Simple Force and Stress Multipoles

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Contents

1. Introduction

E. & F. COSSERAT (1909) developed a theory in which the mechanical interaction between portions of a body across a surface in it is considered to consist not only of forces distributed over the surface, but also of distributed couples. TRUESDELL & TOUPIN (1960) have reformulated and developed this theory in modern notation. In a recent paper, TOUPIN (1963) has derived constitutive equations for finite deformation in which it is assumed that a strain-energy function exists for the material which depends only on the first and second order deformation gradients. A similar constitutive equation was previously derived by GRIOLI (1960). MINDLIN & TIERSTEN (1963) have linearised TOUPIN'S constitutive equations and solved a number of problems in the linear theory of elasticity with couple-stresses.

In the present paper, we develop a theory of greater generality. In this theory, we assume that the force system acting on the body may consist of distributed surface and body forces and surface and body force multipoles of various orders. The latter are defined as force systems whose rate of working in an arbitrary deformation field is given by an expression of the form $F_{i_1 i_2 \ldots i_\beta i}\, v_{i, i_1 \ldots i_\beta}$ where $v_{i, i_1 \ldots i_\beta}$ is the β^{th} gradient of velocity v_i in a rectangular Cartesian coordinate system. $F_{i_1 \ldots i_\beta i}$ is then a tensor describing the multipolar force, which is called a simple force multipole of the first kind*. The definitions of force and stress multipoles given here are effectively special cases of those used by TRUESDELL & TOUPIN (1960, § 232).

* Previous work is concerned with the case when $F_{i_1 \ldots i_\beta i}$ is a skew symmetric second order tensor.

Although, in this paper, we discuss only simple force multipoles of the first kind, this discussion invites generalization to the case of compound force multiples of the first, second, third, ... kinds. A compound force multipole of the first kind may be defined as a force system for which the rate of working is given by an expression of the form $F_{i_1 i_2 \ldots i_\beta i} P_{i i_1 \ldots i_\beta}$, where $P_{i i_1 \ldots i_\beta}$ is a function of the velocity gradients of various orders. In order to define simple and compound force multipoles of a kind higher than the first we must define multipolar deformation fields. These suggested further generalizations will be presented later*.

Notation and definitions of kinematic quantities are given in Section 2. In Section 3 we discuss the effect on these quantities of superposed rigid body motions. Multipolar body forces are defined in Section 4 and multipolar stress fields in Section 5. In Section 6 we postulate the equation of energy and the entropy production inequality. We then systematically apply to the energy equation invariance conditions which arise from consideration of superposed rigid body motions and *deduce* the classical equation of motion and an equation of vector moments. In the classical case in which no multipolar stresses and body forces are present but only the usual stress tensor and body forces, the equations of motion are usually derived from a separate postulate about the balance of linear momentum, and the symmetry of the stress tensor is derived from a postulate about balance of moment of momentum. An energy equation is then assumed in addition to the postulates of linear momentum and moment of momentum. The present work shows that these latter postulates can be *derived* from the equation of energy by making full use of invariance conditions under superposed rigid body motions**.

In Section 7 a particular class of constitutive equations appropriate to generalized elasticity theory are considered, and a complete set of equations is derived from the energy balance equation and the entropy production inequality of Section 6. An alternative form for these equations is given in Section 8. The results of Section 8 are expressed in a more general notation in Section 9, and conditions at the surface of the body are discussed when only stresses and multipolar stresses are present. In Section 10 we examine a more general class of constitutive equations involving a relation between multipolar stress tensors and kinematic gradients at time t of various orders, and we reduce these equations to a canonical form with the help of invariance principles arising from consideration of superposed rigid-body motions.

In Section 11, we demonstrate by means of an example the manner in which we can derive a system of force multipoles acting at a single point, which are energetically equivalent to a system of monopolar forces acting at a number of different points. Finally, in Section 12 we give some consideration to the restrictions imposed on the form of the strain-energy function if the material is isotropic.

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* To be presented in a forthcoming paper in this *Archive*.

** Since writing the above Professor W. NOLL has sent us a proof copy of a paper, written in 1960 and to be published in the proceedings of "Colloque sur l'axiomatique", in which he obtains the classical equations of motion and moments for forces from other postulates, but his ideas do not appear to be the same as those used here.

2. Notation

We refer the motion of the continuum to a fixed system of rectangular Cartesian axes. The position of a typical particle of the continuum at time τ is denoted by $x_i(\tau)$ where

$$x_i(\tau) = x_i(X_1, X_2, X_3, \tau) \qquad (-\infty < \tau \leqq t), \tag{2.1}$$

and X_A is a reference position of the particle. We also use the notation

$$x_i = x_i(t). \tag{2.2}$$

If this deformation is to be possible in a real material, then

$$\det\left[\frac{\partial x_i(\tau)}{\partial X_A}\right] > 0. \tag{2.3}$$

For some purposes it is convenient to express $x_i(\tau)$ in terms of the current position of the particle at time t so that

$$x_i(\tau) = x_i(x_1, x_2, x_3, t, \tau), \tag{2.4}$$

and

$$\det\left[\frac{\partial x_i(\tau)}{\partial x_j}\right] > 0. \tag{2.5}$$

Displacement gradients taken with respect to the position X_A are denoted by

$$x_{i,A_1\ldots A_\beta}(\tau) = \frac{\partial^\beta x_i(\tau)}{\partial X_{A_1}\partial X_{A_2}\ldots\partial X_{A_\beta}} \qquad (\beta = 1, 2, \ldots), \tag{2.6}$$

and we use the notation

$$x_{i,A_1\ldots A_\beta} = x_{i,A_1\ldots A_\beta}(t). \tag{2.7}$$

Displacement gradients taken with respect to the current position x_i at time t are

$$x_{i,i_1 i_2\ldots i_\beta}(\tau) = \frac{\partial^\beta x_i(\tau)}{\partial x_{i_1}\partial x_{i_2}\ldots\partial x_{i_\beta}} \qquad (\beta = 1, 2, \ldots). \tag{2.8}$$

We observe that

$$\begin{aligned} x_{i,i_1}(t) &= \delta_{ii_1}, \\ x_{i,i_1\ldots i_\beta}(t) &= 0 \qquad (\beta > 1), \end{aligned} \tag{2.9}$$

and that the gradients in (2.6) and (2.8) are symmetric with respect to $A_1, A_2, \ldots, A_\beta$ and $i_1, i_2, \ldots, i_\beta$ respectively.

The components of velocity at the point $x_i(\tau)$ are denoted by $v_i^{(1)}(\tau) = v_i(\tau)$ so that

$$v_i^{(1)}(\tau) = \frac{D x_i(\tau)}{D\tau}, \qquad v_i^{(1)}(t) = v_i(t) = v_i,$$

where $D/D\tau$ denotes differentiation with respect to τ holding X_A fixed in (2.1), or $x_j(t)$ and t fixed in (2.4). More generally, n^{th} velocity components may be defined as

$$v_i^{(n)}(\tau) = \frac{D^n x_i(\tau)}{D\tau^n}, \qquad v_i^{(n)}(t) = v_i^{(n)}, \qquad v_i^{(0)}(\tau) = x_i(\tau). \tag{2.10}$$

From (2.8) and (2.10) we have

$$\frac{D^n x_{i,i_1\ldots i_\beta}(\tau)}{D\tau^n} = \frac{\partial^\beta v_i^{(n)}(\tau)}{\partial x_{i_1}\partial x_{i_2}\ldots\partial x_{i_\beta}} = v_{i,i_1\ldots i_\beta}^{(n)}(\tau), \tag{2.11}$$

and we use the notation

$$v_{i,i_1\ldots i_\beta}^{(n)}(t) = v_{i,i_1\ldots i_\beta}^{(n)} \tag{2.12}$$

for gradients of the n^{th} velocity components at time t with respect to coordinates at time t. Also

$$v_{i,i_1\ldots i_\beta}^{(0)}(\tau) = x_{i,i_1\ldots i_\beta}(\tau), \qquad v_{i,i_1\ldots i_\beta}^{(0)} = 0 \qquad (\beta > 1). \tag{2.13}$$

In view of (2.3) we may write $x_{i,A}(\tau)$ in the polar form

$$x_{i,A}(\tau) = R_{iB}(\tau) M_{BA}(\tau), \tag{2.14}$$

where $M_{BA}(\tau)$ is a positive definite symmetric tensor and $R_{iB}(\tau)$ is a rotation tensor, so that

$$R_{iB}(\tau) R_{iA}(\tau) = \delta_{AB}, \qquad R_{iA}(\tau) R_{jA}(\tau) = \delta_{ij}, \qquad \det R_{iA}(\tau) = 1. \tag{2.15}$$

Also

$$R_{iB} = R_{iB}(t), \qquad M_{AB} = M_{AB}(t). \tag{2.16}$$

In general, throughout the paper, lower case Latin indices $i, i_1, \ldots$ are associated with coordinates $x_i(\tau)$ or x_i and take the values 1, 2, 3: upper case Latin indices $A, A_1, \ldots$ are associated with coordinates X_A and take the values 1, 2, 3. The usual Cartesian summation convention is used.

3. Superposed rigid-body motions

We consider motions of the continuum which differ from those given by (2.1) only by superposed rigid-body motions, at different times. Thus

$$x_i^*(\tau^*) = c_i^*(\tau^*) + Q_{ij}(\tau)[x_j(\tau) - c_j(\tau)], \tag{3.1}$$

where $c_i(\tau)$, $c_i^*(\tau^*)$ are vector functions of τ and $\tau^* = (\tau + a)$ respectively, a is an arbitrary constant and $Q_{ij}(\tau)$ is a proper orthogonal tensor which depends on τ. In Section 2 vectors and tensors are defined in terms of the motion (2.1) and we denote corresponding quantities defined from (3.1) by the same letter to which we add an asterisk*. Then

$$x_{m,A_1\ldots A_\beta}^*(\tau^*) = Q_{mn}(\tau)\, x_{n,A_1\ldots A_\beta}(\tau), \tag{3.2}$$

and

$$\frac{\partial^\beta x_m^*(\tau^*)}{\partial x_{i_1}^* \ldots \partial x_{i_\beta}^*} = Q_{mn}(\tau)\, Q_{i_1 j_1} Q_{i_2 j_2} \ldots Q_{i_\beta j_\beta}\, x_{n,j_1\ldots j_\beta}(\tau), \tag{3.3}$$

where $Q_{ij} = Q_{ij}(t)$. Hence

$$E_{A A_1 A_2 \ldots A_\beta}^*(\tau^*) = E_{A A_1 A_2 \ldots A_\beta}(\tau), \tag{3.4}$$

and

$$E_{i i_1 i_2 \ldots i_\beta}^*(\tau^*) = Q_{ij} Q_{i_1 j_1} \ldots Q_{i_\beta j_\beta} E_{j j_1 j_2 \ldots j_\beta}(\tau), \tag{3.5}$$

where

$$\begin{aligned} E_{A A_1 A_2 \ldots A_\beta}(\tau) &= x_{m,A}(\tau)\, x_{m,A_1 A_2 \ldots A_\beta}(\tau), \\ E_{i i_1 i_2 \ldots i_\beta}(\tau) &= x_{m,i}(\tau)\, x_{m,i_1 i_2 \ldots i_\beta}(\tau). \end{aligned} \tag{3.6}$$

Equations (3.4) and (3.5) are valid for all values of τ. In particular

$$E_{ij}^*(\tau^*) = Q_{ir} Q_{js} E_{rs}(\tau)$$

and if we differentiate both sides of this equation μ-times with respect to τ (assuming that the derivatives exist) and then put $\tau^*=\tau=t$, we have

$$A_{ij}^{*(\mu)}=Q_{ir}\,Q_{js}\,A_{rs}^{(\mu)}, \tag{3.7}$$

where $A_{ij}^{(\mu)}=A_{ij}^{(\mu)}(t)$ are the Rivlin-Ericksen tensors given by

$$A_{ij}^{(\mu)}=\sum_{\alpha=0}^{\mu}\binom{\mu}{\alpha}v_{m,i}^{(\alpha)}\,v_{m,j}^{(\mu-\alpha)}=\frac{DA_{ij}^{(\mu-1)}}{Dt}+A_{im}^{(\mu-1)}\,v_{m,j}+A_{mj}^{(\mu-1)}\,v_{m,i}, \tag{3.8}$$

with $v_{m,i}^{(0)}=\delta_{mi}$. Similarly, by differentiating (3.5) μ-times with respect to τ, putting $\tau^*=\tau=t$ and using (2.11)—(2.13), we obtain the relation

$$A_{ii_1\ldots i_\beta}^{*(\mu)}=Q_{ij}\,Q_{i_1j_1}\cdots Q_{i_\beta j_\beta}\,A_{jj_1\ldots j_\beta}^{(\mu)}, \tag{3.9}$$

where

$$A_{ii_1\ldots i_\beta}^{(\mu)}=\sum_{\alpha=1}^{\mu}\binom{\mu}{\alpha}v_{m,i}^{(\mu-\alpha)}\,v_{m,i_1\ldots i_\beta}^{(\alpha)} \qquad (\beta=2,3,\ldots). \tag{3.10}$$

The tensor $A_{ii_1i_2\ldots i_\beta}^{(\mu)}$ is completely symmetric with respect to the indices $i_1, i_2, \ldots, i_\beta$ and is a natural generalization of the Rivlin-Ericksen tensors $A_{ij}^{(\mu)}$. Taking $Q_{ij}=\delta_{ij}$ we see, from (3.9), that the tensors (3.10) are unaltered by superposed rigid body velocities and angular velocities of all orders, the continuum occupying instantaneously the same position at time t. Other tensors with the same property can be defined which are related to those in (3.10) and we mention one other group of such tensors below. We first observe, however, from (3.10), that

$$A_{ii_1\ldots i_\beta}^{(\mu)}=v_{i,i_1\ldots i_\beta}^{(\mu)}+\sum_{\alpha=1}^{\mu-1}\binom{\mu}{\alpha}\,v_{m,i}^{(\mu-\alpha)}\,v_{m,i_1\ldots i_\beta}^{(\alpha)}$$

and hence, by repeated application of this formula for $\mu=1, 2, \ldots$ and given β, we have

$$v_{i,i_1\ldots i_\beta}^{(\mu)}=A_{ii_1\ldots i_\beta}^{(\mu)}+\text{a polynomial in } v_{m,i}^{(\alpha)} \text{ and } A_{ii_1\ldots i_\beta}^{(\alpha)}, \tag{3.11}$$

for $\alpha=1, 2, \ldots, \mu-1$; $\beta=2, 3, \ldots$.

We define $B_{ji_1i_2\ldots i_\beta}(\tau)$ by the equation

$$x_{i,i_1i_2\ldots i_\beta}(\tau)=x_{i,j}(\tau)\,B_{ji_1i_2\ldots i_\beta}(\tau) \qquad (\beta=2,3,\ldots) \tag{3.12}$$

and observe that $B_{ji_1\ldots i_\beta}(t)=0$ $(\beta\geqq 2)$. The definition of $B_{ji_1\ldots i_\beta}(\tau)$ is unique since $x_{i,j}$ is non-singular. We differentiate (3.12) μ-times with respect to τ and put $\tau=t$ to obtain the equation

$$v_{i,i_1\ldots i_\beta}^{(\mu)}=\sum_{\alpha=1}^{\mu}\binom{\mu}{\alpha}v_{i,m}^{(\mu-\alpha)}\,B_{mi_1\ldots i_\beta}^{(\alpha)} \qquad (\beta=2,3,\ldots), \tag{3.13}$$

where $B_{mi_1\ldots i_\beta}^{(\alpha)}$ denotes the value of the α-derivative of $B_{mi_1\ldots i_\beta}(\tau)$ at the time $\tau=t$. In particular,

$$v_{i,i_1\ldots i_\beta}=B_{ii_1\ldots i_\beta}^{(1)}. \tag{3.14}$$

Explicit expressions for the tensors $B_{ii_1\ldots i_\beta}^{(\alpha)}$, for $\alpha=2, 3, \ldots$, which are symmetric in the indices $i_1, i_2, \ldots, i_\beta$, can be obtained by repeated application of

(3.13). Also, from (3.3) and (3.12), we have

$$B^*_{i i_1 \ldots i_\beta}(\tau^*) = Q_{ij} Q_{i_1 j_1} \cdots Q_{i_\beta j_\beta} B_{j j_1 \ldots j_\beta}(\tau), \tag{3.15}$$

and hence

$$B^{*(\alpha)}_{i i_1 \ldots i_\beta} = Q_{ij} Q_{i_1 j_1} \cdots Q_{i_\beta j_\beta} B^{(\alpha)}_{j j_1 \ldots j_\beta} \qquad (\alpha = 1, 2, \ldots). \tag{3.16}$$

A simple relation exists between the tensors $A^{(\mu)}_{i i_1 \ldots i_\beta}$ and $B^{(\alpha)}_{i i_1 \ldots i_\beta}$. Since

$$E_{i i_1 \ldots i_\beta}(\tau) = E_{ij}(\tau) B_{j i_1 \ldots i_\beta}(\tau) \qquad (\beta = 2, 3, \ldots), \tag{3.17}$$

we may differentiate this μ-times with respect to τ and then put $\tau = t$, to obtain the relation

$$A^{(\mu)}_{i i_1 \ldots i_\beta} = \sum_{\alpha=1}^{\mu} \binom{\mu}{\alpha} A^{(\mu-\alpha)}_{ij} B^{(\alpha)}_{j i_1 \ldots i_\beta} \qquad (\beta = 2, 3, \ldots). \tag{3.18}$$

To close this section we repeat one known result which will be used later. From (3.1) we have

$$v^*_i(\tau^*) = \dot{c}^*_i(\tau^*) + Q_{ij}(\tau)\left[v_j(\tau) - \dot{c}_j(\tau)\right] + \Omega_{ir}(\tau)\left[x^*_r(\tau^*) - c^*_r(\tau^*)\right] \tag{3.19}$$

where

$$\dot{Q}_{ij}(\tau) = \Omega_{ir}(\tau) Q_{rj}(\tau), \qquad \Omega_{ij}(\tau) = -\Omega_{ji}(\tau). \tag{3.20}$$

From (3.19) we have

$$\partial v^*_i(\tau^*)/\partial x^*_j(\tau^*) = Q_{ir}(\tau) Q_{js}(\tau)\, \partial v_r(\tau)/\partial x_s(\tau) + \Omega_{ij}(\tau). \tag{3.21}$$

In particular we can recover the result (3.7) from this when $\mu = 1$, where

$$A_{ij} = v_{i,j} + v_{j,i}. \tag{3.22}$$

In addition, if

$$\omega_{ij} = v_{i,j} - v_{j,i}, \tag{3.23}$$

then

$$\omega^*_{ij} = Q_{ir} Q_{js} \omega_{rs} + 2\Omega_{ij}. \tag{3.24}$$

4. Multipolar body forces of the first kind*

If F_i is a vector and v_i an arbitrary velocity field, and if the scalar

$$F_i v_i \tag{4.1}$$

is a rate of work, per unit mass, at time t, then the vector F_i is called the body force vector, per unit mass. The total rate of work of a body force F_i, per unit mass, distributed throughout a volume V of the continuum, is

$$\int_V \varrho F_i v_i \, dV \tag{4.2}$$

where ϱ is the density (at time t).

If $F_{i_1 \ldots i_\nu i}$ is a tensor and $v_{i, i_1 \ldots i_\nu}$ an arbitrary set of velocity gradients, and if the scalar

$$F_{i_1 \ldots i_\nu i} v_{i, i_1 \ldots i_\nu} \tag{4.3}$$

is a rate of work per unit mass, then the tensor $F_{i_1 \ldots i_\nu i}$ is called *a simple distributed*

* A possible motivation for the definitions presented here is given in Appendix 1.

body force 2^ν-pole of the first kind, per unit mass. More briefly, it *is a simple body force 2^ν-pole of the first kind, per unit mass.* We observe that $F_{i_1 \ldots i_\nu i}$ may be taken to be symmetric in the indices $i_1, i_2, \ldots, i_\nu$ without loss of generality, provided the order of differentiation in the velocity gradient is immaterial. When $\nu = 1$ we may also call the force system a *simple body force dipole of the first kind;* when $\nu = 2$, a *simple body force quadripole of the first kind.* For uniformity, when $\nu = 0$, we may call it a *simple body force monopole* which is the same as a body force vector. Generically, we may call simple body force 2^ν-poles of the first kind: *simple body force multipoles of the first kind.* Throughout this paper we shall be restricted to the first kind of multipoles and for brevity the words "first kind" may frequently be omitted.

The total rate of work of a body force 2^ν-pole, per unit mass, distributed throughout a volume V, is

$$\int_V \varrho\, F_{i_1 \ldots i_\nu i}\, v_{i, i_1 \ldots i_\nu}\, dV. \tag{4.4}$$

5. Multipolar stresses of the first kind

Consider a surface A whose unit normal at the point x_i, in a specified direction, is n_i. If t_i is a vector and if, for all arbitrary velocity fields v_i, the scalar

$$t_i\, v_i \tag{5.1}$$

is a rate of work per unit area of A, then the vector t_i is called the distributed force, per unit area. The total rate of work of this surface force over the whole surface A is

$$\int_A t_i\, v_i\, dA. \tag{5.2}$$

If $t_{i_1 \ldots i_\nu i}$ is a tensor and if, for all arbitrary velocity gradients $v_{i, i_1 \ldots i_\nu}$, the scalar

$$t_{i_1 \ldots i_\nu i}\, v_{i, i_1 \ldots i_\nu} \tag{5.3}$$

is a rate of work per unit area of A, then the tensor $t_{i_1 \ldots i_\nu i}$ is called a *simple distributed surface force 2^ν-pole of the first kind,* per unit area or, more briefly, a *simple surface force 2^ν-pole of the first kind,* per unit area. The tensor $t_{i_1 \ldots i_\nu i}$ may be taken to be completely symmetric in the indices $i_1, \ldots, i_\nu$ without loss of generality.

The total rate of work of a surface force 2^ν-pole, over a surface A, is

$$\int_A t_{i_1 \ldots i_\nu i}\, v_{i, i_1 \ldots i_\nu}\, dA. \tag{5.4}$$

When $\nu = 0$ we recover (5.2).

The tensor $t_{i_1 \ldots i_\nu i}$ at x_i is associated with a surface whose unit normal at the point is n_j, so that if n_j is altered the tensor is altered. When n_j is a unit normal to the x_j-plane through the point we denote the corresponding tensor by

$$\sigma_{j i_1 \ldots i_\nu i}. \tag{5.5}$$

These are the components of a simple surface stress 2^ν-pole tensor of the first kind on an element of area at the point, normal to the x_j-axis. In particular, when $\nu=0$, we recover the usual classical stress tensor σ_{ji}. The rate of work of the 2^ν-pole surface tensor (5.5) is

$$\sigma_{j i_1 \ldots i_\nu i}\, v_{i, i_1 \ldots i_\nu} \tag{5.6}$$

per unit area of the surface normal to the x_j-axis.

The first index j is not necessarily a tensor index under change of axes, but indicates the surface on which the stress acts, the surface being fixed.

6. The energy equations and entropy production inequality

We consider an arbitrary material volume V of the continuum bounded by a surface A at time t. We assume that simple body force 2^α-poles ($\alpha=0, 1, \ldots, \nu$) of the first kind, per unit mass, act throughout V and that simple surface force 2^α-poles ($\alpha=0, 1, \ldots, \nu$) of the first kind, per unit area, act across A. We also assume that there is an internal energy function U per unit mass, an entropy function S, per unit mass, a heat supply function* r per unit mass and unit time (absorbed by the material and furnished by radiation from the external world), a local temperature T, which is assumed to be always positive, and a heat flux vector** Q_i, where Q_i is the flux of heat across a plane at x_i perpendicular to the x_i-axis, per unit area, per unit time. All these functions depend on X_1, X_2, X_3, t. We postulate an energy balance in the form***

$$\begin{aligned}\int_V \varrho v_i \dot{v}_i \, dV + \int_V \varrho \dot{U}\, dV = \int_V \Big[\varrho r + \varrho \sum_{\beta=0}^{\nu} F_{i_1 \ldots i_\beta i} v_{i, i_1 \ldots i_\beta}\Big] dV - \\ - \int_A h\, dA + \int_A \sum_{\beta=0}^{\nu} t_{i_1 \ldots i_\beta i}\, v_{i, i_1 \ldots i_\beta}\, dA,\end{aligned} \tag{6.1}$$

where h is the heat flux across the surface A, per unit area, whose unit outward normal is n_i and a dot denotes material time derivative. We also postulate an entropy production inequality

$$\int_V \varrho \dot{S}\, dV - \int_V \varrho \frac{r}{T}\, dV + \int_A \frac{h}{T}\, dA \geqq 0. \tag{6.2}$$

We now take the volume V in (6.1) to be a tetrahedral element bounded by a plane with arbitrary unit normal n_i and by planes through the point x_i parallel to the coordinate planes. If dA is the area of the plane of the tetrahedron normal to n_i, and dA_j is the element of area of the plane of the tetrahedron normal to the x_j-axis, then

$$dA_j = n_j\, dA. \tag{6.3}$$

* See COLEMAN & NOLL (1963).

** We restrict attention here to the usual heat flux vector although it may be possible to define multipolar heat flux tensors.

*** For completeness the kinetic energy should also contain a quadratic form in velocity gradients of all orders up to ν, but this is omitted in the present paper. The resulting inertia terms can, however, be included by replacing multipolar body forces by: multipolar body forces minus the appropriate multipolar inertia terms. This will be assumed throughout the paper even when it is not stated explicitly.

If we apply equation (6.1) to the tetrahedron and let the tetrahedron shrink to zero while preserving the orientation of its faces, we obtain the equation

$$(t_i - n_j \sigma_{ji}) v_i + \sum_{\beta=1}^{\nu} (t_{i_1 \ldots i_\beta i} - n_j \sigma_{j i_1 \ldots i_\beta i}) v_{i, i_1 \ldots i_\beta} - h + n_i Q_i = 0, \tag{6.4}$$

if we use (6.3) and assume that the contributions from the volume integrals tend to zero more rapidly than those from the surface integrals.

Equation (6.4) is valid for all velocity distributions. We assume that the multipolar stress tensors $t_{i_1 \ldots i_\beta i}$, $\sigma_{j i_1 \ldots i_\beta i}$, the heat flux h, and the heat flux vector Q_i are unaltered by *constant* superposed rigid body velocities*. If we use equation (6.4) with v_i replaced by $v_i + a_i$, where a_i is an arbitrary constant velocity vector, we have

$$(t_i - n_j \sigma_{ji}) (v_i + a_i) + \sum_{\beta=1}^{\nu} (t_{i_1 \ldots i_\beta i} - n_j \sigma_{j i_1 \ldots i_\beta i}) v_{i, i_1 \ldots i_\beta} - h + n_i Q_i = 0. \tag{6.5}$$

Hence

$$(t_i - n_j \sigma_{ji}) a_i = 0$$

for all arbitrary a_i, and since $t_i - n_j \sigma_{ji}$ is independent of a_i,

$$t_i = n_j \sigma_{ji}, \tag{6.6}$$

and (6.4) reduces to

$$\sum_{\beta=1}^{\nu} (t_{i_1 \ldots i_\beta i} - n_j \sigma_{j i_1 \ldots i_\beta i}) v_{i, i_1 \ldots i_\beta} - h + n_i Q_i = 0. \tag{6.7}$$

With the help of (3.8), (3.10), (3.22) and (3.23), equation (6.7) becomes

$$\tfrac{1}{2} (t_{i_1 i} - n_j \sigma_{j i_1 i}) (A_{i i_1} + \omega_{i i_1}) + \sum_{\beta=2}^{\nu} (t_{i_1 \ldots i_\beta i} - n_j \sigma_{j i_1 \ldots i_\beta i}) A_{i i_1 \ldots i_\beta} - h + n_i Q_i = 0. \tag{6.8}$$

We next assume that $t_{i_1 \ldots i_\beta i}$, $\sigma_{j i_1 \ldots i_\beta i}$, h and Q_i are unaltered by superposed uniform rigid body angular velocity, the continuum occupying the same position at time t. Under these conditions we see, from Section 3, that $A_{i i_1 \ldots i_\beta}$ are unaltered but that $\omega_{i i_1}$ becomes $\omega^*_{i i_1}$ where, from (3.24),

$$\omega^*_{i i_1} = \omega_{i i_1} + 2\Omega_{i i_1} \tag{6.9}$$

when $Q_{ij} = \delta_{ij}$. Hence, from (6.8) we deduce that

$$(t_{i_1 i} - n_j \sigma_{j i_1 i}) \Omega_{i i_1} = 0$$

for all arbitrary anti-symmetric tensors $\Omega_{i i_1}$, so that

$$t_{i i_1} - t_{i_1 i} - n_j (\sigma_{j i i_1} - \sigma_{j i_1 i}) = 0, \tag{6.10}$$

since this expression is independent of $\Omega_{i i_1}$. Also, equation (6.8) reduces to

$$\tfrac{1}{2} (t_{i_1 i} - n_j \sigma_{j i_1 i}) A_{i i_1} + \sum_{\beta=2}^{\nu} (t_{i_1 \ldots i_\beta i} - n_j \sigma_{j i_1 \ldots i_\beta i}) A_{i i_1 \ldots i_\beta} - h + n_i Q_i = 0. \tag{6.11}$$

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* The independent thermodynamic variable, which can be taken to be either S or T, is assumed to be unchanged.

It appears to be impossible to make any further deductions from (6.11) until constitutive equations have been obtained for the multipolar stresses and the heat conduction vector.

We return to the energy equation (6.1) and use equation (6.4), to obtain

$$\int_V (\varrho v_i \dot{v}_i + \varrho \dot{U})\, dV = \int_V \Big(\varrho r + \varrho \sum_{\beta=0}^{\nu} F_{i_1 \dots i_\beta i} v_{i, i_1 \dots i_\beta}\Big) dV - \int_A n_i Q_i\, dA + \\ + \int_A \sum_{\beta=0}^{\nu} n_j \sigma_{j i_1 \dots i_\beta i} v_{i, i_1 \dots i_\beta}\, dA$$

for all arbitrary volumes V. By transforming the surface integrals to volume integrals in the usual way and making appropriate smoothness assumptions, we obtain the equation*

$$\begin{aligned} &(\sigma_{ji,j} + \varrho F_i - \varrho \dot{v}_i) v_i + \varrho r - Q_{i,i} - \varrho \dot{U} + (\sigma_{j i_1 i, j} + \varrho F_{i_1 i} + \sigma_{i_1 i}) v_{i, i_1} + \\ &+ \sum_{\beta=2}^{\nu}{}' (\sigma_{j i_1 \dots i_\beta i, j} + \sigma_{i_\beta i_1 \dots i_{\beta-1} i} + \varrho F_{i_1 \dots i_\beta i}) v_{i, i_1 \dots i_\beta} + \\ &+ \sigma_{i_{\nu+1} i_1 \dots i_\nu i} v_{i, i_1 \dots i_\nu i_{\nu+1}} = 0. \end{aligned} \tag{6.12}$$

We recall that $\sigma_{j i_1 \dots i_\beta i}$ (and $t_{i_1 \dots i_\beta i}$) are completely symmetric with respect to the indices $i_1, \dots, i_\beta$ ($\beta = 2, 3, \dots$), but not necessarily with respect to the index j.

In addition to the invariance restrictions already imposed on $t_{i_1 \dots i_\beta i}$, $\sigma_{j i_1 \dots i_\beta i}$, h and Q_i when the motion is altered by superposed uniform rigid body velocities and angular velocities, the continuum occupying instantaneously the same position at time t, we assume that $\dot{U}$ is unaltered by such rigid body motions and that the body forces $F_i, F_{i_1 i}, \dots$, and heat supply function r are unaltered by superposed *uniform* rigid body velocities. We observe that $\dot{v}_i$ is unaltered by such velocities so that by considering equation (6.12) for all velocities $v_i + a_i$, where a_i is an arbitrary constant, we see that

$$\sigma_{ji,j} + \varrho F_i = \varrho \dot{v}_i, \tag{6.13}$$

the classical equation of motion. Also, equation (6.12) reduces to

$$\begin{aligned} &\varrho r - Q_{i,i} - \varrho \dot{U} + \tfrac{1}{2} (\sigma_{j i_1 i, j} + \varrho F_{i_1 i} + \sigma_{i_1 i}) (A_{i i_1} + \omega_{i i_1}) + \\ &+ \sum_{\beta=2}^{\nu}{}' (\sigma_{j i_1 \dots i_\beta i, j} + \sigma_{i_\beta i_1 \dots i_{\beta-1} i} + \varrho F_{i_1 \dots i_\beta i}) A_{i i_1 \dots i_\beta} + \\ &+ \sigma_{i_{\nu+1} i_1 \dots i_\nu i} A_{i i_1 \dots i_\nu i_{\nu+1}} = 0. \end{aligned} \tag{6.14}$$

If we make the additional assumption that r and the multipolar body forces $F_{i_1 \dots i_\beta i}$ ($\beta = 2, 3, \dots, \nu$) are unaltered by superposed uniform rigid body angular velocities, the body occupying the same position at time t, then we see that**

* The prime in $\sum'$ denotes that the terms under the summation sign are omitted when $\nu = 1$.

** This is the classical vector moment equation. When dipolar stresses and body forces are absent we recover the usual result that $\sigma_{i i_1}$ is symmetric. When multipolar inertia terms are included then we assume that multipolar body forces minus the appropriate inertia terms are unaltered by superposed uniform rigid body angular velocities.

$$\sigma_{j i i_1, j} + \sigma_{i i_1} + \varrho F_{i i_1} = \sigma_{j i_1 i, j} + \sigma_{i_1 i} + \varrho F_{i_1 i}, \tag{6.15}$$

and

$$\begin{aligned} \varrho r - Q_{i,i} - \varrho \dot{U} + \sigma_{i_{\nu+1} i_1 \ldots i_\nu i} A_{i i_1 \ldots i_{\nu+1}} + \tfrac{1}{2}(\sigma_{j i_1 i, j} + \varrho F_{i_1 i} + \sigma_{i_1 i}) A_{i i_1} + \\ + \sum_{\beta=2}^{\nu}{}' (\sigma_{j i_1 \ldots i_\beta i, j} + \sigma_{i_\beta i_1 \ldots i_{\beta-1} i} + \varrho F_{i_1 \ldots i_\beta i}) A_{i i_1 \ldots i_\beta} = 0. \end{aligned} \tag{6.16}$$

It appears that information about the constitutive equations for stresses, the heat conduction vector and internal energy is required before any further deductions can be made from equation (6.16). A case of some interest arises when these quantities do not depend on velocity gradients of any order and this will be discussed in more detail in Section 7. In general, constitutive equations must be postulated for $t_{i_1 \ldots i_\beta i}$, h, $\sigma_{j i_1 \ldots i_\beta i}$, Q_i and U and then (6.10), (6.11), (6.15) and (6.16) provide restrictions to be imposed on these equations.

7. Generalized elasticity

Here we suppose that x_i and S are specified functions of X_1, X_2, X_3 and t and we define a generalized elastic material as one for which the following constitutive equations hold at each material point X_i and for all time t:

$$U = U(S, x_{i,A_1}, x_{i,A_1 A_2}, \ldots, x_{i,A_1 A_2 \ldots A_\mu}), \tag{7.1}$$

$$\sigma_{j i_1 \ldots i_\beta i} = \sigma_{j i_1 \ldots i_\beta i}(S, x_{i,A_1}, x_{i,A_1 A_2}, \ldots, x_{i,A_1 A_2 \ldots A_\mu}), \tag{7.2}$$

$$t_{i_1 \ldots i_\beta i} = t_{i_1 \ldots i_\beta i}(S, n_j, x_{i,A_1}, x_{i,A_1 A_2}, \ldots, x_{i,A_1 A_2 \ldots A_\mu}), \tag{7.3}$$

$$T = T(S, x_{i,A_1}, x_{i,A_1 A_2}, \ldots, x_{i,A_1 A_2 \ldots A_\mu}), \tag{7.4}$$

$$Q_i = Q_i(S, x_{i,A_1}, x_{i,A_1 A_2}, \ldots, x_{i,A_1 A_2 \ldots A_\mu}, T_{,i_1}, T_{,i_1 i_2}, \ldots, T_{,i_1 i_2 \ldots i_\mu}), \tag{7.5}$$

$$h = h(S, x_{i,A_1}, x_{i,A_1 A_2}, \ldots, x_{i,A_1 A_2 \ldots A_\mu}, T_{,i_1}, T_{,i_1 i_2}, \ldots, T_{,i_1 i_2 \ldots i_\mu}, n_j), \tag{7.6}$$

where $\beta = 0, 1, \ldots, \nu$ and $\mu \geqq \nu + 1$, and all functions are assumed to be single-valued and sufficiently smooth*.

For a given deformation, the rate of deformation tensors $A_{i i_1}, A_{i i_1 i_2}, \ldots, A_{i i_1 \ldots i_\nu}$ in (6.11) may be chosen arbitrarily and independently of each other so that (repeating equation (6.6) for completeness)

$$\begin{aligned} t_i &= n_j \sigma_{j i}, \\ t_{i_1 \ldots i_\beta i} &= n_j \sigma_{j i_1 \ldots i_\beta i}, \end{aligned} \tag{7.7}$$

and

$$h = n_i Q_i. \tag{7.8}$$

Equation (6.10) is now satisfied automatically.

From (7.7) we see that $\sigma_{j i_1 \ldots i_\beta i}$ transforms as a tensor with respect to all indices, including j, under changes of rectangular Cartesian axes, where the multipolar stresses in each coordinate system are associated with the three coordinate planes in that system.

* The multipolar stresses may also depend on the multipolar body forces $F_{i_1 \ldots i_\beta i}$ ($\beta = 1, \ldots, \nu$). See the footnote on p. 349 for an improved form of constitutive assumptions.

Using (7.8) in equation (6.2), transforming the surface integral to a volume integral, and making the usual smoothness assumptions, we have

$$\varrho\dot{S} - \frac{\varrho r}{T} + \left(\frac{Q_i}{T}\right)_{,i} \geqq 0$$

or

$$\varrho\dot{S} - \frac{\varrho r}{T} + \frac{Q_{i,i}}{T} - \frac{Q_i T_{,i}}{T^2} \geqq 0 \tag{7.9}$$

since (6.2) applies for all arbitrary volumes V in the continuum.

Substituting for r from (6.16) into (7.9) and recalling that $T>0$ we obtain the inequality

$$\begin{aligned}\varrho(T\dot{S} - \dot{U}) - \frac{Q_i T_{,i}}{T} + \sigma_{i_{\nu+1} i_1 \dots i_\nu i} A_{i i_1 \dots i_{\nu+1}} + \frac{1}{2}(\sigma_{j i_1 i,j} + \varrho F_{i_1 i} + \sigma_{i_1 i}) A_{i i_1} + \\ + \sum_{\beta=2}^{\nu}{}' (\sigma_{j i_1 \dots i_\beta i,j} + \sigma_{i_\beta i_1 \dots i_{\beta-1} i} + \varrho F_{i_1 \dots i_\beta i}) A_{i i_1 \dots i_\beta} \geqq 0.\end{aligned} \tag{7.10}$$

Before making further deductions from this equation, it is convenient to make use of the invariance property of U under superposed rigid body rotations. Using the notation of Section 3 the function U satisfies the condition

$$U(S, x_{i,A_1}, x_{i,A_1 \dots A_\beta}) = U(S, x^*_{i,A_1}, x^*_{i,A_1 \dots A_\beta})$$

where β takes the values $2, 3, \dots, \mu$. In view of (3.2) this equation becomes

$$U(S, x_{i,A_1}, x_{i,A_1 \dots A_\beta}) = U(S, Q_{ij} x_{j,A_1}, Q_{ij} x_{j,A_1 \dots A_\beta}), \tag{7.11}$$

for all proper orthogonal values of Q_{ij}. It follows directly, as a special case of a result obtained by PIPKIN & RIVLIN (1959), that U must be expressible as a single-valued function of S and $E_{AA_1 \dots A_\beta}$ $(\beta = 1, \dots, \mu)$, thus:

$$U = U(S, E_{AA_1}, \dots, E_{AA_1 \dots A_\mu}). \tag{7.12}$$

Alternatively, the Schmidt orthogonalization procedure may be used to obtain (7.12) from (7.11) in a manner analogous to that employed in a different context by PIPKIN & RIVLIN (1961). We shall sketch here another procedure for obtaining (7.12) which is similar to that used by NOLL (1955) in another connection.

Since we are concerned with the value of U at a particular particle X_A we may take the special value R_{jA} for Q_{Aj} in (7.11), so that

$$\begin{aligned} U &= U(S, M_{AA_1}, R_{jA} x_{j,A_1 \dots A_\beta}) \\ &= U(S, M_{AA_1}, M^{-1}_{AB} x_{j,A} x_{j,A_1 \dots A_\beta}), \end{aligned} \tag{7.13}$$

since M_{AB} is non-singular. We recall * the definitions (3.6) for $E_{AA_1 \dots A_\beta}$ $(\beta = 1, 2 \dots)$ and observe that

$$E_{AB} = M_{AA_1} M_{A_1 B}. \tag{7.14}$$

Since M_{AB} is a positive definite symmetric tensor satisfying (7.14), a single-valued function of M_{AB} can be replaced by a single-valued function of E_{AB}, so that (7.13) can be replaced by the different form (7.12). We can verify that this satisfies the condition (7.11) for arbitrary proper orthogonal values of Q_{ij}.

* E_{AB} is symmetric in A, B and $E_{AA_1 \dots A_\beta}$ is completely symmetric in $A_1, A_2, \dots, A_\beta$.

In order to illustrate the use of equation (7.10) we consider, for simplicity, the case when only monopolar and dipolar stresses and body forces are present. The general case follows in a similar manner apart from extra algebraic complexity. Equation (7.10) reduces to

$$\varrho(T\dot{S}-\dot{U})-\frac{Q_i T_{,i}}{T}+\frac{1}{2}(\sigma_{j i_1 i,j}+\varrho F_{i_1 i}+\sigma_{i_1 i})A_{i i_1}+\sigma_{i_2 i_1 i}A_{i i_1 i_2}\geqq 0, \quad (7.15)$$

where

$$\begin{aligned}\dot{U}=\frac{\partial U}{\partial S}\dot{S}+\frac{\partial U}{\partial E_{AB}}\dot{E}_{AB}+\frac{\partial U}{\partial E_{AA_1A_2}}\dot{E}_{AA_1A_2}+\\ +\cdots+\frac{\partial U}{\partial E_{AA_1\ldots A_\mu}}\dot{E}_{AA_1\ldots A_\mu}\qquad(\mu\geqq 2).\end{aligned}\quad (7.16)$$

To avoid ambiguity we assume that U in (7.16) is arranged as a symmetric function of E_{AA_1} and a symmetric function of $E_{AA_1\ldots A_\beta}$ $(\beta=2,\ldots,\mu)$ as far as the indices $A_1,\ldots,A_\beta$ are concerned. From Section 3 we have

$$\dot{E}_{AB}=A_{ij}\,x_{i,A}\,x_{j,B}, \quad (7.17)$$

$$\dot{E}_{AA_1A_2}=\tfrac{1}{2}A_{ij}(x_{i,A}\,x_{j,A_1A_2}+x_{j,A}\,x_{i,A_1A_2})+A_{i i_1 i_2}\,x_{i,A}\,x_{i_1,A_1}\,x_{i_2,A_2}, \quad (7.18)$$

so that, with the help of (7.16), the inequality (7.15) becomes

$$\begin{aligned}&\varrho\Big(T-\frac{\partial U}{\partial S}\Big)\dot{S}-\frac{Q_i T_{,i}}{T}+\Big(\sigma_{i_2 i_1 i}-\varrho\,x_{i,A}\,x_{i_1,A_1}\,x_{i_2,A_2}\frac{\partial U}{\partial E_{AA_1A_2}}\Big)A_{i i_1 i_2}+\\ &+\frac{1}{2}\Big[\sigma_{j i_1 i,j}+\varrho F_{i_1 i}+\sigma_{i_1 i}-\varrho(x_{i,A}\,x_{i_1,A_1A_2}+x_{i_1,A}\,x_{i,A_1A_2})\frac{\partial U}{\partial E_{AA_1A_2}}-\\ &-2\varrho\,x_{i,A}\,x_{i_1,B}\frac{\partial U}{\partial E_{AB}}\Big]A_{i i_1}-\\ &-\varrho\frac{\partial U}{\partial E_{AA_1A_2A_3}}\dot{E}_{AA_1A_2A_3}-\cdots-\varrho\frac{\partial U}{\partial E_{AA_1\ldots A_\mu}}\dot{E}_{AA_1\ldots A_\mu}\geqq 0.\end{aligned}\quad (7.19)$$

For a given deformation and entropy, at a particular time, this inequality must be satisfied for all arbitrarily* assigned values of $\dot{S}$ and velocity gradients $A_{i i_1}, A_{i i_1 i_2},\ldots,A_{i i_1\ldots i_\mu}$. Now $\dot{E}_{AA_1\ldots A_\mu}$ $(\mu\geqq 3)$ can be expressed in terms of $A_{i i_1\ldots i_\beta}$ $(\beta=\mu,\mu-1,\ldots)$. We choose $\dot{S}$, $A_{i i_1},\ldots,A_{i i_1\ldots i_{\mu-1}}$ to be zero so that (7.19) becomes

$$-\frac{Q_i T_{,i}}{T}-\varrho\frac{\partial U}{\partial E_{AA_1\ldots A_\mu}}x_{i,A}\,x_{i_1,A_1}\cdots x_{i_\mu,A_\mu}A_{i i_1\ldots i_\mu}\geqq 0$$

for all arbitrary values of $A_{i i_1\ldots i_\mu}$, positive or negative. In general this will only be possible if

$$\frac{\partial U}{\partial E_{AA_1\ldots A_\mu}}x_{i,A}\,x_{i_1,A_1}\cdots x_{i_\mu,A_\mu}=0$$

or, since $x_{i,A}$ is non-singular,

$$\frac{\partial U}{\partial E_{AA_1\ldots A_\mu}}=0. \quad (7.20)$$

* Subject to symmetries in the indices which are already taken into account by the manner in which U has been symmetrized. We can choose $\dot{S}$ independently of the velocity gradients and then r, the heat supply, is determined from (6.16).

Hence U is completely independent of $E_{AA_1\ldots A_\mu}$. Similarly, we can show that U is independent of $E_{AA_1\ldots A_{\mu-1}}, \ldots, E_{AA_1A_2A_3}$. Equation (7.19) then reduces to

$$\varrho\left(T-\frac{\partial U}{\partial S}\right)\dot{S}-\frac{Q_i T_{,i}}{T}+\left(\sigma_{i_2 i_1 i}-\varrho\, x_{i,A}\, x_{i_1,A_1}\, x_{i_2,A_2}\frac{\partial U}{\partial E_{AA_1A_2}}\right)A_{i i_1 i_2}+$$
$$+\frac{1}{2}\left[\sigma_{j i_1 i,j}+\varrho\, F_{i_1 i}+\sigma_{i_1 i}-\varrho\,(x_{i,A}\, x_{i_1,A_1A_2}+x_{i_1,A}\, x_{i,A_1A_2})\frac{\partial U}{\partial E_{AA_1A_2}}-\right. \tag{7.21}$$
$$\left.-2\varrho\, x_{i,A}\, x_{i_1,B}\frac{\partial U}{\partial E_{AB}}\right]A_{i i_1}\geqq 0$$

for all arbitrary values of $\dot{S}$, $A_{i i_1}$, $A_{i i_1 i_2}$, at a given deformation and entropy, at a particular time, where now

$$U=U(E_{AB}, E_{AA_1A_2}, S). \tag{7.22}$$

Following an argument similar to that used above we see that*

$$T=\frac{\partial U}{\partial S}, \tag{7.23}$$

$$\sigma_{(i_2 i_1)i}=\varrho\, x_{i,A}\, x_{i_1,A_1}\, x_{i_2,A_2}\frac{\partial U}{\partial E_{AA_1A_2}}, \tag{7.24}$$

$$\sigma_{i_1 i}+\sigma_{j i_1 i,j}+\varrho\, F_{i_1 i}=\varrho\,(x_{i,A}\, x_{i_1,A_1A_2}+x_{i_1,A}\, x_{i,A_1A_2})\frac{\partial U}{\partial E_{AA_1A_2}}+$$
$$+2\varrho\, x_{i,A}\, x_{i_1,B}\frac{\partial U}{\partial E_{AB}}, \tag{7.25}$$

and

$$-Q_i T_{,i}\geqq 0, \tag{7.26}$$

where $\sigma_{(i_2 i_1)i}$ is the part of $\sigma_{i_2 i_1 i}$ symmetric with respect to i_1, i_2. Moreover, if we substitute the results (7.22)—(7.26) into (6.16) for the case when only monopolar and dipolar stresses and body forces are present, we have

$$\varrho r-Q_{i,i}-\varrho T\dot{S}=0. \tag{7.27}$$

We observe that equation (6.15) is now satisfied identically by (7.25).

From (7.24) we see that only the symmetric part $\sigma_{(i_2 i_1)i}$ of $\sigma_{i_2 i_1 i}$ is given in terms of the internal energy function U, while the skew symmetric part $\sigma_{[i_2 i_1]i}$ is undetermined. If the body force $F_{i_1 i}$ is specified then equation (7.25) shows that the stress $\sigma_{i_1 i}$ is undetermined to the extent of an additive stress $-\sigma_{[j i_1]i,j}$. Since

$$\sigma_{[j i_1]i,j i_1}=0 \tag{7.28}$$

it follows that the stress $-\sigma_{[j i_1]i,j}$ makes no contribution to the equations of motion (6.13). Moreover, the rate of working of the stress $-\sigma_{[j i_1]i,j}$ and the dipolar stress $\sigma_{[i_2 i_1]i}$ over any closed surface A inside the body, or over the complete boundary of the body, is

$$\int_A(-n_k\sigma_{[jk]i,j}v_i+n_k\sigma_{[kj]i}v_{i,j})\,dA$$
$$=\int_A n_k(\sigma_{[kj]i}v_i)_{,j}dA=\int_V(\sigma_{[kj]i}v_i)_{,jk}dV=0. \tag{7.29}$$

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* The multipolar body forces $F_{i_1 i}\ldots$ are assumed to be given at time t. The arbitrary choice of velocity gradients of all orders at the particle x_i is possible if the body force F_i is chosen suitably throughout the volume.

In deriving the equations of the present section from the equations of energy and entropy balance, we made assumptions (7.1)—(7.6). When we confine attention to stress and dipolar stress we see, from (7.19), that it is only necessary to assume* that $\sigma_{(i_1 i_2) i}$ and $\sigma_{i_1 i} + \varrho F_{i_1 i} + \sigma_{j i_1 i, j}$ depend on the functions displayed in (7.2). It follows that the undetermined part $\sigma_{[i_1 i_2] i}$ of the dipolar stress may be regarded as an arbitrary function of position and time, with a corresponding contribution $-\sigma_{[j i_1] i, j}$ in the stress $\sigma_{i_1 i}$. We shall see in Section 9 that the surface values of this system of stresses and dipolar stresses play an important role in determining correct boundary conditions.

Formulae for the general case when multipolar surface forces and body forces up to order 2^ν are present may be found by a similar process. It is, however, somewhat more convenient to obtain such results in a different notation, and details of this are given in Section 8.

For some purposes it is useful to express stresses in terms of the Helmholtz free energy function

$$A = U - TS, \tag{7.30}$$

where, with the help of (7.23), S is expressed as a function of T and E_{AB}, $E_{AA_1A_2}$, and A is also expressed as a function of these same quantities, so that

$$A = A(E_{AB}, E_{AA_1A_2}, T). \tag{7.31}$$

From (7.23), (7.30) and (7.31) we then have

$$S = -\frac{\partial A}{\partial T}. \tag{7.32}$$

Also, (7.24) and (7.25) become

$$\sigma_{(i_1 i_2) i} = \varrho\, x_{i,A}\, x_{i_1,A_1}\, x_{i_2,A_2} \frac{\partial A}{\partial E_{AA_1A_2}}, \tag{7.33}$$

$$\begin{aligned} &\sigma_{i_1 i} + \sigma_{j i_1 i, j} + \varrho F_{i_1 i} \\ &= \varrho\,(x_{i,A}\, x_{i_1,A_1A_2} + x_{i_1,A}\, x_{i,A_1A_2}) \frac{\partial A}{\partial E_{AA_1A_2}} + 2\varrho\, x_{i,A}\, x_{i_1,B} \frac{\partial A}{\partial E_{AB}}. \end{aligned} \tag{7.34}$$

8. Elasticity: Alternative form

In this section we give an alternative formulation for the theory of generalized elasticity discussed in Section 7, which is more convenient when multipolar stresses of order greater than 2 are present. As in Section 6 we consider an arbitrary material volume V in the continuum bounded by a surface A at time t, and we suppose that V_0 is the corresponding volume in the initial undeformed state of the continuum, bounded by a surface A_0. Let the outward unit normal at A_0 be n_A, referred to our fixed rectangular frame of reference. We now define a force vector p_i, associated with the surface A but measured per unit area of the surface A_0, in a manner similar to that used in Section 5 in defining t_i, so that the rate of work of this surface stress, per unit area of A_0, is

$$p_i v_i \tag{8.1}$$

and the total rate of work of this stress over the whole surface A is

$$\int_{A_0} p_i v_i \, dA_0. \tag{8.2}$$

* See also Section 10.

Similarly $p_{A_1 \ldots A_\nu i}$ is a distributed surface force 2^ν-pole of the first kind, associated with the surface A but measured per unit area of A_0, if $p_{A_1 \ldots A_\nu i}$ is a tensor such that

$$p_{A_1 \ldots A_\nu i} v_{i, A_1 \ldots A_\nu} \tag{8.3}$$

is the rate of work of the multipole*, per unit area of A_0, and

$$\int_{A_0} p_{A_1 \ldots A_\nu i} v_{i, A_1 \ldots A_\nu} dA_0 \tag{8.4}$$

is the total rate of work of the multipole over the surface A. In (8.3) and (8.4)

$$v_{i, A_1 \ldots A_\nu} = \frac{\partial^\nu v_i}{\partial X_{A_1} \ldots \partial X_{A_\nu}}. \tag{8.5}$$

The surface force multipole $p_{A_1 \ldots A_\nu i}$ is associated with a surface A but measured per unit area of A_0 whose unit normal is n_A. When n_B is a unit normal at X_A to the X_B-plane through this point we denote the corresponding stress multipole by

$$\pi_{B A_1 \ldots A_\nu i}. \tag{8.6}$$

This is a stress multipole associated with an element of area at the point x_i in V whose original position in V_0 was perpendicular to the X_B-axis, and measured per unit area of this surface in V_0. The rate of work of such a stress multipole is

$$\pi_{B A_1 \ldots A_\nu i} v_{i, A_1 \ldots A_\nu} \tag{8.7}$$

per unit area of surface in V_0, normal to the X_B-axis.

Body force F_i per unit mass may be defined as in Section 4 and the total rate of work of F_i throughout the volume V can be put in the alternative form

$$\int_{V_0} \varrho_0 F_i v_i dV_0 \tag{8.8}$$

where ϱ_0 is the density of the initial volume V_0. Similarly, multipolar body forces $F_{A_1 \ldots A_\nu i}$, per unit mass, may be defined so that their rate of work is

$$F_{A_1 \ldots A_\nu i} v_{i, A_1 \ldots A_\nu} \tag{8.9}$$

per unit mass, for all arbitrary $v_{i, A_1 \ldots A_\nu}$, and total rate of work throughout V is

$$\int_{V_0} \varrho_0 F_{A_1 \ldots A_\nu i} v_{i, A_1 \ldots A_\nu} dV_0. \tag{8.10}$$

The energy equation (6.1) is now replaced by

$$\begin{aligned} \int_{V_0} \varrho_0 v_i \dot{v}_i dV_0 + \int_{V_0} \varrho_0 \dot{U} dV_0 = \int_{V_0} \Big[\varrho_0 r + \varrho_0 \sum_{\beta=0}^{\nu} F_{A_1 \ldots A_\beta i} v_{i, A_1 \ldots A_\beta}\Big] dV_0 - \int_{V_0} h_0 dA_0 + \\ + \int_{A_0} \sum_{\beta=0}^{\nu} p_{A_1 \ldots A_\beta i} v_{i, A_1 \ldots A_\beta} dA_0, \end{aligned} \tag{8.11}$$

* $p_{A_1 \ldots A_\nu i}$ is completely symmetric with respect to the indices $A_1, A_2, \ldots, A_\nu$.

where h_0 is the flux of heat across the surface A, measured per unit area of A_0. The entropy production inequality (6.2) becomes

$$\int_{V_0} \varrho_0 \dot{S}\, dV_0 - \int_{V_0} \frac{\varrho_0 r}{T}\, dV_0 + \int_{A_0} \frac{h_0}{T}\, dA_0 \geqq 0. \tag{8.12}$$

We also assume that the heat flux vector in the volume V is q_A such that the flux of heat across a surface in V, whose original position in V_0 is perpendicular to the X_A-axis, is q_A measured per unit area of the surface in V_0.

We now take a volume V which is such that in V_0 it was a tetrahedral element bounded by a plane with arbitrary unit normal n_A, and by planes through the point X_A parallel to the coordinate planes. Then, with an argument similar to that used in obtaining (6.4), we have

$$(p_i - n_A \pi_{A\,i})\, v_i + \sum_{\beta=1}^{\nu} (p_{A_1 \dots A_\beta i} - n_B \pi_{B A_1 \dots A_\beta i})\, v_{i, A_1 \dots A_\beta} - h_0 + n_A q_A = 0. \tag{8.13}$$

We restrict further attention* only to the generalized elastic case in which

$$p_{A_1 \dots A_\beta i} = p_{A_1 \dots A_\beta i}(S, n_B, x_{i,A_1}, \dots, x_{i,A_1 \dots A_\mu}), \tag{8.14}$$

$$\pi_{B A_1 \dots A_\beta i} = \pi_{B A_1 \dots A_\beta i}(S, x_{i,A_1}, \dots, x_{i,A_1 \dots A_\mu}), \tag{8.15}$$

$$q_A = q_A(S, x_{i,A_1}, \dots, x_{i,A_1 \dots A_\mu}, T_{,A_1}, \dots, T_{,A_1 \dots A_\mu}), \tag{8.16}$$

$$h_0 = h_0(S, x_{i,A_1}, \dots, x_{i,A_1 \dots A_\mu}, T_{,A_1}, \dots, T_{,A_1 \dots A_\mu}, n_B), \tag{8.17}$$

in addition to assumption (7.1) for U. Since (8.13) is then true for all v_i, $v_{i,A_1}, \dots, v_{i,A_1 \dots A_\beta}$, which can be chosen arbitrarily and independently of each other subject to symmetries in $A_1, \dots, A_\beta$, at a given state of deformation at time t, we have

$$\begin{aligned} p_i &= n_B \pi_{B i}, \\ p_{A_1 \dots A_\beta i} &= n_B \pi_{B A_1 \dots A_\beta i}, \\ h_0 &= n_A q_A. \end{aligned} \tag{8.18}$$

With the help of (8.18), equations (8.11) and (8.12), applied to an arbitrary volume V_0, yield

$$\begin{aligned} &(\pi_{B i,B} + \varrho_0 F_i - \varrho_0 \dot{v}_i)\, v_i + \varrho_0 r - q_{A,A} - \varrho_0 \dot{U} + \\ &\quad + (\pi_{B A_1 i,B} + \varrho_0 F_{A_1 i} + \pi_{A_1 i})\, v_{i,A_1} + \\ &\quad + \sum_{\beta=2}^{\nu}{}' (\pi_{B A_1 \dots A_\beta i,B} + \pi_{A_\beta A_1 \dots A_{\beta-1} i} + \varrho_0 F_{A_1 \dots A_\beta i})\, v_{i,A_1 \dots A_\beta} + \\ &\quad + \pi_{A_{\nu+1} A_1 \dots A_\nu i}\, v_{i,A_1 \dots A_{\nu+1}} = 0, \end{aligned} \tag{8.19}$$

* Most of the results in the rest of this section are not, in general, valid for other kinds of constitutive equations. Assumptions (8.14) and (8.15) could be replaced by the assumptions that the multipole stresses do not depend explicitly on velocity gradients of all orders up to ν. See also Section 10.

and

$$\varrho_0 \dot{S} - \frac{\varrho_0 r}{T} + \frac{q_{A,A}}{T} - \frac{q_A T_{,A}}{T^2} \geqq 0. \tag{8.20}$$

If we combine (8.19) and (8.20) and use the assumption (7.1) about the form of U, we obtain the inequality

$$\begin{aligned} &\varrho_0\left(T - \frac{\partial U}{\partial S}\right)\dot{S} - \frac{q_A T_{,A}}{T} + (\pi_{B i,B} + \varrho_0 F_i - \varrho_0 \dot{v}_i)\, v_i + \\ &\quad + \left(\pi_{B A_1 i,B} + \varrho_0 F_{A_1 i} + \pi_{A_1 i} - \varrho_0 \frac{\partial U}{\partial x_{i,A_1}}\right) v_{i,A_1} + \\ &\quad + \sum_{\beta=2}^{\nu}{}' \left(\pi_{B A_1 \ldots A_\beta i,B} + \pi_{A_\beta A_1 \ldots A_{\beta-1} i} + \varrho_0 F_{A_1 \ldots A_\beta i} - \varrho_0 \frac{\partial U}{\partial x_{i,A_1 \ldots A_\beta}}\right) v_{i,A_1 \ldots A_\beta} + \\ &\quad + \left(\pi_{A_{\nu+1} A_1 \ldots A_\nu i} - \varrho_0 \frac{\partial U}{\partial x_{i,A_1 \ldots A_{\nu+1}}}\right) v_{i,A_1 \ldots A_{\nu+1}} - \\ &\quad - \varrho_0 \frac{\partial U}{\partial x_{i,A_1 \ldots A_{\nu+2}}} v_{i,A_1 \ldots A_{\nu+2}} \cdots - \varrho_0 \frac{\partial U}{\partial x_{i,A_1 \ldots A_\mu}} v_{i,A_1 \ldots A_\mu} \geqq 0. \end{aligned} \tag{8.21}$$

For a given state of deformation and entropy at time t this inequality must be valid for all values of $\dot{S}$, v_i, $v_{i,A_1}, \ldots, v_{i,A_1 \ldots A_\mu}$ which can be chosen arbitrarily and independently (r then being determined from (8.19)). It follows that U in (7.1) reduces to

$$U = U(S, x_{i,A_1}, \ldots, x_{i,A_1 \ldots A_{\nu+1}}), \tag{8.22}$$

and hence by the discussion of Section 7 it can be further reduced to the (different) form

$$U = U(S, E_{AA_1}, E_{AA_1A_2}, \ldots, E_{AA_1 \ldots A_{\nu+1}}). \tag{8.23}$$

The classical equation of motion

$$\pi_{B i,B} + \varrho_0 F_i = \varrho_0 \dot{v}_i \tag{8.24}$$

follows from (8.19) by the same argument as that used in obtaining (6.13) from (6.12). Also, from (8.21), we have

$$T = \frac{\partial U}{\partial S}, \tag{8.25}$$

$$\pi_{A_1 i} + \varrho_0 F_{A_1 i} + \pi_{B A_1 i,B} = \varrho_0 \frac{\partial U}{\partial x_{i,A_1}}, \tag{8.26}$$

$$\pi_{(A_2 A_1) i} + \varrho_0 F_{A_1 A_2 i} + \pi_{B A_1 A_2 i,B} = \varrho_0 \frac{\partial U}{\partial x_{i,A_1 A_2}}, \tag{8.27}$$

$$\pi_{(A_\nu A_1 \ldots A_{\nu-1}) i} + \varrho_0 F_{A_1 \ldots A_\nu i} + \pi_{B A_1 \ldots A_\nu i,B} = \varrho_0 \frac{\partial U}{\partial x_{i,A_1 \ldots A_\nu}}, \tag{8.28}$$

$$\pi_{(A_{\nu+1} A_1 \ldots A_\nu) i} = \varrho_0 \frac{\partial U}{\partial x_{i,A_1 \ldots A_{\nu+1}}}, \tag{8.29}$$

$$-q_A T_{,A} \geqq 0, \tag{8.30}$$

where $\pi_{(A_{\beta+1} A_1 \ldots A_\beta) i}$ ($\beta = 1, 2, \ldots, \nu$) is the completely symmetric part of $\pi_{A_{\beta+1} A_1 \ldots A_\beta i}$, the multipolar stress already being symmetric in the indices $A_1, \ldots, A_\beta$. With the help of (8.24)—(8.29), equation (8.19) reduces to

$$\varrho_0 r - q_{A,A} - \varrho_0 T \dot{S} = 0. \tag{8.31}$$

The classical equation of moments which corresponds to (6.15) has not been written down explicitly in the present notation. It can, however, be shown that such an equation is satisfied identically by the expressions (8.26)—(8.29) in view of the form (8.23) for U.

When $\nu=1$, so that only stresses and dipolar stresses are present, we have

$$\pi_{(A_2 A_1)i}=\varrho_0 \frac{\partial U}{\partial x_{i,A_1 A_2}}, \tag{8.32}$$

$$\pi_{A_1 i}=-\pi_{[B A_1]i,B}-\pi_{(B A_1)i,B}-\varrho_0 F_{A_1 i}-\varrho_0 \frac{\partial U}{\partial x_{i,A_1}}, \tag{8.33}$$

where $\pi_{[B A_1]}$ is the undetermined anti-symmetric part of $\pi_{B A_1 i}$. These results are equivalent to those obtained in Section 7 and will not be discussed further here. In order to interpret the formulae (8.26)—(8.29) when $\nu>1$ we consider in more detail the case $\nu=2$.

When $\nu=2$ we have

$$\pi_{(A_3 A_1 A_2)i}=\varrho_0 \frac{\partial U}{\partial x_{i,A_1 A_2 A_3}}, \tag{8.34}$$

where $\pi_{(A_3 A_1 A_2)i}$ is the completely symmetric part of $\pi_{A_3 A_1 A_2 i}$. We shall now write this latter quantity as $\pi_{A_3 (A_1 A_2)i}$ to emphasize that it is symmetric in the indices A_1, A_2, and we have

$$\pi_{B(A_1 A_2)i}=\pi_{(B A_1 A_2)i}+\tfrac{1}{3}\{2\pi_{B(A_1 A_2)i}-\pi_{A_1(B A_2)i}-\pi_{A_2(B A_1)i}\}. \tag{8.35}$$

The completely symmetric part $\pi_{(B A_1 A_2)i}$ of the multipolar stress $\pi_{B(A_1 A_2)i}$ is given in terms of U by (8.34) but the part in brackets { } in (8.35) is undetermined. Next, from (8.27), we have

$$\pi_{(A_2 A_1)i}=-\varrho_0 F_{A_1 A_2 i}-\pi_{B A_1 A_2 i,B}-\varrho_0 \frac{\partial U}{\partial x_{i,A_1 A_2}}. \tag{8.36}$$

Since

$$\pi_{A_2 A_1 i}=\pi_{(A_2 A_1)i}+\pi_{[A_2 A_1]i} \tag{8.37}$$

we see, from (8.36) and (8.37), that if the multipolar body force $F_{A_1 A_2 i}$ is prescribed then $\pi_{A_2 A_1 i}$ is given in terms of this and the internal energy U, apart from an undetermined additive multipolar stress

$$\pi_{[A_2 A_1]i}-\tfrac{1}{3}\{2\pi_{B(A_1 A_2)i}-\pi_{A_1(B A_2)i}-\pi_{A_2(B A_1)i}\}_{,B}. \tag{8.38}$$

The stress $\pi_{A_1 i}$ is then given by (8.26) in terms of multipolar body forces and the internal energy U apart from an undetermined additive stress

$$-\pi_{[C A_1]i,C}+\tfrac{1}{3}\{2\pi_{B(A_1 C)i}-\pi_{A_1(B C)i}-\pi_{C(B A_1)i}\}_{,BC}. \tag{8.39}$$

The undetermined additive stress (8.39) makes no contribution to the equations of motion (8.24). Also, the rate of work of the undetermined parts of the stress and multipolar stresses over any closed surface inside the body, or over the complete boundary of the body, is zero. The actual values of these stresses on the surface of the body play an important part in determining correct boundary conditions.

Apart from extra algebraic complexity the general case in which we have multipolar stresses of all orders up to 2^ν yields similar results. The multipolar stresses are given in terms of the internal energy U and multipolar body forces, apart from undetermined multipolar stresses which make no contribution to the rate of work over any closed surface in the body, or over the complete boundary of the body. The surface values of these undetermined multipolar stresses contribute to the surface conditions.

In the next section we examine the question of surface conditions for the case when only stresses and dipolar stresses are present. An examination of the general case will follow similar lines but will be considered later.

9. Boundary conditions

Before considering boundary conditions it is convenient to put the results of the previous section in a more general notation. Let points of the initial body be defined by a general curvilinear system of coordinates ϑ^A. At each point in the initial body we then have base vectors $\boldsymbol{g}_A$ and $\boldsymbol{g}^A$ with corresponding metric tensors g_{AB}, g^{AB} such that

$$\boldsymbol{g}_A \cdot \boldsymbol{g}_B = g_{AB}, \qquad \boldsymbol{g}^A \cdot \boldsymbol{g}^B = g^{AB}, \qquad \boldsymbol{g}^A \cdot \boldsymbol{g}_B = \delta^A_B. \tag{9.1}$$

The displacement and velocity vectors $\boldsymbol{u}, \boldsymbol{v}$ of a point ϑ^A of the body may then be expressed in the forms

$$\boldsymbol{u} = u_A \boldsymbol{g}^A = u^A \boldsymbol{g}_A; \qquad \boldsymbol{v} = \dot{\boldsymbol{u}} = v_A \boldsymbol{g}^A = v^A \boldsymbol{g}_A. \tag{9.2}$$

Since coordinates x_i of points in the deformed body are functions of X_A, t we may also regard them as functions of ϑ^A, t, and $\vartheta^A =$ constant also form surfaces in the deformed body. Associated with a surface A we may define contravariant components of surface force and multipolar surface force $p^K, p^{A_1 \dots A_\beta K}$ ($\beta = 1, 2, \dots, \nu$) which are such that their rate of work per unit area of a surface A_0 in the initial body, whose unit normal is ${}_0\boldsymbol{n}$, is

$$p^K v_K, \quad p^{A_1 \dots A_\beta K} v_{K|A_1 \dots A_\beta}, \tag{9.3}$$

respectively, per unit area of A_0. In (9.3) $v_{K|A_1 \dots A_\beta}$ denotes covariant differentiation with respect to ϑ^B using Christoffel symbols obtained from the metric tensor g_{AB}. Since the space is Euclidean the order of covariant differentiation is immaterial and $p^{A_1 \dots A_\beta i}$ will therefore be completely symmetric in the indices $A_1, \dots, A_\beta$. We shall also put

$${}_0\boldsymbol{n} = n^A \boldsymbol{g}_A = n_A \boldsymbol{g}^A. \tag{9.4}$$

When ${}_0\boldsymbol{n}$ is a unit normal to a ϑ^B-surface in the initial body we denote the corresponding contravariant components of the stress multipole by

$$\pi^{BA_1 \dots A_\beta K}, \tag{9.5}$$

acting on the ϑ^B-surface in the deformed body. If we have an elastic body then equations (8.18), in the present more general notation, give

$$\begin{aligned} p^K &= n_B \pi^{BK}, \\ p^{A_1 \dots A_\beta K} &= n_B \pi^{BA_1 \dots A_\beta K}. \end{aligned} \tag{9.6}$$

If $F^{A_1\dots A_\beta K}$ are contravariant components of body force per unit mass their rate of work is

$$F^{A_1\dots A_\beta K}\, v_{K|A_1\dots A_\beta}. \tag{9.7}$$

The function U in (8.23) can be expressed as a different function

$$U=U\left(S,\gamma_{AB},\gamma_{AA_1A_2},\dots,\gamma_{AA_1\dots A_{\nu+1}};\frac{\partial X^B}{\partial\vartheta^A}\right), \tag{9.8}$$

where

$$\begin{aligned}\gamma_{AB}&=\frac{1}{2}\frac{\partial X^C}{\partial\vartheta^A}\frac{\partial X^D}{\partial\vartheta^B}(E_{CD}-\delta_{CD})\\&=\frac{1}{2}\left[\frac{\partial x_i}{\partial\vartheta^A}\frac{\partial x_i}{\partial\vartheta^B}-\frac{\partial X_D}{\partial\vartheta^A}\frac{\partial X_D}{\partial\vartheta^B}\right],\end{aligned} \tag{9.9}$$

$$\gamma_{AA_1\dots A_\beta}=\frac{\partial X^B}{\partial\vartheta^A}\frac{\partial X^{B_1}}{\partial\vartheta^{A_1}}\cdots\frac{\partial X^{B_\beta}}{\partial\vartheta^{A_\beta}}E_{BB_1\dots B_\beta}, \tag{9.10}$$

for $\beta=2,\dots,\nu+1$.

Formulae (8.26)—(8.29) are now replaced by

$$\pi^{A_1K}+\varrho_0F^{A_1K}+\pi^{BA_1K}|_B=\varrho_0\frac{\partial U}{\partial u_{K|A_1}}, \tag{9.11}$$

$$\pi^{(A_2A_1)K}+\varrho_0F^{A_1A_2K}+\pi^{BA_1A_2K}|_B=\varrho_0\frac{\partial U}{\partial u_{K|A_1A_2}}, \tag{9.12}$$

. . . .

$$\pi^{(A_\nu A_1\dots A_{\nu-1})K}+\varrho_0F^{A_1\dots A_\nu i}+\pi^{BA_1\dots A_\nu K}|_B=\varrho_0\frac{\partial U}{\partial u_{K|A_1\dots A_\nu}}, \tag{9.13}$$

$$\pi^{(A_{\nu+1}A_1\dots A_\nu)K}=\varrho_0\frac{\partial U}{\partial u_{K|A_1\dots A_{\nu+1}}}. \tag{9.14}$$

We restrict our attention here to an examination of surface conditions when only monopolar and dipolar surface forces are present, so that

$$\begin{aligned}\pi^{A_1K}&=-\pi^{[BA_1]K}|_B+f^{A_1K},\\ \pi^{A_2A_1K}&=\pi^{[A_2A_1]K}+\pi^{(A_2A_1)K},\end{aligned} \tag{9.15}$$

where

$$\pi^{(A_2A_1)K}=\varrho_0\frac{\partial U}{\partial u_{K|A_1A_2}}, \tag{9.16}$$

$$f^{A_1K}=-\varrho_0F^{A_1K}-\pi^{(BA_1)K}|_B+\varrho_0\frac{\partial U}{\partial u_{K|A_1}}. \tag{9.17}$$

The dipolar stress $\pi^{(A_2A_1)K}$ is given in terms of U and f^{A_1K} is given in terms of U and F^{A_1K}.

We suppose that the initial body is bounded by a surface A_0. We choose the ϑ^A coordinate system so that the surface A_0 is given by $\vartheta^3=0$ and so that the ϑ^3-curves are normal to the surface. The ϑ^α coordinates, where Greek letters take the values 1, 2, form a curvilinear net on the surface A_0 with corresponding metric tensors $a_{\alpha\beta}$, $a^{\alpha\beta}$ and curvature tensor $b_{\alpha\beta}$, all these tensors being symmetric. Also for the surface A_0,

$$n_1=n_2=0,\qquad n_3=1, \tag{9.18}$$

so that, from (9.6) and (9.15),

$$p^{AK} = \pi^{[3A]K} + \pi^{(3A)K},$$
$$p^{K} = -\pi^{[B3]K}|_B + f^{3K}. \tag{9.19}$$

Covariant differentiation in (9.19) is with respect to ϑ^A using Christoffel symbols formed from g_{AB}, and evaluated on $\vartheta^3=0$. The second expression in (9.19) can be replaced* by

$$p^{3} = -\pi^{[\beta 3]3}|_\beta - b_{\beta\lambda}\pi^{[\beta 3]\lambda} + f^{33},$$
$$p^{\alpha} = -\pi^{[\beta 3]\alpha}|_\beta + b^{\alpha}_{\beta}\pi^{[\beta 3]3} + f^{3\alpha}, \tag{9.20}$$

where the vertical line in (9.20) denotes covariant differentiation** with respect to ϑ^α using Christoffel symbols formed from the surface metric tensor $a_{\alpha\beta}$. In obtaining (9.20) we have also used the fact that $\pi^{[BA]K}$ is anti-symmetric in A, B so that $\pi^{[33]K}$ is zero. From (9.20) and (9.19), we have

$$p^{3} - p^{\beta 3}|_\beta - p^{\beta\alpha} b_{\beta\alpha} = f^{33} - \pi^{(3\beta)3}|_\beta - \pi^{(3\alpha)\beta} b_{\beta\alpha},$$
$$p^{\alpha} - p^{\beta\alpha}|_\beta + p^{\beta 3} b^{\alpha}_{\beta} = f^{3\alpha} - \pi^{(3\beta)\alpha}|_\beta + \pi^{(3\beta)3} b^{\alpha}_{\beta}. \tag{9.21}$$

The right-hand sides of equations (9.21) are known functions of F^{AB} and U, and hence of F^{AB} and derivatives of displacements. Covariant differentiation in (9.21) is still with respect to surface coordinates ϑ^α.

The rate of work of surface forces and dipolar surface forces at the surface A_0 depends on three components of p^K and nine components of p^{AK}. The first of equations (9.19) involves six unknown functions $\pi^{[3\beta]K}$ and equations (9.20) involve these and their surface derivatives. If the twelve quantities p^K, p^{AK} take prescribed values on the surface A_0 then three equations (9.20) and nine equations $(9.19)_1$ are, in general, sufficient to determine the six unknowns $\pi^{[\beta 3]K}$ and also to provide six conditions to be satisfied by derivatives of the displacement. In fact, these latter conditions are given by equations (9.21), and (9.22) (see below) which do not involve $\pi^{[\beta 3]K}$, values of $\pi^{[\beta 3]K}$ then being given by $(9.19)_1$ for $A=1, 2$. Surface values of the undetermined dipolar stress $\pi^{[\beta 3]K}$ thus play an essential part in the surface conditions and enable us to prescribe values for all the components of surface force and dipolar surface force p^K, p^{AK} on A_0.

Instead of prescribing values of the surface force and surface dipolar force we can prescribe values of the surface displacements and the surface dipolar forces. This gives twelve conditions which, with the help of $(9.19)_1$, enables us to find the six surface values of $\pi^{[3\mu]K}$ and six conditions on the surface values of the displacements and their derivatives. Since

$$p^{3K} = \pi^{(33)K} \tag{9.22}$$

do not involve $\pi^{[3\mu]K}$ the six conditions on the displacements and their derivatives at the surface are given by the specification of u_A and $\pi^{(33)K}$ at the surface.

A discussion of the general case when dipolar surface forces up to order 2^ν are present is postponed, but sufficient work has been done to indicate that when $\nu=2$ we have enough unknown functions to enable us to prescribe values for thirty multipolar surface forces p^K, p^{AK}, p^{ABK}.

* See, *e.g.* GREEN & ZERNA (1954), p. 36.

** Covariant differentiation in the surface is *not* independent of its order.

10. Constitutive equations

In this section we discuss the formulation of constitutive equations which may, in certain cases, be appropriate to theories in which multipolar forces are present and which are not included in the restrictive assumptions of the previous Sections 7, 8. For convenience we collect here all the basic equations of Section 6. We have first the equations of motion (6.13) and surface conditions (6.6):

$$\sigma_{ji,j} + \varrho F_i = \varrho \dot{v}_i, \tag{10.1}$$

$$t_i = n_j \sigma_{ji}. \tag{10.2}$$

Next, adopting the notation

$$\begin{aligned} \sigma_{i_1 i} + \varrho F_{i_1 i} + \sigma_{j i_1 i, j} &= \bar{\sigma}_{i_1 i}, \\ \sigma_{(i_\alpha i_1 \ldots i_{\alpha-1}) i} + \varrho F_{i_1 \ldots i_\alpha i} + \sigma_{j i_1 \ldots i_\alpha i, j} &= \bar{\sigma}_{i_\alpha i_1 \ldots i_{\alpha-1} i} \qquad (\alpha = 2, \ldots, \nu), \\ \sigma_{(i_{\nu+1} i_1 \ldots i_\nu) i} &= \bar{\sigma}_{i_{\nu+1} i_1 \ldots i_\nu i}, \end{aligned} \tag{10.3}$$

where $\sigma_{(i_\alpha i_1 \ldots i_{\alpha-1}) i}$ is the completely symmetric part of $\sigma_{i_\alpha i_1 \ldots i_{\alpha-1} i}$ with respect to the indices $i_1, \ldots, i_{\alpha-1}, i_\alpha$, and $\bar{\sigma}_{i_\alpha i_1 \ldots i_{\alpha-1} i}$ $(\alpha = 2, \ldots, \nu+1)$ is completely symmetric with respect to $i_1, \ldots, i_\alpha$, equation (6.16) becomes

$$\varrho r - Q_{i,i} - \varrho \dot{U} + \tfrac{1}{2} \bar{\sigma}_{i_1 i} A_{i i_1} + \sum_{\alpha=2}^{\nu}{}' \bar{\sigma}_{i_\alpha i_1 \ldots i_{\alpha-1} i} A_{i i_1 \ldots i_\alpha} + \bar{\sigma}_{i_{\nu+1} i_1 \ldots i_\nu i} A_{i i_1 \ldots i_{\nu+1}} = 0, \tag{10.4}$$

where, from (6.15), we see that

$$\bar{\sigma}_{i_1 i} = \bar{\sigma}_{i i_1}. \tag{10.5}$$

Also, if

$$\begin{aligned} t_{i_1 \ldots i_\alpha i} - n_j \sigma_{j i_1 \ldots i_\alpha i} &= \bar{t}_{i_1 \ldots i_\alpha i} \qquad (\alpha = 1, \ldots, \nu) \\ h - n_i Q_i &= \bar{h}, \end{aligned} \tag{10.6}$$

then, from (6.11) and (6.10), we have

$$\tfrac{1}{2} \bar{t}_{i_1 i} A_{i i_1} + \sum_{\alpha=2}^{\nu} \bar{t}_{i_1 \ldots i_\alpha i} A_{i i_1 \ldots i_\alpha} - \bar{h} = 0, \tag{10.7}$$

$$\bar{t}_{i_1 i} = \bar{t}_{i i_1}. \tag{10.8}$$

The entropy production inequality remains in the form (6.2)

$$\int_V \varrho \dot{S}\, dV - \int_V \varrho \frac{r}{T}\, dV + \int_A \frac{h}{T}\, dA \geqq 0. \tag{10.9}$$

The quantities $\bar{t}_{i_1 \ldots i_\alpha i}$ are tensors which are completely symmetric with respect to the indices $i_1, \ldots, i_\alpha$ but, in general, $\bar{\sigma}_{j i_1 \ldots i_\alpha i}$ are tensors only with respect to the indices $i_1, \ldots, i_\alpha, i$, being completely symmetric with respect to $j, i_1, \ldots, i_\alpha$.

An inspection of (10.4) and (10.7) suggests that constitutive equations are required for the quantities $\bar{t}_{i_1 \ldots i_\alpha i}$ and $\bar{\sigma}_{j i_1 \ldots i_\alpha i}$. Here we restrict our attention to materials for which $\bar{t}_{i_1 \ldots i_\alpha i}$ $(\alpha = 1, \ldots, \nu)$ and $\bar{h}$ do not depend explicitly on velocity gradients of all orders $1, 2, \ldots, \nu$, and hence do not depend explicitly on $A_{i i_1}, \ldots, A_{i i_1 \ldots i_\nu}$. Since these latter quantities can be chosen arbitrarily and independently of each other, subject to symmetry restrictions, it follows from equation (10.7) that $\bar{t}_{i_1\ldots i_\alpha i}$ $(\alpha=1,\ldots,\nu)$ and $\bar{h}$ are zero and that

$$\begin{aligned} t_{i_1\ldots i_\alpha i} &= n_j\,\sigma_{j i_1\ldots i_\alpha i} \qquad (\alpha=1,\ldots,\nu),\\ h &= n_j\,Q_j. \end{aligned} \tag{10.10}$$

Another way of obtaining equations (10.10) is to assume, as a part of our constitutive equations, that the multipolar forces $t_{i_1\ldots i_\alpha i}$ $(\alpha=1,\ldots,\nu)$ depend linearly on the unit vectors n_k. Since these are $\sigma_{k i_1\ldots i_\alpha i}$ when n_k is normal to the x_k-plane, equations $(10.10)_1$ follow, and then (10.7) yields the result $(10.10)_2$. From equations $(10.10)_1$ we see that $\sigma_{j i_1\ldots i_\alpha i}$ transforms as a tensor with respect to all the indices, including j, under changes of rectangular Cartesian axes, where the multipolar stresses in each coordinate system are associated with the three coordinate planes in that system. It follows from (10.3) that $\bar{\sigma}_{j i_1\ldots i_\alpha i}$ $(\alpha=1,\ldots,\nu)$ and $\bar{\sigma}_{i_1 i}$ transform as tensors with respect to all the indices (we already know that $\sigma_{i_1 i}$ is a tensor).

We now suppose that the multipolar stresses $\sigma_{j i_1\ldots i_\alpha i}$ $(\alpha=0,1,\ldots,\nu)$ associated with the x_j-planes at time t correspond to a deformation of the continuum given by (2.1), and that corresponding to the deformation (3.1) we have multipolar stresses $\sigma^*_{j i_1\ldots i_\alpha i}$. If the superposed rigid body motions for all time do not change the values of multipolar stresses, except for orientation at time t, then*

$$\sigma^*_{r i_1\ldots i_\alpha i} = Q_{ij}\,Q_{i_1 j_1}\cdots Q_{i_\alpha j_\alpha}\,Q_{rs}\,\sigma_{s j_1\ldots j_\alpha j}, \tag{10.11}$$

and hence, from (10.3),

$$\bar{\sigma}^*_{r i_1\ldots i_\alpha i} = Q_{ij}\,Q_{i_1 j_1}\cdots Q_{i_\alpha j_\alpha}\,Q_{rs}\,\bar{\sigma}_{s j_1\ldots j_\alpha j} \tag{10.12}$$

if we assume that the multipolar body forces $F_{i_1 i},\ldots,F_{i_1\ldots i_\nu i}$ are unaltered, except for orientation**.

Suppose now that all the multipolar body forces $F_{i_1\ldots i_\alpha i}$ $(\alpha=1,\ldots,\nu)$ are specified and that constitutive equations have been obtained for $\bar{\sigma}_{i_1\ldots i_\alpha i}$ $(\alpha=1,2,\ldots,\nu+1)$, where $\bar{\sigma}_{i_1\ldots i_\alpha i}$ is completely symmetric with respect to the indices $i_1,\ldots,i_\alpha$. The multipolar stresses $\sigma_{i_\alpha i_1\ldots i_{\alpha-1} i}$ $(\alpha=1,\ldots,\nu+1)$, with stress $\sigma_{i_1 i}$ corresponding to $\alpha=1$, are symmetric with respect to the indices $i_1,\ldots,i_{\alpha-1}$, and in order to see what information we have about these stresses we consider the special case of equations (10.3) with $\nu=2$. Thus, the completely symmetric part of $\sigma_{i_2(i_1 i_2)i}$ which, as indicated by the brackets, is symmetric in i_1, i_2, is

$$\sigma_{(i_3 i_1 i_2)i} = \bar{\sigma}_{i_3 i_1 i_2 i} = \bar{\sigma}_{i_1 i_2 i_3 i} \tag{10.13}$$

and is known. We put

$$\sigma_{i_3(i_1 i_2)i} = \sigma_{(i_3 i_1 i_2)i} + \tfrac{1}{3}\left\{2\sigma_{i_3(i_1 i_2)i} - \sigma_{i_1(i_2 i_3)i} - \sigma_{i_2(i_3 i_1)i}\right\} \tag{10.14}$$

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* We have already used the assumption that $\sigma_{j i_1\ldots i_\alpha i}$ is unaltered by superposed uniform rigid body translations and rotations, the body occupying the same position at time t. This is now included in our present assumption as a special case.

** More generally, multipolar forces minus appropriate inertia terms are unaltered, except for orientation. An alternative approach is to assume that U, Q_i, $\bar{\sigma}_{i_\alpha i_1\ldots i_{\alpha-1}}$ $(\alpha=1,\ldots,\gamma+1)$, $\bar{t}_{i_1\ldots i_\alpha i}$ $(\alpha=1,\ldots,\nu)$ and $\bar{h}$ are unaltered by superposed rigid body motions, apart from orientation at time t, so that the left-hand sides of (10.4) and (10.7) are then unaltered.

so that, from (10.3), we have

$$\sigma_{(i_2 i_1) i} = \bar{\sigma}_{i_2 i_1 i} - \varrho F_{i_1 i_2 i} - \bar{\sigma}_{j i_1 i_2 i, j} - \tfrac{1}{3}\{2\sigma_{j(i_1 i_2) i} - \sigma_{i_1 (j i_2) i} - \sigma_{i_2 (j i_1) i}\}_{,j}. \quad (10.15)$$

Also

$$\sigma_{i_2 i_1 i} = \sigma_{(i_2 i_1) i} + \sigma_{[i_2 i_1] i}, \quad (10.16)$$

where $\sigma_{[i_2 i_1] i}$ is skew symmetric in i_1, i_2. It follows from (10.3) and (10.16) that the stress $\sigma_{i_1 i}$ is determined apart from an additive stress

$$-\sigma_{[j i_1] i, j} + \tfrac{1}{3}\{2\sigma_{j(i_1 k) i} - \sigma_{i_1 (j k) i} - \sigma_{k (j i_1) i}\}_{,jk}, \quad (10.17)$$

which does not contribute to the equations of motion. The rate of work of the undetermined parts of the surface force and multipolar surface force over any closed material surface inside the body, or over the complete boundary of the body, is zero. As in Section 9, the value of these forces on the surface plays an important role in determining correct boundary conditions. Apart from extra algebraic complexity the general case in which we have multipolar surface forces up to some order 2^ν yields similar results. When multipolar body forces and the values of $\bar{\sigma}_{i_1 \dots i_\alpha i}$ $(\alpha = 1, 2, \dots, \nu+1)$ are known, the stress and multipolar stresses can be found, apart from undetermined multipolar stresses which make no contribution to the rate of work over any closed surface in the body, or over the complete boundary of the body. The surface values of these undetermined multipolar stresses contribute to the surface conditions.

We next consider a class of constitutive equations* for $\bar{\sigma}_{i_1 \dots i_\alpha i}$ $(\alpha = 1, \dots, \nu+1)$ where $\bar{\sigma}_{i_1 \dots i_\alpha i}$ is symmetric with respect to $i_1, \dots, i_\alpha$. We restrict our attention here to the assumption that $\bar{\sigma}_{i_1 \dots i_\alpha i}$ depends on the deformation gradients, the velocity gradients, ..., N^{th} velocity gradients (say) of all orders (up to the μ^{th}), all measured at time t, so that

$$\bar{\sigma}_{i_1 \dots i_\alpha i} = \varphi_{i_1 \dots i_\alpha i}(x_{p, A_1 \dots A_\beta}, v^{(1)}_{p, q_1 \dots q_\beta}, \dots, v^{(N)}_{p, q_1 \dots q_\beta}) \quad (10.18)$$

for $\alpha = 1, 2, \dots, \nu+1$; $\beta = 1, \dots, \mu$.

With the help of (3.11) equation (10.18) can be expressed in the different form

$$\bar{\sigma}_{i_1 \dots i_\alpha i} = \varphi_{i_1 \dots i_\alpha i}(x_{p,A}, x_{p, A_1 \dots A_\beta}, v^{(1)}_{p,q}, \dots, v^{(N)}_{p,q}, A^{(1)}_{p q_1 \dots q_\beta}, \dots, A^{(N)}_{p q_1 \dots q_\beta}) \quad (10.19)$$

for $\alpha = 1, \dots, \nu+1$; $\beta = 2, \dots, \mu$, where $\varphi_{i_1 \dots i_\alpha i}$ is a single-valued or polynomial function of its arguments according as $\varphi_{i_1 \dots i_\alpha i}$ in (10.18) is a single-valued or polynomial function of its arguments.

We now consider another motion of the continuum given by (3.1), with $\tau^* = \tau$, which is such that at time t the continuum occupies the same position as for that defined by (2.1). Then, using equation (10.12) and the results of Section 3 with $Q_{ij} = \delta_{ij}$, we have

$$\begin{aligned} &\varphi_{i_1 \dots i_\alpha i}(x_{p,A}, x_{p, A_1 \dots A_\beta}, v^{*(1)}_{p,q}, \dots, v^{*(N)}_{p,q}, A^{(1)}_{p q_1 \dots q_\beta}, \dots, A^{(N)}_{p q_1 \dots q_\beta}) \\ &\quad = \varphi_{i_1 \dots i_\alpha i}(x_{p,A}, x_{p, A_1 \dots A_\beta}, v^{(1)}_{p,q}, \dots, v^{(N)}_{p,q}, A^{(1)}_{p q_1 \dots q_\beta}, \dots, A^{(N)}_{p q_1 \dots q_\beta}), \end{aligned} \quad (10.20)$$

since, from (3.9), $A^{*(\mu)}_{p p_1 \dots p_\beta} = A^{(\mu)}_{p p_1 \dots p_\beta}$ $(\beta \geqq 2)$ when $Q_{ij} = \delta_{ij}$.

* In Section 7 the constitutive assumption (7.2) should, at this point, be applied to $\bar{\sigma}_{i_1 \dots i_\alpha i}$ instead of $\sigma_{i_1 \dots i_\alpha i}$.

The limitations on the function $\varphi_{i_1\ldots i_\alpha i}$ implied by (10.20) can be made explicit by the procedure adopted by RIVLIN & ERICKSEN (1955) or GREEN & RIVLIN (1960). We can conclude that $\varphi_{i_1\ldots i_\alpha i}$ can depend on $v_{p,q}^{(1)}, \ldots, v_{p,q}^{(N)}$ only through the components of N symmetric Rivlin-Ericksen tensors $A_{pq}^{(1)}, \ldots, A_{pq}^{(N)}$ defined in (3.8). Thus (10.19) may be replaced by the form

$$\bar{\sigma}_{i_1\ldots i_\alpha i} = \varphi_{i_1\ldots i_\alpha i}(x_{p,A_1\ldots A_\beta}, A_{pq_1\ldots q_\beta}^{(1)}, \ldots, A_{pq_1\ldots q_\beta}^{(N)}) \tag{10.21}$$

where $\alpha=1, \ldots, \nu+1$, $\beta=1, 2, \ldots, \mu$ and where the functions $\varphi_{i_1\ldots i_\alpha i}$ are, in general, different from those in (10.19), but are single-valued or polynomial according as those in (10.19) are single-valued or polynomial.

We again consider the motion (3.1) but now $Q_{ij} \neq \delta_{ij}$. Using (10.12) and the results of Section 3, we see that

$$\begin{aligned}\varphi_{i_1\ldots i_\alpha i}&(x_{p,A_1\ldots A_\beta}, A_{pq_1\ldots q_\beta}^{(1)}, \ldots, A_{pq_1\ldots q_\beta}^{(N)})\\ &= Q_{j_1 i_1} Q_{j_2 i_2} \cdots Q_{j_\alpha i_\alpha} Q_{ji} \varphi_{j_1\ldots j_\alpha j}(Q_{pm} x_{m,A_1\ldots A_\beta},\\ &\quad Q_{pr} Q_{p_1 s_1} \cdots Q_{p_\beta s_\beta} A_{rs_1\ldots s_\beta}^{(1)}, \ldots, Q_{pr} Q_{p_1 s_1} \cdots Q_{p_\beta s_\beta} A_{rs_1\ldots s_\beta}^{(N)})\end{aligned} \tag{10.22}$$

for all proper orthogonal Q_{ij}. Following a method similar to that used in discussing the internal energy U in Section 7, we choose the value R_{jA} for Q_{Aj} in (10.22) and assume that $\varphi_{i_1\ldots i_\alpha i}$ is a single-valued function of $x_{p,A_1\ldots A_\beta}$ and either a single-valued or polynomial function of $A_{pq_1\ldots q_\beta}^{(1)}, \ldots, A_{pq_1\ldots q_\beta}^{(N)}$. Then using the result

$$R_{jA} = x_{j,B} M_{BA}^{-1}$$

we see that (10.21) reduces to

$$\bar{\sigma}_{i_1\ldots i_\alpha i} = x_{i_1,R_1} \cdots x_{i_\alpha,R_\alpha} x_{i,R} \Phi_{R_1\ldots R_\alpha R}(E_{AA_1\ldots A_\beta}, D_{PQ_1\ldots Q_\beta}^{(M)}), \tag{10.23}$$

where $\alpha=1, \ldots, \nu+1$; $\beta=1, \ldots, \mu$; $M=1, 2, \ldots, N$; $E_{AA_1\ldots A_\beta}$ are defined in (3.6) and

$$D_{PQ_1\ldots Q_\beta}^{(M)} = x_{p,P} x_{p_1,Q_1} \cdots x_{p_\beta,Q_\beta} A_{pp_1\ldots p_\beta}^{(M)}. \tag{10.24}$$

Also $\Phi_{R_1\ldots R_\alpha R}$ is a single valued function of $E_{AA_1\ldots A_\beta}$ and a single valued or polynomial function of $D_{PQ_1\ldots Q_\beta}^{(M)}$, and $\Phi_{R_1\ldots R_\alpha R}$ is completely symmetric with respect to the indices $R_1, \ldots, R_\alpha$.

If, in equation (10.18), the displacement gradients are absent the constitutive equation refers to a fluid. We can obtain this case by taking the reference state to be the state at time t so that $X_i = x_i$. The corresponding final form for $\bar{\sigma}_{i_1\ldots i_\alpha i}$ is found, from (10.23) and (10.24), to be

$$\bar{\sigma}_{i_1\ldots i_\alpha i} = \Phi_{i_1\ldots i_\alpha i}(A_{pp_1\ldots p_\beta}^{(M)}, \varrho) \tag{10.25}$$

where $\alpha=1, \ldots, \nu+1$; $\beta=1, \ldots, \mu$; $M=1, 2, \ldots, N$. Moreover, $\Phi_{i_1\ldots i_\alpha i}$ is now a hemihedral isotropic function of its arguments, and ϱ is the density of the fluid at time t.

11. Appendix 1

Let $\boldsymbol{f}_1, \ldots, \boldsymbol{f}_N$ be a system of forces acting on particles at $Q_1, \ldots, Q_N$ with vector positions $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_N$ relative to a point Q. Let $\boldsymbol{r}$ be the position vector of Q relative to another fixed point O. Let us suppose that the particles at $Q_1, \ldots, Q_N$ are moving with velocities $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_N$. Then the rate R at which work is done by the system of forces is given by

$$R = \sum_{P=1}^{N} \boldsymbol{f}_P \cdot \boldsymbol{v}_P. \tag{11.1}$$

We now suppose that the velocities $\boldsymbol{v}_P$ are functions of position in space and time. Let $\boldsymbol{v}$ be the velocity at Q. Then, using Cartesian tensor notation, we have

$$v_i^{(P)} = \sum_{\alpha=0}^{\nu} \frac{1}{\alpha!} a_{i_1}^{(P)} \dots a_{i_\alpha}^{(P)} v_{i,i_1 \dots i_\alpha} + K_i^{(P)}, \tag{11.2}$$

provided the velocity $\boldsymbol{v}$ at Q has continuous spatial derivatives up to order $\nu+1$ in some neighborhood of Q, where $K_i^{(P)}$ is a remainder term. Introducing (11.2) into (11.1), we have

$$R = \sum_{\alpha=0}^{\nu} F_{i_1 \dots i_\alpha i} v_{i,i_1 \dots i_\alpha} + \sum_{P=1}^{N} f_i^{(P)} K_i^{(P)}, \tag{11.3}$$

where

$$F_{i_1 \dots i_\alpha i} = \frac{1}{\alpha!} \sum_{P=1}^{N} a_{i_1}^{(P)} \dots a_{i_\alpha}^{(P)} f_i^{(P)}. \tag{11.4}$$

If we define $F_{i_1 \dots i_\alpha i}$ as given by (11.4) to be a simple force 2^α-pole of the first kind, then the rate of work R of the system of forces $\boldsymbol{f}_1, \dots, \boldsymbol{f}_N$ is equivalent to the rate of work of simple force 2^α-poles ($\alpha = 0, 1, \dots, \nu$), provided the remainder term in (11.3) can be neglected. In (11.3) the rate of work of a simple force 2^α-pole of the first kind is

$$F_{i_1 \dots i_\alpha i} v_{i,i_1 \dots i_\alpha}. \tag{11.5}$$

We may now generalize the definition of a simple force 2^α-pole of the first kind by assuming that if, for all arbitrary velocity gradients $v_{i,i_1 \dots i_\alpha}$, the expression (11.5) is a scalar which is a rate of work, then $F_{i_1 \dots i_\alpha i}$ is a tensor called a simple force 2^α-pole of the first kind. Without loss of generality it can be taken to be completely symmetric in the indices $i_1, \dots, i_\alpha$.

12. Appendix 2

We consider an elastic material for which the internal energy U is expressible in the form

$$U = U(S, E_{AA_1}, \dots, E_{AA_1 \dots A_\mu}). \tag{12.1}$$

Then from (8.26)—(8.29), we obtain*

$$\begin{gathered}
\pi_{(A_1 \dots A_\mu) i} = \varrho_0 x_{i,A} \frac{\partial U}{\partial E_{AA_1 \dots A_\mu}} \quad (\mu > 1), \\
\pi_{(A_1 \dots A_\nu) i} + \varrho_0 F_{A_1 \dots A_\nu i} + \pi_{B A_1 \dots A_\nu i, B} = \varrho_0 x_{i,A} \frac{\partial U}{\partial E_{AA_1 \dots A_\nu}} \\
(\nu = 2, \dots, \mu - 1;\ \mu > 2), \\
\pi_{Ai} + \varrho_0 F_{Ai} + \pi_{BAi,B} = 2\varrho_0 x_{i,A_1} \frac{\partial U}{\partial E_{AA_1}} + \varrho_0 \sum_{\beta=2}^{\mu} x_{i,A_1 \dots A_\beta} \frac{\partial U}{\partial E_{AA_1 \dots A_\beta}} \quad (\mu \geq 2), \\
\pi_{A_1 i} = 2\varrho_0 x_{i,A} \frac{\partial U}{\partial E_{AA_1}} \quad (\mu = 1).
\end{gathered} \tag{12.2}$$

* Before using formulae (12.2) U must be suitably symmetrized.

If the elastic material is symmetric in its reference state, U must be a scalar invariant of the tensors $E_{AA_1}, \ldots, E_{AA_1 \ldots A_\mu}$ under the group of transformations describing the symmetry. In the particular case when the material is isotropic, the group is the full or proper orthogonal group accordingly as the material does or does not possess a centre of symmetry.

We shall consider that U is a polynomial in the tensors $\tilde{E}_{AA_1 \ldots A_\beta}$ $(\beta=1, \ldots, \mu)$ defined by

$$\begin{aligned} \tilde{E}_{AA_1} &= E_{AA_1} - \delta_{AA_1}, \\ \tilde{E}_{AA_1 \ldots A_\beta} &= E_{AA_1 \ldots A_\beta} \qquad (\beta = 2, \ldots, \mu). \end{aligned} \tag{12.3}$$

Then, it must be expressible in the form

$$U = \sum K_{AA_1 \ldots A_{\mu_1} BB_1 \ldots B_{\mu_2} \ldots CC_1 \ldots C_{\mu_\nu}} \tilde{E}_{AA_1 \ldots A_{\mu_1}} \tilde{E}_{BB_1 \ldots B_{\mu_2}} \ldots \tilde{E}_{CC_1 \ldots C_{\mu_\nu}}, \tag{12.4}$$

where the K's are constant tensors. If the appropriate symmetry group is the full orthogonal group, the K's in (12.4) may be expressed as the sum of outer products of Kronecker deltas with scalar coefficients. Then, only terms for which $\mu_1 + \mu_2 + \cdots + \mu_\nu + \nu = \chi$, say, is even can occur in (12.4) and a typical term in the expression for U is

$$\text{Const.} \times \delta_{p_1 p_2} \delta_{p_3 p_4} \ldots \delta_{p_{\chi-1} p_\chi} \tilde{E}_{AA_1 \ldots A_{\mu_1}} \tilde{E}_{BB_1 \ldots B_{\mu_2}} \ldots \tilde{E}_{CC_1 \ldots C_{\mu_\nu}}, \tag{12.5}$$

where $p_1 p_2 \ldots p_\chi$ is a permutation of $AA_1 \ldots A_{\mu_1} \ldots C_{\mu_\nu}$. In writing down such terms we may bear in mind that $\tilde{E}_{AA_1 \ldots A_\beta}$ is unaltered by permutation of $A_1 \ldots A_\beta$ $(\beta \geqq 2)$ and $\tilde{E}_{AA_1}$ is symmetric in A, A_1.

If we assume $\tilde{E}_{AA_1 \ldots A_\alpha}$ $(\alpha=1, \ldots, \mu)$ to be small enough, we may approximate (12.4) by

$$U = C + \sum_{\beta=1}^{\mu} H_{AA_1 \ldots A_\beta} \tilde{E}_{AA_1 \ldots A_\beta} + \sum_{\beta, \gamma=1}^{\mu} K_{AA_1 \ldots A_\beta BB_1 \ldots B_\gamma} \tilde{E}_{AA_1 \ldots A_\beta} \tilde{E}_{BB_1 \ldots B_\gamma}, \tag{12.6}$$

where C, the H's and K's are constants. We may, without loss of generality, omit the constant term C, since the forces, monopolar or multipolar, involve U only through its derivatives. If, further, we assume that when the deformation gradients and body forces of all orders are zero the stresses of all orders and their spatial gradients are zero, the H's in (12.6) are zero. We then obtain

$$U = \sum_{\beta, \gamma=1}^{\mu} K_{AA_1 \ldots A_\beta BB_1 \ldots B_\gamma} \tilde{E}_{AA_1 \ldots A_\beta} \tilde{E}_{BB_1 \ldots B_\gamma}, \tag{12.7}$$

where we may, without loss of generality, take the K's to be unaltered by interchange of the A's and B's and completely symmetric with respect to $A_1, \ldots, A_\beta$ and $B_1, \ldots, B_\gamma$ for $\beta, \gamma \geqq 2$. Also $K_{AA_1 BB_1}$ is symmetric with respect to A, A_1 and B, B_1.

Introducing (12.7) into $(12.2)_1$, we obtain

$$\pi_{(A_1 \ldots A_\mu) i} = 2 \varrho_0 \, x_{i,A} \sum_{\gamma=1}^{\mu} K_{AA_1 \ldots A_\mu BB_1 \ldots B_\gamma} \tilde{E}_{BB_1 \ldots B_\gamma} \qquad (\mu > 1). \tag{12.8}$$

We now write

$$x_i = X_i + \varepsilon \, u_i \tag{12.9}$$

in the expression for $\tilde{E}_{AA_1 \dots A_\beta}$ and neglect terms of higher degree than the first in ε. We then obtain

$$\begin{aligned} \tilde{E}_{AA_1 \dots A_\beta} &\approx u_{A,A_1 \dots A_\beta} = 2 e_{AA_1 \dots A_\beta} \qquad (\beta > 1), \\ \tilde{E}_{AA_1} &\approx u_{A,A_1} + u_{A_1,A} = 2 e_{AA_1}. \end{aligned} \tag{12.10}$$

Introducing (12.9) and (12.10) into (12.8) and neglecting terms of higher degree than the first in ε, we obtain

$$\pi_{(A_1 \dots A_\mu)i} = 4 \varrho_0 \sum_{\gamma=1}^{\mu} K_{iA_1 \dots A_\mu B B_1 \dots B_\gamma} e_{B B_1 \dots B_\gamma}. \tag{12.11}$$

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