the angles of shear deformations, denoted by β_α, in the following fashion:

$$\sin\beta_\alpha = \boldsymbol{e}_\alpha \cdot \overset{0}{\boldsymbol{e}}_3. \tag{13}$$

Since $|\overset{0}{e}_3| = 1$, we have

$$\overset{0}{e}_3 = \sin\beta_\alpha\, e_\alpha + \beta_3\, e_3\,, \tag{14}$$

where $\beta_3 = \overset{0}{e}_3 \cdot e_3$. (15)

From the definition of covariant base vectors, the deformed unit tangent vector $\overset{0}{e}_3$ takes the natural form, in terms of the displacement components, u^m, as:

$$\overset{0}{e}_3 = (\boldsymbol{X}+\boldsymbol{u})_{,3}/|(\boldsymbol{X}+\boldsymbol{u})_{,3}| = u^m\|_3\, \boldsymbol{E}_m; \qquad u^m\|_3 = (\delta_3^m + u^m|_3)/g;$$

$$g = \sqrt{(u^1|_3)^2 + (u^2|_3)^2 + (1 + u^3|_3)^2} \tag{16a–c}$$

where $\boldsymbol{u}\ (= u^m \boldsymbol{E}_m)$ is the displacement vector at the beam axis and $(\ )|_3$ the covariant differentiation with respect to the metric tensor $E_{ij} = \boldsymbol{E}_i \cdot \boldsymbol{E}_j$. With the use of Eqs. (9) and (16), the angles of shear deformations β_α and β_3 are represented, in terms of the displacement components and the finite rotation tensor, as

$$\sin\beta_\alpha = (\mathbf{R}\cdot\boldsymbol{E}_\alpha)\cdot(u^m\|_3\,\boldsymbol{E}_m); \qquad \beta_3 = (\mathbf{R}\cdot\boldsymbol{E}_3)\cdot(u^m\|_3\,\boldsymbol{E}_m)\,. \tag{17a, b}$$

According to the assumption (1), the deformed base vectors at an arbitrary material point are given by

$$\boldsymbol{a}_m = (\boldsymbol{X} + \boldsymbol{u} + Y^\alpha\, \boldsymbol{e}_\alpha)_{,m}\,, \tag{18}$$

where $(\ )_{,\alpha}$ denotes a differentiation with respect to Y^α. It follows from Eqs. (14), (16) and (18) that

$$\boldsymbol{a}_\alpha = \boldsymbol{e}_\alpha; \qquad \boldsymbol{a}_3 = (g\sin\beta_1 - Y^2k_3)\boldsymbol{e}_1 + (g\sin\beta_2 + Y^1k_3)\,\boldsymbol{e}_2 + (g\beta_3 - Y^1k_2 + Y^2k_1)\boldsymbol{e}_3\,, \tag{19a, b}$$

where

$$\boldsymbol{e}_{m,3} = \boldsymbol{k}\times\boldsymbol{e}_m; \qquad \boldsymbol{k} = k_m\,\boldsymbol{e}_m\,. \tag{20a, b}$$

The components k_m of $\boldsymbol{k}$ are expressed in terms of the finite rotation tensor as

$$k_i = \frac{1}{2}\,\varepsilon_{ijk}\,[(\mathbf{R}\cdot\boldsymbol{E}_j)_{,3}]\cdot[(\mathbf{R}\cdot\boldsymbol{E}_k)]\,, \tag{21}$$

where ε_{ijk} is the permutation symbol. It should be noted that the components k_m are not exactly the curvatures and twist after the deformation since the deformed beam axis has undergone extension.

5 Equilibrium equations and boundary conditions

To derive the equilibrium equations, the so-called "static method" or alternatively, the energy method are often employed. Once the appropriate stress resultants and moments are defined, the static method yield the equilibrium equations from a consideration of the free-body diagram of a differential element of the beam. On the basis of the static method, however, it is difficult to derive the boundary conditions consistent with the resulting equilibrium equations. The energy method, on the other hand, may lead to the equilibrium equations and the associated boundary conditions without difficulty, but with tedious calculations. Since no explicit boundary conditions consistent with a finite beam theory are currently available in literature, the energy method is employed herein to derive them.

The Green strain tensor is defined as

$$\boldsymbol{\varepsilon} = \varepsilon_{ij}\,\boldsymbol{A}^i\boldsymbol{A}^j; \qquad \varepsilon_{ij} = \frac{1}{2}(\boldsymbol{a}_i\cdot\boldsymbol{a}_j - \boldsymbol{A}_i\cdot\boldsymbol{A}_j)\,. \tag{22a, b}$$

Substituting Eqs. (6) and (19) into Eqs. (22) leads to

$$\varepsilon_{13} = \varepsilon_{31} = \frac{1}{2}(g\sin\beta_1 - Y^2\tilde{k}_3); \qquad \varepsilon_{23} = \varepsilon_{32} = \frac{1}{2}(g\sin\beta_2 + Y^1\tilde{k}_3); \qquad \varepsilon_{33} = \frac{1}{2}\Big[(g\sin\beta_1 - Y^2k_3)^2$$
$$+ (g\sin\beta_2 + Y^1k_3)^2 + (g\beta_3 - Y^1k_2 + Y^2k_1)^2 - \varrho(K_3)^2 - (g_0)^2\Big]; \qquad \varepsilon_{\alpha\beta} = 0\,. \tag{23a–d}$$

where

$$\tilde{k}_m = k_m - K_m; \qquad \varrho = (Y^1)^2 + (Y^2)^2. \tag{24a, b}$$

The stress resultants and moments are defined, following Atluri (1984), as:

$$T = \int g_0 \boldsymbol{A}^3 \cdot (\mathbf{S}_1 \cdot \mathbf{F}^T) \mathrm{d}A; \qquad \boldsymbol{M} = \int Y^\alpha \boldsymbol{e}_\alpha \times [g_0 \boldsymbol{A}^3 \cdot (\mathbf{S}_1 \cdot \mathbf{F}^T)] \mathrm{d}A, \tag{25a, b}$$

where $\mathbf{S}_1$ $(= S_1^{mn} \boldsymbol{A}_m \boldsymbol{A}_n)$ is the second Piola-Kirchhoff stress tensor, $\mathbf{F}$ the deformation gradient tensor and $\mathrm{d}A = \mathrm{d}Y^1 \mathrm{d}Y^2$. By using the component representation, we obtain the stress resultants and moments in the form (Appendix I)

$$\boldsymbol{T} = T^j \boldsymbol{e}_j, \qquad \boldsymbol{M} = M^j \boldsymbol{e}_j, \qquad T^j = \int t^{3\bar{j}} g_0 \, \mathrm{d}A, \tag{26a–c}$$

$$M^1 = \int t^{3\bar{3}} Y^2 g_0 \mathrm{d}A, \qquad M^2 = -\int t^{3\bar{3}} Y^1 g_0 \mathrm{d}A, \qquad M^3 = \int (t^{3\bar{2}} Y^1 - t^{3\bar{1}} Y^2) g_0 \mathrm{d}A, \tag{26d–f}$$

where $t^{m\bar{n}}$ are the stress components defined by

$$t^{m\bar{n}} = S_1^{mj} \boldsymbol{a}_j \cdot \boldsymbol{e}_n. \tag{27}$$

The $\overline{(\ )}$ in superscript is used to emphasize that these are not components in convected coordinates Y^m.

The internal virtual work is written as (Washizu 1982)

$$IVW = \int S_1^{ij} \delta \varepsilon_{ij} \mathrm{d}V, \tag{28}$$

where $\mathrm{d}V = g_0 \mathrm{d}Y^1 \mathrm{d}Y^2 \mathrm{d}L$. In this paper, following Atluri (1984), we introduce a tensor $(\delta \mathbf{R} \cdot \mathbf{R}^T)$ as a rotational variation. Since $\mathbf{R} \cdot \mathbf{R}^T = \mathbf{I}$, $\delta \mathbf{R} \cdot \mathbf{R}^T$ is a skewsymmetric tensor. There exists, therefore, a vector $\delta \boldsymbol{\phi}$ satisfying $\delta \mathbf{R} \cdot \mathbf{R}^T = \delta \boldsymbol{\phi} \times \mathbf{I}$. Through some straightforward algebra it may be shown that the variation of the finite rotation vector $\boldsymbol{\omega}$, denoted as $\delta \boldsymbol{\omega}$, is related to the vector $\delta \boldsymbol{\phi}$ as

$$\delta \boldsymbol{\phi} = \frac{\sin \omega}{\omega} \delta \boldsymbol{\omega} - \frac{(1 - \cos \omega)}{(\omega)^2} \delta \boldsymbol{\omega} \times \boldsymbol{\omega} + \left\{ \frac{1}{\omega} - \frac{\sin \omega}{(\omega)^2} \right\} \delta \omega \, \boldsymbol{\omega}. \tag{29}$$

Substituting Eqs. (23) into Eq. (28) and using Eqs. (25), one is lead, after some straightforward algebra, to:

$$IVW = -\int [\boldsymbol{T}_{,3} \cdot \delta \boldsymbol{u} + \{\boldsymbol{M}_{,3} + (\boldsymbol{X} + \boldsymbol{u})_{,3} \times \boldsymbol{T}\} \cdot \delta \boldsymbol{\phi}] \mathrm{d}L + [\underset{S_u + S_\sigma}{\boldsymbol{T}} \cdot \delta \boldsymbol{u} + \boldsymbol{M} \cdot \delta \boldsymbol{\phi}]_{L=0}^{L=l}, \tag{30}$$

where l is the length of the beam axis before the deformation, S_u and S_σ are parts of boundary on which geometrical and mechanical boundary conditions are prescribed respectively.

Let $\boldsymbol{P}_b$ be the vector of body force defined per unit volume of the undeformed beam, $\boldsymbol{P}_c$ the vector of distributed surface traction defined per unit area of the undeformed cylindrical surface of the beam, denoted as S_c; and $\boldsymbol{P}_e$ the vector of distributed surface tractions at the end cross sections denoted as S_e. Then the external virtual work is written as (Washizu 1982):

$$EVW = \int \boldsymbol{P}_b \cdot \delta \boldsymbol{V} \mathrm{d}V + \int \boldsymbol{P}_c \cdot \delta \boldsymbol{V} \mathrm{d}S_c + [\underset{S_\sigma}{\int} \boldsymbol{P}_e \cdot \delta \boldsymbol{V} \mathrm{d}S_e]_{L=0}^{L=l}, \tag{31}$$

where $\boldsymbol{V}$ is the displacement vector at an arbitrary material point defined by $\boldsymbol{V} = \boldsymbol{u} + Y^\alpha (\boldsymbol{e}_\alpha - \boldsymbol{E}_\alpha)$ and $\mathrm{d}S_c = |\boldsymbol{R}_{,s} \times \boldsymbol{R}_{,3}| \mathrm{d}s \mathrm{d}L$ in which s is the coordinate taken along the bounding curve of the cross sections.

At first, we assume that the directions of external forces do not change during the deformation. Therefore we may write the external forces in the form

$$\boldsymbol{P}_b = P_b^j \boldsymbol{E}_j, \qquad \boldsymbol{P}_c = P_c^j \boldsymbol{E}_j, \qquad \boldsymbol{P}_e = P_e^j \boldsymbol{E}_j, \tag{32a–c}$$

where P_b^j, P_c^j and P_e^j are constant. Introducing Eqs. (32) into Eq. (31) yields

$$EVW = \int \boldsymbol{q} \cdot \delta \boldsymbol{u} + \boldsymbol{m} \cdot \delta \boldsymbol{\phi}] \mathrm{d}L + [\underset{S_\sigma}{\bar{\boldsymbol{q}}} \cdot \delta \boldsymbol{u} + \bar{\boldsymbol{m}} \cdot \delta \boldsymbol{\phi}]_{L=0}^{L=l}, \tag{33}$$

where

$$\boldsymbol{q} = q^j \boldsymbol{E}_j, \qquad \bar{\boldsymbol{q}} = \bar{q}^j \boldsymbol{E}_j, \qquad \boldsymbol{m} = m_{\alpha j} \boldsymbol{e}_\alpha \times \boldsymbol{E}_j, \qquad \bar{\boldsymbol{m}} = \bar{m}_{\alpha j} \boldsymbol{e}_\alpha \times \boldsymbol{E}_j, \tag{34a–d}$$

$$q^j = \int P_b^j g_0 dA + \int P_c^j |\boldsymbol{R}_{,s} \times \boldsymbol{R}_{,3}| \mathrm{d}s, \qquad \bar{q}^j = \int P_e^j \mathrm{d}S_e, \tag{34e, f}$$

$$m_{\alpha j} = \int Y^\alpha P_b^j g_0 dA + \int Y^\alpha P_c^j |\boldsymbol{R}_{,s} \times \boldsymbol{R}_{,3}| \mathrm{d}s, \qquad \bar{m}_{\alpha j} = \int Y^\alpha P_e^j \mathrm{d}S_e. \tag{34g, h}$$

From the principle of virtual work, i.e., $IVW = EVW$ (Washizu 1982), we obtain the linear momentum balance *(LMB)* and angular momentum balance *(AMB)* conditions, expressed as:

$$\boldsymbol{T}_{,3} + \boldsymbol{q} = \boldsymbol{0}, \quad \text{(for arbitrary } \delta\boldsymbol{u})$$

$$\boldsymbol{M}_{,3} + (\boldsymbol{X} + \boldsymbol{u})_{,3} \times \boldsymbol{T} + \boldsymbol{m} = \boldsymbol{0} \quad \text{(for arbitrary } \delta\boldsymbol{\phi}). \tag{35a, b}$$

The associated boundary conditions are written as

$$\boldsymbol{T} = \bar{\boldsymbol{q}}; \quad \boldsymbol{M} = \bar{\boldsymbol{m}} \quad \text{on } S_\sigma, \quad \boldsymbol{u} = \bar{\boldsymbol{u}}; \quad \boldsymbol{\phi} = \bar{\boldsymbol{\phi}} \quad \text{on } S_u, \tag{36a–d}$$

where $\bar{\boldsymbol{u}}$ and $\bar{\boldsymbol{\phi}}$ denote the prescribed values on S_u.

It is well known that the displacement field and the variations of variables determine whether the energy method yields the same equilibrium equations as those derived by the static method. In this paper, because of using $\delta\mathbf{R} \cdot \mathbf{R}^T$ as the rotational variation, we can derive the same *LMB* and *AMB* conditions as those derived by the static method (Ericksen and Truesdell 1958; Reissner 1973). The consistent boundary conditions are also obtained in the process. It follows from the above derivation, Eqs. (33–36), that the "external moment" vectors $\boldsymbol{m}$ (Eqs. 35b, 34c) and $\bar{\boldsymbol{m}}$ (Eqs. 36b, 34d] are dependent on deformations even though the components of external moments $m_{\alpha j}$ and $\bar{m}_{\alpha j}$ are independent of deformations. In the existing literature (Ericksen and Truesdell 1958; Reissner 1973, 1981), where static method has been employed, the existence of an "external moment" vector is assumed, a priori, and then the *AMB* condition takes the same form as that given by Eq. (35b).

Next, for later convenience, we consider the "non-conservative" type follower forces defined by

$$\hat{\boldsymbol{P}}_b = \hat{P}_b^j \boldsymbol{e}_j; \quad \hat{\boldsymbol{P}}_c = \hat{P}_c^j \boldsymbol{e}_j; \quad \hat{\boldsymbol{P}}_e = \hat{P}_e^j \boldsymbol{e}_j, \tag{37a–c}$$

where $\hat{\boldsymbol{P}}_b$ is the vector of body force defined per unit volume of the undeformed beam, $\hat{\boldsymbol{P}}_c$ the vector of distributed surface force defined per unit area of the undeformed cylindrical surface of the beam, S_c; and $\hat{\boldsymbol{P}}_e$ the vector of distributed surface tractions at the end cross section, S_e. In this case, the *EVW* is written as

$$EVW = \int [\hat{\boldsymbol{q}} \cdot \delta\boldsymbol{u} + \hat{\boldsymbol{m}} \cdot \delta\boldsymbol{\phi}] \mathrm{d}L + [\hat{\bar{\boldsymbol{q}}} \cdot \delta\boldsymbol{u} + \hat{\bar{\boldsymbol{m}}} \cdot \delta\boldsymbol{\phi}]_{\substack{L=0 \\ S_\sigma}}^{L=l}, \tag{38}$$

where

$$\hat{\boldsymbol{q}} = \hat{q}^j \boldsymbol{e}_j; \quad \hat{\bar{\boldsymbol{q}}} = \hat{\bar{q}}^j \boldsymbol{e}_j; \quad \hat{\boldsymbol{m}} = \hat{m}_{\alpha j} \boldsymbol{e}_\alpha \times \boldsymbol{e}_j; \quad \hat{\bar{\boldsymbol{m}}} = \hat{\bar{m}}_{\alpha j} \boldsymbol{e}_\alpha \times \boldsymbol{e}_j,$$

$$\hat{q}^j = \int \hat{P}_b^j g_0 dA + \int \hat{P}_c^j |\boldsymbol{R}_{,s} \times \boldsymbol{R}_{,3}| \mathrm{d}s; \qquad \hat{\bar{q}}^j = \int \hat{P}_e^j \mathrm{d}S_e, \tag{39e, f}$$

$$\hat{m}_{\alpha j} = \int Y^\alpha \hat{P}_b^j g_0 dA + \int Y^\alpha \hat{P}_c^j |\boldsymbol{R}_{,s} \times \boldsymbol{R}_{,3}| \mathrm{d}s; \qquad \hat{\bar{m}}_{\alpha j} = \int Y^\alpha \hat{P}_e^j \mathrm{d}S_e. \tag{39g–h}$$

Since $IVW = EVW$, the equilibrium equations are obtained as

$$\boldsymbol{T}_{,3} + \hat{\boldsymbol{q}} = \boldsymbol{0}; \quad \boldsymbol{M}_{,3} + (\boldsymbol{X} + \boldsymbol{u})_{,3} \times \boldsymbol{T} + \hat{\boldsymbol{m}} = \boldsymbol{0}. \tag{40, b}$$

The associated boundary conditions are written as

$$\boldsymbol{T} = \hat{\bar{\boldsymbol{q}}}; \quad \boldsymbol{M} = \hat{\bar{\boldsymbol{m}}} \quad \text{on} \quad S_\sigma, \quad \boldsymbol{u} = \bar{\boldsymbol{u}}; \quad \boldsymbol{\phi} = \bar{\boldsymbol{\phi}} \quad \text{on} \quad S_u, \tag{41a–d}$$

As is well known, in the case of follower forces, not only the external moment vector $\hat{\boldsymbol{m}}$ but also the external force vector $\hat{\boldsymbol{q}}$ are dependent on deformations.

In summary, the *IVW* is given by Eq. (30); and *EVW* is given in the case of fixed-directional ('dead') loading by Eq. (33) while in the case of follower loading it is given by Eq. (38). In both the cases of loading, the principle of virtual work is

$$IVW - EVW = 0.$$

The question then arises if the above equation can be written, equivalently, as the stationary condition or the vanishing of the first variation of a well-defined functional. It will be shown in

Sect. 6 and 7 of this paper that when $\boldsymbol{T}$ and $\boldsymbol{M}$ are expressed in terms of appropriate kinematic variables, *IVW* of Eq. (30) can be expressed as the first variation of an internal energy functional, written in terms of $\mathbf{U}$ and $\mathbf{R}$. On the other hand, *even in the case of conservative loading*, the *EVW*, especially of the moments, i.e., the term $\boldsymbol{m} \cdot \delta\boldsymbol{\phi}$ [with the $\boldsymbol{m}$ as defined in Eq. (34c) and $\delta\boldsymbol{\phi}$ as defined through $\delta\boldsymbol{\phi} \times \mathbf{I} = \delta\mathbf{R} \cdot \mathbf{R}^T$] does not, on first sight, appear to correspond to the first variation of an external energy functional. This has certain implications in constructing weak solutions, say based on the finite element method. It is well-known that if the governing equations of the problem can be written as the Euler-Lagrange equations corresponding to the vanishing of the first variation of a well-defined functional, and if similar basis functions are used for the trial and test functions, the tangent-stiffness matrix *at an equilibrium state* is always symmetric. If arbitrary trial functions are used for $\mathbf{R}$, and arbitrary and different test functions are used for $\delta\boldsymbol{\phi}$, it is seen from Eqs. (30), (33), and (34c), that the geometric stiffness matrix derived from the principle of virtual work will be unsymmetric even for conservative loading. Further, the contributions to the unsymmetric stiffness arise not only from the *IVW* of Eq. (30), but also from the $\boldsymbol{m} \cdot \delta\boldsymbol{\phi}$ term in *EVW* of Eq. (33). In the case of non-conservative loading, the *EVW* of Eq. (38) will in any case not correspond to the first variation of an energy functional, and will lead to a contribution to the unsymmetric stiffness matrix at an equilibrium state.

In order to express $(\boldsymbol{m} \cdot \delta\boldsymbol{\phi})$ of Eq. (33) as the first variation of an energy functional, we adopt a strategy wherein $(\boldsymbol{m} \cdot \delta\boldsymbol{\phi})$ can be expressed in terms of components of $\boldsymbol{m}$ and $\delta\boldsymbol{\phi}$ in the *undeformed basis*, $\boldsymbol{E}_j$. Thus, from (34c) and the definition of $\delta\boldsymbol{\phi}$ [i.e., $\delta\boldsymbol{\phi} \times \mathbf{I} = \delta\mathbf{R} \cdot \mathbf{R}^T$), we have:

$$\boldsymbol{m} \cdot \delta\boldsymbol{\phi} = m_{\alpha j}(\boldsymbol{e}_\alpha \times \boldsymbol{E}_j) \cdot \delta\boldsymbol{\phi} = m_{\alpha j}\,\boldsymbol{E}_j \cdot (\delta\boldsymbol{\phi} \times \boldsymbol{e}_\alpha) = m_{\alpha j}\,\boldsymbol{E}_j \cdot (\delta\mathbf{R} \cdot \mathbf{R}^T \cdot \boldsymbol{e}_\alpha) = m_{\alpha j}[\boldsymbol{E}_j \cdot \delta\mathbf{R} \cdot \boldsymbol{E}_\alpha]. \tag{42}$$

If further, one writes $\mathbf{R}$ in terms of Lagrangean components, i.e., $\mathbf{R} = R_{jk}\,\boldsymbol{E}_j\,\boldsymbol{E}_k$; then $\delta\mathbf{R} = \delta R_{jk}\,\boldsymbol{E}_j\,\boldsymbol{E}_k$, and hence,

$$\boldsymbol{m} \cdot \delta\boldsymbol{\phi} = m_{\alpha j}\,\delta R_{j\alpha} = \delta\,[m_{\alpha j} R_{j\alpha}], \tag{43}$$

where $m_{\alpha j}$ are given constants as defined in (34g). We assume that the Lagrangean components R_{jk} of $\mathbf{R}$ are expressed in terms of three arbitrary parameters α^j, such that $(\delta\mathbf{R} = R_{jk;i}\,\boldsymbol{E}_j\,\boldsymbol{E}_k\,\delta\alpha^i)$, where $(\ )_{;i}$ denotes the differentiation with respect to α^i. When a dead-load system of forces as in Eq. (32) is considered, the principle of virtual work $IVW = EVW$ [with *IVW* as in (30) and *EVW* as in (33)] leads to the linear momentum balance conditions as in (35a); however, the angular momentum balance conditions corresponding to arbitrary variations $\delta\alpha^i$ become:

$$N_j + r_j = 0, \qquad [AMB \text{ for } (\delta\alpha^j)], \tag{44}$$

where

$$N_j = Q^1\,(\boldsymbol{e}_3 \cdot \boldsymbol{E}_i)\,R_{i2;j} + Q^2(\boldsymbol{e}_1 \cdot \boldsymbol{E}_i)\,R_{i3;j} + Q^3\,(\boldsymbol{e}_2 \cdot \boldsymbol{E}_i)\,R_{i1;j},$$

$$r_j = m_{\alpha i}\,R_{i\alpha;j}; \qquad Q^i\,\boldsymbol{e}_i = \boldsymbol{M}_{,3} + (\boldsymbol{X} + \boldsymbol{u})_{,3} \times \boldsymbol{T}, \tag{45a–c}$$

in which $m_{\alpha i}$ are defined in Eq. (34g). The associated boundary conditions are written as

$$L_j = \bar{r}_j \quad \text{on } S_\sigma; \qquad \alpha^j = \bar{\alpha}^j \quad \text{on } S_u, \tag{46a, b}$$

where $\bar{\alpha}^j$ denote the prescribed values on S_u and

$$L_j = M^1\,(\boldsymbol{e}_3 \cdot \boldsymbol{E}_i)\,R_{i2;j} + M^2\,(\boldsymbol{e}_1 \cdot \boldsymbol{E}_i)\,R_{i3;j} + M^3\,(\boldsymbol{e}_2 \cdot \boldsymbol{E}_i)\,R_{i1;j}. \tag{47}$$

The external moments denoted by $\bar{r}_j$ are obtained from Eq. (45b) by replacing $m_{\alpha i}$ by $\bar{m}_{\alpha i}$, where $\bar{m}_{\alpha i}$ are defined in Eq. (34h).

To show the equivalence of *AMB* condition given by Eq. (35b), associated with $\delta\boldsymbol{\phi}$, and by Eq. (44), associated with $\delta\alpha^j$, we consider the tensor equations of *AMB* conditions.
Since $\delta\mathbf{R} \cdot \mathbf{R}^T = \delta\boldsymbol{\phi} \times \mathbf{I}$, we have

$$\{\boldsymbol{M}_{,3} + (\boldsymbol{X} + \boldsymbol{u})_{,3} \times \boldsymbol{T} + \boldsymbol{m}\} \cdot \delta\boldsymbol{\phi} = \mathbf{C} : (\delta\mathbf{R} \cdot \mathbf{R}^T), \tag{48}$$

where the use of Eq. (45c) is made and

$$\mathbf{C} = Q^1\,\boldsymbol{e}_3\,\boldsymbol{e}_2 + Q^2\,\boldsymbol{e}_1\,\boldsymbol{e}_3 + Q^3\,\boldsymbol{e}_2\,\boldsymbol{e}_1 + m_{\alpha m}\,\boldsymbol{E}_m\,\boldsymbol{e}_\alpha. \tag{49}$$

On the other hand, with simple manipulation, we have

$$(N_j + r_j)\,\delta\alpha^j = \{\, \mathbf{C} : (\mathbf{R}_{,j} \cdot \mathbf{R}^T) \,\}\, \delta\alpha^j . \tag{50}$$

As shown in Eq. (48), the *AMB* condition associated with $\delta\boldsymbol{\phi}$ is that $\mathbf{C} - \mathbf{C}^T = 0$. While, as shown in Eq. (50), the *AMB* conditions associated with $\delta\alpha^j$ is that $\mathbf{C} : (\mathbf{R}_{,j} \cdot \mathbf{R}^T) = 0$. It is shown consequently that, since $\mathbf{R}_{,j} \cdot \mathbf{R}^T$ is a skewsymmetric tensor, the *AMB* conditions associated with $\delta\boldsymbol{\phi}$ is equivalent to that associated with $\delta\alpha^j$.

6 Constitutive equations

In a finite displacement theory, a variety of stress tensor has been used. As a result, a number of constitutive equations have been proposed. We postulate herein that the present materials are homogeneous, isotropic and linearly elastic.

Equations (26) indicate that the use of the stress tensor $t^{m\bar{n}}$ yields the compact definition for the stress resultants and moments. Therefore, we utilize the stress tensor $t^{m\bar{n}}$ and the conjugate strain tensors $\gamma_{m\bar{n}}$ to construct the constitutive equations. The conjugate strain tensor $\gamma_{m\bar{n}}$ are given by (Appendix 2)

$$\gamma_{m\bar{n}} = \boldsymbol{a}_m \cdot \boldsymbol{e}_n - \boldsymbol{A}_m \cdot \boldsymbol{E}_n . \tag{51}$$

For one-dimensional beams, we assume the following relationships:

$$t^{3\bar{\alpha}} = G\gamma_{3\bar{\alpha}}; \qquad t^{3\bar{3}} = E\gamma_{3\bar{3}}, \tag{52a, b}$$

where G is the shearing modulus and E the Young modulus. Substituting Eq. (51) and (52) into Eqs. (26) and modifying the shear rigitity yields

$$\begin{bmatrix} T^1 \\ T^2 \\ T^3 \\ M^1 \\ M^2 \\ M^3 \end{bmatrix} = \begin{bmatrix} GA_0 & 0 & 0 & 0 & 0 & -GI_1 \\ & GA_0 & 0 & 0 & 0 & GI_2 \\ & & EA & EI_1 & -EI_2 & 0 \\ & & & EI_{11} & -EI_{12} & 0 \\ \text{Sym.} & & & & EI_{22} & 0 \\ & & & & & GJ \end{bmatrix} \begin{bmatrix} h_1 \\ h_2 \\ h_3 \\ \tilde{k}_1 \\ \tilde{k}_2 \\ \tilde{k}_3 \end{bmatrix}, \tag{53}$$

where

$$h_1 = g\sin\beta_1; \quad h_2 = g\sin\beta_2; \quad h_3 = g\beta_3 - 1, \quad A = \int g_0\,\mathrm{d}A; \quad A_0 = kA; \quad I_\alpha = \int Y^\alpha g_0\,\mathrm{d}A, \tag{54a–f}$$

$$I_{12} = \int Y^1 Y^2 g_0\,\mathrm{d}A; \quad I_{11} = \int (Y^2)^2 g_0\,\mathrm{d}A; \quad I_{22} = \int (Y^1)^2 g_0\,\mathrm{d}A, \quad J = \int \varrho\, g_0\,\mathrm{d}A. \tag{54g–j}$$

The factor k is a shear-correction factor (Cowper 1966). It is worth noting that if we introduce the constitutive equations expressed by the second Piola-Kirchhoff stress tensor and the Green strain tensor such that $S_1^{3\alpha} = G\varepsilon_{3\alpha}$ and $S_1^{33} = E\varepsilon_{33}$, the constitutive Eqs. (53) *are not obtained.*

Next we consider the appropriate choices for the origin of coordinates Y^m. It is possible, even for a naturally curved and twisted beam, to choose the origin so that $I_\alpha = I_{12} = 0$. Another way is to choose the origin so that the coordinate Y^3 coincides with the fiber axis of beams. In the latter case, the coefficients I_α and I_{12} do not always become zero.

For later convenience, we introduce the strain energy function W_s expressed as

$$\begin{aligned} W_s = {} & \frac{1}{2} GA_0(h_1)^2 + \frac{1}{2} GA_0(h_2)^2 + \frac{1}{2} EA(h_3)^2 + \frac{1}{2} EI_{11}(\tilde{k}_1)^2 + \frac{1}{2} EI_{22}(\tilde{k}_2)^2 + \frac{1}{2} GJ(\tilde{k}_3)^2 \\ & + EI_1 h_3 \tilde{k}_1 - EI_2 h_3 \tilde{k}_2 - EI_{12} \tilde{k}_1 \tilde{k}_2 - GI_1 h_1 \tilde{k}_3 + GI_2 h_2 \tilde{k}_3 . \end{aligned} \tag{55}$$

It should be emphasized that the present strain energy function is derived from the stress-strain relationships given Eqs. (52).

7 General mixed variational principle

As a basis of a numerical method, a variational principle often plays an important role. In finite elasticity, so far, the principle of stationary potential energy has been more widely used. The pure displacement formulation, however, has now been abandoned by most researchers of finite element formulations (Hibbitt 1986). Therefore, recently, the general mixed variational formulations are receiving a wide attention. In this section, we will derive the functional for general mixed variational principle for finitely deformed beams. Based on the resulting functional, we will present the incremental functionals in the context of a total Lagrangian (TL) formulation and an updated Lagrangian (UL) formulation.

As first shown by Fraeijs de Veubeke (1972), and later generalized by Atluri and Murakawa (1977), a general mixed principle, for a 3-dimensional elastic material, and involving the first Piola-Kirchhoff stress tensor $\mathbf{t}_1$, the right stretch tensor $\mathbf{U}$ the finite rotation tensor $\mathbf{R}$ and the displacement vector $\boldsymbol{v}$ as variables, can be stated as the stationary condition of the functional F_1:

$$F_1(\mathbf{t}_1, \mathbf{U}, \mathbf{R}, \boldsymbol{v}) = \int_{v_0} [W_0(\mathbf{U}) + \mathbf{t}_1^T : \{(\mathbf{I} + \nabla_0\, \boldsymbol{v})^T - \mathbf{R} \cdot \mathbf{U}\} - \varrho_0 \bar{\boldsymbol{b}} \cdot \boldsymbol{v}] \mathrm{d}V_0 - \int_{S_\sigma} \bar{\boldsymbol{t}} \cdot \boldsymbol{v} \,\mathrm{d}s - \int_{S_u} \boldsymbol{t} \cdot (\boldsymbol{v} - \bar{\boldsymbol{v}}) \mathrm{d}s, \tag{56}$$

where W_0 is a strain energy function, ϱ_0 the mass density in the undeformed state, $\bar{\boldsymbol{b}}$ the body force vector per unit mass, $\bar{\boldsymbol{t}}$ the traction on the boundary per unit undeformed area and ∇_0 the gradient operator in the undeformed state.

The functional F_1 for a finitely deformed shell has been derived by Atluri (1984). Based on the resulting modified functional (Atluri, 1984), some numerical results have been obtained by Punch and Atluri (1986). However, to the best of the author's knowledge, no studies exist on the functional F_1 for a finitely deformed beam.

We see that the following relationships hold for the present problem of a beam:

$$\nabla_0(\ ) = \boldsymbol{A}^m (\ )_{,m}; \quad \mathbf{I} = \boldsymbol{A}_m \boldsymbol{A}^m; \quad \mathbf{t}_1 = t_1^{ij} \boldsymbol{A}_i \boldsymbol{a}_j, \tag{57a–c}$$

$$\mathbf{U} = \boldsymbol{A}_\alpha \boldsymbol{A}^\alpha + \{\boldsymbol{h} + \boldsymbol{E}_3 - Y^2 k_3 \boldsymbol{E}_1 + Y^1 k_3 \boldsymbol{E}_2 + (-Y^1 k_2 + Y^2 k_1) \boldsymbol{E}_3\} \boldsymbol{A}^3; \qquad \boldsymbol{h} = h_m \boldsymbol{E}_m. \tag{57d, e}$$

Since $\mathbf{t}_1 = \mathbf{S}_1 \cdot \mathbf{F}^T$; $\mathbf{S}_1 = S_1^{ij} \boldsymbol{A}_i \boldsymbol{A}_j$; $\mathbf{F} = \boldsymbol{a}_j \boldsymbol{A}^j$ (Atluri 1984), the components S_1^{ij} are numerically equal to the components t_1^{ij}. Since $\mathbf{R} \cdot \mathbf{R}^T = \mathbf{I}$, it follows that $\mathbf{R}_{,3} \cdot \mathbf{R}^T$ is a skewsymmetric tensor. There exists, therefore, the vector $\boldsymbol{l}_3$ such that $\mathbf{R}_{,3} \cdot \mathbf{R}^T = \boldsymbol{l}_3 \times \mathbf{I}$. In terms of $\boldsymbol{\omega}$ of Eq. (10c), the vector $\boldsymbol{l}_3$ is represented as (Pietraszkiewicz and Badur 1983)

$$\boldsymbol{l}_3 = \frac{\sin\omega}{\omega} \boldsymbol{\omega}_{,3} - \frac{1-\cos\omega}{(\omega)^2} \boldsymbol{\omega}_{,3} \times \boldsymbol{\omega} + \left\{\frac{1}{\omega} - \frac{\sin\omega}{(\omega)^2}\right\} \omega_{,3} \boldsymbol{\omega}. \tag{58}$$

Since $\boldsymbol{e}_{\alpha,3} = \boldsymbol{l}_3 \times \boldsymbol{e}_\alpha + \mathbf{R} \cdot \boldsymbol{E}_{\alpha,3}$, we have

$$\mathbf{t}_1^T : \{\mathbf{I} + \nabla_0\ \boldsymbol{v})^T - \mathbf{R} \cdot \mathbf{U})\} = t^{3j} \boldsymbol{a}_j \cdot [(\boldsymbol{x} + \boldsymbol{u})_{,3} - \mathbf{R} \cdot (\boldsymbol{h} + \boldsymbol{E}_3) + Y^\alpha \boldsymbol{l}_3 \times \boldsymbol{e}_\alpha - (\mathbf{R} \cdot \bar{\boldsymbol{k}}) \times Y^\alpha \boldsymbol{e}_\alpha], \tag{59}$$

where $\boldsymbol{k} = k_m \boldsymbol{E}_m$. (60)

Using Eqs. (59) and (56), and employing the notations given earlier in this paper, the functional F_1 for a finitely deformed beam is expressed, after some algebra, as

$$F_1(\boldsymbol{T}, \boldsymbol{M}, \boldsymbol{h}, \bar{\boldsymbol{k}}, \boldsymbol{u}, \mathbf{R}) = \int [W_s(\boldsymbol{h}, \bar{\boldsymbol{k}}) + \boldsymbol{T} \cdot \{(\boldsymbol{x} + \boldsymbol{u})_{,3} - \mathbf{R} \cdot (\boldsymbol{h} + \boldsymbol{E}_3)\} + \boldsymbol{M} \cdot \{\boldsymbol{l}_3 - \mathbf{R} \cdot \bar{\boldsymbol{k}}\} - \boldsymbol{q} \cdot \boldsymbol{u}] \mathrm{d}L - \underset{S_u}{[}\boldsymbol{T} \cdot (\boldsymbol{u} - \bar{\boldsymbol{u}}) + \boldsymbol{M} \cdot (\boldsymbol{\phi} - \bar{\boldsymbol{\phi}})]_{L=0}^{L=l} - \underset{S_\sigma}{[}\bar{\boldsymbol{q}} \cdot \boldsymbol{u}]_{L=0}^{L=l}, \tag{61}$$

where $\boldsymbol{l}_3$ is a vector function of $\mathbf{R}$. We now consider the first variation of F_1, which is:

$$\delta F_1 = \int \Big[\Big(\frac{\partial W_s}{\partial h_m} - T^m\Big) \delta h_m + \Big(\frac{\partial W_s}{\partial \bar{k}_m} - M^m\Big) \delta \bar{k}_m + \delta \boldsymbol{T} \cdot \{(\boldsymbol{x} + \boldsymbol{u})_{,3} - R \cdot (\boldsymbol{h} + \boldsymbol{E}_3)\} + \delta \boldsymbol{M} \cdot \{\boldsymbol{l}_3 - \mathbf{R} \cdot \bar{\boldsymbol{k}}\} - (\boldsymbol{T}_{,3} + \boldsymbol{q}) \cdot \delta \boldsymbol{u} - \{\boldsymbol{M}_{,3} + (\boldsymbol{x} + \boldsymbol{u})_{,3} \times \boldsymbol{T}\} \cdot \delta \boldsymbol{\phi}] \mathrm{d}L - \underset{S_u}{[}\delta \boldsymbol{T} \cdot (\boldsymbol{u} - \bar{\boldsymbol{u}}) + \delta \boldsymbol{M} \cdot (\boldsymbol{\phi} - \bar{\boldsymbol{\phi}})]_{L=0}^{L=l} - \underset{S_\sigma}{[}(\bar{\boldsymbol{q}} - \boldsymbol{T}) \cdot \delta \boldsymbol{u} - \boldsymbol{M} \cdot \delta \boldsymbol{\phi}]_{L=0}^{L=l}. \tag{62}$$

It can be seen that the stationary condition, $\delta F_1 = 0$, leads to the constitutive Eqs. (53), the compatibility Eqs. (17) and (21), the *LMB* and *AMB* conditions (35), and the mechanical and geometrical boundary conditions (36). It is reemphasized that *no external moments* are included in F_1.

When external moments $\boldsymbol{m}$, due to dead-load type of forces, as in Eq. (34c) are present, as shown in Sect. 5, the functional F_1, may be modified as:

$$G_1(\boldsymbol{T}, \boldsymbol{M}, \boldsymbol{h}, \tilde{\boldsymbol{k}}, \boldsymbol{u}, \alpha^m, L_j^+) = \int [W_s(\boldsymbol{h}, \tilde{\boldsymbol{k}}) + \mathbf{T} \cdot \{(\boldsymbol{x}+\boldsymbol{u})_{,3} - \mathbf{R}\cdot(\boldsymbol{h}+\boldsymbol{E}_3)\} + \boldsymbol{M}\cdot\{\boldsymbol{l}_3 - \mathbf{R}\cdot\tilde{\boldsymbol{k}}\}$$
$$- \boldsymbol{q}\cdot\boldsymbol{u} - m_{\alpha i} R_{i\alpha}]\,\mathrm{d}L - \underset{S_u}{[}\,\boldsymbol{T}\cdot(\boldsymbol{u}-\bar{\boldsymbol{u}}) + L_j^+(\alpha^j - \bar{\alpha}^j)]_{L=0}^{L=l} - \underset{S_\sigma}{[}\,\bar{\boldsymbol{q}}\cdot\boldsymbol{u} + \bar{m}_{\alpha i} R_{i\alpha}]_{L=0}^{L=l}, \tag{63}$$

where L_j^+ is a Lagrangian multiplier and $\mathbf{R}$ and $\boldsymbol{l}_3$ are functions of α^m. The first variation of G_1 takes the form

$$\delta G_1 = \int\Big[\Big(\frac{\partial W_s}{\partial h_m} - T^m\Big)\delta h_m + \Big(\frac{\partial W_s}{\partial \tilde{k}_m} - M^m\Big)\delta\tilde{k}_m + \delta\boldsymbol{T}\cdot\{(\boldsymbol{x}+\boldsymbol{u})_{,3} - \mathbf{R}\cdot(\boldsymbol{h}+\boldsymbol{E}_3)\}$$
$$+ \delta\boldsymbol{M}\cdot\{\boldsymbol{l}_3 - \mathbf{R}\cdot\tilde{\boldsymbol{k}}\} - (\boldsymbol{T}_{,3}+\boldsymbol{q})\cdot\delta\boldsymbol{u} - (N_j + r_j)\,\delta\alpha^j]\,\mathrm{d}L$$
$$- \underset{S_u}{[}\,\delta\boldsymbol{T}\cdot(\boldsymbol{u}-\bar{\boldsymbol{u}}) + \delta L_j^+(\alpha^j - \bar{\alpha}^j) + (L_j^+ - L_j)\,\delta\alpha^j]_{L=0}^{L=l}$$
$$- \underset{S_\sigma}{[}\,(\bar{\boldsymbol{q}} - \boldsymbol{T})\cdot\delta\boldsymbol{u} + (\bar{r}_j - L_j)\,\delta\alpha^j]_{L=0}^{L=l}. \tag{64}$$

The physical meaning of the Lagrangian multiplier L_j^+ is clear from Eqs. (64). The stationary condition, $\delta G_1 = 0$, yields the constitutive Eqs. (53), the compatibility Eqs. (17) and (21), the *LMB* condition (35 a) and *AMB* condition (44), and the mechanical and geometrical boundary conditions (36 a), (46 a), (36 c) and (46 b). It should be stressed that the effects of the external moments due to a system of conservative forces, are taken into account in G_1.

For an incremental approach, we construct the incremental functionals in the context of TL and UL formulation. Let C_0 be the initial known configuration of beams, and let C_N and C_{N+1}, respectively, be the configuration prior to, and after, the addition of the $(N+1)$-th increment of prescribed loads and/or deformations. In the TL formulation, the fixed metric of C_0 is used to refer to all the state variables in each successive configuration. In the UL formulation, the variables in the state C_{N+1} are referred to the configuration in C_N. Let $(\ )^N$ denote the variable in C_N and $\Delta(\ )$ the incremental variable in passing from C_N to C_{N+1}. Note that all variables are referred to the convected coordinate system Y^m.

(i) TL formulation. At first we consider the incremental functional of F_1. In the TL formulation, the finite rotation tensor is required to satisfy the orthogonality condition that $(\mathbf{R}^N + \Delta\mathbf{R})\cdot(\mathbf{R}^N + \Delta\mathbf{R})^T = \mathbf{I}$; or in a variational sense, $\delta\Delta\mathbf{R}\cdot(\mathbf{R}^N + \Delta\mathbf{R})^T$ is skewsymmetric. As a result, we obtain the incremental functional of F_1 in the form

$$\Delta F_1(\Delta\boldsymbol{T}, \Delta\boldsymbol{M}, \Delta\boldsymbol{h}, \Delta\tilde{\boldsymbol{k}}, \Delta\mathbf{R}, \Delta\boldsymbol{u}) = \int_{c_0} [\Delta W_s + \Delta\boldsymbol{T}\cdot\{\Delta\boldsymbol{u}_{,3} - \mathbf{R}^N\cdot\Delta\boldsymbol{h} - \Delta\mathbf{R}\cdot(\boldsymbol{h}^N + \boldsymbol{E}_3 + \Delta\boldsymbol{h})\}$$
$$- \boldsymbol{T}^N\cdot\{\Delta\mathbf{R}\cdot(\boldsymbol{h}^N + \boldsymbol{E}_3 + \Delta\boldsymbol{h})\} + (\boldsymbol{M}^N + \Delta\boldsymbol{M})\cdot\{\Delta\boldsymbol{l}_3 - \mathbf{R}^N\cdot\Delta\tilde{\boldsymbol{k}} - \Delta\mathbf{R}\cdot(\tilde{\boldsymbol{k}}^N + \Delta\tilde{\boldsymbol{k}})\}$$
$$- \Delta\boldsymbol{q}\cdot\Delta\boldsymbol{u}]\,\mathrm{d}L - \underset{S_u}{[}\,\Delta\boldsymbol{T}\cdot(\Delta\boldsymbol{u} - \Delta\bar{\boldsymbol{u}}) + (\boldsymbol{M}^N + \Delta\boldsymbol{M})\cdot(\Delta\boldsymbol{\phi} - \Delta\bar{\boldsymbol{\phi}})]_{L=0}^{L=l} - \underset{S_\sigma}{[}\,\Delta\bar{\boldsymbol{q}}\cdot\Delta\boldsymbol{u}]_{L=0}^{L=l}, \tag{65}$$

where

$$\Delta W_s = \frac{1}{2}GA_0(\Delta h_1)^2 + \frac{1}{2}GA_0(\Delta h_2)^2 + \frac{1}{2}EA(\Delta h_3)^2 + \frac{1}{2}EI_{11}(\Delta\tilde{k}_1)^2 + \frac{1}{2}EI_{22}(\Delta\tilde{k}_2)^2 + \frac{1}{2}GJ(\Delta\tilde{k}_3)^2$$
$$+ EI_{11}\tilde{k}_1^N\Delta\tilde{k}_1 + EI_{22}\tilde{k}_2^N\Delta\tilde{k}_2 + GJ\tilde{k}_3^N\Delta\tilde{k}_3 + EI_1(h_3^N + \Delta h_3)\Delta\tilde{k}_1 - EI_2(h_3^N + \Delta h_3)\Delta\tilde{k}_2$$
$$- GI_1(h_1^N + \Delta h_1)\Delta\tilde{k}_3 + GI_2(h_2^N + \Delta h_2)\Delta\tilde{k}_3 - EI_{12}\{\Delta\tilde{k}_1(\tilde{k}_2^N + \Delta\tilde{k}_2) + \tilde{k}_1^N\Delta\tilde{k}_2\}. \tag{66}$$

The linear and third order terms with respect to incremental values are included in ΔF_1 to satisfy the orthogonality condition of finite rotation tensor exactly. The variational equations corresponding to $\delta\Delta F_1 = 0$ are, as can be shown easily

$$\frac{\partial \Delta W_s}{\partial \Delta h_m} = \Delta T^m; \qquad \frac{\partial \Delta W_s}{\partial \Delta \tilde{k}_m} = M^m + \Delta M^m; \qquad \Delta \boldsymbol{u}_{,3} = \mathbf{R}^N \cdot \Delta \boldsymbol{h}^n + \Delta \mathbf{R} \cdot (\boldsymbol{h}^N + \boldsymbol{E}_3^N + \Delta \boldsymbol{h}), \tag{67a–c}$$

$$\Delta \boldsymbol{l}_3 = \mathbf{R}^N \cdot \Delta \tilde{\boldsymbol{k}} + \Delta \mathbf{R} \cdot (\tilde{\boldsymbol{k}}^N + \Delta \tilde{\boldsymbol{k}}); \qquad \Delta \boldsymbol{T}_{,3} + \Delta \boldsymbol{q} = 0; \tag{67d, e}$$

$$(\boldsymbol{M}^N + \Delta \boldsymbol{M})_{,3} + (\boldsymbol{X} + \boldsymbol{u}^N + \Delta \boldsymbol{u})_{,3} \times (\boldsymbol{T}^N + \Delta \boldsymbol{T}) = 0, \tag{f}$$

$$\Delta \boldsymbol{T} = \Delta \bar{\boldsymbol{q}}: \quad \boldsymbol{M}^n + \Delta \boldsymbol{M} = 0 \quad \text{on } S_\sigma; \qquad \Delta \boldsymbol{u} = \Delta \bar{\boldsymbol{u}}; \qquad \Delta \boldsymbol{\phi} = \Delta \bar{\boldsymbol{\phi}} \quad \text{on } S_u, \tag{67g–j}$$

where the following relation is used:

$$\delta \Delta \mathbf{R} \cdot (\mathbf{R}^N + \Delta \mathbf{R})^T = \delta \Delta \boldsymbol{\phi} \times \mathbf{I}. \tag{68}$$

The present incremental governing equations, except for Eqs. (67b), (67f) and (67h), are exact ones in the state C_{n+1}. The constitutive Eq. (67b), the AMB condition (67f) and the mechanical boundary condition (67h) contain the constant terms associated with C_N. This is, as shown by Atluri and Murakawa (1977), Atluri (1979, 1980) and Murakawa and Atluri (1978), due to the nonlinear orthogonality condition of finite rotation tensor. These constant terms in Eqs. (67b, f, h) show the governing equations in the state C_N. Therefore, these constant terms vanish if the state variables satisfy the governing equations in C_N. Consequently, the present incremental functional ΔF_1 leads to the exact incremental governing equation in C_{N+1}.

As can be seen from the functionals F_1 and G_1, given in Eqs. (61) and (63), respectively, we can derive the incremental functional of G_1 by a slight modification of ΔF_1. To obtain ΔG_1 from ΔF_1 given in Eq. (65), one may introduce $-(m_{\alpha i}^N + \Delta m_{\alpha i})\,\Delta R_{i\alpha}$ and $(\bar{m}_{\alpha i}^N + \Delta \bar{m}_{\alpha i})\,\Delta R_{i\alpha}$ into the integration and boundary term S_σ, respectively, and replace $\Delta \mathbf{R}$ and $(\boldsymbol{M}^N + \Delta \boldsymbol{M}) \cdot (\Delta \boldsymbol{\phi} - \Delta \bar{\boldsymbol{\phi}})$ on S_u by $\Delta \alpha^m$ and $(L_j^{N+} + \Delta L_j^+)\,(\Delta \alpha^j - \Delta \bar{\alpha}^j)$, respectively. The stationary condition, $\delta \Delta G_1 = 0$, leads to the exact incremental governing equation in C_{N+1}.

(ii) UL formulation. In the UL formulation, the notation $^*(\ )$ is used to emphasize that these values are referred to the configuration in C_N. Since $^*\mathbf{R}^N = {}^*\mathbf{I}$, the orthogonality condition of finite rotation tensor is written as

$$\delta \Delta^* \mathbf{R} \cdot ({}^*\mathbf{I} + \Delta^* \mathbf{R})^T = \delta \Delta^* \boldsymbol{\phi} \times {}^*\mathbf{I}. \tag{69}$$

The incremental functional of F_1 is obtained as

$$\begin{aligned} \Delta^* F_1 (\Delta^* \boldsymbol{T}, \Delta^* \boldsymbol{M}, \Delta^* \boldsymbol{h}, \Delta^* \tilde{\boldsymbol{k}}, \Delta^* \mathbf{R}, \Delta^* \boldsymbol{u},) &= \int_{C_N} [\Delta^* W_s + \Delta^* \boldsymbol{T} \cdot \{\Delta^* \boldsymbol{u}_{,3} - \Delta^* \boldsymbol{h} - \Delta^* \mathbf{R} \cdot ({}^*\boldsymbol{X}_{,3}^N + \Delta^* \boldsymbol{h})\} \\ &- {}^*\boldsymbol{T}^N \cdot \{\Delta^* \mathbf{R} \cdot ({}^*\boldsymbol{X}_{,3}^N + \Delta^* \boldsymbol{h})\} + ({}^*\boldsymbol{M}^N + \Delta^* \boldsymbol{M}) \cdot (\Delta^* \boldsymbol{l}_3 - \Delta^* \tilde{\boldsymbol{k}} - \Delta^* \mathbf{R} \cdot \Delta^* \tilde{\boldsymbol{k}}) - \Delta^* \boldsymbol{q} \cdot \Delta^* \boldsymbol{u}]\,\mathrm{d}L \\ &- \underset{Su}{[}\Delta^* \boldsymbol{T} \cdot (\Delta^* \boldsymbol{u} - \Delta^* \bar{\boldsymbol{u}}) + ({}^*\boldsymbol{M}^N + \Delta^* \boldsymbol{M}) \cdot (\Delta^* \boldsymbol{\phi} - \Delta^* \bar{\boldsymbol{\phi}})]_{L=0}^{L={}^*l} - \underset{S_\sigma}{[}\Delta^* \bar{\boldsymbol{q}} \cdot \Delta^* \boldsymbol{u}]_{L=0}^{L={}^*l}. \end{aligned} \tag{70}$$

where $^*\boldsymbol{X}^N$ is the position vector of the beam axis in the reference state C_N, and

$$\begin{aligned} \Delta^* W_s = {} & \frac{1}{2} G A_0 (\Delta^* h_1)^2 + \frac{1}{2} G A_0 (\Delta^* h_2)^2 + \frac{1}{2} E A (\Delta^* h_3)^2 + \frac{1}{2} E I_{11} (\Delta^* \tilde{k}_1)^2 + \frac{1}{2} E I_{22} (\Delta^* \tilde{k}_2)^2 \\ & + \frac{1}{2} G J (\Delta^* \tilde{k}_3)^2 + E I_1 \Delta^* h_3 \Delta^* \tilde{k}_1 - E I_2 \Delta^* h_3 \Delta^* \tilde{k}_2 - G I_1 \Delta^* h_1 \Delta^* \tilde{k}_3 + G I_2 \Delta^* h_2 \Delta^* \tilde{k}_3 \\ & - E I_{12} \Delta^* \tilde{k}_1 \Delta^* \tilde{k}_2 + {}^*M^{1N} \Delta^* \tilde{k}_1 + {}^*M^{2N} \Delta^* \tilde{k}_2 + {}^*M^{3N} \Delta^* \tilde{k}_3 . \end{aligned}$$

The linear and third order terms with respect to incremental values are included in the incremental functional $\Delta^* F_1$ so as to satisfy the nonlinear orthogonality condition (69). The Euler-Lagrange equations and natural boundary conditions of the statement $\delta \Delta^* F_1 = 0$ are

$$\frac{\partial \Delta^* W_s}{\partial \Delta^* h_m} = \Delta^* T_m; \qquad \frac{\partial \Delta^* W_s}{\partial \Delta^* \tilde{k}_m} = {}^*M^{mn} + \Delta^* M^m; \qquad \Delta^* \boldsymbol{u}_{,3} = \Delta^* \boldsymbol{h} + \Delta^* \mathbf{R} \cdot ({}^*\boldsymbol{X}_{,3}^N + \Delta^* \boldsymbol{h}); \tag{72a–c}$$

$$\Delta^* \boldsymbol{l}_3 = (^*\mathbf{I} + \Delta^* \mathbf{R}) \cdot \Delta^* \bar{\boldsymbol{k}}; \quad \Delta^* \boldsymbol{T}_{,3} + \Delta^* \boldsymbol{q} = 0;$$

$$(^*\boldsymbol{M}^N + \Delta^* \boldsymbol{M})_{,3} + (^*\boldsymbol{X}^N + \Delta^* \boldsymbol{u})_{,3} \times (^*\boldsymbol{T}^N + \Delta^* \boldsymbol{T}) = 0;$$

$$\Delta^* \boldsymbol{T} = \Delta^* \bar{\boldsymbol{q}}; \quad ^*\boldsymbol{M}^N + \Delta^* \boldsymbol{M} = 0 \quad \text{on } S_\sigma; \quad \Delta^* \boldsymbol{u} = \Delta^* \bar{\boldsymbol{u}}; \quad \Delta^* \boldsymbol{\phi} = \Delta^* \bar{\boldsymbol{\phi}} \quad \text{on } S_u. \qquad (72\text{d–j})$$

The constant terms associated with the reference state C_N vanish if the state variables satisfy the governing equations in the reference state. Then Eqs. (72) present the incremental governing equations. Note again that no external moments exist in $\Delta^* F_1$.

The incremental functional of G_1 is obtained by a slight modification of $\Delta^* F_1$. To obtain $\Delta^* G_1$ from $\Delta^* F_1$ given in Eq. (70), one may introduce

$$-(^*m^N_{\alpha i} + \Delta^* m_{\alpha i}) \Delta^* R_{i\alpha} \quad \text{and} \quad (^*\bar{m}^N_{\alpha i} + \Delta^* \bar{m}_{\alpha i}) \Delta^* R_{i\alpha}$$

into the integration and boundary term S_σ, respectively, and replace $\Delta^* \mathbf{R}$ and $(^*\boldsymbol{M}^N + \Delta^* \boldsymbol{M}) \cdot (\Delta^* \boldsymbol{\phi} - \Delta^* \bar{\boldsymbol{\phi}})$ on S_u by $\Delta^* \alpha^m$ and $(^*L^{N+}_j + \Delta^* L^+_j)(\Delta^* \alpha^j - \Delta^* \bar{\alpha}^j)$, respectively. The stationary condition, $\delta \Delta^* G_1 = 0$, yields the exact incremental governing equations in C_{N+1}.

8 Applications

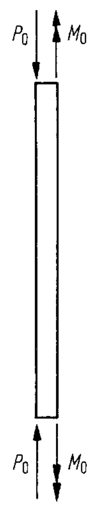
To investigate a validity of the present governing equations, we consider the problem of buckling of an initially straight beam subjected to the action of an axial compressive force P_0 and a twisting couple M_0, as shown in Fig. 2. As described before, the twisting couple is generated by external end forces. When the end forces, as shown in Fig. 3, are conservative, the resulting mechanical boundary condition for the moment is dependent on the deformations (see Appendix 3).

In this example, we assume that the end forces are nonconservative so that the torsional moment along the beam is constant before the buckling. In the case of initially straight beams, it is possible to choose the origin of coordinate Y^m so that $I_\alpha = I_{12} = 0$. As a result, it is easy to show that the nonvanishing stress resultants and moments before buckling are the axial force, $T^3 = EAh_3 = -P_0$, and the torsional moment, $M^3 = GJk_3 = M_0$.

Let $\Delta(\ )$ be the incremental value after the buckling. In the buckling problem, there exists at least one equilibrium position in the vicinity of the original equilibrium position under the same boundary conditions. The equilibrium equations for forces and moments in the direction of $\boldsymbol{e}_\alpha$ are written, from Eqs. (35), as

$$T^1_{,3} - k_3 T^2 + k_2 T^3 = 0; \quad T^2_{,3} + k_3 T^1 - k_1 T^3 = 0; \quad M^1_{,3} - k_3 M^2 + k_2 M^3 - (1 + h_3) T^2 + h_2 T^3 = 0;$$

$$M^2_{,3} + k_3 M^1 - k_1 M^3 + (1 + h_3) T^1 - h_1 T^3 = 0. \qquad (73\text{a–d})$$



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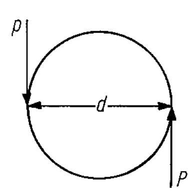

3

Figs. 2 and 3. 2 A straight beam subjected to axial forces and twisting couples. **3** Forces at the end cross section

These component expressions are the same as those of Reissner (1973, 1981). Keeping in mind that T^3, M^3, h_3 and $\tilde{k}_3 (= k_3)$ do not vanish before buckling and retaining the linear terms with respect to the incremental values, we obtain the present buckling equations represented by

$$\Delta T^1_{,3} + T^3 \Delta \tilde{k}_2 - \tilde{k}_3 \Delta T^2 = 0; \qquad \Delta T^2_{,3} + \tilde{k}_3 \Delta T^1 - T^3 \Delta \tilde{k}_1 = 0;$$
$$\Delta M^1_{,3} - \tilde{k}_3 \Delta M^2 + M^3 \Delta \tilde{k}_2 - (1 + h_3) \Delta T^2 + T^3 \Delta h_2 = 0;$$
$$\Delta M^2_{,3} + \tilde{k}_3 \Delta M^1 - M^3 \Delta \tilde{k}_1 + (1 + h_3) \Delta T^1 - T^3 \Delta h_1 = 0. \tag{74a–d}$$

Eliminating the incremental shear forces and using the incremental relations such as $\Delta M^\alpha = EI_{\alpha\alpha} \Delta \tilde{k}_\alpha$ leads to

$$\frac{EI_{11}}{1 - \frac{P_0}{EA} + \frac{P_0}{GA_0}} \Delta \tilde{k}_{1,33} + \left\{ P_0 + \frac{\left(1 - \frac{EI_{11}}{GJ}\right) \frac{(M_0)^2}{GJ}}{1 - \frac{P_0}{EA} + \frac{P_0}{GA_0}} \right\} \Delta \tilde{k}_1 + \frac{\left(1 - 2\frac{EI_{22}}{GJ}\right) M_0}{1 - \frac{P_0}{EA} + \frac{P_0}{GA_0}} \Delta \tilde{k}_{2,3} = 0,$$

$$\frac{EI_{22}}{1 - \frac{P_0}{EA} + \frac{P_0}{GA_0}} \Delta \tilde{k}_{2,33} + \left\{ P_0 + \frac{\left(1 - \frac{EI_{22}}{GJ}\right) \frac{(M_0)^2}{GJ}}{1 - \frac{P_0}{EA} + \frac{P_0}{GA_0}} \right\} \Delta \tilde{k}_2 + \frac{\left(1 - 2\frac{EI_{11}}{GJ}\right) M_0}{1 - \frac{P_0}{EA} + \frac{P_0}{GA_0}} \Delta \tilde{k}_{1,3} = 0, \tag{75a, b}$$

The expressions (75) denote the buckling equations for the present problem. Following the way of Timoshenko and Gere (1961), it is easy to obtain the buckling load under the prescribed boundary conditions.

For comparison with the existing results, we consider a simply supported beam with equal bending rigidities EI. Since $\Delta \tilde{k}_\alpha = 0$ at $Y^3 = 0$ and l, we obtain the following expression:

$$\frac{\left\{\left(1 - 2\frac{EI}{\underline{GJ}}\right) M_0\right\}^2}{4EI} + P_0 \left(1 - \frac{P_0}{\underset{\sim}{EA}} + \frac{P_0}{\underset{\cdots}{GA}}\right) + \left(1 - \frac{EI}{\underline{GJ}}\right) \frac{(M_0)^2}{GJ} = \frac{\pi^2 EI}{l^2}. \tag{76}$$

The terms denoted $\underline{(\ )}$ and $\underset{\sim}{(\ )}$ indicate the effects of the twist and the stretch before buckling respectively and the term $\underset{\cdots}{(\ )}$ is the effects of shear deformations. Neglecting the twisting couple M_0 in Eq. (76) yields the Euler buckling load in which the effects of shear deformations are taken into account. Ziegler (1982) and Reissner (1982) have discussed the Euler buckling load. It should be noted that the difference among those results depends only on the constitutive equations. The well-known Greenhill equation (Timoshenko and Gere 1961) is derived from neglecting the terms $\underline{(\ )}$, $\underset{\sim}{(\ )}$ and $\underset{\cdots}{(\ )}$ in Eq. (76).

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Appendix 1

Following Atluri (1984), the differential force vector acting on the deformed cross-section is given as

$$\mathrm{d}\boldsymbol{T} = g_0 \mathrm{d}A \boldsymbol{A}^3 \cdot (\mathbf{S}_1 \cdot \mathbf{F}^T). \tag{A.1}$$

Since $\mathbf{S}_1 = S_1^{mn} \boldsymbol{A}_m \boldsymbol{A}_n$ and $\mathbf{F}^T = \boldsymbol{A}^i \boldsymbol{a}_i$, integrating Eq. (A.1) leads to

$$\boldsymbol{T} = \int S_1^{3m} \boldsymbol{a}_m g_0 \, \mathrm{d}A . \tag{A.2}$$

Introducing Eq. (27) into Eq. (A.2) yields

$$\boldsymbol{T} = \int t^{3\bar{m}} \boldsymbol{e}_m g_0 \, \mathrm{d}A . \tag{A.3}$$

Equations (26a) and (26c) are derived from Eq. (A.3).

In a similar way, the differential moment vector acting on the deformed cross sections is represented as

$$\mathrm{d}\boldsymbol{M} = Y^\alpha \boldsymbol{e}_\alpha \times \mathrm{d}\boldsymbol{T}. \tag{A.4}$$

After some manipulation, we have

$$\boldsymbol{M} = \int \{ t^{3\bar{3}} Y^2 \boldsymbol{e}_1 - t^{3\bar{3}} Y^1 \boldsymbol{e}_2 + (t^{3\bar{2}} Y^1 - t^{3\bar{1}} Y^2) \boldsymbol{e}_3 \} g_0 \, \mathrm{d}A . \tag{A.5}$$

Equations (26b) and (26d–f) are derived from Eq. (A.5).

Appendix 2

When we express the deformed base vectors as

$$\boldsymbol{a}_m = K_{m\bar{n}} \boldsymbol{e}_n , \tag{A.6}$$

the internal virtual work per unit volume is presented as

$$S_1^{mn} \delta \varepsilon_{mn} = \frac{1}{2} S_1^{mn} \delta (K_{m\bar{l}} K_{n\bar{l}}) . \tag{A.7}$$

Since $S_1^{nm} = S_1^{mn}$ and $t^{m\bar{n}} = S_1^{ml} K_{ln}$, we have

$$S_1^{mn} \delta \varepsilon_{mn} = t^{m\bar{l}} \delta K_{m\bar{l}}. \tag{A.8}$$

Consequently, the conjugate strain tensors $\gamma_{m\bar{n}}$ are defined by Eq. (51).

Appendix 3

We consider, herein, the end forces which give rise to the twisting couple of constant magnitude M_0. For simplicity, we consider a beam with circular cross sections subjected end forces, as shown in Fig. 3. If we treat the end forces as conservative ones defined by Eq. (32b), the mechanical boundary conditions at both end cross section are obtained as

$$M^3 = Pd \, [R_{11} (R_{11} R_{22} - R_{12} R_{21}) + R_{31} (R_{31} R_{22} - R_{21} R_{32})] / \det |R_{ij}| . \tag{A.9}$$

On the other hand, if we treat the end forces as nonconservative ones defined by Eq. (37b), the mechanical boundary conditions at both end cross sections become

$$M^3 = Pd . \tag{A.10}$$

As shown in Eqs. (A.9) and (A.10), the end forces which give rise to the constant twisting couple must be nonconservative ones.

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