Convergence of Solutions of H-Systems or How to Blow Bubbles

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0. Introduction

Let $\Gamma \subset \mathbb{R}^3$ be a Jordan curve. The problem of finding surfaces of constant mean curvature spanned by Γ has been extensively studied, with a lengthy literature, including [2, 7, 8, 9, 18, 21, 22, 23, 26, 28, 29]. In particular, if $\Gamma \subset B_R$ —a ball of radius R—with $R < 1$, it is known that there exist surfaces of mean curvature one spanned by Γ. Here, we deal only with surfaces Σ parametrized on the unit disk

$$\Omega = \{(x, y) \in \mathbb{R}^2;\ x^2 + y^2 < 1\},$$

and thus $\Sigma = u(\bar{\Omega})$ where $u : \bar{\Omega} \to \mathbb{R}^3$ satisfies,

$$\text{(0.1)} \qquad \begin{cases} \Delta u = 2u_x \wedge u_y & \text{on } \Omega, \\ |u_x|^2 - |u_y|^2 = u_x \cdot u_y = 0 & \text{on } \Omega, \\ u(\partial\Omega) = \Gamma. \end{cases}$$

In this paper we investigate the behavior of such surfaces as $\Gamma \to 0$. Let (Γ_n) be a sequence of Jordan curves such that $\Gamma_n \to 0$, that is $\Gamma_n \subset B_{R_n}(0)$ and $R_n \to 0$. Let Σ_n denote a surface of constant mean curvature one spanned by Γ_n. It has been suggested by Professor J. SERRIN (private communication; see also [18]) that under appropriate assumptions Σ_n should converge to a sphere of radius one. Our main results are the following.

Theorem 0.1. *Assume that the areas of the surfaces Σ_n remain bounded. Then a subsequence of the Σ_n converges to $\{0\}$ or to a finite (connected) union of spheres of radius one, such that at least one of them contains* 0.

In general, we do not have more precise information about the limiting configuration. Indeed, it would be interesting to determine whether an arbitrary configuration of spheres may be achieved as a limit of (Σ_n), for some appropriate sequence (Γ_n).

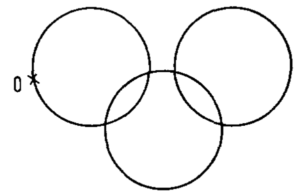

Fig. 1

However, we do have a more refined conclusion when the Σ_n's are chosen in a *special way*. We recall that if $R < 1$ and $\Gamma \subset B_R$, there exists a "small" surface $\underline{\Sigma}$, $\underline{\Sigma} \subset B_R$, of constant mean curvature one, spanned by Γ (see HILDEBRANDT [8]). Another surface $\overline{\Sigma}, \overline{\Sigma} \neq \underline{\Sigma}$, of constant mean curvature one, spanned by Γ, has been constructed by the authors in [2] (see also [23] and [22]); we call it a "large" solution of (0.1). *For such special solutions* we have the following

Theorem 0.2. *Assume that* $\Gamma_n \to 0$. *Let* $\overline{\Sigma}_n$ *be a large solution corresponding to* Γ_n, *obtained through the construction of* [2]. *Then a subsequence of the* $\overline{\Sigma}_n$ *converges to a single sphere of radius one containing* 0.

A similar conclusion for the volume constrained Plateau problem has been obtained earlier by H. WENTE [28].

Such geometric problems are closely related to this question. Let $u^n \colon \overline{\Omega} \to \mathbb{R}^3$ be a solution of the system

$$\begin{cases} \Delta u^n = 2u_x^n \wedge u_y^n & \text{on } \Omega \\ u^n = \gamma^n & \text{on } \partial\Omega. \end{cases} \tag{0.2}$$

Suppose that $\gamma^n \to 0$. What can be said about the sequence (u^n)?

Our approach relies on a kind of "blow-up" analysis. After the "blow-up" has been performed we are led to an equation on all of $\mathbb{R}^2$. Our next lemma plays an important role since it provides a complete description of the solutions on all of $\mathbb{R}^2$.

Lemma 0.1. *Let* $\omega \in L^1_{\mathrm{loc}}(\mathbb{R}^2; \mathbb{R}^3)$ *be such that*

$$\Delta\omega = 2\omega_x \wedge \omega_y \quad \text{on } \mathbb{R}^2, \quad \int_{\mathbb{R}^2} |\nabla\omega|^2 < \infty. \tag{0.3}$$

Then ω *has precisely the form*

$$\omega(z) = \pi\left(\frac{P(z)}{Q(z)}\right) + C, \quad z = (x, y) = x + iy, \tag{0.4}$$

where $\pi \colon \mathbb{C} \to S^2$ *denotes stereographic projection,* P, Q *are polynomials and* C *is a constant. In addition* $\int_{\mathbb{R}^2} |\nabla\omega|^2 = 8\pi \operatorname{Max}\{\deg P, \deg Q\}$.

Note that (0.3) is invariant under translation and dilation. Thus, if ω satisfies (0.3) and if we set

$$u^n = \omega\left(\frac{\cdot - a}{\varepsilon_n}\right) \tag{0.5}$$

where $a \in \Omega$ and $\varepsilon_n \to 0$, then u^n satisfies (0.2). Moreover if $\omega(\infty) = 0$, then $\gamma^n \to 0$.

Our main result asserts that if (u^n) is *any* sequence bounded in H^1 and satisfying (0.2) with $\gamma^n \to 0$, then (u^n) *behaves essentially like a finite superposition of terms of the form* (0.5). More precisely we have

Theorem 0.3. *Suppose* (u^n) *satisfies* (0.2) *with* $\gamma^n \to 0$ *in* $H^{1/2}(\partial\Omega; \mathbb{R}^3)$ *and* $\int_\Omega |\nabla u^n|^2 \leqq C$. *Then there exist*

(i) *a finite number of solutions* $\omega^1, \omega^2, \dots \omega^p$ *of* (0.3),

(ii) *sequences* $(a_n^1), (a_n^2), \dots, (a_n^p)$ *in* Ω, *and*

(iii) *sequences* $(\varepsilon_n^1), (\varepsilon_n^2), \dots (\varepsilon_n^p)$ *with* $\varepsilon_n^i > 0$ $(\forall i, \forall n)$ *and* $\lim_{n\to\infty} \varepsilon_n^i = 0 (\forall i)$, *such that, for a subsequence of the* u^n,

(0.6)
$$\left\| u^n - \sum_{i=1}^{p} \omega^i \left(\frac{\cdot - a_n^i}{\varepsilon_n^i} \right) \right\|_{H^1} \xrightarrow[n\to\infty]{} 0$$

and

(0.7)
$$\int_\Omega |\nabla u^n|^2 = \sum_{i=1}^{p} \int_{\mathbb{R}^2} |\nabla \omega^i|^2 + o(1).$$

Comments. *1.* A variant of Theorem 0.3 (see Theorem 3) asserts that if (u^n) satisfies (0.2) with $\gamma^n \to 0$ in $L^\infty(\partial\Omega; \mathbb{R}^3)$ and $\int |\nabla u^n|^2 \leqq C$, then there exist ω^i, (a_n^i) and (ε_n^i) as in Theorem 0.3 such that

$$\left\| u^n - \sum_{i=1}^{p} \omega^i \left(\frac{\cdot - a_n^i}{\varepsilon_n^i} \right) \right\|_{L^\infty} \xrightarrow[n\to\infty]{} 0.$$

This property is of course very useful for geometrical applications.

2. Under the assumptions of Theorem 0.3 it follows that $(1/8\pi) \int |\nabla u^n|^2$ converges to some *integer*. We deduce in particular that

(a) if $\int_\Omega |\nabla u^n|^2 \leqq 8\pi - \delta$ for some $\delta > 0$, then $\int_\Omega |\nabla u^n|^2 \to 0$;

(b) if $\int |\nabla u^n|^2 = 8\pi + o(1)$, there is *exactly one* non constant solution ω of (0.3) such that

$$\left\| u^n - \omega \left(\frac{\cdot - a_n}{\varepsilon_n} \right) \right\|_{H^1} \to 0.$$

This is precisely what happens when we choose u^n to be the "large" solution of (0.1) constructed in [2].

3. Theorem 0.3 says that the functions (u^n) "concentrate" around a finite number of points $a^i = \lim_{n\to\infty} a_n^i$. In case $a^i \neq a^j$, then the functions

$$\omega_n^i = \omega^i \left(\frac{\cdot - a_n^i}{\varepsilon_n^i} \right) \quad \text{and} \quad \omega_n^j = \omega^j \left(\frac{\cdot - a_n^j}{\varepsilon_n^j} \right)$$

have essentially "disjoint supports". However, it could happen that $a^i = a^j$ and $i \neq j$, say for example if $a_n^i \equiv a_n^j \equiv a$. In such a case we prove that $\varepsilon_n^i/\varepsilon_n^j$ tends to 0 or ∞ as $n \to \infty$. This means that the functions (ω_n^i) and (ω_n^j) concentrate at the same point, but the "speeds of concentration" are very different. For a detailed analysis of the general case, see Theorem 2.

4. The conclusion of Theorem 0.3 still holds if we replace (0.2) by

$$(0.8)\qquad \begin{cases} \Delta u^n = 2u^n_x \wedge u^n_y + f^n & \text{on } \Omega \\ u^n = 0 & \text{on } \partial\Omega, \end{cases}$$

and $f^n \to 0$ in H^{-1}.

This has some implications for the Palais-Smale condition. Consider, for example the functional on $H^1_0(\Omega;\mathbb{R}^3)$ given by

$$E(u) = \int_\Omega |\nabla u|^2 + \tfrac{4}{3}\int_\Omega u\cdot u_x \wedge u_y$$

(critical points of E correspond to solutions of $\Delta u = 2u_x \wedge u_y$). Let (u^n) be a sequence in H^1_0 such that

$$E(u^n) \to c, \qquad E'(u^n) \to 0 \quad \text{in } H^{-1}.$$

In general (u^n) need not be relatively compact in H^1_0 (that is, the (PS) condition is *not* satisfied). However the conclusion of Theorem 0.3 still holds and it follows that $c = (8\pi/3)\,k$ where $k \geqq 0$ is an integer. Theorem 0.3 implies in particular that $u^n \to 0$ strongly in $H^1_{\text{loc}}(\Omega \setminus \cup \{a^i\})$. A similar phenomenon had been observed for the first time by SACKS & UHLENBECK [17] in the context of harmonic maps; subsequently their technique was used by MEEKS-YAU [15] and by SIU-YAU [19]. The general method of concentration compactness due to P. L. LIONS [14] could also be used in our problem. It would show that, under the assumptions of Theorem 0.3, $|\nabla u^n|^2$ converges in the sense of measures on $\overline{\Omega}$ to a finite sum of Dirac masses, $\Sigma\,\alpha_i\,\delta_{a^i}$ with $\alpha_i \geqq 8\pi$. Our conclusion is more precise and leads for example to $\alpha_i = 8\pi k_i$ where k_i is a integer. However, our proof is inspired by the method of concentration compactness and we introduce (as in [13], [14]) the concentration functions $Q_n(t) = \operatorname{Max}_{z\in\overline{\Omega}} \int_{z+t\Omega} |\nabla u^n|^2$ (presumably, one could also use the same compactness device as in [4], [12]).

Related questions have been considered by C. TAUBES [25] for the Yang-Mills equations in dimension four and (independently of our work) by M. STRUWE [24] for the problem:

$$\begin{cases} -\Delta u_n = |u_n|^{p-1}\,u_n + f_n & \text{on } \Omega \subset \mathbb{R}^N \\ u_n = 0 & \text{on } \partial\Omega \end{cases}$$

where $f_n \to 0$ on H^{-1} and $p = (N+2)/(N-2)$—except that the analogue of Lemma 0.1 is still missing (i.e., there is no precise description of the set of solutions of $-\Delta\omega = |\omega|^{p-1}\,\omega$ in $\mathbb{R}^N$ and $\int |\nabla\omega|^2 < \infty$; however all solutions ω with constant sign are known, see [5]).

The paper is organized as follows:

In Section 1 we prove Theorem 0.3.

In Section 2 we describe some additional properties dealing with the "speeds of concentration".

In Section 3 we establish convergence in the L^∞ norm.

In Section 4 we discuss geometrical applications.

The Appendix contains the proof of Lemma 0.1, as well as some technical facts.

The results of this paper were earlier announced in reference [3].

1. Strong convergence in H^1

Let (u^n) be a sequence in $H_0^1\,[= H_0^1(\Omega;\mathbb{R}^3)]$ satisfying

$$\text{(1)}\qquad \begin{cases} \Delta u^n = 2u_x^n \wedge u_y^n + f^n & \text{on } \Omega \\ u^n = 0 & \text{on } \partial\Omega, \end{cases}$$

with

$$\text{(2)}\qquad f^n \to 0 \text{ strongly in } H^{-1}$$

and

$$\text{(3)}\qquad \int_\Omega |\nabla u^n|^2 \leqq C.$$

We claim that $u^n \rightharpoonup 0$ weakly in H_0^1; indeed suppose that $u^n \rightharpoonup u$ weakly in H_0^1. We deduce from Lemma A.9 in [2] that $u_x^n \wedge u_y^n \to u_x \wedge u_y$ in $\mathcal{D}'$, and thus u satisfies

$$\text{(4)}\qquad \begin{cases} \Delta u = 2u_x \wedge u_y & \text{on } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

On the other hand, from a result of Wente [27] we know that $u = 0$ is the only solution of (4).

In general (u^n) does not converge to 0 strongly in H_0^1. Thus our purpose is to obtain a more precise analysis of the behavior of (u^n) as $n \to \infty$. Our method involves a "blow-up" analysis near some singular points. This leads in a natural way to the consideration of functions $\omega \in L^1_{\mathrm{loc}}(\mathbb{R}^2;\mathbb{R}^3)$ satisfying

$$\text{(5)}\qquad \begin{cases} \Delta\omega = 2\omega_x \wedge \omega_y & \text{on } \mathbb{R}^2 \\ \displaystyle\int_{\mathbb{R}^2} |\nabla\omega|^2 < \infty. \end{cases}$$

The solutions of (5) are smooth (see [26]) and they are completely described in Lemma A.1 in the Appendix. In particular they are bounded, $\omega(\infty) = \lim_{|z|\to\infty} \omega(z)$ exists, and

$$\text{(6)}\qquad \int_{\mathbb{R}^2} |\nabla\omega|^2 = 8\pi k$$

where $k \geqq 0$ is an integer. The main result of Section 1 is the following

Theorem 1. *Assume* (u^n) *satisfies* (1), (2), (3) *and that* $\int_\Omega |\nabla u^n|^2$ *does not tend to* 0. *Then there exist*

(i) *a finite number of non constant solutions* $\omega^1, \omega^2, \dots \omega^p$ *of* (5) *with* $\omega^i(\infty) = 0\ (\forall i)$,

(ii) *sequences* $(a_n^1), (a_n^2), \dots (a_n^p)$ *in* Ω, *and*

(iii) *sequences* $(\varepsilon_n^1), (\varepsilon_n^2), \dots (\varepsilon_n^p)$ *with* $\varepsilon_n^i > 0\ (\forall i, \forall n)$ *and* $\lim_{n\to\infty} \varepsilon_n^i = 0\ (\forall i)$, *such that, for a subsequence of the u^n (still denoted by (u^n))*

$$\left\| u^n - \sum_{i=1}^{p} \omega^i\left(\frac{\cdot - a_n^i}{\varepsilon_n^i}\right)\right\|_{H^1} \xrightarrow[n\to\infty]{} 0, \tag{7}$$

$$\int_{\Omega} |\nabla u^n|^2 = \sum_{i=1}^{p} \int_{\mathbb{R}^2} |\nabla \omega^i|^2 + o(1), \tag{8}$$

$$\frac{1}{\varepsilon_n^i} \operatorname{dist}(a_n^i, \partial\Omega) \xrightarrow[n\to\infty]{} \infty \ (\forall i). \tag{9}$$

As an immediate consequence of Theorem 1 we have

Corollary 1. *Let (u^n) be a sequence in H^1 satisfying*

$$\Delta u^n = 2u_x^n \wedge u_y^n \quad on\ \Omega, \qquad u^n = \gamma^n \quad on\ \partial\Omega,$$

with

$$\|\gamma^n\|_{H^{1/2}(\partial\Omega)} \xrightarrow[n\to\infty]{} 0 \tag{10}$$

and

$$0 < \alpha \leqq \int_{\Omega} |\nabla u^n|^2 \leqq C. \tag{11}$$

Then the conclusion of Theorem 1 holds.

Proof of Corollary 1. Let h^n be the solution of

$$\Delta h^n = 0 \quad \text{on } \Omega, \qquad h^n = \gamma^n \quad \text{on } \partial\Omega,$$

and set $v^n = u^n - h^n$. Then v^n satisfies

$$\begin{cases} \Delta v^n = 2v_x^n \wedge v_y^n + f^n & \text{on } \Omega \\ v^n = 0 & \text{on } \partial\Omega, \end{cases}$$

where $f^n = 2[(h_x^n \wedge v_y^n) + (v_x^n \wedge h_y^n) + (h_x^n \wedge h_y^n)]$, and $f^n \to 0$ strongly in H^{-1} by Lemma A.1 in [2] (since $h^n \to 0$ in H^1). Therefore we are reduced to the situation of Theorem 1.

Proof of Theorem 1. We may always assume in addition that

$$\|u^n\|_{L^\infty} \leqq C. \tag{12}$$

Indeed let $\varphi^n \in H_0^1$ be the solution of

$$\Delta\varphi^n = f^n \quad \text{on } \Omega, \qquad \varphi^n = 0 \quad \text{on } \partial\Omega,$$

so that $\varphi^n \to 0$ in H_0^1. Set $v^n = u^n - \varphi^n$; we have

$$\begin{cases} \Delta v^n = 2v_x^n \wedge v_y^n + g^n & \text{on } \Omega \\ v^n = 0 & \text{on } \partial\Omega, \end{cases}$$

where $g^n = 2(v_x^n \wedge \varphi_y^n + \varphi_x^n \wedge v_y^n + \varphi_x^n \wedge \varphi_y^n)$. It follows from Lemma A.1 in [2] that $\|v^n\|_{L^\infty} \leqq C$ and $g^n \to 0$ in H^{-1}. Therefore (v^n) satisfies the assumptions of Theorem 1 and (v^n) is bounded in L^∞.

In what follows we assume systematically that (u^n) satisfies (12), and we extend u^n by 0 outside Ω. The main ingredient in the proof of Theorem 1 is the following

Lemma 1. *Assume that* (u^n) *satisfies* (1), (2), (3), (12) *and*

$$\int_\Omega |\nabla u^n|^2 \geqq \alpha > 0. \tag{13}$$

Then, there exist

(i) *a non constant solution* ω *of* (5),

(ii) *a sequence* (a_n) *in* Ω, *and*

(iii) *a sequence* (ε_n) *with* $\varepsilon_n > 0$ *and* $\lim_{n\to\infty} \varepsilon_n = 0$,

such that (for some subsequence still denoted by (u^n)*)*

$$\tilde{u}^n(z) = u^n(\varepsilon_n z + a_n) \to \omega(z) \text{ for a.e. } z \in \mathbb{R}^2 \tag{14}$$

$$\nabla \tilde{u}^n \rightharpoonup \nabla\omega \text{ weakly in } L^2(\mathbb{R}^2) \tag{15}$$

and in addition

$$\frac{1}{\varepsilon_n} \operatorname{dist}(a_n, \partial\Omega) \to \infty. \tag{16}$$

The proof of Lemma 1 uses the basic inequality Lemma A.8 of [2], which we recall here.

Lemma 2. *There is a constant* c_0 *such that*

$$\left|\int_\Omega u \cdot v_x \wedge v_y\right| \leqq c_0 \|\nabla u\|_{L^2} \|\nabla v\|_{L^2}^2 \quad \forall u \in H^1(\Omega) \cap L^\infty(\Omega), \forall v \in H_0^1(\Omega).$$

In the proof of Lemma 1 we shall also use the following convergence result.

Lemma 3. *Let* (u^n) *be a sequence in* $L^\infty(\Omega) \cap H^1(\Omega)$ *and let* $u \in L^\infty(\Omega) \cap H^1(\Omega)$. *Assume that*

$$\Delta u^n = 2u_x^n \wedge u_y^n + g^n \quad \text{on } \Omega, \tag{17}$$

$$\|\nabla u^n\|_{L^2(\Omega)} \leqq \mu_0 + o(1) \quad \text{with } 2c_0\mu_0 < 1, \tag{18}$$

$$u^n \rightharpoonup u \quad \text{weakly in } H^1(\Omega), \tag{19}$$

$$g^n \to 0 \quad \text{strongly in } H^{-1}(\Omega). \tag{20}$$

Then $u^n \to u$ *strongly in* $H^1(\Omega')$ *for all* $\Omega' \subset\subset \Omega$.

Proof. *Step 1.* We first reduce to the case where $u = 0$. Set $v^n = u^n - u$; then we have

$$\Delta v^n = 2v_x^n \wedge v_y^n + 2(u_x \wedge v_y^n + v_x^n \wedge u_y) + g^n \quad \text{on } \Omega.$$

Let ψ^n be the solution of the problem

$$\begin{cases} \Delta\psi^n = 2(u_x \wedge v_y^n + v_x^n \wedge u_y) & \text{on } \Omega \\ \psi^n = 0 & \text{on } \partial\Omega. \end{cases}$$

We claim that $\psi^n \to 0$ strongly in H_0^1. Indeed from Lemma A.1 in [2] we know that

$$\|\psi^n\|_{L^\infty} \leqq C\,\|\nabla u\|_{L^2}\,\|\nabla v^n\|_{L^2} \leqq C$$
$$\|\nabla\psi^n\|_{L^2} \leqq C\,\|\nabla u\|_{L^2}\,\|\nabla v^n\|_{L^2} \leqq C.$$

On the other hand,

$$-\int_\Omega |\nabla\psi^n|^2 = 2\int_\Omega \psi^n \cdot (u_x \wedge v_y^n + v_x^n \wedge u_y),$$

and (for some subsequence) both $\psi^n \wedge u_x$ and $\psi^n \wedge u_y$ converge strongly in L^2 by dominated convergence. Since v_x^n and v_y^n converge weakly to 0 in L^2 it follows that $\int_\Omega |\nabla\psi^n|^2 \to 0$. Finally we have

$$\Delta v^n = 2v_x^n \wedge v_y^n + h^n \quad \text{on } \Omega$$

for some sequence $h^n \to 0$ strongly in H^{-1}, and moreover

$$\int_\Omega |\nabla v^n|^2 = \int_\Omega |\nabla u^n|^2 - \int_\Omega |\nabla u|^2 + o(1) \leqq \mu_0^2 + o(1).$$

Step 2. We assume now that $u = 0$. Fix $\zeta \in \mathcal{D}(\Omega)$. By (17)

$$-\int \nabla u^n \cdot \nabla(\zeta^2 u^n) = 2\int \zeta^2 u^n \cdot u_x^n \wedge u_y^n + o(1).$$

Therefore, using (19) we find that

$$-\int |\nabla(\zeta u^n)|^2 = 2\int u^n \cdot (\zeta u^n)_x \wedge (\zeta u^n)_y + o(1).$$

We deduce from Lemma 2 and (18) that

$$\begin{aligned}\int |\nabla(\zeta u^n)|^2 &\leqq 2c_0\,\|\nabla u^n\|_{L^2}\,\|\nabla(\zeta u^n)\|_{L^2}^2 + o(1)\\ &\leqq 2c_0\mu_0\,\|\nabla(\zeta u^n)\|_{L^2}^2 + o(1).\end{aligned}$$

Hence

$$\int \zeta^2\,|\nabla u^n|^2 = o(1).$$

Proof of Lemma 1. As in [13] and [14] we introduce the concentration functions

$$Q_n(t) = \operatorname*{Sup}_{z\in\mathbb{R}^2} \int_{z+t\Omega} |\nabla u^n|^2 \quad \text{for } t \geqq 0.$$

Each function $Q_n(t)$ is continuous and non-decreasing in t, and $Q_n(0) = 0$, $Q_n(1) = Q_n(\infty) = \int_\Omega |\nabla u^n|^2 \geqq \alpha$.

We fix a constant ν such that

(21) $$0 < \nu < \operatorname{Min}\{1/4c_0^2, \alpha\}.$$

There exists some $0 < \varepsilon_n < 1$ such that $Q_n(\varepsilon_n) = \nu$ and there exists some $a_n \in \bar{\Omega}$ such that

$$Q_n(\varepsilon_n) = \int_{a_n+\varepsilon_n\Omega} |\nabla u^n|^2 = \nu.$$

Set $\tilde{u}^n(z) = u^n(\varepsilon_n z + a_n)$; by (3) and (12) we have

(22) $$\int_{\mathbb{R}^2} |\nabla \tilde{u}^n|^2 = \int_{\mathbb{R}^2} |\nabla u^n|^2 \leqq C$$

(23) $$\|\tilde{u}^n\|_{L^\infty(\mathbb{R}^2)} = \|u^n\|_{L^\infty(\mathbb{R}^2)} \leqq C.$$

Therefore we may assume that[1]

(24) $$\tilde{u}^n \to \omega \quad \text{a.e. on } \mathbb{R}^2$$

(25) $$\nabla \tilde{u}^n \rightharpoonup \nabla \omega \text{ weakly in } L^2(\mathbb{R}^2).$$

Let $\Omega_n = (1/\varepsilon^n)(\Omega - a_n)$, so that $\Omega_n \to U$. We now distinguish several cases.

Case (a). $\varepsilon_n \to l > 0$,
Case (b). $\varepsilon_n \to 0$ and $(1/\varepsilon_n) \operatorname{dist}(a_n, \partial\Omega) \to m < \infty$, so that U is a half-plane,
Case (c). $\varepsilon_n \to 0$ and $(1/\varepsilon_n) \operatorname{dist}(a_n, \partial\Omega) \to \infty$, so that $U = \mathbb{R}^2$.

We shall establish that cases (a) and (b) cannot occur (a similar phenomenon appears in [1]).

Let θ^n be the solution of

$$\Delta\theta^n = f^n \quad \text{on } \Omega, \qquad \theta^n = 0 \quad \text{on } \partial\Omega,$$

so that $\theta^n \to 0$ in $H_0^1(\Omega)$. We have

$$\Delta(u^n - \theta^n) = 2u_x^n \wedge u_y^n \quad \text{on } \Omega,$$

and thus

(26) $$\Delta(\tilde{u}^n - \tilde{\theta}^n) = 2\tilde{u}_x^n \wedge \tilde{u}_y^n \quad \text{on } \Omega_n,$$

where $\tilde{\theta}^n(z) = \theta^n(\varepsilon_n z + a_n)$. Note that

$$\int_{\Omega_n} |\nabla \tilde{\theta}^n|^2 = \int_{\Omega} |\nabla \theta^n|^2 = o(1).$$

Hence passing to the limit in (26) we obtain

(27) $$\Delta\omega = 2\omega_x \wedge \omega_y \quad \text{on } U$$

and moreover

(28) $$\omega = 0 \quad \text{on } \partial U.$$

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[1] This is valid only for a subsequence; we shall however often extract subsequences without explicitly mentioning this fact.

Suppose that we are in Case (a). We recall that $u^n \rightharpoonup 0$ weakly in $H_0^1(\Omega)$; thus $u^n \to 0$ strongly in $L^2(\Omega)$ and $\int\limits_{\Omega_n} |\tilde{u}^n|^2 \to 0$. We claim that

$$\nabla \tilde{u}^n \to 0 \quad \text{strongly in } L^2(\mathbb{R}^2). \tag{29}$$

This is impossible, however, since

$$\int\limits_{\Omega} |\nabla \tilde{u}^n|^2 = \int\limits_{a_n+\varepsilon_n\Omega} |\nabla u^n|^2 = \nu > 0,$$

and thus Case (a) is excluded.

In order to establish (29) it suffices to prove that

$$\int \zeta^2 \, |\nabla \tilde{u}^n|^2 = o(1) \tag{30}$$

for all $\zeta \in \mathscr{D}(\mathbb{R}^2)$ with $\operatorname{supp} \zeta \subset z + \Omega$ for some $z \in \mathbb{R}^2$. Fix such a ζ. Multiplying (26) through by $\zeta^2 \tilde{u}^n$ we find

$$\int\limits_{\Omega_n} |\nabla(\zeta\tilde{u}^n)|^2 = -2 \int\limits_{\Omega_n} \tilde{u}^n \cdot (\zeta\tilde{u}^n)_x \wedge (\zeta\tilde{u}^n)_y + o(1). \tag{31}$$

We deduce from Lemma 2 and (31) that

$$\begin{aligned} \int\limits_{\Omega_n} |\nabla(\zeta\tilde{u}^n)|^2 &\leqq 2c_0 \, \|\nabla \tilde{u}^n\|_{L^2(z+\Omega)} \int\limits_{\Omega_n} |\nabla(\zeta\tilde{u}^n)|^2 + o(1) \\ &\leqq 2c_0 \sqrt{\nu} \int\limits_{\Omega_n} |\nabla(\zeta\tilde{u}^n)|^2 + o(1). \end{aligned}$$

Since $2c_0 \sqrt{\nu} < 1$ we obtain (30), and hence Case (a) is excluded.

Suppose that we are in Case (b). We deduce from (27) and (28) that $\omega = 0$; this is WENTE's result [27] (WENTE considers the case where U is a disk, but the case where U is a half-plane may be deduced from the case of a disk by a conformal diffeomorphism). Therefore using (24) and (25) we have

$$\tilde{u}^n \to 0 \text{ a.e. on } \mathbb{R}^2$$

$$\nabla \tilde{u}^n \rightharpoonup 0 \text{ weakly in } L^2(\mathbb{R}^2).$$

Exactly as in Case (a) we can prove that

$$\nabla \tilde{u}^n \to 0 \quad \text{strongly in } L^2_{\text{loc}}(\mathbb{R}^2). \tag{32}$$

However this is impossible since

$$\int\limits_{\Omega} |\nabla \tilde{u}^n|^2 = \int\limits_{a_n+\varepsilon_n\Omega} |\nabla u^n|^2 = \nu > 0.$$

Therefore Case (b) is excluded.

Hence the only case which occurs is Case (c). In order to conclude the proof of Lemma 1 we have only to show that ω is not a constant. We claim that

$$\nabla \tilde{u}^n \to \nabla \omega \quad \text{strongly in } L^2_{\text{loc}}(\mathbb{R}^2). \tag{33}$$

Since on the other hand, we know that

$$\int_\Omega |\nabla \tilde{u}^n|^2 = \nu > 0$$

it follows that $\int_\Omega |\nabla \omega|^2 = \nu > 0$; therefore ω is not a constant.

It remains to prove (33). Fix any $z \in \mathbb{R}^2$; since we are in Case (c), we have $\{z + \Omega\} \subset \Omega_n$ for n large enough. Therefore we may apply Lemma 3 to the sequence $(\tilde{u}^n)$ restricted to $\{z + \Omega\}$. It follows that $\tilde{u}^n \to \omega$ strongly in $H^1_{\text{loc}}(\{z + \Omega\})$ and therefore $\nabla \tilde{u}^n \to \nabla \omega$ strongly in $L^2_{\text{loc}}(\mathbb{R}^2)$.

The proof of Theorem 1 consists of iterating the construction of Lemma 1. Our next lemma explains how to carry out this iteration.

Lemma 4. *Assume (u^n) and ω, (a_n), (ε_n), are as in Lemma 1. Set*

$$\omega^n(z) = \omega\left(\frac{z - a_n}{\varepsilon_n}\right)$$

and let h^n be the solution of

$$\begin{cases} \Delta h^n = 0 & \text{on } \Omega \\ h^n = \omega^n & \text{on } \partial\Omega. \end{cases}$$

Set $v^n = u^n - \omega^n + h^n$. Then v^n satisfies

$$\begin{cases} \Delta v^n = 2 v_x^n \wedge v_y^n + k^n & \text{on } \Omega \\ v^n = 0 & \text{on } \partial\Omega \end{cases} \tag{34}$$

with

$$k^n \to 0 \quad \text{strongly in } H^{-1}(\Omega) \tag{35}$$

$$\int_\Omega |\nabla v^n|^2 = \int_\Omega |\nabla u^n|^2 - \int_{\mathbb{R}^2} |\nabla \omega|^2 + o(1) \tag{36}$$

$$\|v^n\|_{L^\infty(\Omega)} \leqq C. \tag{37}$$

It is now clear how to prove Theorem 1 with the help of Lemma 1 and Lemma 4, namely:

Proof of Theorem 1 concluded. First note that if (u^n) satisfies (1), (2), (12), and in addition

$$\int |\nabla u^n|^2 \leqq C < 8\pi \quad \forall n, \tag{38}$$

then, in fact

$$\int |\nabla u^n|^2 = o(1). \tag{39}$$

[Indeed if (39) fails, then (13) holds for some $\alpha > 0$. Applying Lemma 1 and Lemma 4 we see that

$$0 \leqq \int_\Omega |\nabla v^n|^2 = \int_\Omega |\nabla u^n|^2 - \int_{\mathbb{R}^2} |\nabla \omega|^2 + o(1) \leqq C - 8\pi + o(1),$$

which is impossible.]

Suppose now that (u^n) satisfies (1), (2), (12) and

$$\int |\nabla u^n|^2 \geqq 8\pi + o(1).$$

We iterate the constructions of Lemma 1 and Lemma 4 until the iterated function satisfies (38). This requires only a finite number of steps—in fact at most $\sup_n (1/8\pi) \int |\nabla u^n|^2$—and leads to the results

$$\left\| u^n - \sum_{i=1}^p \omega^i\left(\frac{\cdot - a_n^i}{\varepsilon_n^i}\right) + \sum_{i=1}^p h_i^n \right\|_{H^1} \xrightarrow[n\to\infty]{} 0 \tag{40}$$

and

$$\int_\Omega |\nabla u^n|^2 = \sum_{i=1}^p \int_{\mathbb{R}^2} |\nabla \omega^i|^2 + o(1). \tag{41}$$

Finally from Lemma A.2 in the Appendix

$$\|h_i^n - \omega^i(\infty)\|_{H^1} \to 0 \quad \forall\, i,$$

and so from (40) follows

$$\left\| u^n - \sum_{i=1}^p \omega^i\left(\frac{\cdot - a_n^i}{\varepsilon_n^i}\right) + \sum_{i=1}^p \omega^i(\infty) \right\|_{H^1} \to 0.$$

The conclusion of Theorem 1 follows if we replace ω^i by $\omega^i - \omega^i(\infty)$. It remains therefore only to prove Lemma 4:

Proof of Lemma 4. First we recall from Lemma A.2 in the Appendix that

$$\|h^n - \omega(\infty)\|_{H^1(\Omega)} \to 0,$$

and in particular $\int_\Omega |\nabla h^n|^2 = o(1)$. Next, we have

$$\begin{aligned} \Delta v^n &= 2u_x^n \wedge u_y^n + f^n - 2\omega_x^n \wedge \omega_y^n \\ &= 2(v^n + \omega^n - h^n)_x \wedge (v^n + \omega^n - h^n)_y + f^n - 2\omega_x^n \wedge \omega_y^n \\ &\equiv 2v_x^n \wedge v_y^n + f^n + \Delta\varphi^n + \Delta\psi^n, \end{aligned}$$

where φ^n and ψ^n are respectively the solutions of

$$\begin{cases} \Delta\varphi^n = 2[(u_x^n - \omega_x^n) \wedge \omega_y^n + \omega_x^n \wedge (u_y^n - \omega_y^n)] & \text{on } \Omega \\ \varphi^n = 0 & \text{on } \partial\Omega \end{cases}$$

and

$$\begin{cases} \Delta\psi^n = -2(h_x^n \wedge v_y^n + v_x^n \wedge h_y^n - h_x^n \wedge h_y^n) & \text{on } \Omega \\ \psi^n = 0 \quad \text{on } \partial\Omega . \end{cases}$$

Using Lemma A.1 from [2] we see that

$$\|\nabla\psi^n\|_{L^2}^2 \leqq C\,\|\nabla h^n\|_{L^2}\,(\|\nabla v^n\|_{L^2} + \|\nabla h^n\|_{L^2}) = o(1). \tag{42}$$

On the other hand, the φ^n term can be treated by applying Lemma A.3 in the Appendix. Note here that $\alpha^n = u^n - \omega^n$ satisfies

$$\tilde{\alpha}^n(z) = \alpha^n(\varepsilon_n z + a_n) = \tilde{u}^n(z) - \omega(z) \to 0 \quad \text{a.e. on } \mathbb{R}^2$$

and thus

$$\int_\Omega |\nabla\varphi^n|^2 = o(1). \tag{43}$$

Hence (v^n) satisfies (34) and (35).

Finally we prove (36); indeed we have

$$\begin{aligned} \int_\Omega |\nabla v^n|^2 &= \int_\Omega |\nabla u^n|^2 - 2\int_\Omega \nabla u^n\,\nabla\omega^n + \int_\Omega |\nabla\omega^n|^2 + o(1) \\ &= \int_\Omega |\nabla u^n|^2 - 2\int_{\Omega_n} \nabla\tilde{u}^n\,\nabla\omega + \int_{\Omega_n} |\nabla\omega|^2 + o(1) \\ &= \int_\Omega |\nabla u^n|^2 - \int_{\mathbb{R}^2} |\nabla\omega|^2 + o(1) \end{aligned}$$

since $\nabla\tilde{u}^n \rightharpoonup \nabla\omega$ weakly in $L^2(\mathbb{R}^2)$ by (15). This completes the proof.

Remark 1. Given $\varphi \in L^\infty(\mathbb{R}^2)$ with $\nabla\varphi \in L^2(\mathbb{R}^2)$ we set

$$Q(\varphi) = \int_{\mathbb{R}^2} \varphi \cdot \varphi_x \wedge \varphi_y .$$

Similarly if $\varphi \in L^\infty(\Omega) \cap H_0^1(\Omega)$ we also set

$$Q(\varphi) = \int_\Omega \varphi \cdot \varphi_x \wedge \varphi_y$$

(When $\varphi \in H_0^1(\Omega)$ (and $\varphi \notin L^\infty(\Omega)$) it still makes sense to consider $Q(\varphi)$; the precise meaning of Q is explained in [2]). We claim that under the assumptions of Theorem 1 we have

$$Q(u^n) = \sum_{i=1}^{p} Q(\omega^i) + o(1). \tag{44}$$

This has the following implication for the functional E defined on H_0^1 by

$$E(u) = \int_\Omega |\nabla u|^2 + \tfrac{4}{3}\,Q(u).$$

Suppose that (u^n) is a sequence in H_0^1 which satisfies the (PS) condition, namely

$$E'(u^n) \to 0 \quad \text{in } H^{-1}, \tag{45}$$

$$E(u^n) \to c, \tag{46}$$

Then $c = 8\pi k/3$ for some integer $k \geqq 0$.

Indeed we deduce from (45) that

$$\Delta u^n = 2u_x^n \wedge u_y^n + f^n \quad \text{on } \Omega$$

with $f^n \to 0$ in H^{-1}. Moreover

$$-\int_\Omega |\nabla u^n|^2 = 2Q(u^n) + \langle f^n, u^n\rangle$$

and thus

$$E(u^n) = \int_\Omega |\nabla u^n|^2 + \tfrac{4}{3} Q(u^n) = \tfrac{1}{3} \int_\Omega |\nabla u^n|^2 - \tfrac{2}{3} \langle f^n, u^n\rangle = c + o(1).$$

Hence $\int |\nabla u^n|^2 \leqq C$. Applying (8) and (44) we obtain

$$E(u^n) = \sum_{i=1}^{p} \int_{\mathbf{R}^2} |\nabla \omega^i|^2 + \tfrac{4}{3} \sum_{i=1}^{p} Q(\omega^i) + o(1) = \tfrac{1}{3} \sum_{i=1}^{p} \int_{\mathbf{R}^2} |\nabla \omega^i|^2 + o(1).$$

Using (6) and (46) we see that $c = 8\pi k/3$ for some integer $k \geqq 0$.

Proof of (44). Using the notation of Lemma 4, we claim that

$$Q(v^n) = Q(u^n - \omega^n + h^n) = Q(u^n) - Q(\omega) + o(1). \tag{47}$$

Indeed we write (see Lemma A.11 in [2]):

$$Q(v^n) = Q(u^n) + Q(-\omega^n + h^n) + 3 \int_\Omega u^n \cdot (-\omega_x^n + h_x^n) \wedge (-\omega_y^n + h_y^n)$$
$$+ 3 \int_\Omega (-\omega^n + h^n) \cdot u_x^n \wedge u_y^n.$$

By Lemma A.2 in the Appendix we have

$$\|h^n - \omega(\infty)\|_{H^1(\Omega)} \to 0.$$

Using also the fact

$$\|h^n - \omega(\infty)\|_{L^\infty(\Omega)} \to 0,$$

we see easily that

$$Q(-\omega^n + h^n) = -Q(\omega) + o(1),$$

$$\int_\Omega u^n \cdot (-\omega_x^n + h_x^n) \wedge (-\omega_y^n + h_y^n) = \int_{\Omega_n} \tilde{u}^n \cdot \omega_x \wedge \omega_y + o(1)$$
$$= \int_{\mathbf{R}^2} \omega \cdot \omega_x \wedge \omega_y + o(1)$$

and

$$\int_{\Omega} (-\omega^n + h^n) \cdot u_x^n \wedge u_y^n = - \int_{\Omega_n} \omega \cdot \tilde{u}_x^n \wedge \tilde{u}_y^n + o(1)$$
$$= - \int_{\mathbb{R}^2} \omega \cdot \omega_x \wedge \omega_y + o(1).$$

This completes the proof of (47), and (44) is an easy consequence of (47).

2. An additional property of the speeds of concentration

In case Theorem 1 leads to more than one ω the following additional information is very useful.

Theorem 2. *Assume* (u^n), ω^i, (a_n^i), (ε_n^i) *are as in Theorem 1. Then we have*

$$\text{(48)} \qquad \operatorname{Max}\left\{\frac{\varepsilon_n^i}{\varepsilon_n^j}, \frac{\varepsilon_n^j}{\varepsilon_n^i}, \frac{|a_n^i - a_n^j|}{\varepsilon_n^i + \varepsilon_n^j}\right\} \xrightarrow[n\to\infty]{} \infty \quad \forall i \neq j.$$

Remark 2. Property (48) can be understood by considering two extreme cases. Suppose first that $a_n^i \xrightarrow[n\to\infty]{} a^i$ and $a_n^j \xrightarrow[n\to\infty]{} a^j$ with $a^i \neq a^j$; then (48) clearly holds. This means that *the functions*

$$\omega_n^i = \omega^i\left(\frac{\cdot - a_n^i}{\varepsilon_n^i}\right) \quad \textit{and} \quad \omega_n^j = \omega^j\left(\frac{\cdot - a_n^j}{\varepsilon_n^j}\right)$$

concentrate at two different points and thus their supports become "essentially disjoint". On the other extreme, suppose that $a_n^i \equiv a_n^j \equiv a$. From (48) either $\varepsilon_n^i/\varepsilon_n^j \xrightarrow[n\to\infty]{} \infty$ or $\varepsilon_n^j/\varepsilon_n^i \xrightarrow[n\to\infty]{} \infty$. This means that *the functions* ω_n^i *and* ω_n^j *concentrate at the same point, but the speeds of concentration are very different.*

Remark 3. It follows easily from (48) that if $i \neq j$ the functions ω_n^i and ω_n^j are "almost orthogonal" in H^1. More precisely we have

$$\int_{\Omega} |\omega_n^i|\,|\omega_n^j| + \int_{\Omega} |\nabla\omega_n^i|\,|\nabla\omega_n^j| = o(1) \quad \forall i \neq j.$$

Remark 4. The "converse" of Theorems 1 and 2 holds. Namely, let $\omega^1, \omega^2, \dots \omega^p$ be a finite number of solutions of (5), let $(a_n^1), (a_n^2), \dots (a_n^p)$ be sequences in Ω, and let $(\varepsilon_n^1), (\varepsilon_n^2), \dots (\varepsilon_n^p)$ be sequences with $\varepsilon_n^i > 0$ $(\forall i, \forall n)$ and $\lim_{n\to\infty} \varepsilon_n^i = 0$ $(\forall i)$. Assume that (48) holds and set

$$u^n = \sum_{i=1}^{p} \omega^i\left(\frac{\cdot - a_n^i}{\varepsilon_n^i}\right) + \Theta^n$$

with $\Theta^n \to 0$ strongly in H^1. Then u^n satisfies

$$\Delta u^n = 2u_x^n \wedge u_y^n + f^n \quad \text{on } \Omega$$

with $f^n \to 0$ strongly in H^{-1}. This may be proved easily with the help of Lemma A.3 from the Appendix.

The proof of Theorem 2 relies on the following

Lemma 5. *Let* ω *and* $\omega_1, \omega_2, \dots \omega_k$ *be a finite number of solutions of* (5). *Assume that*

$$\omega = \sum_{i=1}^{k} \omega_i. \tag{49}$$

Then

$$\int_{\mathbb{R}^2} |\nabla \omega|^2 \leqq \sum_{i=1}^{k} \int_{\mathbb{R}^2} |\nabla \omega_i|^2. \tag{50}$$

Moreover if equality holds in (50), *that is, if*

$$\int_{\mathbb{R}^2} |\nabla \omega|^2 = \sum_{i=1}^{k} \int_{\mathbb{R}^2} |\nabla \omega_i|^2, \tag{51}$$

then each ω_i *is a constant, with the possible exception of one of them.*

Proof. We write (see Lemma A.1 in the Appendix)

$$\omega = \pi\left(\frac{P}{Q}\right) + C \equiv \overline{\omega} + C, \quad \omega_i = \pi\left(\frac{P_i}{Q_i}\right) + C_i \equiv \overline{\omega}_i + C_i.$$

Thus by (49)

$$\overline{\omega} = \sum_{i=1}^{k} \overline{\omega}_i + \overline{C}$$

where $\overline{C} = \sum_i C_i - C$. On the other hand (see the proof of Lemma A.1)

$$-\Delta\overline{\omega} = \overline{\omega}\,|\nabla\overline{\omega}|^2 \quad \text{on } \mathbb{R}^2$$

$$-\Delta\overline{\omega}_i = \overline{\omega}_i\,|\nabla\overline{\omega}_i|^2 \quad \text{on } \mathbb{R}^2 \quad \forall i,$$

and thus

$$\overline{\omega}\,|\nabla\overline{\omega}|^2 = \sum_{i=1}^{k} \overline{\omega}_i\,|\nabla\overline{\omega}_i|^2.$$

Forming the scalar product with $\overline{\omega}$ and using the fact that $|\overline{\omega}| = 1$, we obtain

$$\int_{\mathbb{R}^2} |\nabla\overline{\omega}|^2 = \sum_{i=1}^{k} \int_{\mathbb{R}^2} \overline{\omega}\cdot\overline{\omega}_i\,|\nabla\overline{\omega}_i|^2.$$

But

$$\overline{\omega}\cdot\overline{\omega}_i = 1 - \tfrac{1}{2}|\overline{\omega} - \overline{\omega}_i|^2 \quad (\text{since } |\overline{\omega}| = |\overline{\omega}_i| = 1),$$

and hence we find

$$\int_{\mathbb{R}^2} |\nabla\overline{\omega}|^2 = \sum_{i=1}^{k} \int_{\mathbb{R}^2} |\nabla\overline{\omega}_i|^2 - \tfrac{1}{2}\int_{\mathbb{R}^2} \sum_{i=1}^{k} |\overline{\omega} - \overline{\omega}_i|^2\,|\nabla\overline{\omega}_i|^2. \tag{52}$$

This proves (50). Suppose now that (51) holds. For each i we have (using (52))

$$|\overline{\omega} - \overline{\omega}_i|\,|\nabla\overline{\omega}_i| = 0 \text{ a.e.}$$

If $\overline{\omega}_i$ is not a constant, then $\nabla\overline{\omega}_i \neq 0$ everywhere except possibly at a finite number of points (see the proof of Lemma A.1); thus $\overline{\omega}_i = \overline{\omega}$. This implies the conclusion of Lemma 5.

Proof of Theorem 2. We introduce the following equivalence relation on the integers $1 \leqq i \leqq p$, $1 \leqq j \leqq p$, namely

$$i \sim j \text{ if and only if } \operatorname{Max}\left\{\frac{\varepsilon_n^i}{\varepsilon_n^j}, \frac{\varepsilon_n^j}{\varepsilon_n^i}, \frac{|a_n^i - a_n^j|}{\varepsilon_n^i + \varepsilon_n^j}\right\} \text{ remains bounded as } n \to \infty.$$

Denote the corresponding equivalence classes by $I_1, I_2, \dots I_l$. We shall prove that each equivalence class contains precisely one element, which is exactly the assertion of Theorem 2. We break the proof into four steps.

Step 1. We claim that

$$\int_{\mathbb{R}^2} |\nabla\omega_n^i|\,|\nabla\omega_n^j| = o(1) \quad \text{if } i \text{ and } j \text{ are not equivalent.} \tag{53}$$

Indeed

$$\int_{\mathbb{R}^2} |\nabla\omega_n^i|\,|\nabla\omega_n^j| = \int_{\mathbb{R}^2} |\nabla\omega^i|\,\varphi^n \tag{54}$$

where

$$\varphi^n(z) = \frac{\varepsilon_n^i}{\varepsilon_n^j}\left|\nabla\omega^j\left(\frac{\varepsilon_n^i z + a_n^i - a_n^j}{\varepsilon_n^j}\right)\right|.$$

Since i and j are not equivalent we may assume that *either*

$$\varepsilon_n^i/\varepsilon_n^j \xrightarrow[n\to\infty]{} 0$$

or

$$\varepsilon_n^i/\varepsilon_n^j \xrightarrow[n\to\infty]{} l \text{ with } 0 < l < \infty \text{ and } |a_n^i - a_n^j|/\varepsilon_n^j \xrightarrow[n\to\infty]{} \infty.$$

The sequence (φ^n) is bounded in $L^2(\mathbb{R}^2)$ and moreover (in both cases) $\varphi^n \to 0$ a.e. Therefore $\varphi^n \rightharpoonup 0$ weakly in L^2. The required conclusion thus follows from (54).

Step 2. When $i \sim j$ we introduce the expressions

$$l_{ij} = \lim_{n\to\infty} \varepsilon_n^i/\varepsilon_n^j \quad \text{and} \quad p_{ij} = \lim_{n\to\infty} (a_n^i - a_n^j)/\varepsilon_n^j.$$

For *each* equivalence class I we *fix* some $i \in I$ and set

$$\omega_I(z) = \sum_{j\in I} \omega^j(l_{ij} z + p_{ij}), \quad z \in \mathbb{R}^2.$$

We claim that for each equivalence class

$$\int_\Omega \Big|\sum_{j\in I} \nabla\omega_n^j\Big|^2 = \int_{\mathbb{R}^2} |\nabla\omega_I|^2 + o(1). \tag{55}$$

Indeed

$$\begin{aligned}\int_\Omega \Big|\sum_{j\in I} \nabla\omega_n^j\Big|^2 &= \int_\Omega \Big|\nabla\omega_n^i + \sum_{j\in I, j\neq i} \nabla\omega_n^j\Big|^2 \\ &= \int_{\Omega_n} \Big|\nabla\omega^i(z) + \sum_{j\in I, j\neq i} \frac{\varepsilon_n^i}{\varepsilon_n^j} \nabla\omega^j\left(\frac{\varepsilon_n^i z + a_n^i - a_n^j}{\varepsilon_n^j}\right)\Big|^2,\end{aligned}$$

where $\Omega_n = \dfrac{\Omega - a_n^i}{\varepsilon_n^i}$, and also

$$\frac{\varepsilon_n^i}{\varepsilon_n^j} \nabla\omega^j\left(\frac{\varepsilon_n^i z + a_n^i - a_n^j}{\varepsilon_n^j}\right) \xrightarrow[n\to\infty]{} l_{ij}\, \nabla\omega^j(l_{ij} z + p_{ij})$$

strongly in $L^2(\mathbb{R}^2)$. Thus (55) holds.

Step 3. We claim that ω_I satisfies (5) for each equivalence class I. Set

$$\Theta^n = u^n - \sum_{j=1}^{p} \omega_n^j \quad \text{on } \mathbb{R}^2$$

(recall that u^n has been extended by 0 outside Ω), so that

$$\int_{\mathbb{R}^2} |\nabla\Theta^n|^2 = o(1) \quad \text{and} \quad \|\Theta^n\|_{L^\infty} \leqq C.$$

Fix $i \in I$ as in Step 2 and set

$$\tilde{u}^n(z) = u^n(\varepsilon_n^i z + a_n^i) = \omega^i(z) + \sum_{j\neq i} \omega^j\left(\frac{\varepsilon_n^i z + a_n^i - a_n^j}{\varepsilon_n^j}\right) + \tilde{\Theta}^n(z) \tag{56}$$

where $\tilde{\Theta}^n(z) = \Theta^n(\varepsilon_n^i z + a_n^i)$. As in the proof of Lemma 1 we have

$$\tilde{u}^n(z) \to \omega(z) \text{ a.e. on } \mathbb{R}^2$$
$$\nabla\tilde{u}^n \rightharpoonup \nabla\omega \quad \text{weakly in } L^2(\mathbb{R}^2),$$

and of course ω satisfies (5). On the other hand

$$\int_{\mathbb{R}^2} |\nabla\tilde{\Theta}^n|^2 = o(1) \quad \text{and} \quad \|\tilde{\Theta}^n\|_{L^\infty} \leqq C$$

and thus $\tilde{\Theta}^n \to C$ a.e. on $\mathbb{R}^2$, where C is a constant. Finally we observe that if $j \notin I$, then

$$\omega^j\left(\frac{\varepsilon_n^i z + a_n^i - a_n^j}{\varepsilon_n^j}\right) \to C_{ij} \quad \text{a.e. on } \mathbb{R}^2$$

for some constant C_{ij}.

Passing to the limit in (56), we obtain $\omega = \omega_I + C$ where C is a constant. Hence ω_I satisfies (5).

Step 4. Proof of Theorem 2 concluded. We deduce from Lemma 5 and Step 3 that for each equivalence class I we have

$$\int_{\mathbb{R}^2} |\nabla \omega_I|^2 \leqq \sum_{j\in I} \int_{\mathbb{R}^2} |\nabla \omega^j|^2. \tag{57}$$

Moreover equality holds if and only if I is reduced to a single element (recall that each ω^j is nonconstant). We deduce from Step 1 that

$$\int_\Omega |\nabla u^n|^2 = \sum_{q=1}^{l} \int_\Omega \left|\sum_{j\in I_q} \nabla \omega_n^j\right|^2 + o(1),$$

and using (55) we find

$$\int_\Omega |\nabla u^n|^2 = \sum_{q=1}^{l} \int_{\mathbb{R}^2} |\nabla \omega_{I_q}|^2 + o(1). \tag{58}$$

On the other hand, by (8),

$$\int_\Omega |\nabla u^n|^2 = \sum_{i=1}^{p} \int_{\mathbb{R}^2} |\nabla \omega^i|^2 + o(1). \tag{59}$$

Combining (57), (58) and (59) we see that equality holds in (57) for each equivalence class I.

3. Convergence in the L^∞ norm

The main result of Section 3 is the following

Theorem 3. *Let (u^n) be a sequence in H^1 satisfying*

$$\begin{cases} \Delta u^n = 2u_x^n \wedge u_y^n & \text{on } \Omega \\ u^n = \gamma^n & \text{on } \partial\Omega, \end{cases} \tag{60}$$

with

$$\|\gamma^n\|_{L^\infty(\partial\Omega)} \xrightarrow[n\to\infty]{} 0 \tag{61}$$

and

$$\int_\Omega |\nabla u^n|^2 \leqq C. \tag{62}$$

Then either $\|u^n\|_{L^\infty(\Omega)} \xrightarrow[n\to\infty]{} 0$ *or there exist* ω^i (a_n^i), (ε_n^i) *as in Theorem 1 such that*

$$\left\| u^n - \sum_{i=1}^{p} \omega^i \left(\frac{\cdot - a_n^i}{\varepsilon_n^i}\right)\right\|_{L^\infty} \xrightarrow[n\to\infty]{} 0. \tag{63}$$

Moreover (9) and (48) hold.

Proof. Let $\underline{u}^n$ denote the "small" solution of the problem

$$(64) \qquad \begin{cases} \Delta \underline{u}^n = 2\underline{u}^n_x \wedge \underline{u}^n_y & \text{on } \Omega \\ \underline{u}^n = \gamma^n & \text{on } \partial\Omega, \end{cases}$$

so that (see [2] or [8]) we have

$$(65) \qquad \|\underline{u}^n\|_{L^\infty(\Omega)} \leqq \|\gamma^n\|_{L^\infty(\partial\Omega)} = o(1).$$

From (61), (62) and the construction of $\underline{u}^n$ it follows that $\|\underline{u}^n\|_{H^1} \leqq C$. Set $v^n = u^n - \underline{u}^n$ so that $v^n \in H^1_0$ and

$$(66) \qquad \begin{aligned} \Delta v^n &= 2(v^n_x + \underline{u}^n_x) \wedge (v^n_y + \underline{u}^n_y) - 2\underline{u}^n_x \wedge \underline{u}^n_y \\ &\equiv 2v^n_x \wedge v^n_y + f^n, \end{aligned}$$

where $f^n = 2(v^n_x \wedge \underline{u}^n_y + \underline{u}^n_x \wedge v^n_y)$. We claim that $f^n \to 0$ in H^{-1}. Indeed let φ^n be the solution of

$$\Delta \varphi^n = f^n \quad \text{on } \Omega, \qquad \varphi^n = 0 \quad \text{on } \partial\Omega.$$

Using Lemma A.4 in [2], we have

$$-\int_\Omega |\nabla \varphi^n|^2 = \int_\Omega f^n \cdot \varphi^n = 2\int_\Omega \underline{u}^n \cdot (v^n_x \wedge \varphi^n_y + \varphi^n_x \wedge v^n_y),$$

and thus

$$\int_\Omega |\nabla \varphi^n|^2 = o(1).$$

Theorem 1 applied to the sequence (v^n) asserts that either $\int |\nabla v^n|^2 \xrightarrow[n\to\infty]{} 0$ or there exist ω^i, (a^i_n) and (ε^i_n) such that

$$\left\| v^n - \sum_{i=1}^p \omega^i \left(\frac{\cdot - a^i_n}{\varepsilon^i_n} \right) \right\|_{H^1} \xrightarrow[n\to\infty]{} 0.$$

In the first case, we deduce from (66) and Lemma A.1 in [2] that

$$\|v^n\|_{L^\infty} \leqq C\|\nabla v^n\|^2_{L^2} + C\|\nabla v^n\|_{L^2}\|\nabla \underline{u}^n\|_{L^2} = o(1),$$

and therefore $\|u^n\|_{L^\infty} \to 0$.

In the second case we set $R^n = v^n - \sum_{i=1}^p \omega^i_n$ so that

$$(67) \qquad \|R^n\|_{H^1} \xrightarrow[n\to\infty]{} 0.$$

We claim that

$$(68) \qquad \|R^n\|_{L^\infty} \xrightarrow[n\to\infty]{} 0$$

(the relation (68) clearly implies (63) and so will complete the proof). Indeed we write

$$\begin{aligned} \Delta R^n &= 2\left[v^n_x \wedge v^n_y + v^n_x \wedge \underline{u}^n_y + \underline{u}^n_x \wedge v^n_y - \sum_i (\omega^i_n)_x \wedge (\omega^i_n)_y \right] \\ &\equiv A^n + B^n + C^n, \end{aligned}$$

where

$$A^n = 2\left[R_x^n \wedge R_y^n + R_x^n \wedge \left(\underline{u}^n + \sum_i \omega_n^i\right)_y + \left(\underline{u}^n + \sum_i \omega_n^i\right)_x \wedge R_y^n\right],$$

$$B^n = 2\sum_i \underline{u}_x^n \wedge (\omega_n^i)_y + (\omega_n^i)_x \wedge \underline{u}_y^n,$$

$$C^n = \sum_{i \neq j} (\omega_n^i)_x \wedge (\omega_n^j)_y + (\omega_n^j)_x \wedge (\omega_n^i)_y.$$

Introduce U^n, V^n and W^n respectively as the solutions of the problems

$$\Delta U^n = A^n \quad \text{on } \Omega, \qquad U^n = 0 \quad \text{on } \partial\Omega,$$

$$\Delta V^n = B^n \quad \text{on } \Omega, \qquad V^n = 0 \quad \text{on } \partial\Omega,$$

$$\Delta W^n = C^n \quad \text{on } \Omega, \qquad W^n = 0 \quad \text{on } \partial\Omega.$$

From (67) and Lemma A.1 in [2]

$$\|U^n\|_{L^\infty} \leqq C\,\|\nabla R^n\|_{L^2}\left(\|\nabla R^n\|_{L^2} + \|\nabla \underline{u}^n\|_{L^2} + \sum_i \|\nabla\omega^i\|_{L^2}\right) = o(1).$$

Also from (65) and Lemma A.3 in the Appendix

$$\|V^n\|_{L^\infty} = o(1).$$

Using Lemma A.3 again we see that

$$\|W^n\|_{L^\infty} = o(1).$$

Indeed observe that if $i \neq j$ then by Theorem 2

$$\omega_n^j(\varepsilon_n^i z + a_n^i) = \omega^j\left(\frac{\varepsilon_n^i z + a_n^i - a_n^j}{\varepsilon_n^j}\right) \to C_{ij} \quad \text{a.e. on } \mathbb{R}^2.$$

Finally note that

$$\begin{cases} \Delta(R^n - U^n - V^n - W^n) = 0 & \text{on } \Omega \\ R^n - U^n - V^n - W^n = -\sum \omega_n^i & \text{on } \partial\Omega, \end{cases}$$

and thus

$$\|R^n - U^n - V^n - W^n\|_{L^\infty(\Omega)} \leqq \sum_i \|\omega_n^i\|_{L^\infty(\partial\Omega)}.$$

Therefore we have

$$\|R^n\|_{L^\infty(\Omega)} \leqq \|U^n\|_{L^\infty(\Omega)} + \|V^n\|_{L^\infty(\Omega)} + \|W^n\|_{L^\infty(\Omega)} + \sum_i \|\omega_n^i\|_{L^\infty(\partial\Omega)} = o(1)$$

(recall that $(1/\varepsilon_n^i)\ \operatorname{dist}(a_n^i, \partial\Omega) \xrightarrow[n\to\infty]{} \infty$ and that $\omega^i(\infty) = 0$). This concludes the proof of (68) and completes the proof of Theorem 3.

We now consider a *special case* of Theorem 3. Suppose that u^n is a large solution of (60) obtained through the construction of [2]. To describe this construction method, let $\gamma \in H^{1/2}(\partial\Omega) \cap L^\infty(\partial\Omega)$ be such that

$$\|\gamma\|_{L^\infty(\partial\Omega)} < 1 \text{ but } \gamma \text{ is not constant.}$$

We consider the problem

$$(69)\qquad \begin{cases} \Delta u = 2u_x \wedge u_y & \text{on } \Omega \\ u = \gamma & \text{on } \partial\Omega \end{cases}$$

and denote by $\underline{u}$ the small solution of (69) obtained by HILDEBRANDT [8] (or [9]). We look for a second solution of (69) of the form

$$u = \underline{u} - v, \quad v \not\equiv 0$$

so that v satisfies

$$(70)\qquad \begin{cases} \mathscr{L}v \equiv -\Delta v + 2(\underline{u}_x \wedge v_y + v_x \wedge \underline{u}_y) = 2v_x \wedge v_y & \text{on } \Omega \\ v = 0 \quad \text{on } \partial\Omega. \end{cases}$$

Note that

$$(\mathscr{L}v, v) = \int |\nabla v|^2 + 4 \int \underline{u} \cdot v_x \wedge v_y \qquad \forall v \in H_0^1.$$

In [2] we have established that

$$(71)\qquad J = \inf_{\substack{v \in H_0^1 \\ Q(v) = 1}} (\mathscr{L}v, v) < S \equiv (32\pi)^{1/3}$$

and that the infimum in (71) is achieved by some v^0 satisfying

$$\mathscr{L}v^0 = J v_x^0 \wedge v_y^0 \quad \text{on } \Omega.$$

Therefore $\bar{u} = \underline{u} - (J/2)\, v^0$ provides another solution of (69); $\bar{u}$ is called a "large" solution of (69).

Theorem 4. *Let (γ^n) be a sequence in $H^{1/2}(\partial\Omega) \cap L^\infty(\partial\Omega)$ such that γ^n is not a constant and*

$$(72)\qquad \|\gamma^n\|_{L^\infty(\partial\Omega)} \xrightarrow[n\to\infty]{} 0.$$

Let $\bar{u}^n$ be a large solution of the problem

$$(73)\qquad \begin{cases} \Delta u = 2u_x \wedge u_y & \text{on } \Omega \\ u = \gamma^n & \text{on } \partial\Omega. \end{cases}$$

Then there exist

(i) *a solution ω of (5) with $\omega(\infty) = 0$ and $\int_{\mathbf{R}^2} |\nabla\omega|^2 = 8\pi$,*
(ii) *a sequence (a_n) in Ω, and*
(iii) *a sequence (ε_n) with $\varepsilon_n > 0 \ \forall n$ and $\lim \varepsilon_n = 0$,*

such that

$$\left\| \bar{u}^n - \omega\left(\frac{\cdot - a_n}{\varepsilon_n}\right) \right\|_{L^\infty} \xrightarrow[n\to\infty]{} 0$$

and

$$\frac{1}{\varepsilon_n}\operatorname{dist}(a_n, \partial\Omega) \xrightarrow[n\to\infty]{} \infty.$$

Proof. Let $\underline{u}^n$ be the "small" solution of (5). Set

$$(\mathcal{L}_n v, v) = \int |\nabla v|^2 + 4 \int \underline{u}^n \cdot v_x \wedge v_y \qquad \forall v \in H_0^1$$

and

$$J_n = \operatorname*{Inf}_{\substack{v\in H_0^1\\ Q(v)=1}} (\mathcal{L}_n v, v).$$

Let v_n^0 be some point where the infimum is achieved, so that

$$\bar{u}^n = \underline{u}^n - \frac{J_n}{2} v_n^0.$$

Since $\|\underline{u}^n\|_{L^\infty} = o(1)$ we have

$$(\mathcal{L}_n v, v) \geqq (1 - o(1)) \int |\nabla v|^2 \qquad \forall v \in H_0^1,$$

and thus

$$J_n \geqq (1 - o(1))\, S;$$

here we have used the inequality $|Q(v)|^{2/3} \leqq (1/s) \int |\nabla v|^2 \forall v \in H_0^1$ which is a consequence of a classical isoperimetric inequality (see [2]).

On the other hand from (71) we have $J_n < S$ and therefore

$$J_n = S + o(1). \tag{74}$$

Set $v^n = \bar{u}^n - \underline{u}^n = -(J_n/2)\, v_n^0$. Then

$$\int |\nabla v^n|^2 = \frac{J_n^2}{4} \int |\nabla v_n^0|^2 = \frac{S^3}{4} + o(1) = 8\pi + o(1) \tag{75}$$

since $\int |\nabla v_n^0|^2 = J_n + o(1) = S + o(1)$.

The proof of Theorem 3 shows that there exist ω, (a_n), (ε_n) satisfying (i), (ii), (iii), such that

$$\left\| v^n - \omega\left(\frac{\cdot - a_n}{\varepsilon_n}\right)\right\|_{H^1} \xrightarrow[n\to\infty]{} 0$$

and

$$\left\| v^n - \omega\left(\frac{\cdot - a_n}{\varepsilon_n}\right)\right\|_{L^\infty} \xrightarrow[n\to\infty]{} 0; \tag{76}$$

there is exactly one ω^i since in general

$$\int_\Omega |\nabla v^n|^2 = \sum_i \int_{\mathbb{R}^2} |\nabla \omega^i|^2 + o(1),$$

while here we have $\int_\Omega |\nabla v^n|^2 = 8\pi + o(1)$. Since $\|\underline{u}_n\|_{L^\infty} = o(1)$ the conclusion of Theorem 4 now follows from (76).

A similar result holds for the Plateau problem

$$(77)\qquad \begin{cases} \Delta u = 2u_x \wedge u_y & \text{on } \Omega \\ |u_x|^2 - |u_y|^2 = u_x \cdot u_y = 0 & \text{on } \Omega \\ u(\partial\Omega) = \Gamma, \end{cases}$$

where Γ is a given Jordan curve (more precisely $\Gamma = \alpha(\partial\Omega)$ for some $\alpha \in C(\partial\Omega; \mathbb{R}^3) \cap H^{1/2}(\partial\Omega; \mathbb{R}^3)$ which is one to one). We know that if $\Gamma \subset B_R$ and $R < 1$ there exists a "small" solution $\underline{u}_P$ of (77) (see [8]) and a "large" solution $\bar{u}^P$ of (77) (see [2]).

Corollary 2. *Let (Γ_n) be a sequence of Jordan curves such that*

$$(78)\qquad \Gamma_n \to 0$$

(that is, $\Gamma_n \subset B_{R_n}(0)$ and $R_n \to 0$). Let $\bar{u}_P^n$ be a large solution of the Plateau problem (77) *corresponding to $\Gamma = \Gamma_n$, obtained via the construction of* [2]. *Then there exist ω, (a_n), (ε_n) as in Theorem* 4, *such that*

$$\left\| \bar{u}_P^n - \omega\left(\frac{\cdot - a_n}{\varepsilon_n}\right) \right\|_{L^\infty} \xrightarrow[n\to\infty]{} 0$$

and

$$\frac{1}{\varepsilon_n} \operatorname{dist}(a_n, \partial\Omega) \xrightarrow[n\to\infty]{} 0.$$

Proof. The construction used in [2] shows that the "large" solution $\bar{u}_P^n$ of the Plateau problem coincides with the "large" solution of the Dirichlet problem (73) for some appropriate $\gamma^n : \partial\Omega \to \mathbb{R}^3$ such that $\gamma^n \in C(\partial\Omega) \cap H^{1/2}(\partial\Omega)$ and $\gamma^n(\partial\Omega) = \Gamma_n$. Therefore $\|\gamma^n\|_{L^\infty(\partial\Omega)} \to 0$ and we may use Theorem 4.

4. Geometrical applications

Consider again a solution u of the Plateau problem (77). The surface $\Sigma = u(\Omega)$ has mean curvature one and is spanned by Γ.

We study the behavior of a sequence of surfaces $\Sigma_n = u^n(\bar{\Omega})$ corresponding to a sequence Γ_n such that $\Gamma_n \to 0$. As a direct consequence of Corollary 2 we obtain

Corollary 3. *Let (Γ_n) be a sequence of Jordan curves such that $\Gamma_n \to 0$. Let $\bar{\Sigma}_n = \bar{u}_P^n(\Omega)$, where $\bar{u}_P^n$ is a large solution of* (77) *corresponding to $\Gamma = \Gamma_n$, obtained via the construction of* [2].

Then a subsequence of the surfaces $\bar{\Sigma}_n$ converges to a sphere of radius one containing 0.

There are possibly other solutions of (77).[1] We now consider the behavior of a sequence of surfaces $\Sigma_n = u^n(\bar{\Omega})$ where u^n is *any* solution of (77) corresponding to $\Gamma = \Gamma_n$. Our main result is the following

Theorem 5. *Let (Γ_n) be a sequence of Jordan curves such that $\Gamma_n \to 0$. Let $\Sigma_n = u^n(\bar{\Omega})$, where u^n is any solution of (77) corresponding to $\Gamma = \Gamma_n$. We assume that*

$$\tfrac{1}{2}\int_{\Omega} |\nabla u^n|^2 = \text{area}\,(\Sigma_n) \leqq C.$$

Then a subsequence of the surfaces Σ_n converges to 0 or to a finite (connected) union of spheres of radius one, and such that at least one of them contains 0.

Proof. We consider an order relation on the sequences of positive numbers tending to 0. Let $\alpha = (\alpha_n)$ and $\beta = (\beta_n)$ be two such sequences. We say that

$$\alpha \leqq \beta \text{ if } \lim_{n\to\infty} \beta_n/\alpha_n < \infty.$$

Without explicit mention we shall systematically extract subsequences, so that we may agree that every sequence of positive numbers has a limit in $[0, +\infty]$. Strictly speaking this relation is not an order relation since $\alpha \leqq \beta$ and $\beta \leqq \alpha$ do not imply $\alpha = \beta$; however they imply that $\alpha \sim \beta$ in the sense that $0 < \lim \alpha_n/\beta_n < \infty$. This order relation is total, that is, given α and β then at least one of the relations $\alpha \leqq \beta$ or $\beta \leqq \alpha$ holds.

Applying Theorem 3 to the sequence (u^n) we obtain the ω^i and the sequences (a_n^i) and (ε_n^i). We order the sequences (ε_n^i) in such a way that $(\varepsilon_n^1) \leqq (\varepsilon_n^2) \leqq \ldots \ldots \leqq (\varepsilon_n^p)$. Then for $i < j$

$$0 \leqq \lim_{n\to\infty} \varepsilon_n^j/\varepsilon_n^i < \infty \quad \text{and} \quad 0 < \lim_{n\to\infty} \varepsilon_n^i/\varepsilon_n^j \leqq \infty.$$

We define for $i \neq j$

$$p_{ij} = \begin{cases} \infty & \text{if } 0 < \lim_{n\to\infty} \varepsilon_n^i/\varepsilon_n^j \leqq \infty \\ \lim_{n\to\infty} (a_n^i - a_n^j)/\varepsilon_n^j & \text{if } \lim_{n\to\infty} \varepsilon_n^i/\varepsilon_n^j = 0. \end{cases} \tag{78}$$

If $0 < \lim_{n\to\infty} \varepsilon_n^i/\varepsilon_n^j < \infty$ it follows from Theorem 2 that

$$\lim_{n\to\infty} (a_n^i - a_n^j)/\varepsilon_n^j = \infty.$$

Here $\lim_{n\to\infty} (a_n^i - a_n^j)/\varepsilon_n^j$ is understood to be in $\mathbb{R}^2 \cup \{\infty\}$, which is identified as S^2.

For each integer i, $1 \leqq i \leqq p$ we consider the sphere

$$S_i = \omega^i(\mathbb{R}^2 \cup \{\infty\}) + \sum_{j\neq i} \omega^j(p_{ij}). \tag{79}$$

[1] It would be very interesting to determine if and when there exist solutions of (77) which are different from the ones constructed in [2].

If $i<j$, then $p_{ij}=\infty$ and thus $\omega^j(p_{ij})=0$. It follows that

$$S_i=\omega^i(\mathbb{R}^2\cup\{\infty\})+\sum_{j<i}\omega^j(p_{ij}). \tag{80}$$

In particular $S_1=\omega^1(\mathbb{R}^2\cup\{\infty\})$ contains 0. We shall prove that

$$u^n(\overline{\Omega})\to\bigcup_{i=1}^{p}S_i. \tag{81}$$

The proof is divided into two steps.

Step 1. For each $\alpha\in\bigcup_{i=1}^{p}S_i$ we construct a sequence (ξ_n) in Ω such that $u^n(\xi_n)\to\alpha$. Clearly it suffices to perform this construction for each $\alpha\in\bigcup_{i=1}^{p}S_i$, except for a finite number of points.

Given α, we may write, for some i and some $z\in\mathbb{R}^2$,

$$\alpha=\omega^i(z)+\sum_{j<i}\omega^j(p_{ij}).$$

Set

$$\xi_n=\varepsilon_n^iz+a_n^i.$$

As a consequence of (9) note that $\xi_n\in\Omega$ for n large enough. Applying Theorem 3 we obtain

$$u^n(\xi_n)=\omega^i(z)+\sum_{j\neq i}\omega^j\left(\frac{\varepsilon_n^iz+a_n^i-a_n^j}{\varepsilon_n^j}\right)+o(1). \tag{82}$$

If $i<j$ we have *either*

$$0<\lim_{n\to\infty}\varepsilon_n^i/\varepsilon_n^j<\infty,\ \text{and then}\ \lim_{n\to\infty}(a_n^i-a_n^j)/\varepsilon_n^j=\infty,$$

or

$$\lim_{n\to\infty}\varepsilon_n^i/\varepsilon_n^j=\infty,\ \text{and then}\ \lim_{n\to\infty}(\varepsilon_n^iz+a_n^i-a_n^j)/\varepsilon_n^j=\infty,$$

except, possibly, for one value of z (indeed, suppose that for some $z_0\in\mathbb{R}^2$ we have

$$|\varepsilon_n^iz_0+a_n^i-a_n^j|/\varepsilon_n^j\leqq C;$$

then

$$\frac{|\varepsilon_n^iz+a_n^i-a_n^j|}{\varepsilon_n^j}\geqq\frac{\varepsilon_n^i}{\varepsilon_n^j}|z-z_0|-C\xrightarrow[n\to\infty]{}\infty\ \text{if}\ z\neq z_0).$$

On the other hand if $j<i$ we have *either*

$$0<\lim_{n\to\infty}\varepsilon_n^i/\varepsilon_n^j<\infty\ \text{and then}\ \lim_{n\to\infty}(a_n^i-a_n^j)/\varepsilon_n^j=\infty=p_{ij}$$

or

$$\lim_{n\to\infty}\varepsilon_n^i/\varepsilon_n^j=0\ \text{and then}\ \lim_{n\to\infty}(a_n^i-a_n^j)/\varepsilon_n^j=p_{ij}.$$

Combining all these cases with (82) we see that

$$u^n(\xi_n) = \omega^i(z) + \sum_{j<i} \omega^j(p_{ij}) + o(1) = \alpha + o(1).$$

Step 2. Let (ξ_n) be any sequence in $\bar{\Omega}$. We claim that (modulo a subsequence) $u^n(\xi_n) \to \alpha$ for some $\alpha \in \bigcup_{i=1}^{p} S_i$. Indeed set

$$I = \{j;\ 1 \leqq j \leqq p \text{ and } (\xi_n - a_n^j)/\varepsilon_n^j \text{ remains bounded as } n \to \infty\}.$$

We distinguish two cases.

Case (*a*). $I = \emptyset$, that is $\lim_{n\to\infty} (\xi_n - a_n^j)/\varepsilon_n^j = \infty \quad \forall j$.
Then using Theorem 3,

$$u^n(\xi_n) = \sum_j \omega^j\left(\frac{\xi_n - a_n^j}{\varepsilon_n^j}\right) + o(1) = o(1).$$

Case (*b*). $I \neq \emptyset$. Let i denote the largest integer in I. We claim that $u^n(\xi_n) \to \alpha$ for some $\alpha \in S_i$. Indeed, we have

$$u^n(\xi_n) = \sum_j \omega^j\left(\frac{\xi_n - a_n^j}{\varepsilon_n^j}\right) + o(1).$$

Moreover

(i) *if* $j > i$ we have $\lim_{n\to\infty} (\xi_n - a_n^j)/\varepsilon_n^j = \infty$ (since $j \notin I$),

(ii) *if* $j < i$ we write

$$\frac{\xi_n - a_n^j}{\varepsilon_n^j} = \frac{\xi_n - a_n^i}{\varepsilon_n^j} + \frac{a_n^i - a_n^j}{\varepsilon_n^j} = \left(\frac{\xi_n - a_n^i}{\varepsilon_n^i}\right)\frac{\varepsilon_n^i}{\varepsilon_n^j} + \frac{a_n^i - a_n^j}{\varepsilon_n^j}$$

and recall that $0 \leqq \lim_{n\to\infty} \varepsilon_n^i/\varepsilon_n^j < \infty$. In all possible cases therefore,

$$\lim_{n\to\infty} (\xi_n - a_n^j)/\varepsilon_n^j = p_{ij} \quad \forall j \neq i,$$

and thus

$$u^n(\xi_n) = \omega^i(z) + \sum_{j\neq i} \omega^j(p_{ij}) + o(1),$$

where $z = \lim_{n\to\infty} (\xi_n - a_n^i)/\varepsilon_n^i$. This concludes the proof of Step 2.

Finally, the set $\bigcup_{i=1}^{p} S_i$ is connected since it is a limit of connected sets ($u^n(\bar{\Omega})$ is connected).

Appendix

We start with the description of the set of solutions of the problem

(A.1) $$\Delta\omega = 2\omega_x \wedge \omega_y \quad \text{on } \mathbb{R}^2,$$

(A.2) $$\int_{\mathbb{R}^2} |\nabla\omega|^2 < \infty.$$

Lemma A.1.[1] *Assume* $\omega \in L^1_{loc}(\mathbb{R}^2; \mathbb{R}^3)$ *satisfies* (A.1), (A.2). *Then* ω *has precisely the form*

$$\omega(z) = \pi\left(\frac{P(z)}{Q(z)}\right) + C, \quad z = (x, y) = x + iy, \tag{A.3}$$

where P, Q *are polynomials,* C *is a constant and* $\pi : \mathbb{C} \to S^2$ *is the stereographic projection from the north pole, that is*

$$\pi(z) = \frac{2}{1 + x^2 + y^2}\begin{pmatrix} x \\ y \\ -1 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Moreover any ω *given by* (A.3) *satisfies* (A.1), (A.2). *In addition*

$$\int_{\mathbb{R}^2} |\nabla\omega|^2 = 8\pi k \quad \text{with } k = \text{Max}\,\{\deg P, \deg Q\} \tag{A.4}$$

provided P/Q *is irreducible.*

Proof. We recall (see WENTE [26]) that if ω satisfies (A.1), (A.2) then ω is smooth and even (real) analytic. We claim that $\omega(\infty) = \lim_{|z| \to \infty} \omega(z)$ exists and that $\omega \circ \pi^{-1}$ is smooth on S^2 (including at the north pole).

Indeed set

$$\tilde{\omega}(x, y) = \omega\left(\frac{x}{x^2 + y^2}, \frac{y}{x^2 + y^2}\right) \quad \text{on } \mathbb{R}^2 \setminus \{0\}$$

so that $\tilde{\omega} \in C^\infty(\mathbb{R}^2 \setminus \{0\})$. An easy computation shows that

$$|\nabla\tilde{\omega}(x, y)|^2 = \frac{1}{(x^2 + y^2)^2}\left|(\nabla\omega)\left(\frac{x}{x^2 + y^2}, \frac{y}{x^2 + y^2}\right)\right|^2$$

and thus

$$\int_{\mathbb{R}^2} |\nabla\tilde{\omega}|^2 = \int_{\mathbb{R}^2} |\nabla\omega|^2 < \infty;$$

also

$$-\Delta\tilde{\omega} = 2\tilde{\omega}_x \wedge \tilde{\omega}_y \quad \text{on } \mathbb{R}^2 \setminus \{0\}.$$

A standard argument leads to $\tilde{\omega} \in H^1_{loc}(\mathbb{R}^2)$ and

$$-\Delta\tilde{\omega} = 2\tilde{\omega}_x \wedge \tilde{\omega}_y \quad \text{in } \mathcal{D}'(\mathbb{R}^2).$$

Therefore $\tilde{\omega}$ is smooth on $\mathbb{R}^2$ (including 0); going back to ω, this implies that $\omega \circ \pi^{-1}$ is smooth on S^2 (including at the north pole).

Next we claim that ω is conformal, that is,

$$|\omega_x|^2 - |\omega_y|^2 = \omega_x \cdot \omega_y = 0 \quad \text{on } \mathbb{R}^2. \tag{A.5}$$

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[1] We thank H. WENTE for some useful indications concerning Lemma A.1.

Indeed set

$$\varphi = |\omega_x|^2 - |\omega_y|^2 - 2i\omega_x \cdot \omega_y \equiv \Phi + i\Psi.$$

A standard computation based on (A.1) shows that

$$\Phi_x = \Psi_y \quad \text{and} \quad \Phi_y = -\Psi_x,$$

and thus φ is holomorphic on $\mathbb{C}$. We conclude that $\varphi \equiv 0$ since $\varphi \in L^1(\mathbb{R}^2)$.

From (A.1) and a result in [6] (see Lemma 2.1) it follows that either $\omega \equiv C$, or $\nabla\omega \neq 0$ everywhere except at some isolated points which are denoted by (z_i) (in fact there can be only a finite number of such points since ω can be considered as defined on S^2). We set

$$\mathcal{O} = \mathbb{R}^2 \setminus \bigcup_i \{z_i\}$$

and consider the Gauss map n defined by

$$n = \frac{\omega_x \wedge \omega_y}{|\omega_x \wedge \omega_y|} \quad \text{on } \mathcal{O}.$$

Note that n is well defined and smooth on $\mathcal{O}$ since by (A.5) we have $|\omega_x \wedge \omega_y| = \frac{1}{2}|\nabla\omega|^2$. It is known (see for example RUH [15] or JOST [10]) that n is harmonic on $\mathcal{O}$, that is,

$$-\Delta n = n|\nabla n|^2 \quad \text{on } \mathcal{O}, \tag{A.6}$$

a result which can also be verified directly using (A.1) and (A.5).

We claim that each isolated singularity of n is removable and thus that n is smooth on all of $\mathbb{R}^2$. Indeed suppose for example that 0 is a singular point of n, that is, $\nabla\omega(0) = 0$. We know from a result of [6] (see Lemma 2.1 and Lemma 2.2) that, in some suitable direct orthonormal basis of $\mathbb{R}^3$, ω may be written as

$$\omega = (\omega^1, \omega^2, \omega^3)$$

where, up to additive constants,

$$\omega^1 + i\omega^2 = az^m + O(|z|^{m+1}) \text{ as } z \to 0,$$
$$\omega^3 = O(|z|^{m+1}) \qquad \text{as } z \to 0;$$

where $a > 0$ is a constant and $m \geqq 2$ is an integer. In such a basis

$$\omega_x \wedge \omega_y = \begin{pmatrix} 0 \\ 0 \\ a^2m^2|z|^{2m-2} \end{pmatrix} + O(|z|^{2m-1}),$$

and thus for z near 0 ($z \neq 0$) we have

$$n = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + O(|z|).$$

It follows that

(A.7) n is continuous at 0.

We claim that

(A.8) $$\int_B |\nabla n|^2 < \infty$$

where B is some small around 0. Indeed by (A.7) there is some neighborhood U of 0 such that

$$n(z) \cdot n(0) \geqq \tfrac{1}{2} \text{ for } z \in U.$$

Let B be some ball contained in U and with center at 0. Choose a sequence (ζ_k) of functions such that $\zeta_k \in \mathscr{D}(U \setminus \{0\})$, $\zeta_k \to 1$ on B, and $\int |\nabla \zeta_k|^2 \leqq C$. Multiplying (A.6) by $n(0)\, \zeta_k^2$, we find

$$\tfrac{1}{2} \int \zeta_k^2 |\nabla n|^2 \leqq 2 \int |\zeta_k|\, |\nabla n|\, |\nabla \zeta_k|$$

and thus

$$\int \zeta_k^2 |\nabla n|^2 \leqq 16 \int |\nabla \zeta_k|^2 \leqq 16C.$$

Letting $k \to \infty$ yields (A.8).

From (A.6), (A.8) and a result of SACKS & UHLENBECK ([17], Theorem 3.6) it follows that 0 is a removable singularity and thus n is smooth on all of $\mathbb{R}^2$.

We assert that in fact n is defined and smooth on S^2. Indeed let $\tilde{\omega}$ as above and set

$$\tilde{n}(x, y) = n\left(\frac{x}{x^2 + y^2}, \frac{y}{x^2 + y^2}\right) \text{ on } \mathbb{R}^2 \setminus \{0\}.$$

An easy computation shows that

$$\tilde{n} = -\frac{\tilde{\omega}_x \wedge \tilde{\omega}_y}{|\tilde{\omega}_x \wedge \tilde{\omega}_y|}$$

and therefore $\tilde{n}$ is a smooth harmonic map on $\mathbb{R}^2$. Consequently n may be considered as a smooth harmonic map from S^2 into S^2. However, all such from S^2 into S^2 are known (see e.g. SPRINGER [18] or LEMAIRE [11]). More precisely, there exist polynomials P and Q such that either

(A.9) $$n(z) = \pi\left(\frac{P(z)}{Q(z)}\right)$$

or

(A.10) $$n(z) = \pi\left(\frac{P(\bar{z})}{Q(\bar{z})}\right).$$

Next, we claim that

(A.11) $$(\omega + n)_x = (\omega + n)_y = 0 \quad \text{on } \mathbb{R}^2.$$

It suffices, of course, to check (A.11) on $\mathbb{R}^2 \setminus \bigcup_i \{z_i\}$. We consider the basis

$$i = \omega_x/|\omega_x|, \quad j = \omega_y/|\omega_y|, \quad k = i \wedge j = n$$

and set $r = |\omega_x| = |\omega_y|$. In this basis we write

$$\omega_x = \begin{pmatrix} r \\ 0 \\ 0 \end{pmatrix}, \quad \omega_y = \begin{pmatrix} 0 \\ r \\ 0 \end{pmatrix}, \quad \omega_{xx} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}, \quad \omega_{xy} = \begin{pmatrix} d \\ e \\ f \end{pmatrix},$$

so that

$$\omega_{yy} = 2\omega_x \wedge \omega_y - \omega_{xx} = \begin{pmatrix} -a \\ -b \\ 2r^2 - c \end{pmatrix}.$$

Differentiating the relation $\omega_x^2 - \omega_y^2 = 0$ with respect to x and y yields

(A.12) $$a = e \quad \text{and} \quad b = -d.$$

On the other hand

$$r_x = \omega_x \cdot \omega_{xx}/r = a, \quad r_y = \omega_x \cdot \omega_{xy}/r = d,$$

and

$$r^2 n_x = \omega_{xx} \wedge \omega_y + \omega_x \wedge \omega_{xy} - 2(\omega_x \wedge \omega_y)\, r_x/r$$

$$r^2 n_y = \omega_{xy} \wedge \omega_y + \omega_x \wedge \omega_{yy} - 2(\omega_x \wedge \omega_y)\, r_y/r.$$

Thus, using (A.12) we find

(A.13) $$n_x = \frac{1}{r}\begin{pmatrix} -c \\ -f \\ 0 \end{pmatrix}, \quad n_y = \frac{1}{r}\begin{pmatrix} -f \\ c - 2r^2 \\ 0 \end{pmatrix}.$$

Since n is harmonic from S^2 into S^2, this gives

$$n_x^2 - n_y^2 = n_x \cdot n_y = 0$$

(recall that $n_x^2 - n_y^2 - 2i\, n_x \cdot n_y$ is holomorphic and belongs to $L^1(\mathbb{R}^2)$). Hence

(A.14) $$f = 0 \quad \text{and} \quad c = r^2.$$

Combining (A.13) and (A.14) we obtain (A.11).

It now follows that there is a constant C such that

$$\omega + n = C.$$

Therefore ω is either of the form

(A.15) $$\omega(z) = -\pi\left(\frac{P(z)}{Q(z)}\right) + C$$

or

(A.16) $$\omega(z) = -\pi\left(\frac{P(\bar{z})}{Q(\bar{z})}\right) + C.$$

However $-\pi(\zeta) = \pi(-1/\bar{\zeta})$ for all $\zeta \in \mathbb{C}$ and thus

$$-\pi\left(\frac{P(z)}{Q(z)}\right) = \pi\left(-\frac{\bar{Q}(z)}{\bar{P}(z)}\right).$$

Functions ω of the form (A.15) satisfy

$$-\Delta\omega = 2\omega_x \wedge \omega_y$$

while functions of the form (A.16) satisfy

$$\Delta\omega = 2\omega_x \wedge \omega_y,$$

as follows at once from the fact that $\omega(z) = \pi(z)$ satisfies $\Delta\omega = 2\omega_x \wedge \omega_y$ and $\omega(z) = \pi(\bar{z})$ satisfies $-\Delta\omega = 2\omega_x \wedge \omega_y$.

On the other hand, if f is any holomorphic function and u satisfies $\Delta u = 2u_x \wedge u_y$, then $v = u \circ f$ also satisfies $\Delta v = 2v_x \wedge v_y$. Hence ω is of the form (A.3). The last assertion in Lemma A.1 may be found for example in [11].

Lemma A.2. *Let $\omega \in L^\infty(\mathbb{R}^2)$ with $\nabla\omega \in L^2(\mathbb{R}^2)$ and $\omega \to 0$ at infinity (in the usual sense). Set*

$$\omega^n(z) = \omega\left(\frac{z - a_n}{\varepsilon_n}\right)$$

where (a_n) is a sequence in Ω and (ε_n) is a sequence of positive numbers such that $\varepsilon_n \xrightarrow[n\to\infty]{} 0$ and $(1/\varepsilon_n)\operatorname{dist}(a_n, \partial\Omega) \xrightarrow[n\to\infty]{} \infty$. Then

$$\|\omega^n\|_{H^{1/2}(\partial\Omega)} \xrightarrow[n\to\infty]{} 0$$

Proof. Given $\varepsilon > 0$ we can find some $\bar{\omega} \in L^\infty(\mathbb{R}^2)$ with compact support such that $\nabla\bar{\omega} \in L^2(\mathbb{R}^2)$ and

$$\|\omega - \bar{\omega}\|_{L^\infty} < \varepsilon, \qquad \|\nabla\omega - \nabla\bar{\omega}\|_{L^2} < \varepsilon.$$

Set

$$\bar{\omega}^n(z) = \bar{\omega}\left(\frac{z - a_n}{\varepsilon_n}\right).$$

Then

$$\|\omega^n\|_{H^{1/2}(\partial\Omega)} \leqq \|\omega^n - \bar{\omega}^n\|_{H^{1/2}(\partial\Omega)} + \|\bar{\omega}^n\|_{H^{1/2}(\partial\Omega)}.$$

Note that $\bar{\omega}^n = 0$ on $\partial\Omega$ for n large enough, while

$$\begin{aligned}\|\omega^n - \bar{\omega}^n\|_{H^{1/2}(\partial\Omega)} &\leqq \|\omega^n - \bar{\omega}^n\|_{H^1(\Omega)} = \|\omega^n - \bar{\omega}^n\|_{L^2(\Omega)} + \|\nabla\omega^n - \nabla\bar{\omega}^n\|_{L^2(\Omega)} \\ &\leqq C\,\|\omega^n - \bar{\omega}^n\|_{L^\infty(\Omega)} + \|\nabla\omega^n - \nabla\omega^n\|_{L^2(\Omega)} \\ &\leqq C\,\|\omega - \bar{\omega}^n\|_{L^\infty(\mathbb{R}^2)} + \|\nabla\omega - \nabla\bar{\omega}\|_{L^2(\mathbb{R}^2)} \\ &\leqq (C + 1)\,\varepsilon,\end{aligned}$$

which completes the proof.

Lemma A.3. *Let* (α^n) *be a sequence in* $H^1(\Omega)$ *such that*

$$\|\alpha^n\|_{L^\infty(\Omega)} \leqq C. \tag{A.17}$$

Let (a_n) *be a sequence in* Ω *and let* (ε_n) *be a sequence of positive numbers such that* $\varepsilon_n \xrightarrow[n\to\infty]{} 0$ *and*

$$\frac{1}{\varepsilon_n} \operatorname{dist}(a_n, \partial\Omega) \xrightarrow[n\to\infty]{} \infty. \tag{A.18}$$

Set $\tilde{\alpha}^n(z) = \alpha^n(\varepsilon_n z + a_n)$ *for* $z \in \mathbb{R}^2$ (1). *We assume that*

$$\tilde{\alpha}^n(z) \xrightarrow[n\to\infty]{} C \quad \textit{for a.e. } z \in \mathbb{R}^2 \tag{A.19}$$

where C *is a constant.*

Also let $\omega \in L^\infty(\mathbb{R}^2)$ *with* $\nabla\omega \in L^2(\mathbb{R}^2)$ *and* $\omega \to \omega(\infty)$ *at infinity (in the usual sense). Set*

$$\omega^n(z) = \omega\left(\frac{z - a_n}{\varepsilon_n}\right).$$

Let β^n *be the solution of the problem*

$$\begin{cases} \Delta\beta^n = \alpha_x^n \wedge \omega_y^n + \omega_x^n \wedge \alpha_y^n & \textit{on } \Omega, \\ \beta^n = 0 & \textit{on } \partial\Omega. \end{cases} \tag{A.20}$$

Then

$$\|\nabla\beta^n\|_{L^2(\Omega)} \xrightarrow[n\to\infty]{} 0 \tag{A.21}$$

and

$$\|\beta^n\|_{L^\infty(\Omega)} \xrightarrow[n\to\infty]{} 0. \tag{A.22}$$

Remark A.1. Assumption (A.19) obviously holds with $C = 0$ if $\|\alpha^n\|_{L^\infty(\Omega)} \to 0$.

Proof. Without loss of generality it can be assumed that $C = 0$ and that $\omega(\infty) = 0$. Using Lemma A.1 from [2] and the same device as in the proof of Lemma A.2 it suffices to consider the case where $\omega \in \mathcal{D}(\mathbb{R}^2)$ (indeed given $\varepsilon > 0$ we can find some $\bar{\omega} \in \mathcal{D}(\mathbb{R}^2)$ such that $\|\nabla\omega - \nabla\bar{\omega}\|_{L^2} < \varepsilon$).
Also, without loss of generality it can be assumed that each α^n is defined on all of $\mathbb{R}^2$ and that

$$\operatorname{Supp} \alpha^n \subset B_2(0), \qquad \|\alpha^n\|_{H^1} \leqq C.$$

Since $\|\nabla\alpha^n\|_{L^2(\mathbb{R}^2)} \leqq C$ and $\tilde{\alpha}^n \to 0$ a.e. on $\mathbb{R}^2$, it follows by standard arguments that

$$\tilde{\alpha}^n \to 0 \quad \text{in} \quad L^p_{\text{loc}}(\mathbb{R}^2) \quad \forall p < \infty. \tag{A.23}$$

---

(1) Assumption (A.18) implies that $\varepsilon_n z + a_n \in \Omega$ for each $z \in \mathbb{R}^2$ and for n large enough.

We first prove (A.21). Using (A.20) and Lemma A.4 from [2] we see that

$$\int_\Omega |\nabla\beta^n|^2 = -\int_\Omega \alpha^n \cdot (\omega_x^n \wedge \beta_y^n + \beta_x^n \wedge \omega_y^n)$$

and thus

$$\|\nabla\beta^n\|_{L^2(\Omega)} \leqq \|\alpha^n \nabla\omega^n\|_{L^2(\Omega)} = \|\tilde{\alpha}^n \nabla\omega\|_{L^2(\Omega_n)} \to 0$$

where $\Omega_n = (\Omega - a_n)/\varepsilon_n$.
We now prove (A.22). Set $r = (x^2 + y^2)^{1/2}$ and

$$\Psi^n = \frac{1}{2\pi}(\log r) * (\alpha_x^n \wedge \omega_y^n + \omega_x^n \wedge \alpha_y^n) \quad \text{on } \mathbb{R}^2$$

so that

$$\Delta\Psi^n = \alpha_x^n \wedge \omega_y^n + \omega_x^n \wedge \alpha_y^n \quad \text{on } \mathbb{R}^2. \tag{A.24}$$

From (A.20), (A.24) and the maximum principle we obtain

$$\|\beta^n\|_{L^\infty(\Omega)} \leqq 2\,\|\Psi^n\|_{L^\infty(\Omega)}.$$

On the other hand we have

$$\Psi^n = \frac{1}{2\pi}(\log) * [(\alpha^n \wedge \omega_y^n)_x + (\omega_x^n \wedge \alpha^n)_y]$$

$$= \frac{1}{2\pi}\left[\frac{x}{r^2} * (\alpha^n \wedge \omega_y^n) + \frac{y}{r^2} * (\omega_x^n \wedge \alpha^n)\right].$$

Therefore for $p \in \mathbb{R}^2$ (and $z = (x, y)$)

$$|\Psi^n(p)| \leqq \left(\frac{1}{r}\right) * (|\alpha^n|\,|\nabla\omega^n|)(p)$$

$$= \int_{\mathbb{R}^2} \frac{1}{|p - z|}|\alpha^n(z)|\left|\nabla\omega\left(\frac{z - a_n}{\varepsilon_n}\right)\right|\frac{dx\,dy}{\varepsilon_n}.$$

It follows that

$$|\Psi^n(p)| \leqq \int_{\mathbb{R}^2} \frac{1}{|q_n - \zeta|}|\tilde{\alpha}^n(\zeta)|\,|\nabla\omega(\zeta)|\,d\xi\,d\eta, \tag{A.25}$$

where

$$\zeta = (\xi, \eta) \quad \text{and} \quad q_n = (p - a_n)/\varepsilon_u.$$

Set

$$\Theta^n = \frac{1}{r} * |\tilde{\alpha}^n|\,|\nabla\omega|.$$

In view of (A.25) it suffices to prove that

$$\|\Theta^n\|_{L^\infty(\mathbb{R}^2)} \xrightarrow[n\to\infty]{} 0. \tag{A.26}$$

But this is clear since $1/r \in L^\infty + L^{3/2}$ while $\|\tilde{\alpha}^n\|\,|\nabla\omega| \to 0$ in L^3 and in L^1 (here we use (A.23) and the fact that ω has compact support).

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