On Material Frame-Indifference, Intrinsic Spin, and Certain Constitutive Relations Motivated by the Kinetic Theory of Gases

A. I. MURDOCH

Communicated by M. E. GURTIN

1. Introduction

The status of material frame-indifference (material objectivity) as a basic principle of continuum mechanics was challenged by MÜLLER (1972) who exhibited, within the framework of the kinetic theory of gases, approximative expressions for the stress and heat flux in a gas composed of identical Maxwellian molecules. These expressions, valid for general frames of reference, are frame-dependent in the sense of involving the spin S of the given frame relative to an (and hence any) inertial frame. The generally-accepted conclusion has been that if these relations are regarded as constitutive equations then material frame-indifference is violated. WANG (1975) could find no basis for such violation within the fundamental tenets of kinetic theory, while TRUESDELL (1976) denied the relevance of kinetic theory (as currently understood) to considerations of material objectivity, arguing that approximative relations derived therefrom cannot be regarded as constitutive relations in the sense of continuum mechanics. EDELEN & MCLENNAN (1973), SÖDERHOLM (1976), and WOODS (1981) have also presented relations, based upon the work of BURNETT (1935), which are claimed to be incompatible with material frame-indifference. However, these authors lack the general approach of MÜLLER in that their relations are associated with inertial frames. SPEZIALE (1981) has claimed that the incompatibility with material objectivity of all such relations must derive from the (MAXWELL & CHAPMAN-ENSKOG) approximation procedures employed.

Here MÜLLER's relations are shown actually to be frame-indifferent while the others, inherently valid only for inertial frames, are rendered objective by the addition of terms which vanish in inertial frames. *It follows that criticisms of material objectivity based upon these relations are without foundation.* The relations in question involve dependence upon **intrinsic spin**[1] $(W+S)$. The admissibility of this parameter as a constitutive variable is used to clarify a possible

[1] Here W denotes the spin field associated with the body, so that $(W+S)$ represents the spin of a material element relative to an inertial frame. This kinematic parameter is objective and hence intrinsic.

ambiguity in the interpretation of the postulate of invariance of material properties under superposed rigid motions, and demonstrates its equivalence to material frame-indifference. It is also indicated how considerations of intrinsic spin may possibly obviate difficulties in the constitutive theory of superfluids.

In Section 2 is presented a somewhat more observer-oriented statement of material frame-indifference than is customary (*cf.* TRUESDELL & NOLL (1965) and TRUESDELL (1977)), while in Section 3 the expressions for stress and heat flux involved in the controversy are exhibited and analysed. Those of MÜLLER are shown to be properly indifferent. The relations for stress and heat flux derived from BURNETT'S work are modified by the addition of scalar multiples of $(\boldsymbol{SD} - \boldsymbol{DS})$ and $\boldsymbol{Sg}$ respectively in such a way as to render these expressions objective. (Here $\boldsymbol{D}$ and $\boldsymbol{g}$ denote respectively the stretching and temperature gradient fields associated with the body.) Since $\boldsymbol{S} = \boldsymbol{0}$ in inertial frames, the modified relations coincide therein with their original versions. Remarks are made in Section 4 on the admissibility and use of intrinsic spin in constitutive relations. Attention is drawn to the possible resolution of a constitutive difficulty encountered in the modelling of Helium II. Further, two possible interpretations of the principle of invariance of material properties under superposed rigid motions are discussed, one being essentially an alternative statement of material objectivity and the other implying the impossibility of material sensitivity to $(\boldsymbol{W} + \boldsymbol{S})$. The latter interpretation is thus at variance with the requirements of material frame-indifference. The objective natures of $(\boldsymbol{W} + \boldsymbol{S})$ and certain other parameters used in this work are established, within the framework of NOLL'S neo-classical space-time, in an Appendix.

2. Material Objectivity

The principle of material frame-indifference (*cf.* NOLL (1958) and TRUESDELL & NOLL (1965)) is a fundamental postulate of continuum mechanics which holds that the properties which characterise any given material (continuous body) should be independent of observer.[2] In application it is usually couched in terms of constitutive relations and equivalent processes, a formulation more in keeping with considerations of invariance under superposed rigid motions than of observer independence.[3] It is instructive to note that there are in essence *two* basic assumptions involved:

Assumption 1: *The physical quantities which characterise the behaviour of a given material are intrinsic*

and

Assumption 2: *All observers agree upon the nature of any given material.*

[2] That is, these properties should be *intrinsic*, or *objective*.

[3] The principles of material frame-indifference and of invariance of material properties under superposed rigid motions have been shown to be equivalent in so far as the restrictions they impose upon response functions. Further, these restrictions essentially involve only proper orthogonal tensors (*cf.* MURDOCH (1982)). See also Remark 4.4.

As a consequence of the first assumption, for example, both the stress tensor $\boldsymbol{T}$ and heat flux vector $\boldsymbol{q}$ take objective values. That is (employing standard notation)

$$\boldsymbol{T}^* = \boldsymbol{Q}\boldsymbol{T}\boldsymbol{Q}^T \quad \text{and} \quad \boldsymbol{q}^* = \boldsymbol{Q}\boldsymbol{q},$$

where $(\boldsymbol{T}, \boldsymbol{q})$ and $(\boldsymbol{T}^*, \boldsymbol{q}^*)$ denote these quantities as they appear to two observers, and $\boldsymbol{Q}$ denotes the orthogonal tensor associated with the (instantaneous) isometry which relates points in space as perceived by these observers. (This matter is treated comprehensively by NOLL (1973). See also the Appendix.) In respect of the second assumption, if $\boldsymbol{p} = \hat{\boldsymbol{p}}(\boldsymbol{l})$ is the constitutive relation by which observer O describes a physical parameter $\boldsymbol{p}$ which helps to characterise a given material ($\boldsymbol{l}$ being a list of independent physical parameters and $\hat{\boldsymbol{p}}$ a relevant response function), then the corresponding relation appropriate to another observer O^* will take the form $\boldsymbol{p}^* = \bar{\boldsymbol{p}}^*(\boldsymbol{l}^*)$. Here $\boldsymbol{l}^*$ is the list of parameters for O^* which is the counterpart of $\boldsymbol{l}$ for O, and $\bar{\boldsymbol{p}}^*$ an appropriate response function.

Taken together, Assumptions 1 and 2 mandate the usual restriction(s) of material frame-indifference upon the response function(s) pertaining to any single observer in respect of any given material. These restrictions are not, however, trivial deductions, and in MURDOCH (1983) the case of an elastic material was discussed to indicate the basic ideas involved. In particular, the response functions employed by different observers are to be regarded as distinct[4] and the foregoing assumptions then used to obtain restrictions upon the response function(s) appropriate to any *one* observer.

If material objectivity is to fail as a principle then one or both of the assumptions must fail. The kinetic theory of gases is consistent with the first assumption, as also is the more general discrete approach of MURDOCH (1982) (modulo agreement by observers upon which quantities are subject to fluctuation or, equivalently, $\boldsymbol{Q}$ as it appears in this section must not vary significantly on a microscopic time scale). It would thus seem likely that if the principle is to fail then there must be a material about whose basic nature there is no consensus. Specifically, the behaviour of some characteristic parameter (such as stress) will be modelled by one observer O as $\boldsymbol{p} = \hat{\boldsymbol{p}}(\boldsymbol{l})$ and by at least one other observer O^* as $\boldsymbol{p}^* = \bar{\boldsymbol{p}}^*(\boldsymbol{m}^*)$ where $\boldsymbol{l}^* \neq \boldsymbol{m}^*$. In such a case the basic nature of the material is clearly frame-dependent.

In the next section are discussed two situations in which frame-dependence is *apparently* encountered and which have been used to cast doubt upon material objectivity. In these cases $\boldsymbol{l}$ includes the spin $\boldsymbol{S}$ of the frame of the observer relative to an inertial frame and/or the spin $\boldsymbol{W}$ of the body. In fact, as will be shown, in both cases the dependence is really upon the spin of the body relative to an inertial frame, namely $(\boldsymbol{W} + \boldsymbol{S})$. In this combination material objectivity is not violated; indeed, it is the only such combination as is observed in Section 4.

[4] For example, in situations requiring reference configurations (as in the case for elastic solids) it is clearly unsatisfactory to assume that different observers will select the same reference configuration, and hence the response functions *must* be regarded as distinct.

While the foregoing presupposes all observers to be cognizant of inertial frames it is to be noted that this assumption is implicit in classical physics, being necessary to establish the concept of force.

3. Results from the Kinetic Theory of Gases

Within the framework of kinetic theory it is possible to derive approximative expressions for the stress tensor $\boldsymbol{T}$ and heat flux vector $\boldsymbol{q}$ in a simple gas. In particular, using the CHAPMAN-ENSKOG procedure BURNETT (1935) derived the third approximation to the velocity distribution function which yields (according to CHAPMAN & COWLING (1970))

$$\begin{aligned}\boldsymbol{T} = &-p\boldsymbol{l} + 2\mu\overline{\boldsymbol{D}} + (\alpha_1\mu^2/p)(\operatorname{tr}\boldsymbol{D})\,\overline{\boldsymbol{D}} + (\alpha_2\mu^2/p)\left\{\overline{\dot{\overline{\boldsymbol{D}}} - (\boldsymbol{L}^T\overline{\boldsymbol{D}} + \overline{\boldsymbol{D}}\boldsymbol{L})}\right\}\\ &+ (\alpha_3\mu^2/\varrho\theta)\,\overline{\nabla\boldsymbol{g}} + (\alpha_4\mu^2/2\varrho p\theta)\left\{\overline{\nabla p\otimes\boldsymbol{g} + \boldsymbol{g}\otimes\nabla p}\right\}\\ &+ (\alpha_5\mu^2/\varrho\theta^2)\,\overline{\boldsymbol{g}\otimes\boldsymbol{g}} + (\alpha_6\mu^2/p)\,\overline{\overline{\boldsymbol{D}}\,\overline{\boldsymbol{D}}}\end{aligned}\tag{1}$$

and

$$\begin{aligned}\boldsymbol{q} = &-\lambda\boldsymbol{g} + (\beta_1\mu^2/\varrho\theta)(\operatorname{tr}\boldsymbol{D})\,\boldsymbol{g} + (\beta_2\mu^2/\varrho\theta)\{\dot{\boldsymbol{g}} - \boldsymbol{L}^T\boldsymbol{g}\}\\ &+ (\beta_3\mu^2/\varrho p)\,\overline{\boldsymbol{D}}\,\nabla p + (\beta_4\mu^2/\varrho)\operatorname{div}\overline{\boldsymbol{D}} + (3\beta_5\mu^2/\varrho\theta)\,\overline{\boldsymbol{D}}\boldsymbol{g}.\end{aligned}\tag{2}$$

Here $\alpha_1, \ldots, \alpha_6, \beta_1, \ldots, \beta_5$ denote dimensionless quantities, $p\,(= -\operatorname{tr}\boldsymbol{T}/3)$ denotes the pressure, ϱ density, μ viscosity, λ thermal conductivity, θ temperature, $\boldsymbol{g}$ temperature gradient, $\boldsymbol{L}$ velocity gradient[5] with symmetric part $\boldsymbol{D}$, and, for any tensor $\boldsymbol{A}$,

$$\overline{\boldsymbol{A}} := \boldsymbol{A} - \frac{1}{3}(\operatorname{tr}\boldsymbol{A})\,\mathbf{1},$$

where $\operatorname{tr}\boldsymbol{A}$ signifies the trace of $\boldsymbol{A}$. A superposed dot denotes a material time derivative.

The appearance in the foregoing (or similar) relations of $\boldsymbol{L}$ and hence of its skew part, the spin $\boldsymbol{W}$, has led a number of authors[6] to conclude that these, regarded as constitutive equations, violate the principle of material frame-indifference. However, the above equations pertain to the stress and heat flux only in *inertial* frames. This becomes clear when comparison is made with the analysis of MÜLLER (1972), who derived approximative expressions for $\boldsymbol{T}$ and $\boldsymbol{q}$ in a general frame. Using an iterative scheme due to MAXWELL and developed by IKENBERRY & TRUESDELL (1956), MÜLLER obtained relations comparable with (1) and (2) in respect of Maxwellian molecules in which the spin $\boldsymbol{S}$ of the frame relative to any inertial frame[7]

[5] CHAPMAN & COWLING (1970) term $\boldsymbol{L}^T$ the velocity gradient.

[6] *Cf.* EDELEN & MCLENNAN (1973), SÖDERHOLM (1976), and WOODS (1981).

[7] The spin tensor W_{ik} defined by MÜLLER (1972, equation (1.3)) is $-\boldsymbol{S}$ in our notation. This is discussed in the Appendix.

is present. These relations take the forms

$$T = -p\mathbf{1} + 2\mu\overline{D} - (2\mu/p)\,\overset{\dot\frown}{(\mu\overline{D})} + (4\mu^2/3p)\,(\overline{D}\cdot\overline{D})\,\mathbf{1}$$

$$- (2\mu^2/p)\,\{(\operatorname{tr} D)\,\overline{D} + 2D\overline{D} + WD - DW\}$$

$$- (3\mu/2\varrho\theta)\,\overline{\{\nabla(\mu g) + \nabla(\mu g)^T\}} - (4\mu^2/p)\,(SD - DS) \qquad (3)$$

and

$$q = -\alpha\mu g + (3\alpha\mu/2p)\,\overset{\dot\frown}{(\mu g)} + (\alpha\mu^2/2p)\,\{5\,(\operatorname{tr} D)\,g + 3Lg + 8\overline{D}g\}$$

$$+ (3\mu/\varrho)\operatorname{div}(\mu\overline{D}) - (3\mu^2/\varrho p)\,\overline{D}\,\nabla p + (45\mu/4\varrho\theta)\,\nabla(\mu\theta\dot{p}/p)$$

$$- (75\mu/4\varrho\theta)\,\nabla(\mu\theta\dot{\varrho}/\varrho) + (2\mu/7p)\operatorname{div}(\mu^2\overline{D}^2/\varrho)$$

$$+ (3\mu/7p)\,\nabla(\mu^2\overline{D}\cdot\overline{D}/\varrho) + (6\mu^2/\varrho p)\operatorname{div}(\mu\overline{D}) + (3\alpha\mu^2/p)\,Sg. \qquad (4)$$

Here $\alpha\mu$ denotes the thermal conductivity. The expression for the stress differs slightly from that in the original paper and is taken from a recent (1982) research note of MÜLLER.

Now suppose that relations (1)–(4) are regarded as constitutive equations with $p, \mu, \alpha_1, \ldots, \beta_5$, and $\lambda\,(=\alpha\mu)$ frame-indifferent scalar quantities. It will be shown that relations (3) and (4) are entirely consistent with the principle of material frame-indifference, while terms can be added to relations (1) and (2) which render them frame-indifferent and which vanish in inertial frames.

The only terms which are not immediately seen to be individually objective in (3) are

$$-(2\mu^2/p)\,\{\dot{D} + WD - DW + 2SD - 2DS\}$$

and in (4) are

$$(3\alpha\mu^2/2p)\,\{\dot{g} + Lg + 2Sg\}.$$

These may be written in the forms

$$-(2\mu^2/p)\,\{(\dot{D} - WD + DW) + 2(W + S)\,D - 2D(W + S)\}$$

and

$$(3\alpha\mu^2/2p)\,\{(\dot{g} + Sg) + (W + S)\,g + Dg\}$$

respectively. However, as is shown in the Appendix, the expressions $(\dot{D} - WD + DW)$, $(W + S)$ and $(\dot{g} + Sg)$ are objective, from which it follows that (3) and (4) are compatible with material objectivity.

In respect of relations (1) and (2) the only terms not clearly individually objective are

$$(\alpha_2\mu^2/p)\{\dot{D} - \overline{(L^T\bar{D} + \bar{D}L)}\}$$

and

$$(\beta_2\mu^2/\varrho\theta)\{\dot{g} - L^T g\}$$

respectively. Noting

$$\overline{(L^T\bar{D} + \bar{D}L)} = 2D\bar{D} - WD + DW - \frac{2}{3}(\mathrm{tr}\,(D\bar{D}))\,\mathbf{1},$$

we may write these expressions as

$$(\alpha_2\mu^2/p)\{(\dot{D} - WD + DW) + 2(W + S)\,D - 2D(W + S) - 2D\bar{D} + \frac{2}{3}(\mathrm{tr}\,(D\bar{D}))\,\mathbf{1}\}$$

and

$$(\beta_2\mu^2/\varrho\theta)\{(\dot{g} + Sg) + (W + S)\,g - Dg\}$$

respectively, with $S = \mathbf{0}$. The objectivity of $(\dot{\bar{D}} - WD + DW)$, $(W + S)$, and $(\dot{g} + Sg)$ render these expressions objective, so enabling (1) and (2) to be written in frame-indifferent form by the addition of terms

$$(2\alpha_2\mu^2/p)\,(SD - DS) \quad \text{and} \quad (2\beta_2\mu^2/\varrho\theta)\,Sg$$

respectively.

4. Intrinsic Spin as a Constitutive Variable

4.1 If constitutive relations involve a functional dependence upon the velocity gradient L but not the spin S of the observer frame relative to an inertial frame, then a standard argument (*cf.* TRUESDELL (1977, § IV.4)) shows that material objectivity implies this dependence can only be upon the symmetric part D of L. However, if S is also included as an independent variable, and it is noted that a dependence upon L and S is equivalent to one upon D, $(W - S)$, and $(W + S)$, then the same argument rules out dependence upon $W - S$. Further, the response functions must also be isotropic in the arguments D and $(W + S)$.

4.2 An apparent conflict of constitutive relations with material objectivity as a consequence of a dependence upon the spin W in inertial frames is encountered in studies of Helium (*cf.* ROBERTS & DONNELLY (1974, § 3(e) and HILLS & ROBERTS (1977)). Reinterpretation of spin as intrinsic spin in constitutive equations relating to the superfluid component would seem to offer a possible resolution of this problem.

4.3 The admissibility of $(W + S)$ as a constitutive variable clarifies the interpretation of the principle of invariance of material properties under superposed rigid motions. Roughly speaking, this principle requires that, for a given observer, two motions perceived to be rigidly related must give rise to material responses that differ only in so far as being appropriately rotated versions of each other. If this is taken to mean that the observer and the material universe without the body are to remain unchanged, but that only the motion of the body is to be rigidly affected, then clearly the spin relative to inertial frames would change. Accordingly, the principle would be violated by the existence of materials whose response depends upon $(W + S)$. However, this principle may be interpreted in terms of a superposed rigid motion of the entire material universe with respect to the given observer, in which case $(W + S)$ will merely change in an appropriately rotated manner. Of course, this second interpretation is equivalent to having a second observer whose frame moves rigidly relative to that of the original observer, and is thus seen to be an alternative statement of the principle of material frame-indifference. It would seem that those who wish to retain the first of the above interpretations must address the physical arguments which motivate the possible sensitivity of material response to $(W + S)$ (*cf.* MÜLLER (1972, § 4), SÖDERHOLM (1976, § 3–5), WOODS (1981, § 1), and ROBERTS & DONNELLY (1974, § 3(e))).

Appendix

According to NOLL (1973), the totality of events (or, loosely speaking, points in space-time) occurring at any given instant is, for any given observer O, taken to be in bijective correspondence with a copy of three-dimensional Euclidean space, $\mathcal{E}$. The point space $\mathcal{E}$ is regarded as 'space' as perceived by O or as the frame of reference of O. Thus if O^* is another observer with an associated copy $\mathcal{E}^*$ of three-dimensional Euclidean space, at each instant t is induced a bijective correspondence

$$\alpha_t : \mathcal{E} \to \mathcal{E}^* .$$

If $x \in \mathcal{E}$ then $x^* (= \alpha_t(x))$ denotes the point associated by O^* with that event deemed by O to have occurred at the point x at the instant t.

We assume that observers agree upon distance between simultaneous events; then α_t is an isometry and hence (*cf.* NOLL (1973), p. 64) there exists an orthogonal tensor $Q_t (= \nabla \alpha_t)$ such that for all $x, y \in \mathcal{E}$

$$y^* - x^* = \alpha_t(y) - \alpha_t(x) = Q_t(y - x).$$

Here $y - x$ denotes the displacement from x to y in $\mathcal{E}$ and so is an element of the translation space $\mathcal{V}$ of $\mathcal{E}$. Similarly, $y^* - x^* \in \mathcal{V}^*$, the translation space of $\mathcal{E}^*$, and Q_t is orthogonal as a linear map from $\mathcal{V}$ onto $\mathcal{V}^*$.

Now suppose $\hat{O}$ is an inertial observer with frame $\hat{\mathcal{E}}$ and that i_t and i_t^* are the isometries at instant t which link $\hat{\mathcal{E}}$ with $\mathcal{E}$ and $\mathcal{E}^*$ respectively. In particular

$$i_t^* = \alpha_t \circ i_t .$$

It follows from the chain rule that

$$\boldsymbol{R}_t^* = \boldsymbol{Q}_t \boldsymbol{R}_t \tag{1}$$

where

$$\boldsymbol{R}_t := \nabla \boldsymbol{i}_t \text{ and } \boldsymbol{R}_t^* := \nabla \boldsymbol{i}_t^*.$$

If $\boldsymbol{u}^*$ is a fixed displacement in $\mathscr{E}^*$ then this will be perceived by O at instant t as $\boldsymbol{u}_t := \boldsymbol{Q}_t^T \boldsymbol{u}^*$ and its time rate of change in frame $\mathscr{E}$ will be given by

$$\dot{\boldsymbol{u}}_t = \dot{\boldsymbol{Q}}_t^T \boldsymbol{Q}_t \boldsymbol{u}_t.$$

For this reason the spin of frame $\mathscr{E}^*$ relative to $\mathscr{E}$, $\boldsymbol{\Omega}_t$ say, is defined to be[8]

$$\boldsymbol{\Omega}_t := \dot{\boldsymbol{Q}}_t^T \boldsymbol{Q}_t. \tag{2}$$

We denote the spins of frame $\mathscr{E}$ and $\mathscr{E}^*$ with respect to $\hat{\mathscr{E}}$ by $\boldsymbol{\Lambda}_t$ and $\boldsymbol{\Lambda}_t^*$ respectively; then from (1) and (2)

$$\begin{aligned} \boldsymbol{\Lambda}_t^* := (\dot{\boldsymbol{R}}_t^*)^T \boldsymbol{R}_t^* &= (\dot{\boldsymbol{Q}}_t \boldsymbol{R}_t + \boldsymbol{Q}_t \dot{\boldsymbol{R}}_t)^T \boldsymbol{Q}_t \boldsymbol{R}_t \\ &= \boldsymbol{R}_t^T \boldsymbol{\Omega}_t \boldsymbol{R}_t + \boldsymbol{\Lambda}_t. \end{aligned} \tag{3}$$

That is, the spin of $\mathscr{E}^*$ relative to $\hat{\mathscr{E}}$ is the sum of the spin of $\mathscr{E}^*$ relative to $\mathscr{E}$ and the spin of $\mathscr{E}$ relative to $\hat{\mathscr{E}}$, since $\boldsymbol{R}_t^T \boldsymbol{\Omega}_t \boldsymbol{R}_t$ is merely the spin of $\mathscr{E}^*$ relative to $\mathscr{E}$ as perceived by $\hat{0}$.

If $\boldsymbol{u}$ is an objective vector field associated with a continuous body as observed by O then this appears to O^* as

$$\boldsymbol{u}^* = \boldsymbol{Q}_t \boldsymbol{u}.$$

Its material time derivative is accordingly given by

$$\dot{\boldsymbol{u}}^* = \boldsymbol{Q}_t \dot{\boldsymbol{u}} + \dot{\boldsymbol{Q}}_t \boldsymbol{u} = \boldsymbol{Q}_t (\dot{\boldsymbol{u}} - \boldsymbol{\Omega}_t \boldsymbol{u}). \tag{4}$$

It follows from (1), (3) and (4) that

$$\begin{aligned} (\dot{\boldsymbol{u}} + \boldsymbol{R}_t \boldsymbol{\Lambda}_t \boldsymbol{R}_t^T \boldsymbol{u})^* &= \dot{\boldsymbol{u}}^* + \boldsymbol{R}_t^* \boldsymbol{\Lambda}_t^* (\boldsymbol{R}_t^*)^T \boldsymbol{u}^* \\ &= \boldsymbol{Q}_t (\dot{\boldsymbol{u}} - \boldsymbol{\Omega}_t \boldsymbol{u} + \boldsymbol{R}_t \{\boldsymbol{R}_t^T \boldsymbol{\Omega}_t \boldsymbol{R}_t + \boldsymbol{\Lambda}_t\} \boldsymbol{R}_t^T \boldsymbol{Q}_t^T \boldsymbol{Q}_t \boldsymbol{u}) \\ &= \boldsymbol{Q}_t (\dot{\boldsymbol{u}} + \boldsymbol{R}_t \boldsymbol{\Lambda}_t \boldsymbol{R}_t^T \boldsymbol{u}). \end{aligned}$$

That is, $\dot{\boldsymbol{u}} + \boldsymbol{S}\boldsymbol{u}$ is an objective vector field, where

$$\boldsymbol{S} := \boldsymbol{R}_t \boldsymbol{\Lambda}_t \boldsymbol{R}_t^T$$

is the spin of frame $\mathscr{E}$ with respect to $\hat{\mathscr{E}}$ as perceived by 0.

---

[8] $\dot{\boldsymbol{Q}}_t^T \boldsymbol{Q}_t$ is a skew tensor defined on $\mathscr{V}$ and so has an axial vector $\boldsymbol{\omega}_t$ in terms of which

$$\dot{\boldsymbol{u}}_t = \boldsymbol{\omega}_t \times \boldsymbol{u}_t.$$

Of course, $\boldsymbol{\omega}_t$ is the angular velocity of $\mathscr{E}^*$ relative to $\mathscr{E}$.

In a similar manner to the derivation of (4), if A denotes an objective tensor field associated with a continuous body, so that

$$A^* = Q_t A Q_t^T,$$

then

$$\dot{A}^* = Q_t(\dot{A} - \Omega_t A + A\Omega_t)\, Q_t^T. \tag{5}$$

If x denotes the location of a material point in $\mathscr{E}$ and $x_0 \in \mathscr{E}$ is a fixed point then

$$x^* := \alpha_t(x) = \alpha_t(x_0) + Q_t(x - x_0)$$

whence

$$v^* = \dot{\alpha}_t(x_0) + \dot{Q}_t(x - x_0) + Q_t v$$

and

$$L^* = (\dot{Q}_t + Q_t L)\, Q_t^T.$$

Here v and L denote the velocity and velocity gradient fields in $\mathscr{E}$ respectively, v^* and L^* their counterparts in $\mathscr{E}^*$. The spin of the body, namely the skew part W of the velocity gradient L, thus transforms as

$$W^* = \dot{Q}_t Q_t^T + Q_t W Q_t^T = Q_t(-\Omega_t + W)\, Q_t^T. \tag{6}$$

From (1), (2), (3) and (6),

$$\begin{aligned}(W + S)^* &= W^* + R_t^* \Lambda_t^* (R_t^*)^T \\ &= Q_t(-\Omega_t + W)\, Q_t^T + Q_t R_t (R_t^T \Omega_t R_t + \Lambda_t)\, R_t^T Q_t^T \\ &= Q_t(W + R_t \Lambda_t R_t^T)\, Q_t^T = Q_t(W + S)\, Q_t^T.\end{aligned}$$

That is, the spin of a body relative to an inertial frame is objective.

Using (5), (6) and the objective nature of A, we have

$$\begin{aligned}(\dot{A} - WA + AW)^* &= \dot{A}^* - W^*A^* + A^*W^* \\ &= Q_t(\dot{A} - \Omega_t A + A\Omega_t)\, Q_t^T - Q_t(-\Omega_t + W)\, Q_t^T Q_t A Q_t^T \\ &\quad + Q_t A Q_t^T Q_t(-\Omega_t + W)\, Q_t^T \\ &= Q_t(\dot{A} - WA + AW)\, Q_t^T.\end{aligned}$$

That is, $\dot{A} - WA + AW$ is an objective tensor field. This is, of course, the so-called co-rotational time derivative of A (*cf.* TRUESDELL & NOLL, equation (36.13)).

Finally it is to be noted that the spin employed by MÜLLER (1972) in equations (3.9) and (3.10) is given by (1.3), which in the notation here employed is $\dot{R}_t R_t^T$. However,

$$\dot{R}_t R_t^T = R_t(R_t^T \dot{R}_t)\, R_t^T = R_t(\dot{R}_t^T R_t)^T\, R_t^T = R_t(-\Lambda_t)\, R_t^T = -S.$$

This explains the apparent discrepancy in sign when these equations are written in direct notation (equations (3) and (4) of Section 3).

Acknowledgments. The author gratefully records his interest in the subject of this note as stemming from correspondence with Professors A. E. GREEN and P. CHADWICK and stimulating discussions with Professor I. MÜLLER, and he thanks Professors TRUESDELL and GURTIN for helpful comments on an earlier manuscript.

References

1935 BURNETT, D., The distribution of molecular velocities and the mean motion in a non-uniform gas. Proc. Lond. Math. Soc. **40**, 382–435.

1956 IKENBERRY, E., & C. TRUESDELL, On the pressures and the flux of energy in a gas according to Maxwell's kinetic theory. I. J. Rational Mech. Anal. **5**, 1–54.

1958 NOLL, W., A mathematical theory of the mechanical behavior of continuous media. Arch. Rational Mech. Anal. **2**, 197–226.

1965 TRUESDELL, C., & W. NOLL, The Non-linear Field Theories. In Handbuch der Physik, Vol. III/3 (ed. S. FLÜGGE) Berlin-Heidelberg-New York: Springer.

1970 CHAPMAN, S., & T. G. COWLING, The Mathematical Theory of Non-Uniform Gases (Revised Third Edition) London: Cambridge University.

1972 MÜLLER, I., On the frame dependence of stress and heat flux. Arch. Rational Mech. Anal. **45**, 241–250.

1973 EDELEN, D. G. B., & J. A. MCLENNAN, Material indifference: a principle or a convenience. Int. J. Eng. Sc. **11**, 813–817.

NOLL, W., Lectures on the foundations of continuum mechanics and thermodynamics. Arch. Rational Mech. Anal. **52**, 62–92.

1974 ROBERTS, P. H., & R. J. DONNELLY, Superfluid mechanics. In Annual Review of Fluid Mechanics Vol. 6 (ed. M. VAN DYKE, W. G. VINCENTI, & J. V. WEHAUSEN) Palo Alto: Annual Reviews.

1975 WANG, C. C., On the concept of frame-indifference in continuum mechanics and in the kinetic theory of gases. Arch. Rational Mech. Anal. **58**, 381–393.

1976 SÖDERHOLM, L. H., The principle of material frame-indifference and material equations of gases. Int. J. Eng. Sc. **14**, 523–528.

TRUESDELL, C., Correction of two errors in the kinetic theory of gases which have been used to cast unfounded doubt upon the principle of material frame-indifference. Meccanica **11**, 196–199.

1977 HILLS, R. N., & P. H. ROBERTS, Superfluid mechanics for a high density of vortex lines. Arch. Rational Mech. Anal. **66**, 43–71.

TRUESDELL, C., A First Course in Rational Continuum Mechanics. New York-San Francisco-London: Academic Press.

1981 SPEZIALE, G., On frame-indifference and iterative procedures in the kinetic theory of gases. Int. J. Eng. Sc. **19**, 63–73.

WOODS, L. C., The bogus axioms of continuum mechanics. Bull. Inst. Math. Applications **17**, 98–102.

1982 MURDOCH, A. I., On material frame-indifference. Proc. R. Soc. Lond. A **380**, 417–426.

1983 MURDOCH, A. I., The motivation of continuum concepts and relations from discrete considerations. To appear in Q. J. Mech. Appl. Math.

Department of Mathematics
University of Strathclyde
Glasgow, Scotland

(Received November 28, 1982)

Reprinted by Proff GmbH & Co. KG · 5340 Bad Honnef