# *On the Existence of Positive Entire Solutions of a Semilinear Elliptic Equation*

WEI-YUE DING & WEI-MING NI

*Dedicated to James Serrin on the occasion of his sixtieth birthday*

**Abstract**

Under suitable hypotheses we obtain various theorems concerning the existence of positive solutions of the equation

$$\Delta u - u + Q(x)\, u^p = 0$$

in $\mathbb{R}^n$, where $p > 1$ and $Q(x)$ is a given potential. If $Q$ is radially symmetric, our result is particularly simple and general. We also study symmetries of solutions of the above equation in a ball with the boundary condition $u = 0$.

## § 1. Introduction

The semilinear elliptic equation in $\mathbb{R}^n$

$$\Delta u - u + |u|^{p-1}\, u = 0, \tag{1.1}$$

$p > 1$, arises in various branches of applied mathematics (see *e.g.* [B, L] and references therein) and has been studied extensively in recent years ([N], [S], [B], [St], [B, L]). A typical existence theorem states that equation (1.1) with $1 < p < (n + 2)/(n - 2)$ has infinitely many solutions which tend to 0 at infinity and one of them is positive. However, all these known solutions are radial and indeed, any positive solutions which are small at infinity are necessarily radial ([G, N, NI]). Whether equation (1.1) has a non-radial solution which is small at infinity remains an open problem. The difficulty in treating equation (1.1) arises because the domain $\mathbb{R}^n$ is unbounded and therefore we lack compact embedding theorems of Sobolev type. However, if we restrict attention to the class of radial functions, "compactness" of such a kind is regained and "standard" variational approaches work (see [N], [S], [B, L]). Naturally, this method does not seem to apply to the more general equation

$$\Delta u - u + Q(x)\,|u|^{p-1}\,u = 0 \tag{1.2}$$

where $Q(x)$ is a non-radial "potential". It should be remarked that if $Q(x) \to 0$ at infinity, once again some "compactness" exists and standard variational arguments deliver existence theorems (see [St]).

In this paper we consider equation (1.2) (and its generalization (2.1)) where the potential $Q(x)$ (or $f(x, u)$ in (2.1), respectively) is neither radial nor necessarily small at infinity. We shall consider *positive* solutions only (the so-called "ground states"). The method we use may be described roughly as follows. First, using the Mountain-Pass Lemma, we solve, for each $k$, the Dirichlet problem

$$(1.2)_k \qquad \begin{cases} \Delta u - u + Q(x)\,u^p = 0 \\ u > 0 \quad \text{in } B_k \quad \text{and} \quad u = 0 \quad \text{on } \partial B_k \end{cases}$$

where $B_k$ is the ball of radius $k$ centered at the origin. We then establish some upper bounds, independent of $k$, of the solutions of $(1.2)_k$ we have obtained; this will ensure that a subsequence of solutions $\{u_k\}$ converges as $k \to \infty$. The major step in this approach is that we then have to show that the limit of this convergent subsequence is positive. We present three different approaches to reach this property under different assumptions on $Q$; this is done in § 3. In § 2 we derive the above-mentioned uniform upper bounds and some preliminaries. In § 4 we consider radial $Q$. A corollary of our main result in § 4 reads as follows: *if* $1 < p < (n+2)/(n-2)$ *and* $0 \leqq Q(r) \leqq C(1 + r^l)$ *where* $r = |x|$, $C$ *is a positive constant and* $0 \leqq l < (n-1)(p-1)/2$, *for all* $r > 0$, *then equation* (1.2) *has a positive radial solution which tends to zero at* $\infty$. This conclusion seems to contain all preceding theorems in the radial case (as far as existence is concerned) and seems to be optimal. We also include in § 5 some discussions of the classical approach to (1.2) and $(1.2)_k$ through maximization, which does not seem to apply to the more general equation (2.1).

We mention that some of the ideas and methods developed in this paper also apply to the more delicate equation

$$\Delta u + K(x)\,u^{(n+2)/(n-2)} = 0, \tag{1.3}$$

$u > 0$ in $\mathbb{R}^n$, $n \geqq 3$, which arises in the problem of finding conformal metrics with prescribed scalar curvatures in Riemannian geometry (see [Ni]). Equation (1.3) with $K$ bounded between two positive constants is treated in [D, N].

While preparing this paper, we were informed that D. ZHANG [Z] had also obtained some results concerning equation (1.2). After this paper was written, we were informed kindly by P.-L. LIONS that he had some related results [L].

The major conclusions reported in this paper were obtained in 1983 while the DING was visiting Minnesota. He acknowledges the warm hospitality he received from the School of Mathematics at the University of Minnesota. He is especially grateful to JAMES SERRIN, to whom this paper is dedicated, for inviting him to Minneapolis.

## § 2. Preliminaries and uniform upper estimates

We consider the following equation in $\mathbb{R}^n$:

$$\Delta u - a(x)\, u + f(x, u) = 0, \tag{2.1}$$

which generalizes (1.2). Let $E_k$ denote $H_0^1(B_k)$, i.e. $E_k$ is the closure of smooth functions with supports contained in $B_k$ in the norm

$$\|u\|_{H^1(\mathbb{R}^n)} = \left[\int_{\mathbb{R}^n} (|\nabla u|^2 + u^2)\right]^{1/2}.$$

Let $\lambda_{1k}$ be the first eigenvalue of $-\Delta + a(x)$ on $B_k$ with Dirichlet boundary data. Assume that $a(x) \geqq 0$ is locally Hölder continuous and that there is a $\bar{\lambda} > 0$ such that

$$\lambda_{1k} \geqq \bar{\lambda} \quad \text{for all } k \geqq 1. \tag{2.2}$$

It is easy to see that (2.2) is satisfied if there is an $a_0 > 0$ such that

$$a(x) \geqq a_0 \quad \text{for all } |x| \geqq \bar{r}, \quad \text{for some } \bar{r} > 0. \tag{2.2$'$}$$

Let $E$ be the closure of smooth functions with compact support in the norm

$$\|u\| = \left[\int_{\mathbb{R}^n} (|\nabla u|^2 + au^2)\right]^{1/2}. \tag{2.3}$$

It is easy to verify that under (2.2) $E$ is a Hilbert space and $\bigcup_{k \geqq 1} E_k$ is dense in $E$. Note the obvious inclusions

$$E_1 \subseteq E_2 \subseteq \ldots \subseteq E.$$

For $f(x, u)$, we assume that $f$ is locally Hölder continuous and that

(2.4)
(a) $\lim_{u \to 0+} \frac{f(x, u)}{u} = 0$ uniformly in $x \in \mathbb{R}^n$, and there is an open set $U \subseteq \mathbb{R}^n$ such that $\lim_{u \to \infty} \frac{f(x, u)}{u} = \infty$ uniformly in $x \in U$.

(b) $0 \leqq f(x, u) \leqq C(1 + u^p)$ for all $x \in \mathbb{R}^n$, $u \in \mathbb{R}^+$, where $C > 0$ is a constant and $1 < p < \frac{n+2}{n-2}$.

(c) there is a number $0 < \theta < \frac{1}{2}$ and a function $0 \leqq A(x) \in L^1(\mathbb{R}^n) \cap C(\mathbb{R}^n)$ such that for all $x \in \mathbb{R}^n$, $u > 0$ we have $\quad A(x) + \theta u f(x, u) \geqq F(x, u) \equiv \int_0^u f(x, t)\, dt.$

*Hypotheses* (2.2) *and* (2.4) (a), (b), (c) *shall be maintained throughout* § 2 *and* § 3. It is clear that the variational functional

$$J(u) \equiv \frac{1}{2} \int_{\mathbb{R}^n} (|\nabla u|^2 + au^2) - \int_{\mathbb{R}^n} F(x, u) = \frac{1}{2} \|u\|^2 - \int_{\mathbb{R}^n} F(x, u)$$

is well defined and is continuously differentiable on $E$. Let $J_k$ denote the restriction of $J$ to the subspace $E_k$.

Since we seek only positive solutions of (2.1), it is convenient to define $f(x, u) \equiv 0$ for $u \leqq 0$ and $x \in \mathbb{R}^n$. Then by the strong maximum principle, any nontrivial critical point $u_k$ of $J_k$ is necessarily a positive solution of

$$(2.1)_k \qquad \begin{aligned} &\Delta u - a(x)\, u + f(x, u) = 0 \quad \text{in } B_k \\ &u = 0 \quad \text{on } \partial B_k. \end{aligned}$$

Likewise, any nontrivial critical point $\bar{u} \geqq 0$ of $J$ is necessarily a positive solution of (2.1).

**Lemma 2.5.** *Assume that* (2.2) *and* (2.4) *hold. Let* $u_k$ *be a critical point of* $J_k$ *and let* $u_k$ *converge weakly to* $\bar{u}$ *in* $E$. *Then* $\bar{u}$ *is a critical point of* $J$.

The proof of this lemma is somewhat standard (assumption (2.4) (b) is used here) and is thus omitted. (See *e.g.* the arguments used in the proof of Lemma 2.19, p. 161, in [R]).

Now we shall solve $(2.1)_k$ and derive a uniform upper bound for the solutions we obtain. From (2.4) (a), (b), we see that for any $\varepsilon > 0$, there is a constant $C_\varepsilon > 0$ such that

$$0 < F(x, u) \leqq \varepsilon u^2 + C_\varepsilon u^{2n/(n-2)} \quad \text{for all } x \in \mathbb{R}^n, \quad u > 0.$$

Choosing $\varepsilon < \bar{\lambda}/4$ ($\bar{\lambda}$ appears in (2.2)), we have, for $u \in E_k$,

$$\begin{aligned} J(u) = J_k(u) &\geqq \frac{1}{4} \int_{B_k} (|\nabla u|^2 + au^2) + \frac{\bar{\lambda}}{4} \int_{B_k} u^2 - \int_{B_k} (\varepsilon u^2 + C_\varepsilon |u|^{2n/(n-2)}) \\ &\geqq \frac{1}{4} \|u\|^2 - C'_\varepsilon \left( \int_{B_k} |\nabla u|^2 \right)^{n/(n-2)} \\ &\geqq \frac{1}{4} \|u\|^2 - C'_\varepsilon \|u\|^{2n/(n-2)} \end{aligned}$$

where the constant $C'_\varepsilon$ is independent of $k$. Since $\bigcup_{k \geqq 1} E_k$ is dense in $E$,

$$J(u) \geqq \frac{1}{4} \|u\|^2 - C'_\varepsilon \|u\|^{2n/(n-2)} \quad \text{for all } u \in E.$$

Therefore, there are positive constants $\delta$ and $\bar{\alpha}$ such that

$$(2.6) \qquad J(u) \geqq \bar{\alpha} > 0 \quad \text{on } \|u\| = \delta \quad \text{in } E.$$

On the other hand, (2.4) (a) implies that (with no loss of generality, we may assume that $U \cap B_1$ is not empty) there is an $e \in E_1$ such that $e \geqq 0$, supp $e \subseteq U \cap B_1$, $\|e\| > \delta$ and $J(te) < 0$ for all $t > 1$. Now define $\Gamma(\Gamma_k)$ as the set of all continuous paths in $E(E_k$, respectively) connecting 0 and $e$, and let

$$\alpha = \inf_{\gamma \in \Gamma} \max_{u \in \gamma} J(u) \tag{2.7}$$

and

$$\alpha_k = \inf_{\gamma \in \Gamma_k} \max_{u \in \gamma} J(u). \tag{2.8}$$

Since $\Gamma_k \subseteq \Gamma_{k+1} \subseteq \Gamma$, we have

$$\alpha_k \geqq \alpha_{k+1} \geqq \alpha \geqq \bar{\alpha} > 0 \quad \text{for all } k \geqq 1. \tag{2.9}$$

Moreover, it is not hard to prove that $\alpha_k \to \alpha$ as $k \to \infty$ since $\bigcup_{k \geqq 1} E_k$ is dense in $E$. By the well known Mountain-Pass Lemma [A, R], $\alpha_k$ is a critical value of $J_k$. Let $u_k \in E_k$ be a critical point of $J_k$ corresponding to $\alpha_k$, i.e. $J_k(u_k) = \alpha_k$ and $J_k'(u_k) = 0$. We have, in particular

$$J_k'(u_k)\, u_k = \|u_k\|^2 - \int_{B_k} u_k f(x, u_k) = 0 \tag{2.10}$$

and

$$J(u_k) = \tfrac{1}{2} \|u_k\|^2 - \int_{B_k} F(x, u_k) = \alpha_k. \tag{2.11}$$

(2.10) and (2.4) (c) together imply

$$\theta \|u_k\|^2 = \int_{B_k} \theta u_k f(x, u_k) \geqq \int_{B_k} F(x, u_k) - \int_{B_k} A(x).$$

Combining with (2.11) and (2.9), we obtain

$$(\tfrac{1}{2} - \theta) \|u_k\|^2 \leqq \alpha_k + A \leqq \alpha_1 + A$$

where $A \equiv \int_{\mathbf{R}^n} A(x)$. Thus

$$\|u_k\|^2 \leqq \frac{\alpha_1 + A}{(\frac{1}{2} - \theta)}. \tag{2.12}$$

Passing to a subsequence of $\{u_k\}$ if necessary, we may assume that $u_k$ converges to $\bar{u}$ *weakly* in $E$. Lemma 2.5 then implies that $\bar{u}$ is a critical point of $J$; thus $\bar{u} \geqq 0$ is a classical solution of (2.1) by standard theorems on elliptic regularity. Showing that $\bar{u} \not\equiv 0$ under various circumstances is the major theme of our next section. Summing up, we have the following

**Theorem 2.13.** *Assume* (2.2) *and* (2.4) *hold. For each $k$, the Dirichlet problem* $(2.1)_k$ *possesses a positive solution $u_k$ with $J(u_k) = \alpha_k$ and $\alpha_k \to \alpha \geqq \bar{\alpha} > 0$, in which $\alpha_k$ and $\alpha$ are given by* (2.7) *and* (2.8). *Moreover, $\{u_k\}$ is uniformly bounded in $E$ and so it contains a subsequence that converges weakly to $\bar{u} \geqq 0$ in $E$.*

A natural question arises: is $J(\bar{u}) = \alpha$? and, is $\alpha$ achieved by a path containing $\bar{u}$ in $\Gamma$? An obvious necessary condition is $\bar{u} \not\equiv 0$ (since $\alpha > 0$). The following proposition gives a sufficient condition. It will be useful in § 3 and such a result seems to be of interest in its own right.

**Proposition 2.14.** *In addition to* (2.2) *and* (2.4), *assume that*

$$\tag{2.15} \tfrac{1}{2}uf(x,u) \geqq F(x,u) \quad \textit{for all } x \in \mathbb{R}^n, \quad u \geqq 0, \quad \textit{and}$$

$$\tag{2.16} \frac{f(x,u)}{u} \textit{ is nondecreasing in } u > 0, \textit{ for each } x \in \mathbb{R}^n.$$

*Then, if the weak limit* $\bar{u}$ *of* $\{u_k\}$ *is nontrivial, there is a path* $\bar{\gamma} \in \Gamma$ *with* $\bar{u} \in \bar{\gamma}$ *such that*

$$\tag{2.17} J(\bar{u}) = \max_{u \in \bar{\gamma}} J(u) = \alpha.$$

**Proof.** Since $J'(\bar{u})\,\bar{u} = 0$, i.e.

$$\tag{2.18} \|\bar{u}\|^2 = \int_{\mathbb{R}^n} \bar{u}f(x,\bar{u})$$

we conclude that

$$\tag{2.19} J(\bar{u}) = \int [\tfrac{1}{2}\bar{u}f(x,\bar{u}) - F(x,\bar{u})].$$

Similarly

$$\tag{2.20} J(u_k) = \int [\tfrac{1}{2}u_k f(x,u_k) - F(x,u_k)] = \alpha_k.$$

Now, the integrands in (2.19) and (2.20) are nonnegative by (2.15). Since $u_k \to \bar{u}$ a.e. in $\mathbb{R}^n$, we conclude by Fatou's Lemma that

$$\tag{2.21} J(\bar{u}) \leqq \liminf J(u_k) = \lim \alpha_k = \alpha.$$

Set, for $t \geqq 0$,

$$g(t) \equiv J(t\bar{u}) = \tfrac{1}{2}t^2\|\bar{u}\|^2 - \int F(x,t\bar{u}).$$

Differentiating with respect to $t$, we have

$$\begin{aligned} g'(t) &= t\|\bar{u}\|^2 - \int \bar{u}f(x,t\bar{u}) \\ &= \int [t\bar{u}f(x,\bar{u}) - \bar{u}f(x,t\bar{u})] \\ &= \int t\bar{u}^2\left[\frac{f(x,\bar{u})}{\bar{u}} - \frac{f(x,t\bar{u})}{t\bar{u}}\right], \end{aligned}$$

where the second equality follows from (2.18). Now, (2.16) implies that $g'(t) \geqq 0$ if $t \in (0,1)$ and $g'(t) \leqq 0$ if $t \geqq 1$. Thus $g(1) = J(\bar{u})$ is the (absolute) maximum on the half-line $l = \{t\bar{u} \mid t \geqq 0\}$.

Let $V^+$ be the set $\{a\bar{u} + be \mid a \geqq 0, b \geqq 0\}$ and let $V$ be the 2-dimensional subspace of $E$ spanned by $\bar{u}$ and $e$. Let $S$ be a circle on $V$ with radius $R$ so large that $J \leqq 0$ on $S \cap V^+$ (this follows from (2.4) (a) and a standard compactness argument on $V^+$) and $\bar{u}$ and $e$ lie inside $S$. Suppose $l$ and $l_1 = \{te \mid t \geqq 0\}$ intersect $S$ at $v$ and $v_1$ respectively. Then, let $\bar{\gamma}$ be the path that consists of the segment on $l$ with endpoints 0 and $v$, the arc $S \cap V^+$ (connecting $v$ and $v_1$), and the segment on $l_1$ with endpoints $v_1$ and $e$. It is clear that $\bar{u} \in \bar{\gamma}$ and $\bar{\gamma} \in \Gamma$ and

$$J(\bar{u}) = \max_{u \in \bar{\gamma}} J(u).$$

Thus, $J(\bar{u}) \geqq \alpha$ by the definition of $\alpha$. This, together with (2.21), gives $J(\bar{u}) = \alpha$.
*Q.e.d.*

## § 3. Main existence theorems

In this section we shall prove several existence theorems concerning positive solutions (in $E$) of equation (2.1). By Theorem 2.13 we have only to show that $\bar{u} \not\equiv 0$ (where $\bar{u}$ is given by Theorem 2.13). The methods we use to achieve this are diverse, and so we divide this section into several subsections, in each of which we present one method.

§§ 3.1. In this subsection we shall prove some existence theorems by using comparison arguments and the variational approach of § 2.

Let $h(x, u) \geqq 0$, $\not\equiv 0$ (therefore with no loss of generality we assume that $h(x, u) > 0$ for some $(x, u) \in B_1 \times \mathbb{R}^+$) and that it is Hölder continuous and satisfies the following conditions:

$$uh_u(x, u) \geqq (1 + \varepsilon)\, h(x, u) \geqq 0 \quad \text{for } x \in \mathbb{R}^n, \quad u > 0, \tag{3.1}$$

$$h(x, u) \leqq C(1 + u^p),\ 1 < p < \frac{n+2}{n-2} \quad \text{for } x \in \mathbb{R}^n, \quad u > 0, \tag{3.2}$$

where $\varepsilon > 0$, $C > 0$ are constants independent of $x \in \mathbb{R}^n$. Again, we set $h(x, u) \equiv 0$ for $x \in \mathbb{R}^n$, $u \leqq 0$ for convenience. Set

$$I(u) = \tfrac{1}{2} \|u\|^2 - \int_{\mathbb{R}^n} H(x, u)$$

where

$$H(x, u) = \int_0^u h(x, t)\, dt.$$

We choose $e \geqq 0$ in $E_1$ (defined in § 2) so that $J(te) < 0$ and $I(te) < 0$ for all $t > 1$. Observe that this change does not alter the value $\alpha$. (See the last paragraph in the proof of Proposition 2.14.) Let

$$\beta = \inf_{\gamma \in \Gamma} \max_{u \in \gamma} I(u),$$

$$M_f = \left\{ u \in E \setminus \{0\} \mid \|u\|^2 = \int_{\mathbb{R}^n} uf(x, u) \right\}$$

and

$$M_h = \left\{ u \in E \setminus \{0\} \mid \|u\|^2 = \int_{\mathbb{R}^n} uh(x, u) \right\}.$$

From (3.1) we see that $h(x, u)/u$ is strictly increasing in $u > 0$ if $h(x, u) > 0$; in fact, $h(x, u)/u^{1+\varepsilon}$ is nondecreasing. Thus it is easy to verify that if $\{x \in \mathbb{R}^n \mid$

$h(x, u(x)) > 0\}$ is not empty, then the half-line $l_u = \{tu \mid t \geqq 0\}$ intersects $M_h$ at exactly one point $\bar{t}u$. Moreover, $I(\bar{t}u) > 0$ is the (absolute) maximum of $I$ on $l_u$. Using the arguments in the last paragraph of the proof of Proposition 2.14, we conclude that

$$\beta \leqq \beta^*, \tag{3.3}$$

where $\beta^* = \inf_{u \in M_h} I(u)$. Similarly, we define $\alpha^* = \inf_{u \in M_f} J(u)$.

**Theorem 3.4.** *Suppose* (2.2), (2.4) *and* (2.16) *hold. Suppose in addition that there is an* $R > 0$ *such that*

$$f(x, u) \leqq h(x, u) \quad \text{for all } x \in \mathbb{R}^n \setminus B_R, u > 0. \tag{3.5}$$

*Then* $\bar{u} \equiv 0$ *implies that* $\alpha \geqq \beta^*$, *where* $\bar{u}, \alpha$ *are given in* § 2.

**Theorem 3.6.** *Under the hypotheses of Theorem* 3.4, *if* $\alpha < \beta^*$ *then* $0 < \bar{u} \in E$ *is a solution of* (2.1).

Theorem 3.6 follows immediately from Theorem 3.4. We shall give some useful corollaries and examples to illustrate Theorem 3.6 later.

**Proof of Theorem 3.4.**

Since $\bar{u} \equiv 0$, $\{u_k\}$ converges to 0 weakly in $E$. Thus as $k \to \infty$, (a subsequence of) $u_k \to 0$ uniformly on compact sets by elliptic regularity estimates. Hence

$$0 \leqq \varepsilon_k \equiv \int_{B_R} u_k f(x, u_k) \to 0.$$

(Note that $R$ is fixed.) It was observed earlier that for each $k$, there is a unique $t_k > 0$ such that $t_k u_k \in M_h$; i.e.

$$t_k^2 \|u_k\|^2 = \int_{\mathbb{R}_n} t_k u_k h(x, t_k u_k).$$

On the other hand, by (3.5),

$$\|u_k\|^2 = \int_{\mathbb{R}_n} u_k f(x, u_k) = \varepsilon_k + \int_{|x|>R} u_k f(x, u_k)$$

$$\leqq \varepsilon_k + \int_{|x|>R} u_k h(x, u_k).$$

Therefore, if $t_k \geqq 1$, then

$$t_k^2 \varepsilon_k + t_k^2 \int_{|x|>R} u_k h(x, u_k)$$

$$\geqq \int_{|x|>R} t_k u_k h(x, t_k u_k)$$

$$\geqq \int_{|x|>R} t_k^{2+\varepsilon} u_k h(x, u_k);$$

the last inequality comes from the fact that

$$h(x, tu) \geqq t^{1+\varepsilon} h(x, u)$$

for all $x \in \mathbb{R}^n$, $t \geqq 1$ and $u \geqq 0$, which is a consequence of (3.1). Hence

$$\begin{aligned} t_k^2 \varepsilon_k &\geqq (t_k^{2+\varepsilon} - t_k^2) \int_{|x|>R} u_k h(x, u_k) \\ &\geqq (t_k^{2+\varepsilon} - t_k^2) \int_{|x|>R} u_k f(x, u_k) \\ &= (t_k^{2+\varepsilon} - t_k^2)(\|u_k\|^2 - \varepsilon_k) \\ &\geqq \alpha(t_k^{2+\varepsilon} - t_k^2), \quad \text{for large } k, \end{aligned}$$

since $\varepsilon_k \to 0$ as $k \to \infty$ and $\|u_k\|^2 \geqq 2J_k(u_k) = 2\alpha_k \geqq 2\alpha$. Thus,

$$\varepsilon_k \geqq \alpha(t_k^\varepsilon - 1),$$

and $t_k \to 1$ as $k \to \infty$. In particular, $\{t_k\}$ is a bounded sequence. *Consequently, $\{t_k u_k\}$ converges to* 0 *weakly in E.* Next, observe that $J(u_k)$ is the maximum of $J$ restricted to $l_{u_k}$ (by (2.16), see the proof of Proposition 2.14), and then

$$\begin{aligned} \alpha_k = J(u_k) &\geqq J(t_k u_k) \\ &= \tfrac{1}{2} t_k^2 \|u_k\|^2 - \int_{|x|>R} F(x, t_k u_k) - \int_{B_R} F(x, t_k u_k) \\ &\geqq \tfrac{1}{2} t_k^2 \|u_k\|^2 - \int_{|x|>R} H(x, t_k u_k) - \int_{B_R} F(x, t_k u_k) \\ &\geqq I(t_k u_k) - \int_{B_R} F(x, t_k u_k) \\ &\geqq \beta^* - \int_{B_R} F(x, t_k u_k) \end{aligned}$$

by (3.3) and the fact that $t_k u_k \in M_h$. As before, the integral

$$\int_{B_R} F(x, t_k u_k) \to 0$$

as $k \to \infty$ since $t_k u_k$ converges weakly to 0 in $E$. Letting $k \to \infty$, we obtain $\alpha \geqq \beta^*$.

*Q.e.d.*

In practical situations, the quantities $\alpha^*, \beta^*$ are often much easier to compute than $\alpha, \beta$ thus it seems desirable to have the following

**Corollary 3.7.** *Under the hypotheses of Theorem* 3.6, *suppose in addition that f satisfies* (3.1) *also. If* $\alpha^* < \beta^*$, *then* (2.1) *has a positive solution* $\bar{u}$ *in E.*

**Proof.** Since under the additional hypothesis $uf_u(x, u) \geqq (1+\varepsilon) f(x, u) \geqq 0$ for $x \in \mathbb{R}^n$, $u > 0$, we also have $\alpha \leqq \alpha^*$ (see the arguments which lead to (3.3)). Thus $\alpha^* < \beta^*$ implies $\alpha < \beta^*$ and then Theorem 3.6 applies.

*Q.e.d.*

Now we come back to the equation

$$\Delta u - u + Q(x)\, u^p = 0, \quad u > 0, \tag{3.8}$$

in $\mathbb{R}^n$, assuming that $Q(x) \geqq 0$ and $\not\equiv 0$, $1 < p < (n+2)/(n-2)$. For purposes of comparison we also consider

$$\Delta u - u + K(x)\, u^p = 0, \quad u > 0 \tag{3.9}$$

in $\mathbb{R}^n$, where $K(x) \geqq 0$. That is, we set $f(x, u) = Q(x)\, u^p$, and $h(x, u) = K(x)\, u^p$, both $Q$ and $K$ are *bounded* functions on $\mathbb{R}^n$, and $Q(x) \leqq K(x)$ for all $|x| \geqq R$, for some $R > 0$. Thus all the hypotheses of Corollary 3.7 are satisfied, and

$$I(u) = \tfrac{1}{2}\|u\|^2 - \frac{1}{p+1} \int_{\mathbb{R}^n} K u_+^{p+1}$$

$$J(u) = \tfrac{1}{2}\|u\|^2 - \frac{1}{p+1} \int_{\mathbb{R}^n} Q u_+^{p+1}$$

$$M_f = \left\{ u \in E \setminus \{0\} \mid \|u\|^2 = \int_{\mathbb{R}^n} Q u_+^{p+1} \right\}$$

$$M_h = \left\{ u \in E \setminus \{0\} \mid \|u\|^2 = \int_{\mathbb{R}^n} K u_+^{p+1} \right\}$$

where

$$\|u\|^2 = \int_{\mathbb{R}^n} (|\nabla u|^2 + u^2), \quad u_+(x) = \max\{u(x), 0\},$$

and $E = H_0^1(\mathbb{R}^n)$. Now, for any $u \in E$, there is at most one $t > 0$ such that $tu \in M_f$. This number $t$ may be calculated explicitly as follows

$$t = \left[ \frac{\|u\|^2}{\int_{\mathbb{R}^n} Q u_+^{p+1}} \right]^{\frac{1}{p-1}}. \tag{3.10}$$

Therefore

$$\alpha^* = \inf_{u \in M_f} \left( \tfrac{1}{2} - \frac{1}{p+1} \right) \int_{\mathbb{R}^n} Q u_+^{p+1},$$

*i.e.*

$$\alpha^* = \left( \tfrac{1}{2} - \frac{1}{p+1} \right) \inf_{u \in E \setminus \{0\}} \left[ \frac{\|u\|}{\left( \int_{\mathbb{R}^n} Q u_+^{p+1} \right)^{\frac{1}{p+1}}} \right]^{\frac{2(p+1)}{p-1}} \tag{3.11}$$

Similarly,

$$\beta^* = \left(\tfrac{1}{2} - \frac{1}{p+1}\right) \inf_{u \in E \setminus \{0\}} \left[ \frac{\|u\|}{\left(\int_{\mathbb{R}^n} K u_+^{p+1}\right)^{\frac{1}{p+1}}} \right]^{\frac{2(p+1)}{p-1}}. \tag{3.12}$$

Thus we have

**Corollary 3.13.** *Suppose that* $Q(x) \geqq 0$, $K(x) \geqq 0$ *in* $\mathbb{R}^n$ *with* $Q(x) \leqq K(x)$ *for* $|x| > R$, *R being some positive constant. Then* (3.8) *possesses a positive solution in* $H_0^1(\mathbb{R}^n)$ *if*

$$\inf_{0 \neq u \in H_0^1(\mathbb{R}^n)} \frac{\|u\|}{\left(\int_{\mathbb{R}^n} Q u_+^{p+1}\right)^{\frac{1}{p+1}}} < \inf_{0 \neq u \in H_0^1(\mathbb{R}^n)} \frac{\|u\|}{\left(\int_{\mathbb{R}^n} K u_+^{p+1}\right)^{\frac{1}{p+1}}}. \tag{3.14}$$

**Corollary 3.15.** *Given* $Q(x) \geqq 0$ *in* $\mathbb{R}^n$, *equation* (3.8) *possesses a positive solution in* $H_0^1(\mathbb{R}^n)$ *if*

$$\inf_{0 \neq u \in H_0^1(\mathbb{R}^n)} \frac{\|u\|}{\left(\int_{\mathbb{R}^n} Q u_+^{p+1}\right)^{\frac{1}{p+1}}} < \lim_{R \to \infty} \left\{ \inf_{0 \neq u \in H_0^1(\mathbb{R}^n)} \frac{\|u\|}{\left(\int_{|x|>R} Q u_+^{p+1}\right)^{\frac{1}{p+1}}} \right\}. \tag{3.16}$$

**Proof.** Take $K(x) = \begin{cases} Q(x) & \text{if } |x| > R, \\ 0 & \text{if } |x| \leqq R; \end{cases}$

the result follows from the preceding corollary.

*Q.e.d.*

It is clear that the following corollary is another reformulation of Corollary 3.15.

**Corollary 3.17.** *Given* $Q(x) \geqq 0$ *in* $\mathbb{R}^n$, *equation* (3.8) *has a positive solution in* $H_0^1(\mathbb{R}^n)$ *if*

$$\sup_{\|u\|=1} \int_{\mathbb{R}^n} Q u_+^{p+1} > \lim_{R \to \infty} \left[ \sup_{\|u\|=1} \int_{|x|>R} Q u_+^{p+1} \right]. \tag{3.18}$$

We should remark that the case $Q \equiv 1$ is not included in (3.18), but $\lim_{|x| \to \infty} Q(x) = 0$ is obviously included. Also, if $Q(x) \geqq a_0 > 0$ for $|x| < R_0$ and $Q(x) \leqq a_1$ for $|x| > R_1$ where $a_0 \gg a_1$, then (3.18) holds and equation (3.8) can be solved.

**Corollary 3.19.** *If*

$$\lim_{|x| \to \infty} Q(x) = \inf_{x \in \mathbb{R}^n} Q(x), \tag{3.20}$$

*then* (3.8) *has a positive solution in* $H_0^1(\mathbb{R}^n)$.

**Proof.** Set

$$m = \lim_{|x|\to\infty} Q(x) = \inf_{x\in\mathbb{R}^n} Q(x).$$

If $Q \not\equiv$ constant, then it is clear that

$$\tilde{\alpha} \equiv \sup_{\|u\|=1} \int_{\mathbb{R}^n} Qu_+^{p+1} > \alpha_m \equiv \sup_{\|u\|=1} \int_{\mathbb{R}^n} mu_+^{p+1}$$

since $\alpha_m$ is attained by some positive function. On the other hand, given any $\varepsilon > 0$, there is an $r$ large enough that

$$\alpha_r \equiv \sup_{\|u\|=1} \int_{|x|>r} Qu_+^{p+1} \leqq (m+\varepsilon) \sup_{\|u\|=1} \int_{|x|>r} u_+^{p+1}$$

$$\leqq \frac{m+\varepsilon}{m}\alpha_m;$$

thus

$$\lim_{r\to\infty} \alpha_r \leqq \frac{m+\varepsilon}{m}\tilde{\alpha}.$$

Since $\varepsilon > 0$ is arbitrary and $\alpha_m < \tilde{\alpha}$, (3.18) holds.

If $Q \equiv$ constant, the result is well known (see Theorems 3.23, 4.8).

*Q.e.d.*

Let

$$\bar{Q} = \limsup_{|x|\to\infty} Q(x), \qquad Q_r = \min_{|x|\leqq r} Q,$$

$$\tau = \lim_{r\to\infty} \tau_r, \qquad \tau_r = \sup_{\|u\|_{H_0^1(B_r)}=1} \int_{B_r} u_+^{p+1}$$

**Corollary 3.21.** *If* $Q_r\tau_r > \bar{Q}\tau$ *for some* $r > 0$, *then* (3.18) *holds and* (3.8) *has a positive solution in* $H_0^1(\mathbb{R}^n)$.

The proof is straightforward since

$$\tilde{\alpha} \geqq Q_r\tau_r > \bar{Q}\tau \geqq \lim_{r\to\infty} \alpha_r.$$

*Remark. Observe that in* (3.11), (3.12), (3.14), (3.16), (3.18) *and the definitions of* $\tilde{\alpha}, \alpha_m, \tau_r, \ldots$ etc., *one can replace* $u_+$ *by* $|u|$ *without altering any of those numbers since* $u$ *and* $|u|$ *have the same* $H^1$*-norm.*

§§ 3.2. In this subsection we use the well known technique of the "moving parallel plane" to obtain a uniform positive lower bound on $u_k$ in Theorem 2.13 and therefore we have $\bar{u} > 0$. This technique goes back to A. D. ALEXANDROV, and has been used by various authors; see, *e.g.* [Se], [G, N, N1] and [G, N, N2].

We start with a few definitions. Let $\Omega$ be a convex domain in $\mathbb{R}^n$ which is symmetric with respect to the hyperplanes $\Sigma_i \equiv \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_i = 0\}$, $i = 1, \ldots, n$. Let $e_i$ denote the unit vector pointing along the positive $x_i$-axis. Let $\varrho \geqq 0$ be a real number. Define $E(\varrho, \Omega)$ to be the set of all functions $u$ on $\Omega$ satisfying $u(y + te_i) \leqq u(y + (2\lambda - t)\, e_i)$ for all $t \geqq \lambda \geqq \varrho$ or $t \leqq -\lambda \leqq -\varrho$, $y \in \Sigma_i \cap \Omega$, $1 \leqq i \leqq n$, provided $y + te_i$ and $y + (2\lambda - t)\, e_i \in \Omega$. The following lemma is an easy consequence of Theorem 2.1′ in [G, N, N2].

**Lemma 3.22.** *Let $\Omega$ be a bounded convex domain in $\mathbb{R}^n$ which is symmetric with respect to $\Sigma_i$, for all $1 \leqq i \leqq n$, and let $g$ be a continuous function which is locally Lipschitz in $u$. Let $u \in C^2(\Omega) \cap C(\overline{\Omega})$ be a solution of $\Delta u + g(x, u) = 0$ with $u > 0$ in $\Omega$ and $u = 0$ on $\partial\Omega$. If $g(\cdot, s) \in E(\varrho, \Omega)$ for all $s > 0$ and $g(x, 0) \geqq 0$ for all $x \in \partial\Omega$, then $u \in E(\varrho, \Omega)$. In particular, $u$ attains its maximum in the cube $C_\varrho = \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n \mid |x_i| \leqq \varrho,\ 1 \leqq i \leqq n\}$.*

**Theorem 3.23.** *Let* (2.2) *and* (2.4) *hold, and let $f(x, u)$ be locally Lipschitz-continuous in $u$. Assume that there is a $\varrho \geqq 0$ such that*

$$(3.24) \qquad -a(x) \in E(\varrho, \mathbb{R}^n) \quad \text{and} \quad f(x, s) \in E(\varrho, \mathbb{R}^n) \quad \text{for all } s > 0.$$

*Then equation* (2.1) *has a positive solution $\bar{u} \in E \cap E(\varrho, \mathbb{R}^n)$.*

**Proof.** Let $u_k$ be a solution of $(2.1)_k$ obtained in Theorem 2.13. By (3.24) and thus Lemma 3.22, we have $u_k \in E(\varrho, B_k)$ and $M_k = \max u_k$ is attained in the cube $C_\varrho$, say, at $P_k$. We claim that *there is a positive constant $\delta$ such that $M_k \geqq \delta$ for all $k$.* Indeed, from $J_k'(u_k)\, u_k = 0$ we conclude that

$$\int u_k f(x, u_k) = \|u_k\|^2 \geqq \bar{\lambda} \int u_k^2.$$

Thus

$$\int u_k(f(x, u_k) - \bar{\lambda} u_k) \geqq 0.$$

Now our assertion follows from (2.4) (a).

Suppose that $C_\varrho \subseteq B_{\bar{k}}$ for some $\bar{k}$. By standard arguments on regularity (and (2.4) (b)), we conclude that there is a constant $M$, depending only on $p$, $\varrho$ and $\bar{k}$, such that

$$\|u_k\|_{C^{1,\gamma}(C_\varrho)} \leqq M$$

for all $k \geqq \bar{k}$. Thus, some subsequence $\{u_{k_j}\}$ converges in $C^1(C_\varrho)$ to $u^* \in C^1(C_\varrho)$. Since $u_{k_j} \to \bar{u}$ weakly in $E$ by Theorem 2.13, $u_{k_j} \to \bar{u}$ almost everywhere, and $u^* = \bar{u}$. It is obvious that $\bar{u} \in E(\varrho, \mathbb{R}^n)$ and is a classical solution of (2.1) with $\max \bar{u} \geqq \delta$.

*Q.e.d.*

In particular, this theorem covers the well-known special case $Q \equiv 1$ in equation (3.8).

§§ 3.3. The "geometrical" theorem in §§ 3.2 is somewhat "rigid"; we shall extend it in this subsection.

**Theorem 3.25.** *Let* (2.2), (2.4) *and* (2.16) *hold. Suppose* $a_1(x)$ *and* $f_1(x, u)$ *satisfy* (2.15), (2.16) *and the hypotheses of Theorem* 3.23. *If there are functions* $b_1(x)$ *and* $b_2(x)$ *such that*

$$a_1(x) - b_1(x) \leqq a(x) \leqq a_1(x) \tag{3.26}$$

$$f_1(x, u) \leqq f(x, u) \leqq f_1(x, u) + b_2(x)\,(u + u^p) \tag{3.27}$$

*for all* $x \in \mathbb{R}^n$, $u > 0$, *where* $b_i(x) \in C(\mathbb{R}^n)$ *with*

$$b_i(x) \to 0 \ \text{as} \ |x| \to \infty, \qquad i = 1, 2, \tag{3.28}$$

*then equation* (2.1) *has a positive solution in E.*

**Proof.** The basic idea is to compare functionals $J(u)$ (defined in § 2) and

$$J_1(u) \equiv \tfrac{1}{2} \int_{\mathbb{R}^n} (|\nabla u|^2 + a_1 u^2) - \int_{\mathbb{R}^n} F_1(x, u)$$

where

$$F_1(x, u) = \int_0^u f_1(x, s)\, ds.$$

Note that the induced norm

$$\|u\|_1 \equiv \left[\int_{\mathbb{R}^n} (|\nabla u|^2 + a_1 u^2)\right]^{\frac{1}{2}}$$

is equivalent to the norm $\|u\|$ defined by (2.3) in view of (3.26) and (3.28). Choose $e \geqq 0$ in $E_1$ (as in § 2) so that $J(te) < 0$ and $J_1(te) < 0$ for all $t > 1$. Observe once again that this new choice of $e$ does not change the value $\alpha$ defined in (2.7) (although $\Gamma_k$ and $\Gamma$ are changed, and we shall denote also the new ones by $\Gamma_k$ and $\Gamma$). This is clear by the last paragraph in the proof of Proposition 2.14. Let

$$\beta_k = \inf_{\gamma \in \Gamma_k} \max_{u \in \gamma} J_1(u),$$

$$\beta = \inf_{\gamma \in \Gamma} \max_{u \in \gamma} J_1(u).$$

By Theorem 3.23 and Proposition 2.14, we see that $\beta$ is a critical value of $J_1$ and there is a $\gamma_1 \in \Gamma$ such that

$$\beta = \max_{u \in \gamma_1} J_1(u). \tag{3.29}$$

It is clear that $J(u) \leqq J_1(u)$ for all $u \in E$; thus $\alpha \leqq \beta$.

*Case* 1. $\alpha = \beta$.
Then

$$\alpha \leqq \max_{u\in\gamma_1} J(u) \leqq \max_{u\in\gamma_1} J_1(u) = \beta = \alpha$$

*i.e.*

$$\alpha = \max_{u\in\gamma_1} J(u),$$

that is, also $\alpha$ is achieved in $\gamma_1 \in \Gamma$. Next we assert that *there is a* $u_1 \in \gamma_1$ *such that* $J'(u_1) = 0$ *and* $J(u_1) = \alpha$. Suppose this is not so. Then, we can find an $\varepsilon > 0$ and a $\delta > 0$ such that $\|J'(u)\| \geqq \delta$ on $\gamma_\varepsilon \equiv \gamma_1 \cap J^{-1}[\alpha - \varepsilon, \alpha + \varepsilon]$. Thus $\|J'(u]\| \geqq \delta/2$ on some neighborhood $U_\varepsilon$ of $\gamma_\varepsilon$. Consider the deformation $\eta(t, x)$ defined by the "gradient flow"

$$(3.30)\qquad \begin{aligned} &\eta_t = -\psi(\eta)\, J'(\eta)/\|J'(\eta)\| \quad \text{and} \\ &\eta(0, x) \equiv x \quad \text{for all } x \in E, \end{aligned}$$

where $\psi$ is a cut-off function so chosen that $\psi$ is positive in $U_\varepsilon$ and vanishes identically outside $U_\varepsilon$. Now, on the deformed path $\eta(1, \gamma_1)$, we have $J < \alpha$; thus

$$\max_{u\in\eta(1,\gamma_1)} J(u) < \alpha,$$

a contradiction. This proves our assertion.

A few remarks are in order. First, our argument concerning the deformation (3.30) is rather sketchy since such a deformation is somewhat standard; see, for example, the proof of Theorem 1.9 in [R], pp. 148–153. Also observe that some "compactness" conditions, such as the well-known "Palais-Smale condition", are *usually* assumed in treating this kind of deformation. In our case, the functional $J$ does not enjoy compactness of that kind in $E$. However, we get around this difficulty by constructing an explicit path $\gamma_1$ which achieves $\alpha$ and is naturally *compact*.

From our assertion above, we see that $\alpha$ is a critical value of $J$ and $u_1$ is a nontrivial critical point of $J$. Furthermore, the construction of $\gamma_1$ (as in the proof of Proposition 2.14) makes $u_1$ nonnegative in $\mathbb{R}^n$; thus, by the maximum principle, $u_1$ must be positive in $\mathbb{R}^n$.

*Case* 2. $\alpha < \beta$. Let $u_k$, $\alpha_k$ and $\bar{u}$ be as in Theorem 2.13. It suffices to show that $\bar{u} \not\equiv 0$ in $E$. Suppose the contrary, $\bar{u} \equiv 0$; *i.e.*, suppose that $u_k$ converges to 0 weakly in $E$. From (3.26) and (3.27) we deduce

$$(3.31)\qquad \begin{aligned} J(u) &\geqq \tfrac{1}{2}\int (|\nabla u|^2 + a_1 u^2) - \int F_1(x, u) - \tfrac{1}{2}\int b_1 u^2 - \int b_2 \left(\frac{u^2}{2} + \frac{|u|^{p+1}}{p+1}\right) \\ &\geqq J_1(u) - \int b(u^2 + |u|^{p+1}) \end{aligned}$$

where $b \equiv b_1 + b_2 \to 0$ at $\infty$.

Let $l_k = \{tu_k \mid t \geqq 0\}$. Since $f$ satisfies (2.16), we see from the proof of Proposition 2.14 that

$$(3.32)\qquad J(u_k) = \max_{u\in l_k} J(u).$$

Suppose $J_1|_{l_k}$ attains its maximum at $t_k u_k$. We claim that *the sequence* $t_k > 0$ *is bounded.* Set $g_k(t) = J_1(tu_k)$, $t \geqq 0$. Using (2.4) (c) for $f_1$, we have

$$\begin{aligned} (3.33) \qquad g_k'(t) &= t\,\|u_k\|_1^2 - \int u_k f_1(x, tu_k) \\ &\leqq t\,\|u_k\|_1^2 - \frac{1}{t\theta_1}\int F_1(x, tu_k) + \frac{1}{t\theta_1}\int A_1(x) \\ &= -\left(\frac{1}{2\theta_1} - 1\right) t\,\|u_k\|_1^2 + \frac{1}{t\theta_1} J_1(tu_k) + \frac{1}{t\theta_1}\int A_1(x) \\ &\leqq -Ct + \frac{1}{t\theta_1} g_k(t) + \frac{1}{t\theta_1}\int A_1(x) \end{aligned}$$

where $C = \left(\frac{1}{2\theta_1} - 1\right) C'$ and $C' > 0$ is a uniform lower bound for $\|u_k\|$, for all $k$, i.e. $\|u_k\| \geqq C' > 0$ for all $k$ since $J(u_k) = \alpha_k \geqq \alpha > 0$. On the other hand, by (2.16) applied to $f_1$, when $t \geqq 1$ we have

$$\begin{aligned} (3.34) \qquad g_k'(t) &\leqq t\,\|u_k\|_1^2 - t\int u_k f_1(x, u_k) \\ &\leqq t\,[\|u_k\|^2 + \int b_1 u^2 - \int u_k f(x, u_k) + \int b_2(u_k^2 + u_k^{p+1})] \\ &\leqq (p+1)\, t \int b(u_k^2 + u_k^{p+1}) \end{aligned}$$

since $u_k \geqq 0$ and

$$\|u_k\|^2 = \int u_k f(x, u_k).$$

Next, *we claim that*

$$(3.35) \qquad \varepsilon_k \equiv \int b(u_k^2 + u_k^{p+1}) \to 0 \quad \text{as } k \to \infty.$$

We postpone the proof of (3.35) till the end of this subsection. From (3.34) and (3.35) we observe that if $t \geqq 1$,

$$g_k(t) \leqq g_k(1) + (p+1)\,\varepsilon_k t^2.$$

By (3.31),

$$g_k(1) = J_1(u_k) \leqq J(u_k) + \varepsilon_k = \alpha_k + \varepsilon_k.$$

Now, (3.33) gives, if $t \geqq 1$

$$g_k'(t) \leqq -\left(C - \frac{(p+1)\,\varepsilon_k}{\theta_1}\right) t + \bar{C}$$

where the constant $C$ is given in (3.33) and $\bar{C}$ is some constant. By (3.35) we can find $k_0 \geqq 1$, $T > 1$ such that $g_k'(t) < 0$ for $k \geqq k_0$, $t \geqq T$, and $T$ is independent of $k \geqq k_0$. Since $g_k'(t_k) = 0$, $t_k \leqq T$ for all $k \geqq k_0$. This proves our assertion.

In view of (3.31), (3.32) and (3.35), we see that

$$J_1(t_k u_k) \leqq J(u_k) + T^{p+1} \int b(u_k^2 + u_k^{p+1})$$
$$= \alpha_k + \varepsilon_k T^{p+1}.$$

Since $\alpha_k$ converges to $\alpha$, which is smaller than $\beta$, we have

$$\max_{u \in l_k} J_1(u) \leqq \beta - \bar{\varepsilon}$$

for $k$ sufficiently large, where $\bar{\varepsilon} < \beta$ is some positive constant. Now we can construct a path $\gamma_k \in \Gamma_k$ based on $l_k$ such that

$$\max_{u \in \gamma_k} J_1(u) \leqq \beta - \bar{\varepsilon}.$$

The construction of such a path $\gamma_k$ is similar to the arguments in the last paragraph of the proof of Proposition 2.14 and is therefore omitted here. By the definition of $\beta_k$, we then have $\beta_k \leqq \beta - \bar{\varepsilon}$ for all $k$ sufficiently large. Letting $k \to \infty$, we see that $\beta \leqq \beta - \bar{\varepsilon}$, a contradiction. To finish the proof, it remains to show (3.35) (assuming $\bar{u} \equiv 0$). Indeed, given any $\varepsilon > 0$, there is an $R > 0$ such that $b(x) \leqq \varepsilon/2C$, for $|x| \geqq R$ where $C > 0$ is a constant such that

$$\int (u_k^2 + u_k^{p+1}) \leqq C$$

for all $k \geqq 1$. (The existence of such a constant follows from (2.12) and the following well known Sobolev-type inequality

$$\|u\|_{L^{p+1}} \leqq C \, \|\nabla u\|_{L^2}^{1-a} \, \|u\|_{L^2}^{a} \quad \text{for all } u \in C_0^\infty(\mathbb{R}^n),$$

where

$$a = 1 - \left(\tfrac{1}{2} - \frac{1}{p+1}\right) n \text{ .)}$$

Then

$$\varepsilon_k = \left( \int_{|x| \geqq R} + \int_{|x| \leqq R} \right) (b(u_k^2 + u_k^{p+1}))$$
$$\leqq \frac{\varepsilon}{2} + \int_{|x| \leqq R} b(u_k^2 + u_k^{p+1}).$$

Since $u_k$ converges to zero weakly in $E$, the second integral over the domain $|x| \leqq R$ converges to zero as $k \to \infty$ by the Sobolev compact embedding theorem. This completes the proof.

*Q.e.d.*

## § 4. The radial case

We consider the following equation

$$(4.1) \qquad \Delta u - a(r)\, u + f(r, u) = 0$$

in $\mathbb{R}^n$, where $r = |x|$. We still need to assume that $a(r)$ and $f(r, u)$ are locally Hölder continuous and (2.2) holds. Instead of working in $E$, we shall deal with the subspace $E_r$ of $E$ which consists in all radial functions in $E$. For $f$, we assume that $f(r,u) \equiv 0$ if $u \leqq 0$ and

(4.2)
(a) $f \in C(\mathbb{R} \times \mathbb{R}^+)$ and $f(r,u)$ is locally Lipschitz-continuous in $u$, and $f \not\equiv 0$,

(b) there is a $p$ such that $1 < p < (n+2)/(n-2)$ and

$$0 \leqq f(r,u) \leqq C_1(r)(1 + u^p)$$

for all $r \geqq 0$, $u > 0$, where $C_1(r) \geqq 0$ is continuous,

(c) there are constants $q > 1$, $\delta > 1$ such that

$$f(r,u) \leqq C_2(r)\, u^q$$

for all $r \geqq 0$, $u \in (0, \delta)$, where $C_2(r)$ is continuous and satisfies

$$0 < C_2(r) \leqq C(1 + r^l)$$

for all $r \geqq 0$, where $C$ is a positive constant and $0 \leqq l < \frac{1}{2}(n-1)(q-1)$,

(d) there are a $\theta \in (0, \frac{1}{2})$ and a continuous $A(r) \geqq 0$ such that

$$A_1 \equiv \int_0^\infty r^{n-1} A(r)\, dr < \infty$$

and

$$A(r) + \theta u f(r,u) \geqq F(r,u)$$

for all $r \geqq 0$, $u > 0$.

Note that there is no restriction on the growth of $C_1(r)$. An example of a nonlinearity satisfying (4.2) is

$$f(r,u) = (1 + r^2)^{l/2} u^p, \quad 0 \leqq l < \tfrac{1}{2}(n-1)(p-1).$$

Therefore it is not clear that the variational functional

$$J(u) = \tfrac{1}{2} \int_{\mathbb{R}^n} [|\nabla u|^2 + a(|x|)\, u^2] - \int_{\mathbb{R}^n} F(|x|, u)$$

where, as before

$$F(|x|, u) \equiv \int_0^u f(|x|, s)\, ds,$$

is well defined on $E_r$. However, $J$ is well defined on each subspace $E_{kr}$, where $E_{kr}$ is the subspace of $E_k$ which consists of all radial functions in $E_k$. (The spaces $E_k$ and $E$ are defined in § 2.) Let

$$J_k = J|_{E_{kr}};$$

then the analysis (in § 2) in deriving (2.12) and in proving Theorem 2.13 carries over. That is, the sequence $\{\|u_k\|\}$, $u_k$ being a critical point of $J_k$ obtained by use of the Mountain-Pass Lemma (as in § 2), is still uniformly bounded. Thus, by passing to a subsequence if necessary, we may assume that $u_k$ converges weakly to $\bar{u}$ in $E_r$. Let $B$ be a (fixed) ball in $\mathbb{R}^n$. Using (4.2) (b) and standard *interior $L^p$-estimates* and Schauder estimates for solutions $u_k$, we conclude that (passing to a subsequence again if necessary) $u_k$ converges to $\bar{u}$ in the space $C^2(\bar{B})$. Therefore $\bar{u}$ is a nonnegative classical solution of (4.1) in $B$. Since $B$ is arbitrary, $\bar{u} \geqq 0$ is a solution of (4.1) in $\mathbb{R}^n$.

The remaining task is to show that $\bar{u} \not\equiv 0$, for then the strong maximum principle will guarantee that $\bar{u} > 0$ in $\mathbb{R}^n$ and $\bar{u}$ will be a desired solution of (4.2). Condition (4.2) (c) will be used in establishing this fact. We shall also need the following estimates due to WALTER STRAUSS (see *e.g.* [S]).

**Radial Lemma.** *For any radial function $u$ in $H^1(\mathbb{R}^n)$, $n \geqq 2$,*

$$|u(r)| \leqq C \|u\|_{H^1(\mathbb{R}^n)} r^{(1-n)/2} \quad \textit{for } r \geqq 1,$$

*in which $C$ depends only on $n$.*

Note that under (2.2), $E$ is continuously embedded into $H^1(\mathbb{R}^n)$. Since $\{\|u_k\|\}$ is bounded above,

$$u_k(r) \leqq C r^{(1-n)/2} \tag{4.3}$$

for $r \geqq 1$ and for all $k$. On the other hand, we observe, just as in the proof of Theorem 3.23, that

$$\int_0^k r^{n-1} u_k(r) \, [f(r, u_k(r)) - \bar{\lambda} u_k(r)] \, dr > 0.$$

Thus, there is an $r_k \in [0, k)$ such that $f(r_k, u_k(r_k)) > \bar{\lambda} u_k(r_k)$.

Let $S = \{k \mid u_k(r_k) < \delta\}$. Then, by (4.2) (c), we have, for $k \in S$,

$$\begin{aligned} \bar{\lambda} u_k(r_k) &\leqq C_2(r_k) \, [u_k(r_k)]^q \\ &\leqq C(1 + r_k^l) \, [u_k(r_k)]^q; \end{aligned}$$

*i.e.*, for $k \in S$,

$$u_k(r_k) \geqq \left[\frac{\bar{\lambda}}{C(1 + r_k^l)}\right]^{\frac{1}{q-1}}. \tag{4.4}$$

If $r_{k_j} \to \infty$, $k_j \in S$, then by the range of $l$, we deduce from (4.4) that

$$r_{k_j}^{\frac{n-1}{2}} u_{k_j}(r_{k_j}) \geqq \text{constant} \cdot r_{k_j}^{\left(\frac{n-1}{2} - \frac{l}{q-1}\right)} \to \infty \quad \text{as } k_j \to \infty,$$

contradicting (4.3). Thus there is an $R > 0$ such that $r_k \leqq R$ for all $k \in S$. This, together with (4.4), implies that there is a positive uniform lower bound $\beta$ for $u_k(r_k)$, $k \in S$, i.e.

$$u_k(r_k) \geqq \beta > 0 \quad \text{for all } k \in S. \tag{4.5}$$

Next, we wish to obtain similar bounds for $r_k$'s and $u_k(r_k)$'s for $k \notin S$. We proceed as follows. For $k \notin S$, we have $u_k(r_k) \geqq \delta$. It follows from (4.3) immediately that there is an $R > 0$ such that $r_k \leqq R$ for all $k \notin S$. Therefore, there is a constant $R$ such that $r_k \leqq R$ for all $k$ and $u_k(r_k) \geqq \min\{\beta, \delta\} > 0$ for all $k$. By the compactness of $B_R$, there is a subsequence $r_{k_j}$ that converges to $\bar{r} \leqq R$ and

$$u_{k_j}(r_{kj}) \to \bar{u}(\bar{r})$$

as $j \to \infty$ since $u_k$ converges to $\bar{u}$ uniformly on $B_R$. Thus, $\bar{u}(\bar{r}) \geqq \min\{\beta, \delta\} > 0$ and $\bar{u} \not\equiv 0$. We have thus proved the following

**Theorem 4.6.** *Suppose* (2.2) *and* (4.2) *hold. Then equation* (4.1) *has a positive radial solution in* $E_r$.

We now return to our model equation

$$\Delta u - u + Q(r)\, u^p = 0, \quad 1 < p < \frac{n+2}{n-2}. \tag{4.7}$$

From Theorem 4.6, we have

**Corollary 4.8.** *If* $Q \geqq 0$ *in* $[0, \infty)$ *and satisfies the growth condition*

$$Q(r) \leqq C(1 + r^l) \text{ for } r \geqq 0, \tag{4.9}$$

$C > 0$ *being some constant and* $l < (n-1)(p-1)/2$, *then equation* (4.7) *has a positive radial solution in* $H_0^1(\mathbb{R}^n)$.

The solutions obtained in Corollary 4.8 must tend to zero exponentially fast at infinity. This standard fact follows from the Radial Lemma and a well known result of T. KATO [K].

## § 5. Maximization

For equation (3.8) a natural approach exploits the classical method by maximization (or, equivalently, minimization). *That is, let*

$$M \equiv \sup_{\|u\|_{H^1(\mathbb{R}^n)}=1} \int_{\mathbb{R}^n} Q(x)\, |u|^{p+1} \tag{5.1}$$

*and try to show that $M$ is attained by some element in $H^1(\mathbb{R}^n)$, which would then be a solution of* (3.8). For equation (4.7), the same idea applies, except we now restrict attention to the class of radial functions, letting

$$M_r = \sup_{\|u\|_{H_r^1(\mathbb{R}^n)}=1} \int_{\mathbb{R}^n} Q(|x|)\, |u|^{p+1} \tag{5.2}$$

where $H_r^1(\mathbb{R}^n)$ is the set of all radial functions in $H^1(\mathbb{R}^n)$.

We first look at the radial case. For equation (4.7), we assume that (as in Corollary 4.8) $Q$ could actually grow at infinity. Thus it is not clear that $M_r$, given by (5.2), is finite. Even if $M_r$ is finite, it is not at all clear that $M_r$ is always assumed.

Corollary 4.8 follows from Theorem 4.6 directly; however, the approach we have been using in this paper employs the Mountain-Pass Lemma, which is quite general and therefore is good for obtaining general theorems of existence. We shall sketch a second proof for Corollary 4.8 by use of the maximization method which is simpler although the *exact homogeneity* of the term $u^p$ is used (to eliminiate a Lagrange multiplier). Furthermore, this second proof also has several by-products and one of them is the finiteness of (5.2) under the growth condition (4.9) on $Q$.

*A sketch of the proof of Corollary* 4.8 *by maximization*

Define

$$M_{k,r} = \sup_{\|u\|_{H^1_{0,r}(B_k)}=1} \int_{B_k} Q(x)\,|u|^{p+1}. \tag{5.3$_k$}$$

It is not hard to see that

(i) $M_{k,r} < \infty$ and $M_{k,r}$ is assumed by an element $u_k > 0$ in $H^1_{0,r}(B_k)$ with $\|u_k\|_{H^1_{0,r}(B_k)} = 1$, where $H^1_{0,r}(B_k)$ is the set of all radial functions in $H^1_0(B_k)$.

(ii) $u_k$ is a solution of

$$\begin{aligned} &\Delta u - u + \lambda Q u^p = 0 \quad \text{in } B_k, \\ &u > 0 \quad \text{in } B_k \quad \text{and} \quad u = 0 \quad \text{on } \partial B_k, \end{aligned} \tag{4.7$'_k$}$$

where the Lagrange multiplier $\lambda = 1/M_{k,r}$. Thus

$$w_k \equiv \left(\frac{1}{M_{k,r}}\right)^{\frac{1}{p-1}} u_k \tag{5.4}$$

solves

$$\begin{aligned} &\Delta u - u + Q u^p = 0 \quad \text{in } B_k, \\ &u > 0 \quad \text{in } B_k, \quad u = 0 \quad \text{on } \partial B_k. \end{aligned} \tag{4.7$_k$}$$

(iii) $M_{k,r}$ increases monotonically with $k$ and thus

$$\|w_k\|_{H^1_r(\mathbb{R}^n)} = \left(\frac{1}{M_{k,r}}\right)^{\frac{1}{p-1}} \leqq \left(\frac{1}{M_{1,r}}\right)^{\frac{1}{p-1}}, \tag{5.5}$$

i.e. $\{w_k\}$ is uniformly bounded (above) in $H^1_r(\mathbb{R}^n)$. Standard elliptic estimates guarantee that (by passing to a subsequence which we still denote by $\{w_k\}$) the sequence $\{w_k\}$ converges to a limit $w$ in $H^1_r(\mathbb{R}^n)$ and the convergence is uniform in $C^2(\Omega)$ for any compact subset $\Omega$ in $\mathbb{R}^n$, and $w \geqq 0$ is a solution of (4.7).

(iv) It remains to show that $w \not\equiv 0$. To this end, set

$$A_k = \{x \in B_k \mid Q(x)\, w_k^{p-1}(x) > 1\}.$$

Clearly, $A_k$ is not empty. For $x \in A_k$, we have

$$w_k(x) \geqq \left(\frac{1}{Q(x)}\right)^{\frac{1}{p-1}} \geqq \frac{C}{|x|^{\frac{l}{p-1}}} \tag{5.6}$$

for $|x| \geqq 1$, by (4.9). On the other hand, by (5.5) and the Radial Lemma, we have

$$w_k(x) \leqq \frac{C}{|x|^{\frac{n-1}{2}}} \tag{5.7}$$

for $|x| \geqq 1$. Combining (5.6), (5.7) and the condition on $l$, we see immediately that $A_k$ is contained in a *fixed* ball of radius $s$. Thus

$$|w_k|_{L^\infty(B_s)} \geqq \underset{B_s}{\mathrm{Min}} \left(\frac{1}{Q}\right)^{\frac{1}{p-1}} > 0$$

for all $k$. Thus $w \not\equiv 0$ and the proof is complete.

As a consequence, we have the following

**Proposition 5.8.** *Let $Q \geqq 0$ in $[0, \infty)$ with $Q(r) \leqq C(1 + r^l)$ for $r \geqq 0$ where $l < (n-1)(p-1)/2$. Then the number $M_r$ defined by (5.2) is finite.*

**Proof.** Since $M_{k,r}$ (defined in $(5.3)_k$) tends monotonically to $M_r$ as $k \to \infty$, *it suffices to show that $M_{k,r}$ is bounded above.* Otherwise $M_{k,r} \to \infty$ as $k \to \infty$. Then, by (5.4)

$$\| w_k \|_{H_r^1(\mathbb{R}^n)} \to 0 \quad \text{as } k \to \infty.$$

Since $w_k$ converges to $w$ in $C^2(\Omega)$ ($\Omega$ is a compact set), we see that

$$\| w \|_{H_r^1(\Omega)} = \lim_{k \to \infty} \| w_k \|_{H_r^1(\Omega)} \leqq \lim \| w_k \|_{H_r^1(\mathbb{R}^n)} = 0,$$

i.e., $w \equiv 0$ on $\Omega$. Since $\Omega$ is arbitrary, $w \equiv 0$ on $\mathbb{R}^n$, a contradiction.

*Q.e.d.*

A natural question arises: does the conclusion of Proposition 5.8 still hold if $l > (n-1)(p-1)/2$? The answer is negative as the following example shows.

*Example* 5.9. For each integer $m > 0$, define

$$u_m(r) = \begin{cases} \dfrac{1}{m^{(n-1)/2}} \cdot e^{(r-m)/2}, & r \leqq m, \\[2ex] \dfrac{1}{r^{(n-1)/2}}, & m \leqq r \leqq m+1, \\[2ex] \dfrac{1}{(m+1)^{(n-1)/2}} \cdot e^{(m+1-r)/2}, & r \geqq m+1. \end{cases}$$

It is routine to verify that $u_m \in H^1(\mathbb{R}^n)$ and that there is a *uniform upper bound* for $\|u_m\|_{H^1(\mathbb{R}^n)}$. However, if $Q(r) = (1 + r^l)$,

$$\int_{\mathbb{R}^n} Q(x)\,|u_m|^{p+1}\,dx \geqq \int_m^{m+1} r^{l-\frac{n-2}{2}(p+1)+n-1}\,dr$$

$$= \int_m^{m+1} r^{l-(n-1)(p-1)/2}\,dr$$

$$\geqq m^{l-(n-1)(p-1)/2} \int_m^{m+1} dr \to \infty \quad \text{as } m \to \infty;$$

thus $M_r = \infty$. Incidentally, this example also shows that the Radial Lemma is sharp.

Our second consequence of the above proof of Corollary 4.8 is the following

**Proposition 5.10.** *Suppose that* $Q \geqq 0$ *in* $[0, \infty)$, $Q \to \infty$ *as* $r \to \infty$ *with*

$$Q(r) \leqq C(1 + r^l) \quad \textit{for } r \geqq 0,$$

*where* $l < (n-1)(p-1)/2$. *Then for every large* $k$ *the problem* $(4.7)_k$ *has a positive solution which is not radially symmetric.*

**Proof.** Define

$$M_k = \sup_{\|u\|_{H_0^1(B_k)}=1} \int_{B_k} Q(x)\,|u|^{p+1}.$$

It is clear that $M_k \to \infty$ as $k \to \infty$ since $Q(r) \to \infty$ as $r \to \infty$. Thus, for large $k$, with $M_{k,r}$ given by $(5.3)_k$, $M_{k,r} \neq M_k$. Since both $M_{k,r}$ and $M_k$ are assumed (this is a rather standard fact whose proof we omit here), say, by $u_k$ and $u_k^*$ respectively, we see that $u_k^*$ can not be radially symmetric. Rescaling, we have

$$w_k \equiv \left(\frac{1}{M_{k,r}}\right)^{\frac{1}{p-1}} u_k, \qquad w_k^* \equiv \left(\frac{1}{M_k}\right)^{\frac{1}{p-1}} u_k^*,$$

and so for large $k$ there are two positive solutions of $(4.7)_k$, one of which is radial while the other is not. (The fact that both $u_k$ and $u_k^*$ are positive in $B_k$ is also standard.)

*Q.e.d.*

*Remark* 5.11. The procedure (i)–(iii) in the above proof of Corollary 4.8 does not have to be restricted to the class of radial functions. Everything carries over without change if we simply drop the restriction to radial functions.

Finally, we take up the question whether (5.1) is assumed for bounded $Q$'s. As an easy example, we state that *if* $Q(x)$ *is bounded, radial and monotonically* *increasing, then M in* (5.1) *is not realized.* This is essentially a special case of the following result, which is not restricted to radial functions.

**Proposition 5.12.** *Suppose there is an* $R_0 > 0$ *such that*

$$\min_{2R \leqq |x| \leqq 4R} Q(x) \geqq \max_{|x| \leqq R} Q(x) \tag{5.13}$$

*for all* $R \geqq R_0$. *If* $Q \not\equiv$ *constant, then the number*

$$M \equiv \sup_{|u|_{H^1(\mathbb{R}^n)}=1} \int_{\mathbb{R}^n} Q(x)\,|u|^{p+1}\,dx$$

*is not attained.*

*Remark* 5.14. Condition (5.13) may be replaced by the following slightly more general one:

There are two sequences $R_k \uparrow \infty$, $S_k \uparrow \infty$ such that

$$\min_{R_k - S_k \leqq |x| \leqq R_k + S_k} Q(x) \geqq \max_{|x| \leqq S_k} Q(x) \quad \text{for all } k.$$

In fact, one can easily formulate various weaker assumptions from the sufficient but not necessary ones used in the proof.

**Proof of Proposition 5.12.** Suppose $M$ is attained by $u$. Define $\tilde{u}_R(x) \equiv u(x - x_R)$, i.e., $\tilde{u}_R$ is a translation of $u$ by $x_R$, where $x_R = (3R, 0, \ldots, 0)$. Then $\tilde{u}_R$ also has $H^1$-norm one and for $R \geqq R_0$,

$$\begin{aligned} \int_{\mathbb{R}^n} Q(x)\,|\tilde{u}_R|^{p+1} &\geqq \int_{B_R(x_R)} Q\,|\tilde{u}_R|^{p+1} \\ &\geqq \left[\min_{B_R(x_R)} Q\right] \cdot \int_{B_R(x_R)} |\tilde{u}_R|^{p+1} \\ &\geqq \min_{2R \leqq |x| \leqq 4R} Q \cdot \int_{B_R(0)} |u|^{p+1} \\ &\geqq \max_{B_R(0)} Q \cdot \int_{B_R(0)} |u|^{p+1} \\ &= \int_{B_R(0)} Q(x)\,|u|^{p+1} + \delta(R), \end{aligned} \tag{5.15}$$

in which $\delta(R)$ is defined by the last equality. It is clear that

$$\delta(R) = \int_{B_R(0)} \left[\max_{B_R(0)} Q - Q(x)\right] |u|^{p+1} \geqq 0$$

is nondecreasing in $R$ and $\delta \not\equiv 0$ since $Q \not\equiv$ constant. Let

$$\delta(\infty) = \lim_{R\to\infty} \delta(R);$$

then there is an $R_1 \geqq R_0$ such that $\delta(R) > \frac{1}{2}\,\delta(\infty)$ and

$$\int_{B_R(0)} Q\,|u|^{p+1} > M - \tfrac{1}{2}\,\delta(\infty)$$

for $R \geqq R_1$. By (5.15), we see that

$$\int_{\mathbb{R}^n} Q\,|\tilde{u}_R|^{p+1} > M,$$

a contradiction.

*Q.e.d.*

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[Z] ZHANG, D., Private communication.

Institute of Mathematics
Academia Sinica
Beijing, P.R.C.

and

School of Mathematics
University of Minnesota
Minneapolis

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