General Existence Theorems for Unilateral Problems in Continuum Mechanics

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Abstract

The problem of minimizing a possibly non-convex and non-coercive functional is studied. Either necessary or sufficient conditions for the existence of solutions are given, involving a generalized recession functional, whose properties are discussed thoroughly. The abstract results are applied to find existence of equilibrium configurations of a deformable body subject to a system of applied forces and partially constrained to lie inside a possibly unbounded region.

Contents

1. Introduction

Many problems in Mathematical Physics can be formulated in terms of minimum problems: an integral functional describing the total energy associated to a configuration has to be minimized over the set of all admissible configurations.

Particularly interesting is the study of the equilibrium configurations of a body Ω subject to a system of applied forces and constrained to lie inside a given, possibly unbounded, region. If we denote by u a vector valued function which describes the configuration of the system, then the corresponding energy may be expressed in the form

$$F(u) - \langle L, u \rangle,$$

where $F(u)$ is the *stored energy functional*, depending on the nature of the body, and the linear form $\langle L, u \rangle$ describes the action of the applied forces. The physical constraints and the regularity properties of the admissible configurations are taken into account by imposing that u varies in a suitable subset $\mathbf{K}$ of a Banach space X. In this way, the minimization problem we deal with is

$$\min \{F(u) - \langle L, u \rangle : u \in \mathbf{K}\}. \tag{1.1}$$

By introducing the *indicator function* of $\mathbf{K}$

$$\chi_{\mathbf{K}}(x) = \begin{cases} 0 & \text{if } x \in \mathbf{K}, \\ +\infty & \text{if } x \notin \mathbf{K}, \end{cases}$$

we see that problem (1.1) is a particular case of the abstract minimization problem

$$\min \{G(u) : u \in X\}, \tag{1.2}$$

when $G(u) = F(u) - \langle L, u \rangle + \chi_{\mathbf{K}}(u)$. The simplest case in which existence theorems for problem (1.2) are well known occurs when the functional G is weak* lower semicontinuous and *coercive*, in the sense that $G(u)$ goes to $+\infty$ as $\|u\|$ goes to $+\infty$ (TONELLI's Direct Method [To1], [To2]). Unfortunately, in many applications coerciveness is too restrictive. A first instance is the case of the Signorini problem in linear elasticity: u is the displacement of the body and the stored energy functional takes the form

$$F(u) = \int_{\Omega} \sum_{i,j,h,k} a_{ijhk}(x)\, \varepsilon_{ij}(u)\, \varepsilon_{hk}(u)\, dx,$$

where $\varepsilon(u)$ is the linearized strain tensor and a_{ijhk} are the components of the elasticity tensor. Existence theorems for this case have been given by FICHERA in [F1].

When dealing with problem (1.1) in the framework of non-linear elasticity, the model proposed by BALL (see [Ba]) for the stored energy functional of a hyperelastic material is the following:

$$F(u) = \int_{\Omega} \mathbf{W}(x, u, \nabla u)\, dx,$$

where u denotes the deformed configuration and the integrand $\mathbf{W}$ is assumed to be polyconvex. In this framework, the unilateral problem was studied for the first time by CIARLET & NECAS in [CN1], [CN2].

Another mechanical situation leading to a minimum problem of the form (1.1) is the class of *masonry-like problems* in which the body is supposed not to react to traction but to behave elastically under compression. These problems have been recently studied by several authors (see, for instance, GAIQUINTA & GIUSTI [GG], ANZELLOTTI [A], ANZELLOTTI, BUTTAZZO & DAL MASO [ABD]), without unilateral constraints.

Thus, in several interesting cases we cannot expect the functional G in (1.2) to be coercive. On the other hand, pure elimination of coerciveness may cause problem (1.2) to have, in general, no solution. Therefore some supplementary hypotheses have to be added in order to get an existence theorem. For instance, existence of a solution of problem (1.1) has been established when F is a quadratic, non-coercive functional, under suitable compatibility conditions involving F, L, $\mathbf{K}$ (see, for instance, FICHERA [F1] and [F2], LIONS & STAMPACCHIA [LS], SCHATZMANN [Sc], BAIOCCHI, GASTALDI & TOMARELLI [BGT2]).

In this paper, we give conditions (weaker than coerciveness) on the behavior at infinity of the functional G in (1.2). If G is convex, these conditions are based on the notion of *recession functional* G^∞ associated with G (see ROCKAFELLER [R]); in the general case, we introduce the new notion of *topological recession functional* $G_{\infty,\sigma}$ associated with G and with a suitable topology σ on X (usually, the weak* topology).

In Section 2 we give the definition and the general properties of the functional $G_{\infty,\sigma}$. In Section 3 we consider problem (1.2) in a quite general framework and we derive a necessary condition for its solvability in terms of $G_{\infty,\sigma}$. Again by means of the topological recession functional, we give a general existence theorem based on some assumptions of semicontinuity, compactness and compatibility (see Theorems 3.4 and 3.9). We also treat in detail some particular cases (for instance, when G is convex) in which these assumptions simplify.

Sections 4, 5, 6 are devoted to the applications of the abstract existence theorems for the above-mentioned unilateral problems in linear elasticity, nonlinear elasticity and masonry-like materials, respectively. In each of these theories our method enables us to consider the equilibrium of a body constrained to lie inside a non-convex region, possibly containing entire directions.

Special attention is devoted to the set $\mathbf{K}$ of admissible configurations: in particular, in Sections 4 and 5 we allow the constraint to be imposed only on a part E of the elastic body (usually, a subset of its boundary). This requires us to work "up to subsets of zero capacity": hence we introduce a *capacitary essential representative* E_{ess} of E (which coincides with E when this is the closure of an open set or a smooth manifold of codimension 1). The properties of the set E_{ess}, along with a capacitary version of Korn's inequality, are proved in the Appendix.

2. Preliminaries and notations

In this section we introduce the tools we will need in what follows, along with their main properties.

Consider a

Hausdorff topological vector space (X, σ)

and a functional $G: X \to]-\infty, +\infty]$. As usual, set

$$\operatorname{dom} G = \{x \in X: G(x) < +\infty\}:$$

if $\operatorname{dom} G \neq \emptyset$, the functional G is said to be *proper*.

If G is proper, convex and σ lower semi-continuous (abbreviated σ-l.s.c.), then its behavior at infinity can be described in terms of what is called the *recession function*, defined as follows.

Definition 2.1. *Let* $G: X \to]-\infty, +\infty]$ *be proper, convex and* σ*-l.s.c. The* **recession function** G^∞ *of* G *is defined by*

$$G^\infty(x) = \lim_{\lambda \to +\infty} \frac{1}{\lambda} G(x_0 + \lambda x), \tag{2.1}$$

where x_0 *is any element of* dom G.

We remark that the limit in (2.1) exists and that the definition above is independent of x_0. Moreover, the value of G^∞ at any x of X can be expressed by (see [R], Theorem 8.5 for the finite dimensional case; see [Bou] for arbitrary topological vector spaces)

$$G^\infty(x) = \lim_{\lambda \to +\infty} \frac{1}{\lambda} [G(x_0 + \lambda x) - G(x_0)] = \sup \{G(z + x) - G(z) \colon z \in \operatorname{dom} G\}$$

$$= \sup_{\lambda > 0} \frac{1}{\lambda} [G(x_0 + \lambda x) - G(x_0)],$$

where x_0 is any element of dom G.

The functional G^∞ turns out to be proper, convex, σ-l.s.c. and positively homogeneous of degree 1, say

$$G^\infty(\lambda x) = \lambda G^\infty(x), \quad \forall \lambda \geqq 0, \quad \forall x \in X.$$

The definition we have recalled is not suitable for general functionals; the one we are going to give requires neither convexity nor semi-continuity; rather it depends on the topology of X.

Definition 2.2. *Let* G: $X \to]-\infty, +\infty]$ *be any functional; we call* **topological recession function** of G the *function* $G_{\infty,\sigma}$ *defined by*

$$G_{\infty,\sigma}(x) = \Gamma^-(\sigma) \liminf_{\lambda \to +\infty} \frac{1}{\lambda} G(x_0 + \lambda x), \quad x \in X,$$

where x_0 *is any element of* X *and the* Γ*-limit is defined by*

$$\Gamma^-(\sigma) \liminf_{\lambda \to +\infty} \frac{1}{\lambda} G(x_0 + \lambda x) = \sup_{E \in N_\sigma(x)} \liminf_{\lambda \to +\infty} \inf_{y \in E} \frac{1}{\lambda} G(x_0 + \lambda y)$$

($N_\sigma(x)$ *denotes the family of neighborhoods of* x *in the topology* σ).

For the definition and properties of Γ limits, see for instance [D], [DF], [Bu]. Here we just give an explicit representation of $G_{\infty,\sigma}$ as follows. For a fixed x of X, introduced the set of nets

$$S(x) = \{(\lambda_\xi, x_\xi)_{\xi \in \Xi} \colon \lambda_\xi \to +\infty, x_\xi \xrightarrow{\sigma} x\},$$

where Ξ denotes an arbitrary direct set. Then

$$G_{\infty,\sigma}(x) = \inf \left\{ \liminf_{\xi \in \Xi} \frac{1}{\lambda_\xi} G(x_0 + \lambda_\xi x_\xi) \ : \ (\lambda_\xi, x_\xi)_{\xi \in \Xi} \in S(x) \right\}. \tag{2.2}$$

Remark 2.3. The functional $G_{\infty,\sigma}$ is σ-l.s.c. and positively homogeneous of degree 1. Moreover, it is not difficult to see that the definition of $G_{\infty,\sigma}$ does not depend on the choice of $x_0 \in X$ (for this reason, we often choose $x_0 = 0$).

Throughout this paper we use the following notation:

$$\operatorname{Ker} G_{\infty,\sigma} = \{x \in X \colon G_{\infty,\sigma}(x) = 0\}.$$

Note that $\operatorname{Ker} G_{\infty,\sigma}$ is not a subspace of X, in general. Anyway, due to the homogeneity of degree 1, it is a set of semi-directions.

Remark 2.4. It is possible to prove (see the references quoted above) that in (2.2) the infimum is actually a minimum, that is, for all x, x_0 of X there exists an element $(\lambda_\xi, x_\xi)_{\xi\in\Xi} \in S(x)$ such that

$$G_{\infty,\sigma}(x) = \liminf_{\xi\in\Xi} \frac{1}{\lambda_\xi} G(x_0 + \lambda_\xi x_\xi).$$

The recession function may be compared with the topological recession function only for convex functionals. In this case, they actually coincide, as the following proposition shows.

Proposition 2.5. *Let* $G \colon X \to]-\infty, +\infty]$ *be proper convex and* σ*-l.s.c. Then*

$$G^\infty(x) = G_{\infty,\sigma}(x), \qquad \forall x \in X.$$

Proof. Fix an element $x \in X$ and $x_0 \in \operatorname{dom} G$. Take an element $(\lambda_\xi, x_\xi)_{\xi\in\Xi}$ of $S(x)$ with $x_\xi = x + \dfrac{x_0}{\lambda_\xi}$; by (2.2), we get

$$G_{\infty,\sigma}(x) \leqq \liminf_{\xi\in\Xi} \frac{1}{\lambda_\xi} G(\lambda_\xi x_\xi) = \liminf_{\xi\in\Xi} \frac{1}{\lambda_\xi} G(x_0 + \lambda_\xi x) = G^\infty(x).$$

Let us prove the opposite inequality. Consider an element $(\lambda_\xi, x_\xi)_{\xi\in\Xi}$ of $S(x)$. Due to the convexity and σ-l.s.c. of G, for all $\lambda > 0$ we get

$$\begin{aligned} G(x_0 + \lambda x) &\leqq \liminf_{\xi\in\Xi} G\left[\left(1 - \frac{\lambda}{\lambda_\xi}\right) x_0 + \frac{\lambda}{\lambda_\xi} \lambda_\xi x_\xi\right] \\ &\leqq \liminf_{\xi\in\Xi} \left[\left(1 - \frac{\lambda}{\lambda_\xi}\right) G(x_0) + \frac{\lambda}{\lambda_\xi} G(\lambda_\xi x_\xi)\right] \\ &= G(x_0) + \lambda \liminf_{\xi\in\Xi} \frac{1}{\lambda_\xi} G(\lambda_\xi x_\xi). \end{aligned}$$

Since this inequality holds for all elements of $S(x)$,

$$\frac{1}{\lambda} G(x_0 + \lambda x) \leqq \frac{1}{\lambda} G(x_0) + G_{\infty,\sigma}(x).$$

Taking the limit as $\lambda \to +\infty$, we get $G^\infty(x) \leqq G_{\infty,\sigma}(x)$ and the proof is complete. □

Besides functionals defined on X, we will be interested in subsets of X and particularly in their behavior at infinity. Again, the simplest possibility is provided by a convex subset; the following definition is well known.

Definition 2.6. *Let* $\mathbf{K}$ *be a non-empty, convex, σ-closed subset of X. We call* **recession cone of** $\mathbf{K}$ *the set (σ-closed, convex cone)*

$$\mathbf{K}^{\infty} = \bigcap_{\lambda>0} \frac{1}{\lambda}(\mathbf{K} - k_0),$$

where k_0 is any element of $\mathbf{K}$ *and* $\mathbf{K} - k_0$ *denotes the set* $\{k - k_0 : k \in \mathbf{K}\}$.

Remark 2.7. The asymptotic behavior of a convex, σ-closed subset $\mathbf{K}$ of X can also be described in terms of the recession function of the *indicator* of $\mathbf{K}$, say the function

$$\chi_{\mathbf{K}}(x) = \begin{cases} 0 & \text{if } x \in \mathbf{K} \\ +\infty & \text{if } x \notin \mathbf{K}. \end{cases}$$

Indeed, it is easy to see that

$$\mathbf{K}^{\infty} = \operatorname{dom}(\chi_{\mathbf{K}})^{\infty}, \quad \text{say} \quad \chi_{\mathbf{K}^{\infty}} = (\chi_{\mathbf{K}})^{\infty}.$$

For the general case we give the following definition.

Definition 2.8. *Let* $\mathbf{K}$ *be a subset of X. We call the* **set of topologically unbounded directions of** $\mathbf{K}$ *the Kuratowski upper limit (as $\lambda \to +\infty$) of the family* $\dfrac{(\mathbf{K} - x_0)}{\lambda}$, *say*

$$\mathbf{K}_{\infty,\sigma} = \bigcap_{\mu>0} \mathrm{cl}_{\sigma}\left[\bigcup_{\lambda>\mu} \frac{1}{\lambda}(\mathbf{K} - x_0)\right],$$

where x_0 is any point of X and cl_σ *denotes the topological closure with respect to σ.*

As in the case of the functional $G_{\infty,\sigma}$, the definition of the set $\mathbf{K}_{\infty,\sigma}$ does not depend on the choice of $x_0 \in X$; moreover, as for convex $\mathbf{K}$ the unbounded directions of a subset $\mathbf{K}$ of X may be characterized in terms of the recession function of its indicator. Indeed the following lemma holds.

Lemma 2.9. *Let* $\mathbf{K}$ *be a subset of X. Then*

$$\mathbf{K}_{\infty,\sigma} = \operatorname{dom}[(\chi_{\mathbf{K}})_{\infty,\sigma}], \quad \textit{say} \quad \chi_{\mathbf{K}_{\infty,\sigma}} = (\chi_{\mathbf{K}})_{\infty,\sigma},$$

or equivalently

$$\mathbf{K}_{\infty,\sigma} = \{x \in X : \exists (\lambda_\xi, x_\xi)_{\xi \in \Xi} \in S(x) \quad \textit{with} \quad x_0 + \lambda_\xi x_\xi \in \mathbf{K},\ \forall \xi \in \Xi\} \tag{2.3}$$

for any $x_0 \in X$.

Proof. The result is trivial if $\mathbf{K} = \emptyset$; hence we prove it only for a non-empty $\mathbf{K}$. Let x be an element of $\mathbf{K}_{\infty,\sigma}$ and let E be a neighborhood of x in the topology σ.

Then, for every $\mu > 0$ there is a $\lambda > \mu$ such that $E \cap \left[\frac{1}{\lambda}(\mathbf{K} - x_0)\right]$ is not empty. By Definition 2.2, this implies that $(\chi_{\mathbf{K}})_{\infty,\sigma}(x) = 0$; hence $\mathbf{K}_{\infty,\sigma} \subset \operatorname{dom}[(\chi_{\mathbf{K}})_{\infty,\sigma}]$. Conversely, let x be an element of $\operatorname{dom}[(\chi_{\mathbf{K}})_{\infty,\sigma}]$. Then there is a net $\{\lambda_\xi, x_\xi\} \in S(x)$ with $x_0 + \lambda_\xi x_\xi \in \mathbf{K}$, for all $\xi \in \Xi$. Thus, for every σ-neighborhood E of x and for every $\mu > 0$ we can find $\xi \in \Xi$ such that $\lambda_\xi > \mu$ and $x_\xi \in E$. Hence $E \cap \left[\frac{1}{\lambda_\xi}(\mathbf{K} - x_0)\right]$ is not empty and this proves that $x \in \mathbf{K}_{\infty,\sigma}$. Thus $\mathbf{K}_{\infty,\sigma} = \operatorname{dom}[(\chi_{\mathbf{K}})_{\infty,\sigma}]$. By (2.2), this is equivalent to (2.3). □

Remark 2.10. The set $\mathbf{K}_{\infty,\sigma}$ is a σ-closed cone (not convex, in general), *i.e.* $\lambda x \in \mathbf{K}_{\infty,\sigma}$ for all $\lambda \geqq 0$, whenever $x \in \mathbf{K}_{\infty,\sigma}$.

Further properties of $\mathbf{K}_{\infty,\sigma}$ are the following.

Lemma 2.11. *Let* $\mathbf{K}$ *be a non-empty subset of* X. *Then*

(*i*) *if* $\mathbf{K}$ *is convex and* σ-*closed, then* $\mathbf{K}^\infty = \mathbf{K}_{\infty,\sigma}$;

(*ii*) $\mathbf{K}_{\infty,\sigma} = (\mathbf{K} - T)_{\infty,\sigma} = (\mathbf{K} \cup T)_{\infty,\sigma}$, *for all bounded subset* T *of* X, $T \neq \emptyset$;

(*iii*) *if* $\mathbf{K}$ *is bounded, then* $\mathbf{K}_{\infty,\sigma} = \{0\}$.

Proof. Property (*i*) follows from Lemma 2.9, Remark 2.7 and Proposition 2.5. The proof of (*ii*), (*iii*) follows from the definition of $\mathbf{K}_{\infty,\sigma}$ and of boundedness in a topological vector space. □

The following lemma lists some properties of the functional $G_{\infty,\sigma}$.

Lemma 2.12. *Let* G, H *be functionals defined on* X *with values in* $]-\infty, +\infty]$. *Then*

(*i*) $\operatorname{dom} G_{\infty,\sigma} \subset (\operatorname{dom} G)_{\infty,\sigma}$;

(*ii*) $(G + H)_{\infty,\sigma} \geqq G_{\infty,\sigma} + H_{\infty,\sigma}$;

(*iii*) *if* H *is positively homogeneous of degree 1 and* σ-*continuous, then* $(G + H)_{\infty,\sigma} = G_{\infty,\sigma} + H$;

(*iv*) *if* G *is non-negative, positively homogeneous of degree greater than 1 and* σ-*l.s.c., then*

$$G_{\infty,\sigma}(x) = \begin{cases} +\infty & \text{if } G(x) \neq 0 \\ 0 & \text{if } G(x) = 0. \end{cases}$$

Proof. The proof follows very easily from Definitions 2.2 and 2.8. □

Remark 2.13. From Lemma 2.12(*i*) and Lemma 2.11(*iii*) it follows that, if $\operatorname{dom} G$ is bounded, then

$$G_{\infty,\sigma}(x) = +\infty \quad \forall x \in X, x \neq 0.$$

The inequality in Lemma 2.12(*ii*) may be strict; this we will show in the example below. On the other hand, in some cases the inequality becomes an equality, as shown in case (*iii*). Another interesting case is considered in the following proposition.

Proposition 2.14. *Let* $J: X \to]-\infty, +\infty]$ *be proper, convex and* σ*-l.s.c.; let* $\mathbf{K}$ *be a non-empty convex* σ*-closed set. Then*

$$(J + \chi_{\mathbf{K}})_{\infty,\sigma} = (J + \chi_{\mathbf{K}})^{\infty} = J^{\infty} + (\chi_{\mathbf{K}})^{\infty} = J^{\infty} + \chi_{\mathbf{K}^{\infty}}.$$

Proof. The proof is an easy consequence of Proposition 2.5, of Remark 2.7 and of Definition 2.1. □

Example 2.15. Let $X = \mathbf{H}^1(0, \pi)$ be the usual Sobolev space of the (classes of) functions which are square integrable over the interval $]0, \pi[$ along with their first derivative. Take as $\mathbf{K}$ the following set:

$$\mathbf{K} = \{v \in X: v(x) = n + \sin nx \text{ for some } n \in \mathbb{N}\}$$

and consider the functional $G = J + \chi_{\mathbf{K}}$, where

$$J(v) = \int_0^{\pi} |v'(x)|^2 \, dx.$$

Then, denoting by σ the weak topology of X, we obtain the function $v_0 = 1$ belongs to $\mathbf{K}_{\infty,\sigma}$ and $J^{\infty}(v_0) = J(v_0) = 0$, while $G_{\infty,\sigma}(v_0) = +\infty$. This shows at once that both the inclusion in Lemma 2.12(i) and the inequality in Lemma 2.12 (ii) may be strict even if J is convex. □

Finally, we give an example of integral functionals for which the evaluation of the recession functional reduces to evaluating the recession function of the integrand.

Example 2.16. Let N, M be positive integers; let Ω be an open subset of $\mathbb{R}^N$ and let $[\mathbf{W}^{1,p}(\Omega)]^M$, $p > 1$, be the Sobolev space of the (classes of) vector valued functions which belong to $\mathbf{L}^p(\Omega)$ along with their first derivatives. Finally, let $f: \Omega \times \mathbb{R}^{NM} \to [0, +\infty]$ be a Borel function.

For all $u \in [\mathbf{W}^{1,p}(\Omega)]^M$ set

$$F(u) = \int_{\Omega} f(x, \nabla u(x)) \, dx.$$

Assume that:

(a) for almost all x of Ω the map "$q \to f(x, q)$" is convex and l.s.c. on $\mathbb{R}^{NM}$;

(b) there is a $u_0 \in [\mathbf{W}^{1,p}(\Omega)]^M$ such that $F(u_0) < +\infty$.

It is well known that under these assumptions the functional F turns out to be proper, convex and l.s.c. with respect to the weak topology of $[\mathbf{W}^{1,p}(\Omega)]^M$. Moreover,

$$F^{\infty}(u) = \int_{\Omega} f^{\infty}(x, \nabla u(x)) \, dx, \qquad u \in [\mathbf{W}^{1,p}(\Omega)]^M,$$

where $f^{\infty}(x, \cdot)$ is the recession function of $f(x, \cdot)$.

In fact, because of the convexity of f, for all $u_0, u \in [\mathbf{W}^{1,p}(\Omega)]^M$ the function

$$g(x, \lambda) = \frac{1}{\lambda} [f(x, \nabla u_0(x) + \lambda \nabla u(x)) - f(x, \nabla u_0(x))]$$

is non-decreasing with respect to λ, for almost all $x \in \Omega$. Therefore, the Beppo Levi theorem gives

$$F^{\infty}(u) = \lim_{\lambda \to +\infty} \frac{1}{\lambda} [F(u_0 + \lambda u) - F(u_0)] = \lim_{\lambda \to +\infty} \int_{\Omega} g(x, \lambda)\, dx$$
$$= \int_{\Omega} f^{\infty}(x, \nabla u(x))\, dx. \qquad \square$$

Remark 2.17. When working with sequences instead of nets, we can define a *sequential recession functional* $G^{\text{seq}}_{\infty,\sigma}$ of a given functional $G: X \to]-\infty, +\infty]$:

$$G^{\text{seq}}_{\infty,\sigma}(x) = \inf\left\{\liminf_{n \to +\infty} \frac{1}{\lambda_n} G(x_0 + \lambda_n x_n) \;:\; \lambda_n \to +\infty, x_n \xrightarrow{\sigma} x\right\} \; x, x_0 \in X,$$

where $\{\lambda_n\}_{\mathbb{N}}, \{x_n\}_{\mathbb{N}}$ are sequences.

The inequality

$$G_{\infty,\sigma}(x) \leqq G^{\text{seq}}_{\infty,\sigma}(x), \qquad x \in X$$

is immediate. Moreover, the functional $G^{\text{seq}}_{\infty,\sigma}$ is positively homogeneous of degree 1. By analogous proofs, we may obtain for $G^{\text{seq}}_{\infty,\sigma}$ results similar to those given in Proposition 2.5 and Lemma 2.12 for $G_{\infty,\sigma}$.

Let $\mathbf{K}$ be a non-empty subset of X: we call *set of sequentially unbounded directions of* $\mathbf{K}$ the set

$$\mathbf{K}^{\text{seq}}_{\infty,\sigma} = \operatorname{dom}\, [(\chi_{\mathbf{K}})^{\text{seq}}_{\infty,\sigma}].$$

It is easy to see that

$$\mathbf{K}^{\text{seq}}_{\infty,\sigma} = \{x \in X\colon \; \exists \{\lambda_n\}_{\mathbb{N}}, \exists \{x_n\}_{\mathbb{N}}, \quad \text{with} \quad \lambda_n \to +\infty, x_n \xrightarrow{\sigma} x$$
$$\text{and} \quad x_0 + \lambda_n x_n \in \mathbf{K},\ \forall n \in \mathbb{N}\},$$

for any $x_0 \in X$; moreover, a statement analogous to Lemma 2.11 still holds.

3. The abstract existence theorems

In this section we give some necessary or sufficient conditions for the existence of solutions of quite general minimum problems.

Let (X, σ) be a topological vector space and let $G: X \to]-\infty, +\infty]$ be a functional. Consider the following problem:

$$\textit{find } y \in X \textit{ such that } G(y) = \min\{G(x) : x \in X\}. \tag{3.1}$$

We begin by stating some necessary conditions for Problem (3.1) to have a solution. More precisely, the conditions we are going to give are necessary for G to have a lower bound. In general, they do not suffice to guarantee either that Problem (3.1) has a solution or that G is bounded from below, as we will show.

Proposition 3.1. *Assume that* $\inf\{G(x): x \in X\} > -\infty$. *Then*

$$G_{\infty,\sigma}(x) \geqq 0 \quad \forall x \in X. \tag{3.2}$$

Proof. For all x of X and for all nets $(\lambda_\xi, x_\xi)_{\xi \in \Xi} \in S(x)$ we have

$$\liminf_{\xi \in \Xi} \frac{1}{\lambda_\xi} G(\lambda_\xi x_\xi) \geqq \liminf_{\xi \in \Xi} \left[\frac{1}{\lambda_\xi} \inf_{z \in X} G(z)\right] = 0.$$

Taking the infimum over all nets of $S(x)$ and recalling (2.2) we get (3.2). □

Remark 3.2. In view of Lemma 2.12(*i*), condition (3.2) imposes restrictions upon the behavior of G only at the elements of $(\operatorname{dom} G)_{\infty,\sigma}$; elsewhere the inequality (3.2) is automatically satisfied. Furthermore, if $G = J + \chi_K$, where J is a proper, convex and σ-l.s.c. functional, the condition (3.2), is satisfied if

$$J^\infty(x) \geqq 0 \quad \text{for all} \quad x \in \mathbf{K}_{\infty,\sigma};$$

this follows from Lemma 2.12(*ii*) and from Proposition 2.5. Note that this condition may be easier to verify than (3.2).

The following example shows that the functional G may be unbounded from below, even if it satisfies condition (3.2).

Example 3.3. Consider the functional

$$G(x) = \begin{cases} -\log x & \text{if } x > 0 \\ +\infty & \text{if } x \leqq 0 \end{cases}$$

defined on the space $X = \mathbb{R}$, endowed with the usual Euclidean topology e. Clearly

$$G_{\infty,\sigma}(x) = G^\infty(x) = \begin{cases} 0 & \text{if } x \geqq 0 \\ +\infty & \text{if } x < 0. \end{cases}$$

but G is not bounded from below. □

A sufficient condition is obtained by adding to the necessary one further requirements on the functional G, namely semicontinuity, compactness and compatibility, in a sense which we will specify.

We begin by stating an existence theorem in quite an abstract framework. We point out that the hypotheses and the steps of the proof generalize the arguments used in the literature to prove several existence theorems under convexity assumptions (see, for instance, [F1], [F2], [LS], [Sc], [BGT1], [BGT2], [GT]).

Theorem 3.4. *Let* $(X, \|\cdot\|)$ *be a normed space (denote by* τ *the topology associated with the norm). Let* σ *be another linear (Hausdorff) topology on* X, *coarser than* τ, *such that for every* $R > 0$ *the ball* $\{x \in X : \|x\| \leqq R\}$ *is* σ-*compact. Let* G:

$X \to]-\infty, +\infty]$ *be a proper functional. Finally, assume*

$$\textbf{semicontinuity:}\ G \text{ is } \sigma\text{-l.s.c. on all } \tau\text{-bounded sets;} \tag{3.3}$$

$$\textbf{compactness:} \begin{cases} \text{for all nets } \{\lambda_\xi\}_\Xi \text{ with } \lambda_\xi \to +\infty \text{ and all} \\ \tau\text{-bounded nets } \{x_\xi\}_\Xi \text{ with } x_\xi \xrightarrow{\sigma} x, \\ \text{if } G(\lambda_\xi x_\xi) \text{ is bounded from above, then } x_\xi \xrightarrow{\tau} x; \end{cases} \tag{3.4}$$

$$\textbf{compatibility:} \begin{cases} (i)\ G_{\infty,\sigma}(x) \geqq 0 \quad \text{for all } x \in X; \\ (ii) \text{ for all } z \in \operatorname{Ker} G_{\infty,\sigma} \text{ there exists } \mu = \mu(z) > 0 \\ \qquad \text{such that } G(x - \mu z) \leqq G(x) \text{ for all } x \in X. \end{cases} \tag{3.5}$$

Then Problem (3.1) *has at least one solution.*

Proof. We divide the proof into several steps.

Step 1. For every $R > 0$, consider the problem

(P_R) *to find* $x_R \in B_R$ *such that* $G(x_R) = \min\{G(x): x \in B_R\}$,

where $B_R = \{x \in X: \|x\| \leqq R\}$. In view of the semicontinuity hypothesis (3.3), (P_R) has a solution x_R. Furthermore, again by (3.3), we can choose x_R such that

$$\|x_R\| = \min\{\|y_R\|: y_R \text{ solves } (P_R)\}. \tag{3.6}$$

Step 2. We distinguish two cases, that in which $\{x_R\}_{\mathbb{R}}$ is bounded (in norm) and that in which it is not. In Step 3 we will prove that only the former case may occur; we claim that in this case there is a solution of (3.1). For, if $\{x_R\}_{\mathbb{R}}$ is τ-bounded, then there is a subnet of $\{x_R\}_{\mathbb{R}}$ (which we still denote by $\{x_R\}_{\mathbb{R}}$), σ-converging to some $z \in X$. Due to the σ-l.s.c. of G,

$$G(z) \leqq \liminf_{R \to +\infty} G(x_R) = \inf\{G(x): x \in X\};$$

hence z solves (3.1).

Step 3. It remains to show that the case $\{x_R\}_{\mathbb{R}}$ unbounded cannot occur. For contradiction, assume that (a subsequence of) $\|x_R\|$ tends to infinity. In this case, the normalized vectors $y_R = \dfrac{x_R}{\|x_R\|}$ are bounded; hence there exists a subnet of $\{y_R\}_{\mathbb{R}}$ (which we still denote by $\{y_R\}_{\mathbb{R}}$), σ-converging to some $y \in X$. We will get a contradiction with the compactness and compatibility assumptions. First, we note that

$$y \in \operatorname{Ker} G_{\infty,\sigma}. \tag{3.7}$$

In fact, since

$$G(x_{R_1}) \leqq G(x_{R_2}) \text{ for all } R_1, R_2 \text{ with } R_1 \geqq R_2, \tag{3.8}$$

we see that $G(x_R)$ is bounded from above; hence

$$G_{\infty,\sigma}(y) \leqq \liminf_{R \to +\infty} \frac{1}{\|x_R\|} G(\|x_R\|\, y_R) = \liminf_{R \to +\infty} \frac{1}{\|x_R\|} G(x_R) \leqq 0.$$

Recalling the necessary condition (3.5)(i), we get (3.7).

Step 4. Let us exploit the compactness assumption. We have $\|x_R\| \to +\infty$, $y_R \xrightarrow{\sigma} y$, and $G(\|x_R\|\, y_R) = G(x_R)$ is bounded from above (again, by (3.8)). Thus, by assumption (3.4), we conclude that the convergence of y_R to y is actually strong, say $\|y_R - y\| \to 0$. This prevents y from being zero, because $\|y_R\| = 1$ for all R.

Step 5. From (3.7) and (3.5)(*ii*) we get that there exists $\mu = \mu(y) > 0$ such that

$$G(x_R - \mu y) \leqq G(x_R). \tag{3.9}$$

By using the strong convergence of y_R to y and the unboundedness of $\{x_R\}_{\mathbb{R}}$, we obtain

$$\begin{aligned} \|x_R - \mu y\| &= \left\| x_R \left(1 - \frac{\mu}{\|x_R\|}\right) + \mu(y_R - y) \right\| \\ &\leqq \left(1 - \frac{\mu}{\|x_R\|}\right) \|x_R\| + \mu\, \|y_R - y\| = \|x_R\| + \mu(\|y_R - y\| - 1). \end{aligned}$$

The right hand side of the last equality is eventually strictly less than $\|x_R\|$ and this is impossible because of (3.6), of (3.9) and because $y \neq 0$. The proof of the theorem is now complete. □

Remark 3.5. In view of Proposition 3.1, the condition (3.5)(*i*) is necessary for the inferior boundedness of G.

The compactness hypothesis (3.4) is used *only* in Step 4. As shown in Step 3, the condition (3.5)(*i*) makes the normalized net σ-converge to an element of Ker $G_{\infty,\sigma}$; it follows that we may impose the compactness hypothesis only for nets σ-converging to elements of Ker $G_{\infty,\sigma}$. In Step 5 we might use weaker forms of the compatibility condition (3.5)(*ii*); for instance

for all $z \in$ Ker $G_{\infty,\sigma}$ *there is a* $\varrho = \varrho(z) > 0$ *such that for all* $x \in X$, *with* $\|x\| \geqq \varrho$,

there is a $\mu = \mu(x, z) > 0$ *such that* $G(x - \mu z) \leqq G(x)$ *and* $\mu < \|x\|$.

Remark 3.6. Theorem 3.4 includes the classical results of Tonelli's type, which provide a solution of problem (3.1) under the following assumptions:

(*a*) *G is σ-l.s.c. on all τ-bounded and proper sets*;
(*b*) *there are* $\alpha > 0$, $b \in \mathbb{R}$, *such that* $G(x) \geqq \alpha\, \|x\| + b$ *for all* $x \in X$.

In fact, (*b*) implies that

$$\lambda_\xi \to +\infty, \quad x_\xi \xrightarrow{\sigma} x, \quad G(\lambda_\xi x_\xi) \leqq C \Rightarrow \|x_\xi\| \to 0 \quad \text{and} \quad x = 0.$$

Thus the compactness hypothesis (3.4) is satisfied. On the other hand, it is $G_{\infty,\sigma}(x) \geqq \alpha\, \|x\|$ for all $x \in X$, and so Ker $G_{\infty,\sigma} = \{0\}$ and the compatibility conditions (3.5) hold trivially.

In particular, (*b*) holds if dom G is bounded. More generally, assume (*a*) and

(*c*) *for some* $\beta \in \mathbb{R}$, $G_\beta = \{x \in X\colon G(x) \leqq \beta\}$ *is non-empty and bounded,* in this case, the minimum problem may be handled in the same way, since

$$\min\{G(x): x \in X\} = \min\{G(x): x \in G_\beta\}.$$

Remark 3.7. By using sequences instead of nets, we obtain a necessary condition weaker than (3.2):

$$\text{if} \quad \inf\{G(x): x \in X\} > -\infty, \text{ then } G^{\text{seq}}_{\infty,\sigma}(x) \geqq 0 \quad \forall x \in X.$$

Analogously, we may give a sequential version of Theorem 3.4, by requiring that

(*i*) *the* τ*-bounded subsets of* X *are sequentially* σ*-relatively compact*;
(*ii*) G *is sequentially* σ*-l.s.c. on all* τ*-bounded sets*;
(*iii*) *the compactness hypothesis* (3.4) *holds for sequences*;
(*iv*) *the compatibility hypothesis* (3.5) *is given on* $G^{\text{seq}}_{\infty,\sigma}$ *instead of* $G_{\infty,\sigma}$.

Remark 3.8. In some applications (see Sections 4 and 5), X is the dual of a normed space V, τ is the strong topology of X and σ is the *weak** topology of X. In this case, by Theorem 3.4 and by Remark 3.7, we get the following theorem.

Theorem 3.9. *Let the following assumptions hold*:

$$\begin{gathered}\textit{either } X \textit{ is reflexive or } X = V', \textit{ with } V \textit{ separable}\\ (\textit{denote by } \sigma \textit{ the weak* topology of } X);\end{gathered} \tag{3.10}$$

semicontinuity: G *is sequentially* σ*-l.s.c. and proper*; (3.11)

compactness: *for all sequences* $\{\lambda_n\}_{\mathrm{N}}$ *with* $\lambda_n \to +\infty$ *and all sequences* $\{x_n\}_{\mathrm{N}}$ *with* $x_n \xrightarrow{\sigma} x$, *if* $G(\lambda_n x_n)$ *is bounded from above, then* $x_n \xrightarrow{\tau} x$; (3.12)

compatibility: (*i*) $G^{\text{seq}}_{\infty,\sigma}(x) \geqq 0$ *for all* $x \in X$;
(*ii*) *for all* $z \in \operatorname{Ker} G^{\text{seq}}_{\infty,\sigma}$ *there is a* $\mu = \mu(z) > 0$ *such that*

$$G(x - \mu z) \leqq G(x) \textit{ for all } x \in X. \tag{3.13}$$

Then problem (3.1) *has at least one solution.* □

An immediate consequences of Theorem 3.4 is the following corollary, which gives a "condition of Lions-Stampacchia type" [LS]: more precisely, when this corollary applies, the set of solutions of (3.1) is bounded.

Corollary 3.10. *Let* X, τ, σ, *and* G *be as in Theorem 3.4. Assume* (3.3), (3.4) *and*

$$G_{\infty,\sigma}(x) > 0 \quad \textit{for all } x \in X, x \neq 0. \tag{3.14}$$

Then problem (3.1) *has at least one solution.* □

Again, we state the corresponding sequential version as follows.

Corollary 3.11. *Let X, τ, σ, and G be as in Theorem* 3.9. *Assume* (3.11), (3.12) *and*

$$G^{\text{seq}}_{\infty,\sigma}(x) > 0 \quad \textit{for all } x \in X, x \neq 0. \tag{3.15}$$

Then problem (3.1) *has at least one solution.* □

Convexity assumptions on G allow us to give sufficient conditions for the existence of minima, easier to verify than (3.5)(*ii*).

Theorem 3.12. *Let* $G: X \to]-\infty, +\infty]$ *be a proper, convex, σ-l.s.c.* (*respectively, sequentially σ-l.s.c.*) *functional. The condition* (3.5)(*ii*) (*respectively,* (3.13)(*ii*)) *is satisfied if we assume that*

$$\operatorname{Ker} G^{\infty} \textit{ is a subspace of } X. \tag{3.16}$$

Proof. We shall prove that (3.5)(*ii*) holds with $\mu = 1$, say

$$G(x - z) \leqq G(x) \quad \text{for all } x \in X, \quad \text{for all } z \in \operatorname{Ker} G^{\infty}.$$

Fix $x_0 \in \operatorname{dom} G$, $z \in \operatorname{Ker} G^{\infty}$, $x \in X$. By (3.16) we conclude that $-z$ belongs to $\operatorname{Ker} G^{\infty}$; hence

$$\lim_{\lambda \to +\infty} \frac{1}{\lambda} G(x_0 - \lambda z) = 0.$$

Since G is convex and σ-l.s.c., we find that

$$G(x - z) \leqq \liminf_{\lambda \to +\infty} G\left[\left(1 - \frac{1}{\lambda}\right)x + \frac{1}{\lambda}(x_0 - \lambda z)\right]$$

$$\leqq \liminf_{\lambda \to +\infty} \left[\left(1 - \frac{1}{\lambda}\right) G(x) + \frac{1}{\lambda} G(x_0 - \lambda z)\right] = G(x).$$

The sequential statement works in the same way. □

If G is not convex, condition (3.16) does not make sense. Its natural substitute would be:

$$\text{if } v \in \operatorname{Ker} G_{\infty,\sigma}, \text{ then } -v \in \operatorname{Ker} G_{\infty,\sigma}.$$

However, this property may hold even if G has *no minimum*, as the following example shows.

Example 3.13. Let $X = \mathbb{R}^2$ with $\tau = \sigma =$ Euclidean topology; let $G = J + \chi_{\mathbf{K}}$, with

$$J(x, y) = y \quad \text{and} \quad \mathbf{K} = \left\{(x, y) \in \mathbb{R}^2 : y \geqq \frac{1}{1 + x^2}\right\}.$$

The function G satisfies (3.3) and (3.4); moreover,

$$\operatorname{Ker} G_{\infty,\sigma} = \{(x, y) \in \mathbb{R}^2 : y = 0\}$$

is a subspace of X. Nevertheless, G has no minimum in X. □

From now on, we consider a situation which arises for instance, in elasticity problems (see Sections 4, 5 and 6). Precisely, we suppose that the space X satisfies hypothesis (3.10) and that the functional G is given by

$$G(x) = F(x) - \langle L, x\rangle + \chi_{\mathbf{K}}(x), \qquad x \in X, \tag{3.17}$$

where

$$\begin{aligned} &F\colon X \to [0, +\infty] && \text{is a proper, sequentially } \sigma\text{-l.s.c. functional,}\\ &L\colon X \to \mathbb{R} && \text{is a linear, } \sigma\text{-continuous functional,}\\ &\mathbf{K} \subset X && \text{is non-empty and sequentially } \sigma\text{-closed.}\end{aligned} \tag{3.18}$$

Then Problem (3.1) takes the following form:

to find $y \in \mathbf{K}$ *such that* $F(y) - \langle L, y\rangle \leqq F(x) - \langle L, x\rangle$ *for all* $x \in \mathbf{K}$. (3.19)

When F and $\mathbf{K}$ are convex, the abstract compatibility assumption (3.13) becomes a "Fichera type condition" (see [F1], [F2]), as shown in the following theorem.

Theorem 3.14. *Assume* (3.17) *and* (3.18), *with both* F *and* $\mathbf{K}$ *convex. Let the compactness hypothesis* (3.12) *hold for the functional* G *defined in* (3.17). *Finally, assume that*

$$F^\infty(x) \geqq \langle L, x\rangle \qquad \forall x \in \mathbf{K}^\infty; \tag{3.20}$$

$$\operatorname{Ker}(F^\infty - L) \cap \mathbf{K}^\infty \textit{ is a subspace.} \tag{3.21}$$

Then Problem (3.19) *has at least one solution.*

Proof. We use Theorem 3.9: we just have to verify that (3.13) holds. Since $G^{\mathrm{seq}}_{\infty,\sigma}(x) = F^\infty(x) - \langle L, x\rangle + \chi_{\mathbf{K}^\infty}(x)$ (see Lemma 2.12(*iii*) and Proposition 2.15), (3.20) implies (3.13)(*i*) and (3.21) implies (3.16) and hence (3.13)(*ii*). □

Remark 3.15. Theorem 3.14 contains in particular the case of quadratic functionals studied by BAIOCCHI, GASTALDI & TOMARELLI in [BGT1] and [BGT2], that is

$$F(x) = \tfrac{1}{2}\, a(x, x), \qquad x \in X,$$

where $a(\cdot, \cdot)$ is a bilinear, continuous, non-negative form defined on $X \times X$. In this case, the positive homogeneity of degree 2 of F implies that

$$\operatorname{Ker}(F^\infty - L) = \operatorname{Ker} F \cap \ker L.$$

Note that this equality also holds when F is positively homogeneous of degree strictly greater than one.

Theorem 3.16. *Assume* (3.17), (3.18) *and the following growth condition for* F:

$$\begin{aligned}&\textit{there exist } \alpha \in \mathbb{R}, \textit{ a seminorm } P : X \to [0, +\infty[\textit{ and a}\\ &\textit{convex, l.s.c. function } \Phi\colon [0, +\infty[\to [0, +\infty] \textit{ such that}\\ &\lim_{\lambda \to +\infty} \frac{\Phi(\lambda)}{\lambda} = +\infty \quad \textit{and} \quad F(x) \geqq \alpha + \Phi[P(x)].\end{aligned} \tag{3.22}$$

Let the compactness hypothesis (3.12) *hold for the functional G defined in* (3.17). *Finally, assume that*

$$\langle L, x\rangle \leqq 0 \quad \textit{for all } x \in \operatorname{Ker} P \cap \mathbf{K}^{\text{seq}}_{\infty,\sigma}; \tag{3.23}$$

$$\begin{gathered} \textit{for all } \; z \in \operatorname{Ker} P \cap \operatorname{Ker} L \cap \mathbf{K}^{\text{seq}}_{\infty,\sigma} \; \textit{ there exists } \; \mu = \mu(z) > 0 \\ \textit{such that, for all } \; x \in \mathbf{K}, \; F(x - \mu z) \leqq F(x) \; \textit{ and } \; x - \mu z \in \mathbf{K}. \end{gathered} \tag{3.24}$$

Then Problem (3.19) *has at least one solution.*

Proof. As in the previous proof, we use Theorem 3.9 and we just have to verify (3.13). Recalling Proposition 2.5 and Lemma 2.12, by (3.22) we have

$$\begin{aligned} G^{\text{seq}}_{\infty,\sigma}(x) &\geqq F^{\text{seq}}_{\infty,\sigma}(x) - \langle L, x\rangle + \chi_{\mathbf{K}^{\text{seq}}_{\infty,\sigma}}(x) \geqq [\Phi \circ P]^{\infty}(x) - \langle L, x\rangle + \chi_{\mathbf{K}^{\text{seq}}_{\infty,\sigma}}(x) \\ &= -\langle L, x\rangle \text{ if } x \in \operatorname{Ker} P \cap \mathbf{K}^{\text{seq}}_{\infty,\sigma} \quad \textit{and} \quad +\infty \; \textit{otherwise}. \end{aligned} \tag{3.25}$$

In particular, (3.23) implies (3.13)(i). Finally, (3.13)(ii) follows from (3.24) because (3.25) yields $\operatorname{Ker} G^{\text{seq}}_{\infty,\sigma} \subset \operatorname{Ker} P \cap \operatorname{Ker} L \cap \mathbf{K}^{\text{seq}}_{\infty,\sigma}$. □

4. Applications to unilateral problems in linear elasticity

In this section we apply the abstract results of the previous one to unilateral problems in linear elasticity.

Let Ω be a non-empty bounded connected open subset of $\mathbb{R}^N$, with a Lipschitz boundary $\partial\Omega$. Let $X = [\mathbf{H}^1(\Omega)]^N$ be the usual Sobolev space of the real vector-valued distributions which belong to $\mathbf{L}^2(\Omega)$ along with their first derivatives; let σ denote the weak topology of X.

Consider a fourth order tensor $\{a_{ijhk}(x)\}$ $(i, j, h, k = 1, \ldots, N)$ of real valued functions $a_{ijhk} \in \mathbf{L}^{\infty}(\Omega)$ with

$$a_{ijhk}(x) = a_{jihk}(x) = a_{hkji}(x)$$

for all x in Ω. Assume that the following ellipticity condition is satisfied (here and in the following the summation convention over repeated indices is adopted): there is strictly positive α such that

$$a_{ijhk}(x)\, \xi_{ij}\xi_{hk} \geqq \alpha\, |\xi|^2 \qquad \forall x \in \Omega \tag{4.1}$$

and for all $N \times N$ symmetric matrices ξ.

As usual, for every $v \in X$ we denote by $v_{i,j}$ the derivative of v_i with respect to the variable x_j and by $\varepsilon(u)$, $\sigma(u)$ the linearized strain and stress tensors given by

$$\varepsilon_{ij}(v) = \tfrac{1}{2}\,(v_{i,j} + v_{j,i}), \qquad \sigma_{ij}(v) = a_{ijhk}\varepsilon_{hk}(v). \tag{4.2}$$

The elastic energy is then given by the functional

$$F(v) = \tfrac{1}{2} \int_{\Omega} \sigma_{ij}(v)\, \varepsilon_{ij}(v)\, dx = \tfrac{1}{2} \int_{\Omega} a_{ijhk}(x)\, \varepsilon_{ij}(v)\, \varepsilon_{hk}(v)\, dx. \tag{4.3}$$

Before introducing the set of constraints, for every subset E of $\mathbb{R}^N$ we define the capacity of E by (see for instance [DaL])

$$\operatorname{cap} E = \inf\{\|u\|_{\mathbf{H}^1(\mathbb{R}^N)} \; : u \in \mathbf{H}^1(\mathbb{R}^N), \quad u \text{ is l.s.c., } u(x) \geqq 1 \quad \forall x \in E\}.$$

We remark that this definition of capacity is slightly different from the usual one (see, for instance, [LeS], [FZ]), although both definitions provide the same class of Borel sets of zero capacity.

We say that a property $P(x)$ holds *quasi-everywhere* (in short, *q.e.*) if the set $\{x \in \mathbb{R}^N: P(x) \;\; \textit{is false}\}$ has zero capacity. It is well known (see for instance [DaL]) that for every $u \in X$ the limit

$$u^*(x) = \lim_{\varepsilon \to 0} \frac{1}{|\Omega \cap B_\varepsilon(x)|} \int_{\Omega \cap B_\varepsilon(x)} u(y)\, dy$$

exists for q.e. $x \in \bar{\Omega}$. The function u^* is called the *quasi-continuous representative* of u.

Now consider two subsets Q, E of $\mathbb{R}^N$ such that Q is closed, cap $E > 0$ and

$$E \subset \bar{\Omega} \subset Q.$$

The set of constraints is then defined by

$$\mathbf{K}(Q, E) = \{v \in X: x + v^*(x) \in Q, \; \textit{for q.e. } x \in E\}. \tag{4.4}$$

In the following, we will render the dependence of $\mathbf{K}$ upon Q and E explicit only when necessary, otherwise we will write $\mathbf{K}$. We note that $\mathbf{K}(Q, E)$ is not empty; in particular

$$0 \in \mathbf{K}(Q, E). \tag{4.5}$$

Let L be an element of X' (the dual space of X). Throughout this chapter we consider the functional

$$G(v) = F(v) - \langle L, v\rangle + \chi_{\mathbf{K}}(v),$$

where F is given by (4.3). G is called the *total energy functional*, and we shall study the following minimization problem:

$$\textit{to find } u \in X \textit{ such that} \quad G(u) = \min\{G(v): v \in X\}. \tag{4.6}$$

We will refer to Problem (4.6) as *Generalized Signorini-like Problem in Linear Elasticity*, or **GSP**. The physical interpretation is the following:

(a) Ω is the reference (initial) configuration of an elastic body whose part E is constrained to lie inside the box Q. We assume that the reference configuration is a natural state, there being no stress in absence of deformation.

(b) $u(x) = (u_1(x), \ldots, u_N(x))$ is the displacement of the particle $x \in \Omega$ (that is, after deformation x becomes $x + u(x)$) and $\varepsilon(u)$, $\sigma(u)$ are the linearized strain and stress tensors respectively. In the framework of the linear approximation, $\varepsilon(u)$ and $\sigma(u)$ are given by (4.2).

(c) $F(u)$ is the elastic energy corresponding to the displacement u; note that $F(0) = 0$, according to the assumption made on the reference configuration.

(d) L is a vector field of *dead forces*, a field of applied forces acting on the body, whose direction does not depend on the deformation but only on the initial coordinates. For instance, we assume that $L = f + g$, where $f \in [\mathbf{L}^2(\Omega)]^N$ are the body forces and $g \in [\mathbf{H}^{-\frac{1}{2}}(\partial\Omega)]^N$ are the surface stresses.

Remark 4.1. In the literature (see [Si], [F1], [F2], [DL], [K1]), the set of admissible displacements in the actual Signorini problem is given by

$$\{u \in X: \; u(x) \cdot \nu(x) \leqq 0 \quad \text{on} \quad E\},$$

where E (the *potential contact area*) is a subset of $\partial\Omega$ and ν is the outward unit vector, normal to $\partial\Omega$: this set is always convex. In our approach we use a different geometric condition expressed through the set **K** defined in (4.4): in particular, we note that **K** is convex if and only if Q is convex (a particular case, the "box condition", was considered by KINDERLEHRER in [K2], where Q is a half space and $E = \partial\Omega$). Our approach will be adopted also in the next sections, devoted to unilateral problems of non-linear type. As a matter of fact, from the mechanical point of view, condition (4.4) suits non-linear elasticity better than linearized elasticity. Nevertheless, from the mathematical point of view, **GSP** has an interest in itself and the sufficient conditions for the existence of solutions may be easily interpreted in terms of physically meaningful quantities (see below). At the end of this section a conjecture concerning a statement of non-existence is reported.

Let us come to the question of existence of solutions for **GSP**. The difficulty is due to the failure of the functional F to be coercive; indeed, this provides information only upon the $\mathbf{L}^2$ norm of the strain tensor $\varepsilon(u)$. However, in some special cases existence is very easy to prove, notwithstanding the non-coerciveness. For instance, if $L = 0$ then the null deformation is obviously a minimizer of the total energy G.

Remark 4.2. A direct argument gives existence for any L when $E = \Omega$ and Q is *bounded*. For this provides an *a priori* estimate on the $\mathbf{L}^\infty$ norm (hence, on the $\mathbf{L}^2$ norm) of a minimizing sequence. Hence we get a uniform bound for the $\mathbf{H}^1$ norm of the sequence (and the weak limit turns out to be a solution of **GSP**), by using the fundamental Korn inequality (see [Te], p. 20):

$$\|v\|_{[\mathbf{W}^{1,p}(\Omega)]^N} \leqq c(\Omega, p)\,[\|v\|_{[\mathbf{L}^p(\Omega)]^N} + \|\varepsilon(v)\|_{[\mathbf{L}^p(\Omega)]^{N\times N}}] \tag{4.7}$$

valid for all $p \in]1, +\infty[$ and all $v \in [\mathbf{W}^{1,p}(\Omega)]^N$.

A refinement of this argument suffices to prove that **GSP** is solvable *for all* systems of applied forces L if Q is bounded. In the Appendix (see Proposition A.5) the proof of this assertion is derived by means of the "capacitary Korn's inequality" in $\mathbf{H}^1$:

$$\|v\|^2 \leqq C\{F(v) + [v]_E^2\},$$

valid for all v of X, where the "capacitary essential supremum" $[\cdot]_E$ is a seminorm defined as

$$[v]_E = \inf\{\lambda: |v^*(x)| \leqq \lambda, \quad \text{for q.e.} \quad x \in E\}.$$

In what follows particular attention will be paid to the set

$$RBM = \{v \in X: F(v) = 0\}$$
$$= \{v \in X: v_i(x) = a_{ij}x_j + b_i, \quad a_{ij}, b_i \text{ constant with } a_{ij} = -a_{ji}\}.$$

RBM is usually called the space of the *rigid body motions*, though it is actually the finite version of the space of the infinitesimal rigid displacements. Indeed, also *large* displacements belong to *RBM*. We remark that an element $v \in RBM$ corresponds to a rigid displacement of the body if and only if it is a translation, i.e. v is a constant.

If $N = 3$, we may equivalently represent *RBM* as follows:

$$RBM = \{v \in X: v(x) = \alpha \times x + \beta, \alpha, \beta \in \mathbb{R}^3\}. \tag{4.8}$$

To unify the notations, we still represent *RBM* by (4.8) also for $N = 2$. In this case, β is a 2-dimensional vector, while α may be interpreted as a vector orthogonal to $\mathbb{R}^2$, when embedded into $\mathbb{R}^3$.

Now we are going to state two existence theorems for **GSP**: the first requires the convexity of the box Q and permits the presence of entire directions in Q; the second applies to possibly non-convex boxes, provided they have no entire directions. We note that a unified, more general treatment could be effected, which applies, in particular, if the box is non-convex *and* has some bilateral recession directions, provided an invariance condition along these directions is satisfied. This approach will be adopted in the following sections.

In our framework, the constrained region E is a general subset of $\bar{\Omega}$ with positive capacity. In view of the existence theorems, it is convenient to introduce a canonical representative of E, which coincides with E whenever it is, say, a smooth closed manifold of $\partial\Omega$ or the closure of an open subset of Ω.

Definition 4.3. *For every subset E of $\mathbb{R}^N$ we set*

$$E_{\text{ess}} = \bigcap \{C: C \textit{ is closed and } \operatorname{cap}(E \setminus C) = 0\}.$$

Proposition 4.4. *The set E_{ess} satisfies the following properties:*

(*i*) *E_{ess} is a closed subset of $\bar{E}$;*
(*ii*) $\operatorname{cap}(E \setminus E_{\text{ess}}) = 0$;
(*iii*) *the three following statements are equivalent:* $\operatorname{cap} E = 0$, $E_{\text{ess}} = \emptyset$, $\operatorname{cap} E_{\text{ess}} = 0$;
(*iv*) $\operatorname{cap}(E \,\Delta\, E_{\text{ess}}) = 0$, *whenever E is closed;*
(*v*) *for all closed sets $M \subset \mathbb{R}^N$ and all continuous functions $v: \mathbb{R}^N \to \mathbb{R}^N$ it is $v(x) \in M$ for q.e. $x \in E$ if and only if $v(x) \in M$ for all $x \in E_{\text{ess}}$;*
(*vi*) *for all continuous functions $v: \mathbb{R}^N \to \mathbb{R}^N$*

$$v(x) = 0 \textit{ for q.e. } x \in E \textit{ if and only if } v(x) = 0 \textit{ for all } x \in E_{\text{ess}}.$$

Proof. See the Appendix. □

Now we are able to state our first existence theorem for **GSP** which we prove by applying the results of Section 3; actually, the same procedure could be used starting from the results of [BGTZ]. By co A and ri A we denote respectively the convex hull of a subset A of $\mathbb{R}^N$ (say, the intersection of the convex sets containing A) and its relative interior, that is, the interior of A with respect to the affine hull of A (see [R]).

Theorem 4.5. *Assume* $N = 2, 3$. *Furthermore, assume that*

$$Q \text{ is a closed convex set}; \tag{4.9}$$

there exists a point $P \in \operatorname{ri}(\operatorname{co} E_{\text{ess}})$ *such that* $\langle L, \alpha \times (x - P)\rangle = 0$, *for all* $\alpha \in \mathbb{R}^3$;

$$\tag{4.10}$$

$$(i)\ \langle L, \beta\rangle \leqq 0 \quad \forall \beta \in Q^\infty,$$

$$(ii)\ \text{if } \beta \in Q^\infty \text{ and } \langle L, \beta\rangle = 0, \quad \text{then} \quad -\beta \in Q^\infty. \tag{4.11}$$

Then **GSP** *has a solution.* □

We point out that $\operatorname{ri}(\operatorname{co} E_{\text{ess}})$ is not empty because E has positive capacity (see Proposition 4.4(*iii*)).

Let us state an essentially new existence theorem for the case of a non-convex box Q. By $Q_{\infty,e}$ we denote the set of topologically unbounded directions of Q (see Definition 2.8; e denotes the Euclidean topology in $\mathbb{R}^N$).

Theorem 4.6. *Assume* $N = 2, 3$. *Furthermore, assume* (4.10) *and*

$$\langle L, \beta\rangle < 0 \quad \forall \beta \in Q_{\infty,e}, \quad \beta \neq 0. \tag{4.12}$$

Then **GSP** *has a solution.* □

Remark 4.7. The condition (4.12) is equivalent to the existence of a point c_0 of $\mathbb{R}^N$ and of a cone C of $\mathbb{R}^N$ such that

$$Q \subset c_0 + C \quad \text{and} \quad \langle L, c\rangle < 0 \quad \forall c \in C, c \neq 0.$$

In particular, if c belongs to C and does not vanish, then $-c$ cannot lie in C.

Let us provide a mechanical interpretation of the assumptions on the system of forces we made in the two existence theorems 4.5 and 4.6.

Let us call *resultant* of the system L the vector $R \in \mathbb{R}^N$ such that $R_i = \langle L, e_i\rangle$, with $\{e_i\}_{i=1,\dots,N}$ the canonical basis in $\mathbb{R}^N$. Then the condition (4.11) requires that the angle between all recession directions of Q and a non-vanishing resultant R is obtuse, allowing it to be a right angle only if the recession direction is entire. Actually, Theorem 4.5 applies to systems of forces with zero resultant *only if* the recession cone of the box Q is itself a subspace of $\mathbb{R}^N$. On the other hand, for non-convex boxes the case $R = 0$ can be studied in Theorem 4.6 when Q is bounded and hence $Q_{\infty,e} = \{0\}$.

Assume that $R \neq 0$; then the *central axis* of the system is defined as the line parallel to R such that the momentum of the system L with respect to each point of it is constant and has the smallest possible modulus. Thus condition (4.10) states that the resultant momentum with respect to the central axis vanishes and there is no couple effect; moreover, it prescribes that the central axis intersects the relative interior of the closed convex hull of E_{ess}. In [F2] it is shown that this intersection condition is necessary for **GSP** to have a solution when Q is a half space and E is a subset of ∂Q.

The proof of Theorem 4.5 (respectively 4.6) will be carried out in several steps, showing that under its assumptions Theorem 3.14 (respectively Corollary 3.11) may be applied.

We begin by considering the set of constraints. The study of the properties of that set can be carried out in a more general scheme, say, substituting $\mathbf{K}(Q, E)$ with the set

$$K = \{v \in X: x + v^*(x) \in Q(x) \quad \text{for q.e. } x \in E\}, \tag{4.13}$$

where, for every $x \in E$, $Q(x)$ is a given closed subset of $\mathbb{R}^N$. Obviously, $\mathbf{K}(Q, E) = K$ by choosing $Q(x) \equiv Q$.

Proposition 4.8. *The set K given by* (4.13) *is sequentially weakly closed in X.*

Proof. See the Appendix. $\square$

Proposition 4.9. *The set K given by* (4.13) *satisfies*

$$K^{\text{seq}}_{\infty,\sigma} \subset \{v \in X: v^*(x) \in Q_{\infty,e}(x) \quad \textit{for q.e. } x \in E\}.$$

Proof. By definition we know that v belongs to $K^{\text{seq}}_{\infty,\sigma}$ if and only if there are $\{\lambda_n\}_{\mathbb{N}}$ and $\{v_n\}_{\mathbb{N}}$ such that $\lambda_n \to +\infty$, $v_n \xrightarrow{\sigma} v$ and $\lambda_n v_n \in \mathbf{K}$ for all $n \in \mathbb{N}$, that is,

$$x + \lambda_n v_n^*(x) \in Q(x) \quad \text{for q.e. } x \in E \quad \text{and for all } n \in \mathbb{N}. \tag{4.14}$$

By Lemma A.1, for q.e. $x \in E$ there is a subsequence $\{v_{n_k}\}_{k \in \mathbb{N}}$, such that $v_{n_k}^*(x)$ converges to $v^*(x)$. This, together with (4.14), implies that $v^*(x) \in Q_{\infty,e}(x)$, for q.e. $x \in E$. $\square$

Proposition 4.10. *Assume that, for q.e. $x \in E$, the set $Q(x)$ is convex. Then K is convex, closed and*

$$K^\infty = \{v \in X: v^*(x) \in Q^\infty(x) \quad \textit{for q.e. } x \in E\}.$$

Proof. By Propositions 2.5, 4.8, and 4.9 we have only to prove that

$$v^*(x) \in Q^\infty(x) \quad \text{for q.e. } x \in E \Rightarrow v \in K^\infty.$$

Let v be such that $v^*(x) \in Q^\infty(x)$ for q.e. $x \in E$; it follows that

$$x + \lambda v^*(x) \in Q(x) \quad \text{for q.e. } x \in E \quad \text{and for all } \lambda > 0.$$

Taking a sequence $\lambda_n \to +\infty$, we find that $\lambda_n v \in K$ for all $n \in \mathbb{N}$; hence $v \in K^\infty$. $\square$

Lemma 4.11. *The compactness assumption* (3.12) *holds.*

Proof. Let $\{\lambda_n\}_{\mathbb{N}}$, $\{x_n\}_{\mathbb{N}}$ and x be such that $\lambda_n \to +\infty$, $x_n \xrightarrow{\sigma} x$ and $G(\lambda_n x_n) \leqq c$ for some constant c; we claim that $x_n \to x$ strongly in $[\mathbf{H}^1(\Omega)]^N$. In fact

$$c \geqq G(\lambda_n x_n) \geqq F(\lambda_n x_n) - \langle L, \lambda_n x_n \rangle .$$

Because F is positively homogeneous of degree 2, sequentially σ-l.s.c. and non-negative, we conclude that

$$0 \leqq F(x) \leqq \liminf_{n\to+\infty} F(x_n) \leqq \limsup_{n\to+\infty} F(x_n) = \limsup_{n\to+\infty} \frac{1}{\lambda_n^2} F(\lambda_n x_n)$$
$$\leqq \limsup_{n\to+\infty} \frac{1}{\lambda_n^2} [c + \langle L, \lambda_n x_n \rangle] = 0 .$$

Hence $x \in RBM$ and $F(x_n - x) = F(x_n) \to 0$. On the other hand, since $x_n \rightharpoonup x$ weakly in $[\mathbf{H}^1(\Omega)]^N$, then $\|x_n - x\|_{[\mathbf{L}^2(\Omega)]^N} \to 0$. Using Korn's inequality (4.7), we conclude that

$$\|x_n - x\|^2_{[\mathbf{H}^1(\Omega)]^N} \leqq c(\Omega) \{\|x_n - x\|^2_{[\mathbf{L}^2(\Omega)]^N} + F(x_n - x)\} \to 0$$

and the proof is complete. □

Remark 4.12. Let us briefly summarize some properties of rigid body motions we will need in the following.

(*i*) If v belongs to RBM, then $v(x) = \alpha \times x + \beta$, with α, β constant vectors (see (4.8)).

(*ii*) If v belongs to $RBM \cap \mathbf{K}^{\text{seq}}_{\infty,\sigma}$, then $v(x) \in Q_{\infty,e}$, **for all** $x \in E_{\text{ess}}$ (see Propositions 4.9 and 4.4(v)); since v is linear, this implies that $v(x) \in \operatorname{co} Q_{\infty,e}$, for all $x \in \operatorname{co} E_{\text{ess}}$.

(*iii*) Assume that there exists a point $P \in \mathbb{R}^N$ such that $\langle L, \alpha \times (x - P) \rangle = 0$, for all $\alpha \in \mathbb{R}^N$. If $v(x) = \alpha \times x + \beta$ belongs to RBM, then $\langle L, v \rangle = \langle L, \alpha \times P + \beta \rangle$. Notice that $\langle L, \alpha \times (x - P) \rangle = 0$, for all $\alpha \in \mathbb{R}^3$, if and only if $L \times (x - P)$ is the null element of X'.

(*iv*) Let v belong to RBM and satisfy $v(x) = 0$ for all $x \in \operatorname{co} E_{\text{ess}}$. Then v vanishes identically because $\operatorname{cap} E > 0$, and hence $\operatorname{cap} E_{\text{ess}} > 0$ (Proposition 4.4(*iii*)); thus the affine space spanned by E_{ess} has at least $N - 1$ dimensions.

Lemma 4.13. *Assume that*

$$\text{there exists a point } P \in \operatorname{co} E_{\text{ess}} \text{ such that}$$
$$\langle L, \alpha \times (x - P) \rangle = 0 \text{ for all } \alpha \in \mathbb{R}^N; \tag{4.15}$$

$$\langle L, b \rangle \leqq 0 \quad \forall b \in Q_{\infty,e} . \tag{4.16}$$

Then

$$\langle L, v \rangle \leqq 0 \quad \forall v \in RBM \cap \mathbf{K}^{\text{seq}}_{\infty,\sigma} . \tag{4.17}$$

Proof. Let v belong to $RBM \cap \mathbf{K}^{\text{seq}}_{\infty,\sigma}$. In view of Remark 4.12(*i*) and (*ii*), $\alpha \times x + \beta \in \text{co}\, Q_{\infty,e}$, for all $x \in \text{co}\, E_{\text{ess}}$. Thanks to (4.15), we may take $x = P$, whence $\alpha \times P + \beta \in \text{co}\, Q_{\infty,e}$, and (4.16) implies $\langle L, \alpha \times P + \beta \rangle \leqq 0$. Recalling Remark 4.12(*iii*), we obtain (4.17). □

Proof of Theorem 4.5. By a translation it is not restrictive to assume $P = 0$ in hypothesis (4.10). Let us show that the abstract result of Theorem 3.14 applies.

(a) X is reflexive and F is convex and sequentially l.s.c. with respect to the weak topology; all we have to verify is the convexity and closedness of $\mathbf{K}$. This follows from Proposition 4.10.

(b) The compactness assumption (3.12) is a consequence of Lemma 4.11.

(c) Necessary condition (3.20). We have to prove that $F^{\infty}(v) \geqq \langle L, v \rangle$, for all $v \in \mathbf{K}^{\infty}$. Since F is positively homogeneous of degree 2 (see Lemma 2.12),

$$F^{\infty}(v) = \begin{cases} 0 & \text{if } F(v) = 0, \\ +\infty & \text{if } F(v) \neq 0. \end{cases}$$

Thus, all we have to verify is that

$$\langle L, v \rangle \leqq 0 \qquad \forall v \in RBM \cap \mathbf{K}^{\infty}.$$

According to Lemma (4.13), this follows from (4.10) and from (4.11)(*i*).

(d) Condition (3.21). In view of Remark 3.15, recalling that $\mathbf{K}^{\infty}$ is a cone, this condition is equivalent to the following:

$$v \in RBM \cap \operatorname{Ker} L \cap \mathbf{K}^{\infty} \Rightarrow -v \in \mathbf{K}^{\infty}.$$

Let v belong to $RBM \cap \operatorname{Ker} L \cap \mathbf{K}^{\infty}$. Because of Remark 4.12(*i*), (*ii*),

$$\alpha \times x + \beta \in \text{co}\, Q_{\infty,e} = Q^{\infty} \quad \text{for all } x \in \text{co}\, E_{\text{ess}}. \tag{4.18}$$

Thanks to (4.10), we may take $x = 0$, whence $\beta \in Q^{\infty}$. Now, since v belongs to $\operatorname{Ker} L$, Remark 4.12(*iii*) entails that $\langle L, \beta \rangle = 0$; hence (4.11)(*ii*) implies that

$$-\beta \in Q^{\infty}. \tag{4.19}$$

To complete the proof, it is enough to show that

$$-\alpha \times x \in Q^{\infty}, \quad \text{for all } x \in \text{co}\, E_{\text{ess}}. \tag{4.20}$$

Indeed, since Q^{∞} is a cone, (4.19) and (4.20) imply that $-\alpha \times x - \beta \in Q^{\infty}$, for all $x \in \text{co}\, E_{\text{ess}}$; hence $-v \in \mathbf{K}^{\infty}$.

Let us prove (4.20). From (4.18) and (4.19) it follows that

$$\alpha \times x \in Q^{\infty} \quad \text{for all } x \in \text{co}\, E_{\text{ess}}. \tag{4.21}$$

Let y belong to $\text{co}\, E_{\text{ess}}$. Due to (4.10) there is a $\mu = \mu(y) > 0$ such that $-\mu y$ lies in $\text{co}\, E_{\text{ess}}$; hence (4.21) gives $\mu\alpha \times (-y) \in Q^{\infty}$. Because Q^{∞} is a cone, it follows that $-\alpha \times y \in Q^{\infty}$. This holds for all $y \in \text{co}\, E_{\text{ess}}$; hence (4.20) is true. The proof is thus complete. □

Proof of Theorem 4.6. Again, we may assume $P = 0$ in hypothesis (4.10). Let us show that we may apply Corollary 3.11.

Semicontinuity and compactness hold as in the proof of Theorem 4.5. Let us prove (3.15), say $G^{\text{seq}}_{\infty,\sigma}(v) > 0$, for all $v \in X$, with $v \neq 0$. To this end, we notice (see Lemma 2.12) that

$$G^{\text{seq}}_{\infty,\sigma} = (F + \chi_{\mathbf{K}})^{\text{seq}}_{\infty,\sigma} - L.$$

On the other hand, using the 2-homogeneity of F, we easily find

$$(F + \chi_{\mathbf{K}})^{\text{seq}}_{\infty,\sigma}(v) = +\infty \quad \text{if either } v \notin \mathbf{K}^{\text{seq}}_{\infty,\sigma} \quad \text{or} \quad F(v) \neq 0.$$

Hence

$$G^{\text{seq}}_{\infty,\sigma}(v) = +\infty \qquad \text{if } v \notin RBM \cap \mathbf{K}^{\text{seq}}_{\infty,\sigma},$$

$$G^{\text{seq}}_{\infty,\sigma}(v) \geqq -\langle L, v\rangle \qquad \text{if } v \in RBM \cap \mathbf{K}^{\text{seq}}_{\infty,\sigma}.$$

From Lemma 4.13 we know that under our assumptions

$$G^{\text{seq}}_{\infty,\sigma}(v) \geqq -\langle L, v\rangle \geqq 0 \qquad \forall v \in RBM \cap \mathbf{K}^{\text{seq}}_{\infty,\sigma}.$$

It remains to show that $\langle L, v\rangle \neq 0$ whenever $v \in RBM \cap \mathbf{K}^{\text{seq}}_{\infty,\sigma}$ does not vanish. Taking such a v and recalling Remark 4.12(*ii*), we see that

$$v(x) = \alpha \times x + \beta \in \text{co}\, Q_{\infty,e} \quad \forall x \in \text{co}\, E_{\text{ess}}. \tag{4.22}$$

Taking $x = 0$, we see that $\beta \in \text{co}\, Q_{\infty,e}$. Since $\langle L, v\rangle = \langle L, \beta\rangle$ (Remark 4.12(*iii*)) the desired result follows by (4.12) if we show that $\beta \neq 0$. By contradiction, assume $\beta = 0$; this implies that $\alpha \neq 0$, for otherwise $v = 0$. By Remark 4.12 (*iv*), there is an $x_\alpha \in \text{co}\, E_{\text{ess}}$ such that $\alpha \times x_\alpha \neq 0$. From (4.22), we find that $\alpha \times x_\alpha \in \text{co}\, Q_{\infty,e}$, so that (4.12) implies that $\langle L, \alpha \times x_\alpha\rangle < 0$. Finally, we use (4.10) to find a $\mu > 0$ such that $-\mu x_\alpha \in \text{co}\, E_{\text{ess}}$. The preceding argument may be repeated to conclude that $-\langle L, \alpha \times x_\alpha\rangle < 0$, which yields the contradiction. Thus $\beta \neq 0$ and the proof is complete. □

Let us end this section with some curious results. Consider a heavy, homogeneous, isotropic elastic ball in $\mathbb{R}^3$ and two different boxes: the half space

$$Q_1 = \{(x_1, x_2, x_3): x_3 \geqq 0\}$$

and a cone with vertex at the origin, lying on the plane $\{(x_1, x_2, x_3): x_3 = 0\}$, for instance, the cone

$$Q_2 = \{(x_1, x_2, x_3): x_2 \geqq 0, x_3 \geqq 0, x_2 x_3 \geqq x_1^2\}.$$

With this choice, let the reference configuration of the ball be

$$\{(x_1, x_2, x_3): x_1^2 + (x_2 - 2)^2 + (x_3 - 1)^2 < 1\}$$

and consider the unilateral condition with $E = \partial\Omega$.

The following alternative holds:

(i) either the solutions of the equilibrium problem for the half space are "irregular" in the sense that a large subset of $\partial\Omega$ is mapped into a segment,
(ii) or the equilibrium problem for the cone cannot have solutions (this we conjecture to happen).

To prove this alternative, let $(\mathbf{GSP})_{1,2}$ denote the equilibrium problem with admissible set $\mathbf{K}_{1,2}$, corresponding to the box $Q_{1,2}$, respectively. A theorem of KINDERLEHRER [K2] states that, if u solves $(\mathbf{GSP})_1$, then the corresponding "contact region" $\Gamma = \{x \in \partial\Omega \colon x_3 + u_3(x) = 0\}$ has positive capacity. In particular, if the "final contact region" $\{x + u(x), x \in \Gamma\}$ is part of a straight line, then (i) must occur. Thus the alternative follows as soon as we prove that, if $(\mathbf{GSP})_2$ is solvable, then $(\mathbf{GSP})_1$ has solutions whose final contact region (in the above sense) lies on a straight line. Obviously, to prove this it is enough to show that

$$\textit{every solution of } (\mathbf{GSP})_2 \textit{ must solve } (\mathbf{GSP})_1. \tag{4.23}$$

As a matter of fact, if u solves $(\mathbf{GSP})_2$, then $u \in \mathbf{K}_1$. Let $w \in \mathbf{K}_1 \cap \mathbf{C}(\bar{\Omega})$; clearly, for all $\mu > 0$ there is a $\lambda > 0$ such that $v = \mu u + w + (0, \lambda, 0)$ belongs to $\mathbf{K}_2$. Since $\langle L, (0, \lambda, 0)\rangle = 0$,

$$F(u) - \langle L, u\rangle \leqq F(v) - \langle L, v\rangle = F(w) - \langle L, w\rangle + \mu^2 F(u) + \mu \int_\Omega a_{ijhk}(x)\, \varepsilon_{ij}(u)\, \varepsilon_{hk}(w)\, dx - \mu\langle L, u\rangle.$$

Letting $\mu \to 0$, we find that

$$F(u) - \langle L, u\rangle \leqq F(w) - \langle L, w\rangle, \quad \textit{for all } w \in \mathbf{K}_1 \cap \mathbf{C}(\bar{\Omega}).$$

By a density argument this implies that u solves $(\mathbf{GSP})_1$; hence (4.23) is proved and the alternative follows.

5. Applications to unilateral problems in non-linear elasticity

In this section, we consider a unilateral problem of the same kind as in the preceding one, apart from the elastic energy functional. Instead of (4.3), we will consider integral functionals of the form

$$F(u) = \int_\Omega f(x, \nabla u(x))\, dx,$$

where f need not be quadratic in ∇u. We refer to [Ba], [C] for the discussion of the physical motivations of the model.

Let Ω be a non-empty, bounded connected open subset of $\mathbb{R}^N$ with a Lipschitz boundary $\partial\Omega$. Ω represents a hyperelastic body, a particle of which is labeled by x; let $u(x)$ be the displacement of the particle x and let $\psi(x) = x + u(x)$ be its final position.

We denote by X the usual Sobolev space $[\mathbf{W}^{1,p}(\Omega)]^N$, *by* X' its dual and by σ the weak topology of X. In this section we assume

$$p > 1.$$

In the following, q.e. will stand for *quasi-everywhere* with respect to the p-capacity cap_p defined as follows

$$\mathrm{cap}_p(E) = \inf\{\|u\|_{\mathbf{W}^{1,p}(\mathbb{R}^N)} \colon u \in \mathbf{W}^{1,p}(\mathbb{R}^N),\ u \text{ is l.s.c., } u(x) \geqq 1 \quad \forall x \in E\}.$$

Let Q, E be subsets of $\mathbb{R}^N$ such that Q is closed, E has positive p-capacity and

$$E \subset \bar{\Omega} \subset Q.$$

Define the set of constraints in the following way:

$$\mathbf{K} = \{\psi \in X : \psi^*(x) \in Q \quad \text{for q.e.} \quad x \in E\}, \tag{5.1}$$

where ψ^* is the quasi-continuous representative of ψ.

Finally, consider the elastic energy

$$F(\psi) = \int_\Omega \mathbf{W}(x, \nabla\psi(x))\, dx.$$

We point out that the stored energy $\mathbf{W}$ depends on the gradient of the final position instead of on the gradient of the displacement. On the functional F we assume that

$$F \text{ is sequentially } \sigma\text{-l.s.c. on } X; \tag{5.2}$$

there exist $\alpha > 0$ and $b \in \mathbf{L}^1(\Omega)$ such that

$$\mathbf{W}(x, s) \geqq \alpha\, |s|^p - b(x), \qquad \text{for all } s \in \mathbb{R}^{N\times N} \quad \text{a.e. } x \in \Omega. \tag{5.3}$$

Let $L \in X'$ be a system of applied forces acting on the body. The *total energy* is then given by

$$G(\psi) = F(\psi) - \langle L, \psi\rangle + \chi_{\mathbf{K}}(\psi).$$

The problem we are interested in is the following:

$$\text{to find } \varphi \in X \text{ such that } G(\varphi) = \min\{G(\psi) : \psi \in X\}. \tag{5.4}$$

We will refer to Problem (5.4) as the *Nonlinear Signorini-type Problem in Elasticity*, briefly **NSP**.

Remark 5.1. If one imposes the additional constraint $\psi = \psi_0$ on a subset Γ of $\partial\Omega$ with positive p-capacity (that is, the mixed displacement-traction problem) then the energy functional turns out to be coercive and existence for **NSP** follows directly from (5.2) and (5.3). This result has been proved in [Ba] for the model of non-linear elasticity with polyconvex stored energy. Here we are interested in the non-coercive situation; that is, we consider Signorini or Neumann conditions.

Remark 5.2. In the framework of BALL's model, problem (5.4) has been considered recently by CIARLET & NEČAS in [CN1] and [CN2] for a non-bilateral box. In particular, in [CN2] the question of the global invertibility of solutions of **NSP** is studied.

Let us discuss sufficient conditions for the solvability of **NSP**. As usual, some compatibility conditions on the data Q and L are needed. A comparison with the preceding section suggests imposing a condition of the following type:

$$\begin{aligned} &(i)\ \langle L, \beta\rangle \leqq 0 \quad \forall \beta \in Q_{\infty,e};\\ &(ii)\ \text{if } \beta \in Q_{\infty,e} \text{ and } \langle L, \beta\rangle = 0, \text{ then } -\beta \in Q_{\infty,e}. \end{aligned} \tag{5.5}$$

As we are going to see, (5.5) actually implies existence of a solution for **NSP**, at least when Q is convex. But, if we admit complete arbitrariness on Q, (5.5) is no longer sufficient, as shown in the following example.

Example 5.3. Let $N = 3$, $p > 3$, $L = (0, 0, -1)$ and

$$Q = \left\{ x = (x_1, x_2, x_3) : x_3 \geqq \frac{1}{1 + x_1^2 + x_2^2}, \quad x_1, x_2 \in \mathbb{R} \right\}.$$

Assume that **NSP** has a solution, say φ. We claim that, for some $\gamma > 0$ large enough and some $\varepsilon > 0$ small enough, it is

$$G(\varphi + \beta) < G(\varphi),$$

where $\beta = (\gamma, 0, -\varepsilon)$. Note that this inequality contradicts the fact that φ is a solution. Since

$$G(\varphi + \beta) = G(\varphi) + \chi_{\mathbf{K}}(\varphi + \beta) - \langle L, \beta \rangle$$

and $\langle L, \beta \rangle = \varepsilon \operatorname{meas}(\Omega) > 0$, then the inequality $G(\varphi + \beta) < G(\varphi)$ follows as soon as we prove that $\chi_{\mathbf{K}}(\varphi + \beta) = 0$, say $\varphi + (\gamma, 0, -\varepsilon) \in \mathbf{K}$. The existence of γ, ε such as to render this condition satisfied follows from the fact that φ_3 has a strictly positive minimum (indeed, φ is continuous on $\bar{\Omega}$, because of the Sobolev embeddings). Thus a contradiction is reached, hence **NSP** has no solution. □

Thus we add some geometric restrictions on Q. Specifically, we assume that

$$\text{if } \beta \in Q_{\infty,e} \text{ and } -\beta \in Q_{\infty,e}, \text{ then } q + \beta \in Q, \text{ for all } q \in Q. \tag{5.6}$$

This condition is obviously satisfied either if Q is convex or if $Q_{\infty,e}$ contains no entire directions or, more generally, if Q is invariant along its entire directions of recession. We just note that (5.6) is *not* satisfied in Example 5.3.

Remark 5.4. The pair of assumptions (5.5) and (5.6) is equivalent to

(*i*) $\langle L, \beta \rangle \leqq 0 \quad \forall \beta \in Q_{\infty,e}$,

(*ii*) if $\beta \in Q_{\infty,e}$ and $\langle L, \beta \rangle = 0$, then $q + \lambda\beta \in Q$ for all $q \in Q$ and all $\lambda \in \mathbb{R}$.

(5.7)

The following existence theorem holds.

Theorem 5.5. *Assume* (5.2), (5.3), (5.7). *If* G *is proper, then* **NSP** *has at least one solution.*

Proof. It is obvious that **NSP** has a solution if and only if the following problem is solvable:

$$\psi \to \min \left\{ F(\psi) + \int_\Omega b(x)\, dx - \langle L, \psi \rangle + \chi_{\mathbf{K}}(\psi) \ :\ \psi \in X \right\}.$$

Then in (5.3) we may assume $b = 0$. In order to apply Theorem 3.16, we have to show that conditions (3.12), (3.18), (3.22)–(3.24) are satisfied. Indeed, (3.22) follows from (5.3), if we take $P(\psi) = \|\nabla\psi\|_{L^p(\Omega)}$ and $\Phi(\lambda) = \lambda^p$. Furthermore, the semicontinuity assumption (3.18) is guaranteed by (5.2) and by Proposition 4.8 (adapted to the case $X = [\mathbf{W}^{1,p}(\Omega)]^N$, $p > 1$).

Let us prove the compactness assumption (3.12). Let $\{\lambda_n\}_{\mathrm{N}}$, $\{\varphi_n\}_{\mathrm{N}}$ be two sequences such that $\lambda_n \to +\infty$, $\varphi_n \xrightarrow{\sigma} \varphi$ and $G(\lambda_n \varphi_n) \leqq c$, for some constant c. Due to the compact embedding of $[\mathbf{W}^{1,p}(\Omega)]^N$ in $[\mathbf{L}^p(\Omega)]^N$, we know that

$$\|\varphi_n - \varphi\|_{[\mathbf{L}^p(\Omega)]^N} \to 0. \tag{5.8}$$

Recalling (5.3), we also conclude that

$$\alpha \int_\Omega |\nabla\varphi_n(x)|^p \, dx \leqq \lambda_n^{1-p}\langle L, \varphi_n\rangle + \lambda_n^{-p} c.$$

From this inequality, taking (5.8) into account and recalling that $p > 1$, we find that φ is constant and that the convergence of φ_n to φ is strong in X.

It remains to verify the compatibility assumptions (3.23), (3.24). First, we note that $\psi \in \operatorname{Ker} P$ if and only if ψ is a constant. Due to Proposition 4.9 (again, adapted to the case $X = [\mathbf{W}^{1,p}(\Omega)]^N$, $p > 1$), if ψ is constant, and belongs to $\mathbf{K}^{\mathrm{seq}}_{\infty,\sigma}$, then it lies in $Q_{\infty,e}$. Thus (3.23) follows from (5.7)(*i*).

Finally, let $\psi \in \operatorname{Ker} P \cap \operatorname{Ker} L \cap \mathbf{K}^{\mathrm{seq}}_{\infty,\sigma}$. As before, we conclude that the constant ψ belongs to $Q_{\infty,e}$; hence (by (5.7)(*ii*)),

$$q - \psi \in Q \quad \text{for all } q \in Q. \tag{5.9}$$

Let $v \in \mathbf{K}$; from (5.9) it follows that $v^*(x) - \psi \in Q$, for q.e. $x \in E$; hence $v - \psi \in \mathbf{K}$. Since $F(v - \psi) = F(v)$, we find that (3.24) holds with $\mu = 1$ and the proof is complete. □

Remark 5.6. The assumption (5.7)(*ii*) is motivated by the abstract assumption of Theorem 3.16. In particular, we may weaken it by requiring that
for all $\beta \in Q_{\infty,e}$, *with* $\langle L, \beta\rangle = 0$, *there exists* $\mu = \mu(z) > 0$ *such that* $q - \mu\beta \in Q$, *for all* $q \in Q$.

The following corollary deals with a special case.

Corollary 5.7. *Let Q be given by*

$$Q = \{x \in \mathbb{R}^N : x_N \geqq 0\}. \tag{5.10}$$

Let G be proper and assume (5.2), (5.3). *If*

$$\langle L, \psi\rangle < 0 \text{ for all constants } \psi \text{ with } \psi_N > 0, \tag{5.11}$$

then **NSP** *has a solution.*

Proof. The proof is very easy and hence it is omitted. □

Remark 5.8. Suppose that Q and L satisfy the following condition:

$$\langle L, \beta\rangle < 0 \quad \text{for all } \beta \in Q_{\infty,e} \text{ with } \beta \neq 0. \tag{5.12}$$

We see at once that (5.7) is satisfied, and so we have the following conclusion.

Let G be proper; if (5.2), (5.3) and (5.12) hold, then **NSP** *has a solution.*

This is essentially the theorem given in [CN1]. Note that (5.12) is equivalent to the existence of a point c_0 of $\mathbb{R}^N$ and of a convex cone C containing no entire directions such that $Q \subset c_0 + C$.

Some discussion about hypothesis (5.2) is necessary. The sequential σ-l.s.c. of functionals of the form $\int_\Omega \mathbf{W}(x, \nabla\psi(x))\,dx$ has been widely studied in the literature. When $\mathbf{W}$ is finite-valued (and satisfies some growth conditions depending on p), a necessary and sufficient condition for (5.2) to occur is called *quasiconvexity* (see, for instance, [M1], [M2], [AF]) of the function $\mathbf{W}(x, z)$ with respect to z, that is

$$\int_D \mathbf{W}(x, z + \nabla\zeta(y))\,dy \geqq \operatorname{meas}(D)\,\mathbf{W}(x, z)$$

for almost all $x \in \mathbb{R}^N$, for all $z \in \mathbb{R}^{N\times N}$, for all bounded open sets $D \subset \mathbb{R}^N$ and for all $\zeta \in [\mathbf{C}_0^\infty(D)]^N$.

In non-linear elasticity the energy functional cannot be finite-valued; as a matter of fact, this would prevent any singular behavior of $\mathbf{W}(x, z)$ when $\det z$ tends to zero (compressible materials), or when $\det z \neq 1$ (incompressible materials).

To overcome these difficulties, in [Ba] the notion of *polyconvexity* is introduced. For instance, in the case $N = 3$, the function $\mathbf{W}(x, z)$ is said to be polyconvex with respect to z if

$$\mathbf{W}(x, z) = \Phi(x, z, \operatorname{adj} z, \det z) \quad \text{for all } x \in \mathbb{R}^3, z \in \mathbb{R}^9,$$

where $\Phi(x, z, a, \delta)$ is convex in (z, a, δ); by $\operatorname{adj} z$ we denote the adjugate matrix of z, *i.e.* the transpose of the matrix of cofactors of z. We recall that, when $\mathbf{W}$ is finite-valued, polyconvexity is a particular case of quasiconvexity (see [Ba]).

More precisely, the three dimensional non-linear elasticity model proposed in [Ba] for the energy integral is the following:

$$F(\psi) = \int_\Omega \Phi(x, \nabla\psi(x), \operatorname{Adj}\nabla\psi(x), \operatorname{Det}\nabla\psi(x))\,dx, \tag{5.13}$$

where $\operatorname{Adj}\nabla\psi$ and $\operatorname{Det}\nabla\psi$ are the adjugate and the determinant of $\nabla\psi$ in the sense of distributions (see [Ba]); the function $\Phi\colon \Omega\times\mathbb{R}^9\times\mathbb{R}^9\times\mathbb{R} \to [0, +\infty]$ satisfies the following properties:

(a) $\Phi(x, z, a, \delta)$ is a Carathéodory function (*i.e.* measurable in x and continuous in (z, a, δ));

(b) $\Phi(x, z, a, \delta) = +\infty$ if $\delta \leqq 0$ and $\Phi(x, z, a, \delta) \to +\infty$ as $z \to z_0$, $a \to \alpha_0$, $\delta \to 0^+$, for all x, z_0, α_0;

(c) (polyconvexity) for all $x \in \Omega$, the function $\Phi(x, \cdot, \cdot, \cdot)$ is convex on its domain (which is not necessarily convex; thus convexity has to be understood in the sense of Busemann, Ewald & Shepard [BES]);

(d) (growth condition)

$$\Phi(x, z, a, \delta) \geqq \alpha(|z|^p + |a|^q + \delta^r) - b(x),$$

for some $\alpha > 0$, $b \in \mathbf{L}^1(\Omega)$, $p > \frac{3}{2}$, $q > 1$, $r > 1$ with $\dfrac{1}{p} + \dfrac{1}{q} < \frac{4}{3}$.
Under these assumptions, the functional (5.13) is sequentially σ-l.s.c. on $[\mathbf{W}^{1,p}(\Omega)]^3$ (see [Ba]); hence our theorem of existence for **NSP** applies.

Remark 5.9. Let ψ be such that $F(\psi) < +\infty$ (in particular, this is true if G is proper and ψ solves **NSP**). By (d), the distributions Adj $\nabla\psi$ and Det $\nabla\psi$ are actually in $[\mathbf{L}^q(\Omega)]^9$ and $\mathbf{L}^r(\Omega)$ respectively (and Det $\nabla\psi > 0$ a.e. in Ω); however, whether they coincide with the functions adj $\nabla\psi$ and det $\nabla\psi$ defined pointwise is not known.

Remark 5.10. If in assumption (d) we have $p \geqq 2$ and $\dfrac{1}{p} + \dfrac{1}{q} \leqq 1$, then the sequential σ-l.s.c. result still holds for the functional (5.13) if Adj $\nabla\psi$ and Det $\nabla\psi$ are replaced respectively by adj $\nabla\psi$ and det $\nabla\psi$ defined pointwise (see [Ba], Lemma 6.1 and Theorem 7.7).

Remark 5.11. Theorem (5.5) applies also to incompressible materials; for them assumption (b) must be substituted by

$$\text{(b}'\text{)} \quad \Phi(x, z, a, \delta) = +\infty \text{ if } \delta \neq 1.$$

6. A non-reflexive example: masonry-like materials

Our aim in this section is to minimize, under some unilateral constraints, the stored energy functional of masonry-like materials (see, for instance, [GG], [A], [ABD]). These materials have the characteristic feature of not resisting to traction though they behave elastically under compression.

In this section we use the following symbols (here and in the following the summation convention over repeated indices is adopted):

$\mathbf{A} : \mathbf{B} = \mathbf{A}_{ij}\mathbf{B}_{ij}$	denotes the scalar product of the $N \times N$ matrices $\mathbf{A}$, $\mathbf{B}$;
sym $\mathbf{A}$	denotes the symmetrized matrix of $\mathbf{A}$: $(\text{sym}\,\mathbf{A})_{ij} = \frac{1}{2}(A_{ij} + A_{ji})$;
$u \cdot v = u_i v_i$	denotes the scalar product of the vectors $u, v \in \mathbb{R}^N$;
$u \otimes v$	denotes the tensor product of the vectors $u, v \in \mathbb{R}^N$: $(u \otimes v)_{ij} = u_i v_j$;
$u \odot v = \text{sym}\, u \otimes v$	denotes the symmetrized tensor product of the vectors $u, v \in \mathbb{R}^N$: $(u \odot v)_{ij} = \frac{1}{2}(u_i v_j + u_j v_i)$;
$\lvert\cdot\rvert$	denotes the Euclidean norm either in $\mathbb{R}^N$ or in $\mathbb{R}^{N \times N}$;
S	denotes the set of symmetric $N \times N$ matrices;
S_-	denotes the cone of negative semidefinite symmetric $N \times N$ matrices;

S_+	denotes the cone of positive semidefinite symmetric $N \times N$ matrices; S_+ coincides with the polar cone of S_-, say the set of all matrices $\mathbf{A} \in S$ such that $\mathbf{A} : \mathbf{B} \leqq 0$ for all $\mathbf{B} \in S_-$;
$\mathbf{P}$	is the orthogonal projection onto S_- and satisfies $\lvert\mathbf{P}(\mathbf{A})\rvert^2 = \lvert \operatorname{dist}(\mathbf{A}, S_+)\rvert^2$;
Ω	is a non-empty, bounded, connected open subset of $\mathbb{R}^N$ with a Lipschitz boundary $\partial\Omega$;
co $\mathbf{A}$	is the convex hull of $\mathbf{A}$, $\mathbf{A} \subset \mathbb{R}^N$;
H_{N-1}	denotes the $(N-1)$-dimensional Hausdorff measure on $\partial\Omega$;
M	is the space of ($N \times N$ matrix valued) measures on Ω with bounded total variation;
$\int_\Omega \lvert\mu\rvert$	denotes the total variation of the measure $\mu \in M$ on Ω;
$\varepsilon(u)$	is the linearized strain tensor, with components $\varepsilon_{ij}(u) = \frac{1}{2}(u_{i,j} + u_{j,i})$.

Consider the Banach space

$$X = BD(\Omega) = \{u \in [\mathbf{L}^1(\Omega)]^N : \varepsilon(u) \in M\},$$

endowed with the norm

$$\|u\|_{BD(\Omega)} = \|u\|_{[\mathbf{L}^1(\Omega)]^N} + \int_\Omega \lvert\varepsilon(u)\rvert .$$

In the following we denote by σ the strong $[\mathbf{L}^1(\Omega)]^N$ topology on $BD(\Omega)$. We recall that

$$\int_\Omega \lvert\varepsilon(u)\rvert = \sup\left\{\tfrac{1}{2}\int_\Omega [u_i\varphi_{ij,j} + u_j\varphi_{ij,i}]\,dx \ : \varphi \in [\mathbf{C}^\infty(\Omega)]^{N\times N}; \varphi = 0 \text{ on } \partial\Omega, \lvert\varphi\rvert \leqq 1\right\}.$$

We also mention the following properties of $X = BD(\Omega)$ (see, for instance, [Te]).

(a) *Sobolev embedding*: $X \subset [\mathbf{L}^p(\Omega)]^N$, $\forall p \in \left[1, \frac{N}{N-1}\right]$, with compact injection if $p < \frac{N}{N-1}$.

(b) *Trace theorem*: there is a linear, continuous, surjective operator $\gamma_0 : X \to [\mathbf{L}^1(\partial\Omega)]^N$ such that

$$\gamma_0(u) = u|_{\partial\Omega} \text{ for all } u \in X \cap [\mathbf{C}^0(\bar{\Omega})]^N.$$

(c) *Korn type inequality*: there is a linear continuous operator r from $BD(\Omega)$ into RBM (notation of Section 4) and a constant $c(\Omega)$ depending only on Ω such that

$$\|u - r(u)\|_{BD(\Omega)} \leqq c(\Omega) \int_\Omega \lvert\varepsilon(u)\rvert .$$

As a consequence of this property,

$$\|u\|_{BD(\Omega)} \sim \int_\Omega \lvert\varepsilon(u)\rvert + p(u),$$

where p is an arbitrary semi-norm, which is a norm on RBM. By the way, we recall that $[\mathbf{C}^\infty(\bar{\Omega})]^N$ is not dense in $BD(\Omega)$ with the strong topology.

(d) *Green formula*: for all $v \in BD(\Omega)$ and all $\mathbf{H} \in [\mathbf{C}^1(\overline{\Omega})]^{N\times N}$

$$\int_\Omega \mathbf{H}: \varepsilon(v) = -\int_\Omega v \cdot \operatorname{div}(\operatorname{sym}\mathbf{H})\, dx + \int_{\partial\Omega} \mathbf{H}: [\gamma_0(v) \odot \nu]\, dH_{N-1}, \qquad (6.1)$$

where ν denotes the outward unit vector orthogonal to $\partial\Omega$.
The model proposed in [A] and [GG] for the stored energy functional of masonry-like materials is

$$\int_\Omega |\mathbf{P}(\varepsilon(u))|^2,$$

where, for $u \in BD(\Omega)$, the correct definition of the integral is in the sense of convex function of a measure (see, for instance, [Te], [A], [ABD]).

For completeness, we briefly recall this definition. Letting f: $\mathbb{R}^k \to [0, +\infty]$ be a convex, l.s.c. function and letting μ be a k-vector valued measure on Ω, we set

$$\int f(\mu) = \int_\Omega f(\mu^a(x))\, dx + \int_\Omega f^\infty \left(\frac{d\mu^s}{d\,|\mu^s|}(x)\right) d\,|\mu^s|\,(x)$$

where $\mu = \mu^a(x)\, dx + \mu^s$ is the Lebesgue decomposition of μ in terms of the absolutely continuous part μ^a and of the singular part μ^s, $|\mu^s|$ is the total variation of μ^s, $\dfrac{d\mu^s}{d|\mu^s|}$ is the Radon-Nikodym derivative of μ^s with respect to $|\mu^s|$ and f^∞ is the recession function of f.

Let Q be a closed subset of $\mathbb{R}^N$ such that $\Omega \subset Q$. Consider the set

$$\mathbf{K} = \{u \in X: x + u(x) \in Q \text{ for a.e. } x \in \Omega\},$$

where a.e. stands for *almost everywhere* in the sense of the Lebesgue measure in $\mathbb{R}^N$. Thus we are led to the following minimization problem

$$\min \left\{ \int_\Omega |\mathbf{P}(\varepsilon(u))|^2 - \int_\Omega h \cdot u\, dx - \int_\Omega \mathbf{H}: \varepsilon(u) \quad : \quad u \in \mathbf{K} \right\}, \qquad (6.2)$$

where $h \in [\mathbf{L}^N(\Omega)]^N$ and $\mathbf{H} \in [\mathbf{C}(\overline{\Omega})]^{N\times N}$. Note that, by Green's formula (6.1), for the load term we have the equality

$$\int_\Omega h \cdot u\, dx + \int_\Omega \mathbf{H}: \varepsilon(u) = \int_\Omega (h - \operatorname{div}(\operatorname{sym}\mathbf{H})) \cdot u\, dx + \int_{\partial\Omega} \mathbf{H}: [\gamma_0(u) \odot \nu]\, dH_{N-1};$$

in this form $(h - \operatorname{div}(\operatorname{sym}\mathbf{H}))$ and $\mathbf{H}$ represent respectively the body forces and the surface stresses. Note that the following analysis applies when the body forces are derivatives of continuous functions and the surface stresses are continuous. Setting

$$F(v) = \int_\Omega |\mathbf{P}(\varepsilon(v))|^2 - \int_\Omega \mathbf{H}: \varepsilon(v),$$

$$\langle L, v \rangle = \int_\Omega h \cdot v\, dx,$$

$$G(v) = F(v) - \langle L, v \rangle + \chi_{\mathbf{K}}(v),$$

we see that problem (6.2) becomes the following:

$$\text{find } u \in X \text{ such that } G(u) = \min\{G(v): v \in X\}. \qquad (6.3)$$

We will refer to Problem (6.3) as the *Signorini-type Problem for Masonry-like Materials*, briefly **MSP**.

In order to apply the abstract existence theorems of Section 3 to **MSP**, we make the following assumption:

There are $\gamma > 0$ *and* $b \in \mathbf{L}^1(\Omega)$ *such that for all* $\zeta \in S$

$$|\mathbf{P}(\zeta)|^2 - \mathbf{H}\colon \zeta \geqq \gamma\,|\zeta| - b(x) \quad \text{for a.e. } x \in \Omega. \tag{6.4}$$

It is not difficult to see that condition (6.4) requires that

$$\operatorname{sym} \mathbf{H}(x) \in S_- \quad \text{for all } x \in \Omega.$$

Conversely, (6.4) is satisfied if

$$\sup_{x \in \Omega} \max_i \lambda_i(x) < 0,$$

where $\lambda_i(x)$, $i = 1, 2, 3$, are the eigenvalues of the matrix sym $\mathbf{H}(x)$. Thus a *necessary* condition for the validity of (6.4) is that the matrix sym $\mathbf{H}(x)$ is negative semidefinite for all $x \in \Omega$, and a *sufficient* condition is that it is negative definite, uniformly in $x \in \Omega$ (in other words, the load $\mathbf{H}$ is *safe*, in the sense of [GG] and [A]). A refinement of condition (6.4) is given in [ABD] by use of the notion of *demicoercivity*.

Theorem 6.1. *Let* $N = 2, 3$. *Assume that condition* (6.4) *holds and that*

there is a point $P \in \operatorname{co} \Omega$ *such that* $\langle L, \alpha \times (x - P)\rangle = 0$ *for all* $\alpha \in \mathbb{R}^3$; (6.5)

$$h(x) \cdot \beta \leqq 0 \text{ for a.e. } x \in \Omega \text{ and for all } \beta \in Q_{\infty,e}; \tag{6.6}$$

if $\beta \in Q_{\infty,e}$ *and* $\langle L, \beta\rangle = 0$, *then* $q + \lambda\beta \in Q$, *for all* $q \in Q$ *and all* $\lambda \in \mathbb{R}$. (6.7)

Then **MSP** *has at least one solution.*

Proof. A translation enables us, with no loss in generality, to assume $P = 0$ in hypothesis (6.5). Our aim is to apply the abstract result of Theorem 3.4. Let us verify the hypotheses.

(a) By the Sobolev embedding theorem, every bounded subset of X is σ-relatively compact.
(b) The lower semicontinuity assumption (3.3) follows from [A], [ABD], [Te] and from (6.1).
(c) Compactness assumption (3.4). Let $\{\lambda_n\}_{\mathrm{N}}$, $\{v_n\}_{\mathrm{N}}$ and v be such that $\lambda_n \to +\infty$, $v_n \xrightarrow{\sigma} v$ and $G(\lambda_n v_n)$ is bounded from above. Then $\lambda_n v_n \in \mathbf{K}$ and by (6.4)

$$\lambda_n \gamma \int_\Omega |\varepsilon(v_n)| - \lambda_n \int_\Omega h(x) \cdot v_n(x)\,dx - \int_\Omega b(x)\,dx \leqq c,$$

with c constant. Hence $v(x) \in Q_{\infty,e}$, for a.e. $x \in \Omega$ and

$$\gamma \int_\Omega |\varepsilon(v)| - \int_\Omega h(x) \cdot v(x)\,dx \leqq 0.$$

By (6.6), we find that $v \in RBM$. Moreover,

$$\limsup_{n\to+\infty} \gamma \int_\Omega |\varepsilon(v_n)| \leqq \limsup_{n\to+\infty} \gamma \int_\Omega |\varepsilon_n(v)| - \int_\Omega h(x) \cdot v(x)\, dx$$

$$= \limsup_{n\to+\infty} \left[\gamma \int_\Omega |\varepsilon(v_n)| - \int_\Omega h(x) \cdot v_n(x)\, dx\right]$$

$$\leqq \limsup_{n\to+\infty} \frac{1}{\lambda_n} \left[c + \int_\Omega b(x)\, dx\right] = 0,$$

and so $v_n \to v$ strongly in $BD(\Omega)$.

(d) Necessary condition (3.5)(i). For all $w \in X$ we have $G_{\infty,\sigma}(w) \geqq F^\infty(w) - \langle L, w\rangle + \chi_{\mathbf{K}_{\infty,\sigma}}(w)$ (see Lemma 2.12(ii) and Proposition 2.5); hence it is enough to show that $F^\infty(w) \geqq \langle L, w\rangle$ for all $w \in \mathbf{K}_{\infty,\sigma}$. Let $w \in \mathbf{K}_{\infty,\sigma}$; then $w(x) \in Q_{\infty,e}$ for a.e. $x \in \Omega$ and, by (6.4),

$$F^\infty(w) \geqq \lim_{\lambda\to+\infty} \frac{1}{\lambda} \left[\gamma \int_\Omega |\varepsilon(\lambda w)| - \int_\Omega b(x)\, dx\right]$$

$$= \gamma \int_\Omega |\varepsilon(w)| \geqq 0 \geqq \int_\Omega h(x) \cdot w(x)\, dx = \langle L, w\rangle. \qquad (6.8)$$

(e) Condition (3.5)(ii). From step (d), if $w \in \operatorname{Ker} G_{\infty,\sigma}$, then $w \in \mathbf{K}_{\infty,\sigma}$ and $F^\infty(w) = \langle L, w\rangle$. Hence by (6.8) we conclude that $w \in RBM$ and $\langle L, w\rangle = 0$, so that, by (6.6)

$$\operatorname{Ker} G_{\infty,\sigma} \subset RBM \cap \{v \in X: v(x) \in Q_{\infty,e} \text{ and } h(x)\cdot v(x) = 0 \text{ for a.e. } x \in \Omega\}. \qquad (6.9)$$

We will prove (3.5)(ii) with $\mu = 1$, that is $G(v - w) \leqq G(v)$ for all $v \in X$ and all $w \in \operatorname{Ker} G_{\infty,\sigma}$. The inequality being trivial if $v \notin \mathbf{K}$, we let $v \in \mathbf{K}$ and $w \in \operatorname{Ker} G_{\infty,\sigma}$. By (6.9) we obtain $w(x) = \alpha \times x + \beta$ and $G(v - w) = G(v) + \chi_{\mathbf{K}}(v - w)$. Thus we have only to prove that $v - w \in \mathbf{K}$, say

$$x + v(x) - (\alpha \times x + \beta) \in Q \quad \text{for a.e. } x \in \Omega. \qquad (6.10)$$

Since $x + v(x) \in Q$ for a.e. $x \in \Omega$, (6.10) follows if we show that

$$q - (\alpha \times x + \beta) \in Q \quad \text{for a.e. } x \in \Omega \text{ and all } q \in Q. \qquad (6.11)$$

We will use the following result, proof of which is given below.

Lemma 6.2. (6.6) and (6.7) hold if and only if the following conditions are satisfied:

$$h(x) \cdot b \leqq 0 \textit{ for a.e. } x \in \Omega \textit{ and for all } b \in \operatorname{co} Q_{\infty,e}; \qquad (6.12)$$

if b and $-b \in \operatorname{co} Q_{\infty,e}$, *then* $q + \lambda b \in Q$, *for all* $q \in Q$ *and all* $\lambda \in \mathbb{R}$; (6.13)

if $b \in \operatorname{co} Q_{\infty,e}$ *and* $\langle L, b\rangle = 0$, *then* $-b \in \operatorname{co} Q_{\infty,e}$. (6.14)

By (6.9), it is $\alpha \times x + \beta \in Q_{\infty,e}$, for all $x \in \Omega$, thus, use of (6.13) shows that (6.11) is a consequence of

$$-(\alpha \times x + \beta) \in \operatorname{co} Q_{\infty,e}, \text{ for all } x \in \Omega. \tag{6.15}$$

Since $\operatorname{co} Q_{\infty,e}$ is a convex cone, (6.15) follows from

(1) $-\beta \in \operatorname{co} Q_{\infty,e}$;

(2) $-\alpha \times x \in \operatorname{co} Q_{\infty,e}$ for all $x \in \Omega$.

(1) By linearity, it is $\alpha \times x + \beta \in \operatorname{co} Q_{\infty,e}$, for all $x \in \operatorname{co} \Omega$, so (6.5) (with $P = 0$ and $x = 0$) implies that $\beta \in \operatorname{co} Q_{\infty,e}$. Since $\langle L, \beta \rangle = 0$, by (6.14) we get that $-\beta \in \operatorname{co} Q_{\infty,e}$.

(2) For every $\bar{x} \in \operatorname{co} \Omega$ it is $\alpha \times \bar{x} \in \operatorname{co} Q_{\infty,e}$: in fact, it is $\alpha \times \bar{x} + \beta \in \operatorname{co} Q_{\infty,e}$ and, by (1), $-\beta \in \operatorname{co} Q_{\infty,e}$; finally, $\operatorname{co} Q_{\infty,e}$ is a convex cone. Let $x \in \Omega$: by (6.5), there exists $\lambda > 0$ such that $-\lambda x \in \operatorname{co} \Omega$, hence $\alpha \times (-\lambda x) \in \operatorname{co} Q_{\infty,e}$. This implies that $-\alpha \times x \in \operatorname{co} Q_{\infty,e}$, hence (2). The proof of the theorem is then complete. □

Proof of Lemma 6.2. The equivalence between (6.6) and (6.12) is trivial. (6.13) *and* (6.14) *imply* (6.7). In fact, let $\beta \in Q_{\infty,e}$ be such that $\langle L, \beta \rangle = 0$. By (6.14) we get $-\beta \in \operatorname{co} Q_{\infty,e}$ and, by (6.13), $q + \lambda\beta \in Q$, for all $q \in Q$ and all $\lambda \in \mathbb{R}$.

To complete the proof, assume (6.6) and (6.7). First, we show that

$$\text{if } b \in \operatorname{co} Q_{\infty,e} \text{ and } \langle L, b \rangle = 0, \text{ then } q - b \in Q \text{ for all } q \in Q. \tag{6.16}$$

In fact, if $b \in \operatorname{co} Q_{\infty,e}$, then there are $\lambda_i \geqq 0$ and $b_i \in Q_{\infty,e}$, $i = 1, \dots, N+1$, such that $\sum_{i=1}^{N+1} \lambda_i = 1$ and $b = \lambda_i b_i$. By (6.6) we find that $\langle L, b_i \rangle \leqq 0$, for all i; hence, if $\langle L, b \rangle = 0$, then $\langle L, b_i \rangle = 0$, for all i. Thus (6.7) implies that $q - \lambda_i b_i \in Q$ for all $q \in Q$ and all i. Hence, $q - b \in Q$, for all $q \in Q$, and (6.16) is proved.

(6.6) *and* (6.7) *imply* (6.13). Let b be such that $\pm b \in \operatorname{co} Q_{\infty,e}$. By (6.6) $\langle L, b \rangle = 0$; thus from the fact that $\operatorname{co} Q_{\infty,e}$ is a set of directions, (6.16) gives $q + \lambda b \in Q$, for all $q \in Q$ and all $\lambda \in \mathbb{R}$.

(6.6) *and* (6.7) *imply* (6.14). Let $b \in \operatorname{co} Q_{\infty,e}$ be such that $\langle L, b \rangle = 0$. By (6.16) we find that $q - nb \in Q$, for all $q \in Q$ and all $n \in \mathbb{N}$. Setting $b_n = \frac{q}{n} - b$, we conclude that $-b = \lim_{n \to +\infty} b_n$ and $nb_n \in Q$, so that $-b \in Q_{\infty,e}$. □

Remark 6.3. Roughly speaking, assumption (6.4) means that the system of applied forces does not allow mutual separation between portions of the body. Note that the term **H** does not affect the resultant force nor the resultant momentum: both of them depend only upon h. In particular, assumptions (6.6) and (6.7) assert that, at every point, the resultant must be orthogonal to the bilateral recession directions of Q and it must be strictly opposite to the non-bilateral ones.

Remark 6.4. We claim that (6.6) is *necessary* for the existence of minima (still better, for boundedness of G below), at least when Q is convex and $H \equiv 0$. Indeed, assume that there is a $\beta \in Q^\infty = Q_{\infty,e}$ and a set $B = \{x \in \Omega: h(x) \cdot \beta > 0\}$ with finite perimeter and positive measure. Let v be in $\mathbf{K}$; the function

$$v_\lambda = \begin{cases} v & \text{in } \Omega \setminus B \\ v + \lambda\beta & \text{in } B \end{cases} = v + \lambda\beta\chi_B$$

belongs to $BD(\Omega)$ for all $\lambda \geqq 0$. Since $\langle L, \beta\chi_B \rangle > 0$, we conclude that $\lim_{\lambda \to +\infty} G(v_\lambda) = -\infty$ whenever $\overline{B + \lambda\beta} \cap \overline{\Omega \setminus B} = \emptyset \quad \forall \lambda > 0$

Appendix

This Appendix is devoted to the study of some properties of the capacity. The notations are those of Section 4.

We begin by giving the following result, convenient in the proof of the properties of the set of constraints K defined in (4.13).

Lemma A.1. *Let $\{v_n\}_{\mathbb{N}}$ be a sequence in $[\mathbf{H}^1(\Omega)]^N$, weakly converging to some v. Then*

$$\liminf_{n \to +\infty} |v_n^*(x) - v^*(x)| = 0 \quad \textit{for q.e. } x \in \overline{\Omega}.$$

Proof. Set $y_n = |v_n^* - v^*|$. The sequence $\{y_n\}_{\mathbb{N}}$ converges weakly to zero in $\mathbf{H}^1(\Omega)$ and so there is a subsequence $\{y_{n_k}\}_{\mathbb{N}}$ such that $w_k = \frac{1}{k} \sum_{i=1}^{k} y_{n_i}$ converges to 0 strongly in $\mathbf{H}^1(\Omega)$ (Banach-Saks theorem). By [DaL] (Proposition 1.6), there is a subsequence $\{w_{k_j}\}_{\mathbb{N}}$ such that

$$\lim_{j \to +\infty} w_{k_j}(x) = 0 \quad \text{for q.e. } x \in \overline{\Omega}.$$

Thus for q.e. $x \in \overline{\Omega}$ we have

$$0 \leqq \liminf_{n \to +\infty} y_n(x) \leqq \liminf_{j \to +\infty} y_{n_{k_j}}(x) \leqq \liminf_{j \to +\infty} w_{k_j}(x) = 0,$$

and the proof is complete. □

We just note that an analogous result holds if we replace $[\mathbf{H}^1(\Omega)]^N$ by $[\mathbf{W}^{1,p}(\Omega)]^N$, $p > 1$; this is used in Section 5.

Proof of Proposition 4.8. Let $\{v_n\}_{\mathbb{N}}$ be a sequence in $[\mathbf{H}^1(\Omega)]^N$ which converges weakly to some $v \in [\mathbf{H}^1(\Omega)]^N$. Assume that for all $n \in \mathbb{N}$, $x + v_n^*(x)$ belongs to $Q(x)$, for q.e. $x \in E$; we must show that $x + v^*(x)$ belongs to $Q(x)$ for q.e. $x \in E$.

Since $Q(x)$ is a closed subset of $\mathbb{R}^N$, it is enough to prove that for q.e. $x \in E$ there is a subsequence $\{v_{n_k}\}_{k \in \mathbb{N}}$ such that $v_{n_k}^*(x)$ converges to $v^*(x)$, say

$$\liminf_{n \to +\infty} |v_n^*(x) - v^*(x)| = 0 \quad \text{for q.e. } x \in E.$$

This follows from Lemma A.1. □

Let us come to the set E_{ess} and to its properties.

Proof of Proposition 4.4. (*i*) E_{ess} is intersection of closed sets; moreover, in Definition 4.3 we may take $C = \bar{E}$, hence $E_{\text{ess}} \subset \bar{E}$.

(*ii*) To shorten notations, for all $A \subset \mathbb{R}^N$ set $A^c = \mathbb{R}^N \setminus A$. Then

$$E \setminus E_{\text{ess}} = E \cap [\cap \{C: C \text{ is closed and } \operatorname{cap}(E \setminus C) = 0\}]^c$$
$$= E \cap [\cup \{C^c: C \text{ is closed and } \operatorname{cap}(E \setminus C) = 0\}].$$

By the Lindelöf theorem, the union between brackets may be effected over a countable family, say $\{C_n\}_{\mathbb{N}}$, such that

$$E \setminus E_{\text{ess}} = E \cap \left[\bigcup_{n=1}^{+\infty} \{(C_n)^c: C_n \text{ is closed and } \operatorname{cap}(E \setminus C_n) = 0\}\right]$$
$$= \bigcup_{n=1}^{+\infty} \{E \setminus C_n: C_n \text{ is closed and } \operatorname{cap}(E \setminus C_n) = 0\}.$$

Since capacity is countably subadditive, we get (*ii*).

(*iii*) Let E have zero capacity. In Definition 4.3 we may take $C = \emptyset$; thus, $E_{\text{ess}} = \emptyset$ and hence $\operatorname{cap} E_{\text{ess}} = 0$. Conversely, by (*ii*) it follows that if $\operatorname{cap} E_{\text{ess}} = 0$, then $\operatorname{cap} E \leqq \operatorname{cap} [E \setminus E_{\text{ess}}] + \operatorname{cap} [E_{\text{ess}} \cap E] = 0$. Thus, all equivalences are proved.

(*iv*) Since E is closed, by (*i*) we have $E_{\text{ess}} \subset E$ and hence $\operatorname{cap}(E_{\text{ess}} \setminus E) = 0$. Recalling (*ii*), we find that

$$\operatorname{cap}(E \triangle E_{\text{ess}}) = \operatorname{cap}[(E \setminus E_{\text{ess}}) \cup (E_{\text{ess}} \setminus E)] \leqq \operatorname{cap}(E \setminus E_{\text{ess}}) + \operatorname{cap}(E_{\text{ess}} \setminus E) = 0.$$

(*v*) The *if* part follows immediately from (*ii*). Conversely, let v be continuous and assume by contradiction that $v(y) \notin C$ for some y of E_{ess}. Since v is continuous and C is closed, $v(x) \notin C$ for all x of an open neighborhood $B(y)$ of y. Since, by hypothesis, $v(z) \in C$ for all z of $E \setminus N$ with $\operatorname{cap} N = 0$, we get $B(y) \subset (E \setminus N)^c$. Hence $E \setminus N \subset [B(y)]^c$, say $\operatorname{cap}\{E \setminus [B(y)]^c\} \leqq \operatorname{cap} N = 0$. Since $[B(y)]^c$ is closed, in Definition 4.3 we may take $C = [B(y)]^c$; hence $E_{\text{ess}} \subset [B(y)]^c$. This is impossible, because y belongs to $E_{\text{ess}} \cap B(y)$.

(*vi*) follows immediately from (*v*), taking $C = \{0\}$. □

If E is not closed, then E_{ess} may be considerably different from E or from $(\bar{E})_{\text{ess}}$. For instance, let $E \subset \mathbb{R}$ be the set of all rational points in $[0, 1]$; then all closed sets C such that $\operatorname{cap}(E \setminus C) = 0$ must contain E; hence $E_{\text{ess}} = (\bar{E})_{\text{ess}} = [0, 1]$. Clearly, $(E \triangle E_{\text{ess}}) = (E_{\text{ess}} \setminus E)$ is the set of irrational points in $[0, 1]$ and it has positive capacity.

On the other hand, the set $E \subset \mathbb{R}^2$ of points in $[0, 1] \times [0, 1]$ with rational coordinates has zero capacity; hence $E_{\text{ess}} = \emptyset$, while $(\bar{E})_{\text{ess}} = \bar{E} = [0, 1] \times [0, 1]$ has positive capacity.

Finally, let us detail the argument mentioned in Remark 4.2 in order to show that **GSP** is solvable for all L when Ω is bounded and E is any subset of $\bar{\Omega}$ with positive capacity.

As already noted, we need a capacitary version of Korn's inequality; this requires in turn the definition of a "capacitary essential supremum" along with a technical lemma, as follows.

Definition A.2. *Let u belong to $[\mathbf{H}^1(\Omega)]^N$ and let u^* be its quasi-continuous representative. Set*

$$[u]_E = \inf \{\lambda \in \mathbb{R} : |u^*(x)| \leqq \lambda \quad \textit{for q.e. } x \in E\}.$$

Lemma A.3. *Assume* $\operatorname{cap} E > 0$. *Let $\{w_n\}_{\mathbb{N}}$ be a sequence in $[\mathbf{H}^1(\Omega)]^N$, such that*

(*a*) *$\{w_n\}_{\mathbb{N}}$ converges strongly in $[\mathbf{H}^1(\Omega)]^N$ to some w,*
(*b*) *$[w_n]_E$ converges to zero.*

Then $w^*(x) = 0$ for q.e. $x \in E$. If in addition w is continuous in $\bar{\Omega}$, then

$$w(x) = 0 \quad \textit{for all } x \in E_{\text{ess}}. \tag{A.1}$$

In particular, if w belongs to RBM, then it vanishes identically in Ω.

Proof. By (*a*) and by [DaL] (Proposition 1.6), there is a subsequence, which we still denote by $\{w_n\}_{\mathbb{N}}$, such that $w_n^*(x) \to w^*(x)$, for q.e. $x \in E$. On the other hand, from (*b*) we easily conclude that $w_n^*(x) \to 0$, for q.e. $x \in E$, so that $w^*(x)=0$ for q.e. $x \in E$. Furthermore, if w is continuous in $\bar{\Omega}$, then $w^* = w$ and (A.1) follows from Proposition 4.4(*vi*). The last assertion follows from Remark 4.12(*iv*). □

The following lemma gives the capacity version of the Korn inequality suitable for our purposes.

Lemma A.4. *Suppose* $\operatorname{cap} E > 0$. *Then there is a $C > 0$ such that for all v of $[\mathbf{H}^1(\Omega)]^N$*

$$\|v\|^2 \leqq C\{\|\varepsilon(v)\|^2_{L^2(\Omega)} + [v]_E^2\}. \tag{A.2}$$

Proof. For contradiction, assume that for all $n \in \mathbb{N}$ there exists $v_n \in [\mathbf{H}^1(\Omega)]^N$ with

$$\|v_n\|^2 > n\{\|\varepsilon(v_n)\|^2_{L^2(\Omega)} + [v_n]_E^2\}.$$

Setting $w_n = \dfrac{v_n}{\|v_n\|}$, we find that:

(1) $\|w_n\| = 1$, hence (without relabeling subsequences) there is a w such that $w_n \to w$ weakly in $[\mathbf{H}^1(\Omega)]^N$;
(2) $\|\varepsilon(w_n)\|_{L^2(\Omega)} \to 0$, hence w is a rigid body motion;
(3) $[w_n]_E \to 0$.

By (1), (2) and by the Korn inequality, we have $w_n \to w$ strongly in $[\mathbf{H}^1(\Omega)]^N$. Using (3) and recalling Lemma A.3, we find that w vanishes identically. This is impossible, because the strong convergence and (1) imply $\|w\| = 1$. □

Thus we are able to prove the theorem we stated.

Proposition A.5. *Assume* cap $E > 0$. *If* Q *is bounded, then* **GSP** *is solvable for all systems of applied forces* L.

Proof. Let β be any positive number. Then $G_\beta = \{v \in [\mathbf{H}^1(\Omega)]^N : G(v) \leqq \beta\} \neq \emptyset$. We claim that G_β is bounded. Indeed, G_β is obviously a subset of $\mathbf{K}$, so that for all $v \in G_\beta$ it is $G(v) = F(v) - \langle L, v \rangle$; hence

$$F(v) \leqq \beta + |\langle L, v \rangle|. \tag{A.3}$$

On the other hand, boundedness of Q implies existence of a number M such that, for all $v \in \mathbf{K}$, $|v^*(x)| \leqq M$ for q.e. $x \in E$; hence $[v]_E \leqq M$. Thus, recalling (A.3) and Lemma A.4, we find that, for all $v \in G_\beta$,

$$\|v\|^2 \leqq C\{F(v) + [v]_E^2\} \leqq C\{\beta + M^2 + |\langle L, v \rangle|\} \leqq C_1 + C_2 \|v\|,$$

with C_1, C_2 constant. Therefore, G_β is bounded and Remark 3.6 gives immediately the existence of a minimizer of G. □

By means of analogous proofs, we can give similar results for the Signorini-like problem in non-linear elasticity **NSP** (the notations are those of Section 5).

Lemma A.6. *Assume that* $p > 1$ *and* $\mathrm{cap}_p\, E > 0$. *Then there is a* $C > 0$ *such that for all* v *of* $[\mathbf{W}^{1,p}(\Omega)]^N$

$$\|v\|^p \leqq C\{\|\nabla v\|^p_{\mathbf{L}^p(\Omega)} + [v]_E^p\}. \qquad \square$$

Proposition A.7. *Assume* $\mathrm{cap}_p\, E > 0$ *and* G *proper. If* Q *is bounded then* **NSP** *is solvable* for all *system of applied forces* L. □

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