Continua with Latent Microstructure

GIANFRANCO CAPRIZ

Dedicated to Walter Noll on his sixtieth birthday

1. Introduction

In [1] a general dynamic model of a continuum with microstructure was proposed which is inspired by (and contains as special cases) known models for continua with voids, liquid crystals, multipolar media, *etc.* The elementary background which supports the proposal was described at length in a course of lectures [2] at the Banach Institute, Warsaw, and reported later in a paper [3]. Further arguments in favour of the model can be found in [4], where, in particular, the case of a fluid with microstructure is studied in some detail.

Here I suggest that a related concept of latent microstructure can be useful in offering an interpretation of constitutive prescriptions involving displacement gradients of higher order, an interpretation which allows one to circumvent a known incompatibility of some of those prescriptions with the Clausius-Duhem inequality. The suggestion is in line with an early remark of TOUPIN (see [5]) on the possible identification of certain materials of second order with Cosserat continua whose microrotations are constrained.

I give complete developments only for a few special cases, where the essential ideas are easily illustrated, avoiding too complex formal developments. Also, in those cases, developments are strictly linked with recent work by DUNN & SERRIN; and, actually, direct inspiration for this paper came from their report [6]. The introduction they propose of an additional term (the interstitial working) in the equation which expresses the balance of energy finds below another justification. Their end result (a subcase of Korteweg fluids is compatible with the Clausius-Duhem inequality) finds also support; but, besides, here a path is open to discuss more complex cases excluded by their premisses.

In Section 2 and 3 I recall briefly the approach proposed in [3] and [4] in a form most suitable for the present developments. In Section 4 I illustrate the concept of latent microstructure and deduce some immediate consequences. In Section 5 I introduce the simplest case of internal constraints and derive reduced balance equations, to mirror closely the developments of DUNN & SERRIN. Finally, in Section 6 and 7 I consider two special cases of elastic fluids.

At a recent meeting (Oberwolfach, Jan. 1984) Professor MAUGIN asserted that the subject matter of Sections 2 and 3 is exhaustively treated in some of his papers, in particular in [7].

2. Continua with Lagrangian microstructure

A body $\mathfrak{B}$ (a set whose members $\mathfrak{X}$ are material elements) is said to be a *continuum with microstructure* if the following properties apply.
(P1) There is a differentiable manifold $\mathcal{Q}$ of finite dimension q whose members $\underset{\sim}{\nu}$ are the microstates (or sets of order parameters). Over $\mathcal{Q}$ a group operation of rigid rotation is defined; *i.e.*, given any proper orthogonal tensor Q and any $\underset{\sim}{\nu} \in \mathcal{Q}$ a unique $\underset{\sim}{\nu}^{(Q)}$ is determined with the properties.

$$\underset{\sim}{\nu}^{(Q^{(1)}Q^{(2)})} = (\underset{\sim}{\nu}^{(Q^{(2)})})^{(Q^{(1)})}, \qquad \underset{\sim}{\nu}^{(1)} = \underset{\sim}{\nu},$$

where $Q^{(1)}, Q^{(2)} \in \mathrm{Orth}^+$ and 1 is the identity tensor.
(P2) A class $\mathscr{C}$ of mappings (called here *complete placements*) $\mathfrak{B} \to \mathscr{E} \times \mathcal{Q}$ ($\mathscr{E}$, three-dimensional Euclidean space) exists

$$x = x(\mathfrak{X}), \qquad \underset{\sim}{\nu} = \underset{\sim}{\nu}(\mathfrak{X}), \qquad x \in \mathscr{E}, \qquad \underset{\sim}{\nu} \in \mathcal{Q},$$

such that:

(i) the *apparent placement* $x = x(\mathfrak{X})$ is a one-to-one mapping of $\mathfrak{B}$ into $\mathscr{E}$, as in the case of placements of ordinary continua; the range $\mathscr{B} = x(\mathfrak{B})$ is an open subset of $\mathscr{E}$;

(ii) any couple of apparent placements $x'(\mathfrak{X})$, $x''(\mathfrak{X})$ is such that the induced bijection of $\mathscr{B}' = x'(\mathfrak{B})$ onto $\mathscr{B}'' = x''(\mathfrak{B})$ is smooth, again as in the case of ordinary continua;

(iii) each complete placement $(x, \underset{\sim}{\nu}) \in \mathscr{C}$ is such that the mapping $\underset{\sim}{\nu} \circ x^{-1}$ of $\mathscr{B} = x(\mathfrak{B})$ into $\mathcal{Q}$ is smooth;

(iv) if $(x, \underset{\sim}{\nu}) \in \mathscr{C}$, then also all the placements $(x^{(Q)}, \underset{\sim}{\nu}^{(Q)})$ belong to $\mathscr{C}$, if

$$x^{(Q)}(\mathfrak{X}) = x(\mathfrak{X}) + c + Q(x(\mathfrak{X}) - x(\mathfrak{X}')), \tag{2.1}$$

with c any vector, $\mathfrak{X}'$ a fixed element of $\mathfrak{B}$, $Q \in \mathrm{Orth}^+$, and $\underset{\sim}{\nu}^{(Q)}$ is specified in accordance with the rule under P1.

Remark 1. Any manifold $\mathcal{Q}'$ which can be put into a sufficiently smooth one-to-one correspondence with $\mathcal{Q}$ can be used to describe the microstates. Correspondingly the way different observers relate their measures of the order parameters (*i.e.*, the specification of the operation of rigid rotation) must be appropriately adapted. The topological properties of $\mathcal{Q}$ are of the essence when, for instance, defects in the microstructure are classified; these matters are not pursued here.

Remark 2. Examples exist of different continua, for all of which the microstructure may be described by the same manifold $\mathcal{Q}$, but to each of which a different group of rigid rotations over $\mathcal{Q}$ is associated.

Examples:

(i) Continua with voids. For $\mathcal{Q}$ the interval [0, 1) of real numbers may be taken; $\underset{\sim}{\nu}$ is the scalar which specifies the void fraction. Rigid rotations leave the microstate unaltered: $\underset{\sim}{\nu}^{(Q)} = \underset{\sim}{\nu}$.

(ii) Standard model for nematic liquid crystals. For $\mathcal{Q}$ the set of diads $c \otimes c$ (c, a unit vector) is taken; $\mathcal{Q}$ is isomorphic to the projective plane, or to a spherical surface where the antipodal points are identified. As a consequence of a rotation specified by a tensor Q the diad $c \otimes c$ goes into $Q(c \otimes c)\, Q^T$.

For the developments below, the introduction of further notation is appropriate. Let r be the vector associated with the orthogonal tensor Q, so that $Q = \exp \mathbf{e}\, r$ ($\mathbf{e}$, the Ricci commutator); then I write $\underset{\sim}{\nu}^{(r)}$ as an alternative to $\underset{\sim}{\nu}^{(Q)}$, and call $\aleph(\underset{\sim}{\nu})$ the Fréchet derivative of $\underset{\sim}{\nu}^{(r)}$ at $\underset{\sim}{\nu}$:

$$\aleph(\underset{\sim}{\nu})\, r := \left.\frac{d\underset{\sim}{\nu}^{r)}}{dr}\right|_{r=0} [r], \tag{2.2}$$

so that

$$\underset{\sim}{\nu}^{(r)} = \underset{\sim}{\nu} + \aleph(\underset{\sim}{\nu})\, r + o(|r|). \tag{2.3}$$

$\aleph(\underset{\sim}{\nu})$ is a linear map of $\mathcal{V}$ (the translation space of $\mathcal{E}$) into the tangent space $\mathcal{T}\mathcal{Q}_{\underset{\sim}{\nu}}$ at $\underset{\sim}{\nu}$.

Most developments involve only values of $\underset{\sim}{\nu}$ in a 'small' subset of $\mathcal{Q}$, which can be imagined covered by one chart only. Thus I use freely a set of local coordinates ν_α ($\alpha = 1, 2, \ldots q$) and also, at each $\underset{\sim}{\nu}$, a corresponding set $\{r_\alpha\}$ of vectors such that

$$(\aleph(\underset{\sim}{\nu})\, r)_\alpha = r_\alpha \cdot r. \tag{2.4}$$

A motion of $\mathfrak{B}$ is a mapping of a real interval $[\tau_0, \tau_1)$ into $\mathcal{C}$:

$$x = x(\mathfrak{X}, \tau), \qquad \underset{\sim}{\nu} = \underset{\sim}{\nu}(\mathfrak{X}, \tau).$$

Velocity v and generalized velocity $\underset{\sim}{v}$ in that motion are

$$v := \dot{x}(\mathfrak{X}, \tau), \qquad \underset{\sim}{v} := \dot{\underset{\sim}{\nu}}(\mathfrak{X}, \tau);$$

notice that v may happen to be any element of $\mathcal{V}$, whereas $\underset{\sim}{v}(\mathfrak{X}, \tau)$ must belong to $\mathcal{T}\mathcal{Q}_{\underset{\sim}{\nu}}(\mathfrak{X}, \tau)$.

Of course, velocity and generalized velocity, at a given instant τ, can be also considered as fields on $\mathcal{B}_\tau := x(\mathfrak{B}, \tau)$, as tacitly implied by most formulae below. The connected notion of virtual velocities, as fields on $\mathcal{B}_\tau$, will also be of use later.

A velocity distribution is rigid (with translational speed $\dot{c}$ and angular speed w) if it has the form

$$\begin{gathered} v^{(R)}(\mathfrak{X}, \tau) = \dot{c}(\tau) + w(\tau) \times (x(\mathfrak{X}, \tau) - x(\mathfrak{X}', \tau)), \\ \underset{\sim}{v}^{(R)}(\mathfrak{X}, \tau) = \aleph(\underset{\sim}{\nu}(\mathfrak{X}, \tau))\, w(\tau). \end{gathered} \tag{2.5}$$

When a local chart is introduced and the notation (2.4) is used, the components of $\underset{\sim}{v}^{(R)}$ can be written as follows

$$v_\alpha^{(R)} = r_\alpha \cdot w. \tag{2.6}$$

As any ordinary continuum, $\mathfrak{B}$ is assumed to have a mass, which can be expressed as a total of a mass density ϱ and which is preserved; during any motion the usual equation of balance of mass

$$\dot{\varrho} + \varrho \operatorname{div} v = 0 \tag{2.7}$$

is presumed valid.

On the contrary one can expect that micromotions contribute to the the kinetic energy of the body; it is a matter of convenience to define the order parameters in such a way that the kinetic energy of $\mathfrak{B}$ reduces to the classical expression when $\underset{\sim}{v}$ vanishes. Here I assume explicitly that the kinetic energy is measured, in general, by

$$\int_{\mathscr{B}} \varrho(\tfrac{1}{2}v^2 + \varkappa(\underset{\sim}{\nu}, \underset{\sim}{v})), \tag{2.8}$$

where $\varkappa$ is a mapping of the tangent bundle $\mathscr{T}\mathscr{Q}$ of $\mathscr{Q}$ into $\mathbb{R}^+$, such that

$$\varkappa(\underset{\sim}{\nu}, \underset{\sim}{v}) \geqq 0, \qquad \varkappa(\underset{\sim}{\nu}, \boldsymbol{0}) = 0.$$

Usually $\varkappa$ is assumed to be a positive (semi) definite quadratic form on each $\mathscr{T}\mathscr{Q}_{\underset{\sim}{\nu}}$; in terms of local coordinates

$$\varkappa = \tfrac{1}{2} \sum_{1}^{q}{}_{\alpha\beta}\, \mu^{\alpha\beta}(\underset{\sim}{\nu})\, v_\alpha v_\beta. \tag{2.9}$$

Mechanical actions on $\mathfrak{B}$ comprise the usual external body force of density ϱb (per unit volume of $\mathfrak{B}$) and the usual surface traction t. Besides, one must account for actions on the microstructure; they can be appropriately described around any element $\mathfrak{X}$ by members of the cotangent space $\mathscr{T}^*\mathscr{Q}_{\underset{\sim}{\nu}}(\mathfrak{X})$: an external body force $\varrho\underset{\sim}{\beta}$, a resultant of internal actions $-\underset{\sim}{\zeta}$ (both per unit volume of $\mathscr{B}$) and a surface traction $\underset{\sim}{\tau}$ (with components, respectively, $\varrho\beta^\alpha$, $-\zeta^\alpha, \tau^\alpha$).

Mechanical balance is expressed by the usual equation for momentum

$$\left(\int_{\mathscr{E}} \varrho v\right)^{\cdot} = \int_{\partial\mathscr{E}} t + \int_{\mathscr{E}} \varrho b, \tag{2.10}$$

(valid for any subset $\mathscr{E}$ of $\mathscr{B}$ with smooth boundary $\partial\mathscr{E}$) and by an equation modelled on the Lagrange equation

$$\left(\int_{\mathscr{E}} \varrho \frac{\partial\varkappa}{\partial \underset{\sim}{v}}\right)^{\cdot} - \int_{\mathscr{E}} \varrho \frac{\partial\varkappa}{\partial \underset{\sim}{\nu}} = \int_{\partial\mathscr{E}} \underset{\sim}{\tau} + \int_{\mathscr{E}} (\varrho\underset{\sim}{\beta} - \underset{\sim}{\zeta}). \tag{2.11}$$

Remark 3. Eqn. (2.11) is correct only if $\varkappa$ is a quadratic form in $\underset{\sim}{v}$; otherwise the function $\varkappa$ appearing in it must not be identified with the kinetic energy density; rather the latter is the Legendre transform $\varkappa'$ of the former, or in terms of components

$$\varkappa' := \sum_{1}^{q}{}_{\alpha} \frac{\partial\varkappa}{\partial v_\alpha} v_\alpha - \varkappa.$$

Below, as in (2.11), we presume that $\varkappa'$ and $\varkappa$ coincide.

Sufficient regularity of t and $\underset{\sim}{\tau}$ implies the existence of the tensor of stress T and of a linear operator from $\mathscr{V}$ into $\mathscr{T}^*\mathscr{Q}_{\underset{\sim}{\nu}}$, the microstress $\beth$, such that

$$t = Tn, \quad \underset{\sim}{\tau} = \beth n, \tag{2.12}$$

where n is the unit vector of the exterior normal to $\partial\mathscr{C}$. Local balance equations ensue: Cauchy's balance equation

$$\varrho\ddot{x} = \varrho b + \operatorname{div} T, \tag{2.13}$$

and a new balance equation for micromomentum

$$\varrho\left(\left(\frac{\partial\varkappa}{\partial\underset{\sim}{v}}\right)^{\cdot} - \frac{\partial\varkappa}{\partial\underset{\sim}{\nu}}\right) = \varrho\underset{\sim}{\beta} - \zeta + \operatorname{div}\beth; \tag{2.14}$$

on a local chart the latter can be written

$$\varrho\left(\left(\frac{\partial\varkappa}{\partial v_\alpha}\right)^{\cdot} - \frac{\partial\varkappa}{\partial \nu_\alpha}\right) = \varrho\beta^\alpha - \zeta^\alpha + \operatorname{div} t^\alpha, \tag{2.15}$$

where t^α is defined so that (see (2.12))

$$\tau^\alpha = t^\alpha \cdot n. \tag{2.16}$$

Under appropriate circumstances (2.13), (2.14) can be deduced also from a variational principle; I refer to [4] for details.

3. Balance of moment of momentum, energy and entropy

A kinetic energy theorem follows from (2.13), (2.14); if $\mathscr{C}$ is any portion of $\mathscr{B}$ where the motion is smooth, then

$$\begin{aligned}\left(\int_{\mathscr{C}} \varrho\left(\tfrac{1}{2}v^2 + \varkappa\right)\right)^{\cdot} &= \int_{\mathscr{C}} \varrho(b\cdot v + \underset{\sim}{\beta}\cdot\underset{\sim}{v}) + \int_{\partial\mathscr{C}} (t\cdot v + \underset{\sim}{\tau}\cdot\underset{\sim}{v}) + \\ &\quad - \int_{\mathscr{C}} (T\cdot\operatorname{grad} v + \beth\cdot\operatorname{grad}\underset{\sim}{v} + \underset{\sim}{\zeta}\cdot\underset{\sim}{v}).\end{aligned} \tag{3.1}$$

The first two terms in the right-hand side express the power of external actions; thus one is led to interpret the quantity

$$-(T\cdot\operatorname{grad} v + \beth\cdot\operatorname{grad} v + \underset{\sim}{\zeta}\cdot\underset{\sim}{v}) \tag{3.2}$$

as the density per unit volume of the power of internal actions.

In simple continua the symmetry of T is a necessary and sufficient condition for the vanishing of that density for all virtual rigid velocity distributions. In the present context that condition can be expressed using formulae (2.5) and reads (with an obvious interpretation of the transposition symbol)

$$\mathbf{e}\,T = \aleph^T\underset{\sim}{\zeta} + (\operatorname{grad}\aleph^T)\,\beth. \tag{3.3}$$

Alternatively, and with the use of a local chart, eqn. (3.3) can be written as follows

$$\operatorname{skw} T = \tfrac{1}{2}\,\mathbf{e}\left(\sum_{1}^{q}{}_{\alpha}\,(\zeta^{\alpha} r_{\alpha} + (\operatorname{grad} r_{\alpha})\, t^{\alpha})\right). \tag{3.4}$$

This equation is interpreted as the expression of the balance of moment of momentum for $\mathfrak{B}$; I refer to [7] and to earlier papers cited there for the derivation of conditions of the type (3.3) in many specific theories of non-simple continua.

With the use of (3.3), the power density (3.2) can be put into a form whose invariance under superposed rigid body motions is immediately apparent

$$-(T \cdot D + \underset{\sim}{\zeta} \cdot (\underset{\sim}{v} - \aleph s) + \beth \cdot (\operatorname{grad} \underset{\sim}{v} - (\operatorname{grad} \aleph)^{t} s)), \tag{3.5}$$

or

$$-\left(T \cdot D + \sum_{1}^{q}{}_{\alpha}\,(\zeta^{\alpha}(v_{\alpha} - r_{\alpha} \cdot s) + t^{\alpha} \cdot (\operatorname{grad} v_{\alpha} - (\operatorname{grad} r_{\alpha})^{T} s)\right);$$

here D is the strain rate tensor and s is the spin in the macromotion:

$$D := \operatorname{sym} \operatorname{grad} v, \qquad s := -\tfrac{1}{2}\,\mathbf{e}\,(\operatorname{skw} \operatorname{grad} v), \tag{3.6}$$

and the exponent t affixed to grad $\aleph$ indicates transposition of the last two of the three indices.

The balance of energy is expressed by an appropriate modification of the classical relation, which involves the extra kinetic energy and the power of actions on the microstructure

$$\left(\int_{\mathscr{E}} \varrho(\varepsilon + \tfrac{1}{2}v^{2} + \varkappa)\right)^{\cdot} = \int_{\partial\mathscr{E}} (t \cdot v + \underset{\sim}{\tau} \cdot \underset{\sim}{v} + q \cdot n) + \int_{\mathscr{E}} \varrho(b \cdot v + \underset{\sim}{\beta} \cdot v + \chi); \tag{3.7}$$

here ε is internal energy per unit mass, q is heat flux into $\mathscr{E}$, χ is radiant heating per unit mass.

When the functions involved are sufficiently smooth one can deduce from (3.7) the local relation

$$\varrho\dot{\varepsilon} = T \cdot \operatorname{grad} v + \underset{\sim}{\zeta} \cdot \underset{\sim}{v} + \beth \cdot \operatorname{grad} \underset{\sim}{v} + \operatorname{div} q + \varrho\chi. \tag{3.8}$$

Use will also be made later of some consequences of the Clausius-Duhem inequality, which will be accepted here in the standard form

$$\left(\int_{\mathscr{E}} \varrho\eta\right)^{\cdot} \geqq \int_{\partial\mathscr{E}} \frac{q \cdot n}{\theta} + \int_{\mathscr{E}} \varrho\,\frac{\chi}{\theta}, \tag{3.9}$$

where traditional notation is used for entropy η and temperature θ. The local consequence of (3.9) will be called upon below

$$\varrho\dot{\eta} \geqq \operatorname{div} \frac{q}{\theta} + \varrho\,\frac{\chi}{\theta}. \tag{3.10}$$

If one eliminates χ in (3.10) using (3.8) and expresses ε in terms of the Helmholtz free energy ψ

$$\psi := \varepsilon - \theta\eta,$$

then one obtains the reduced inequality

$$\varrho\,(\dot{\psi} + \eta\dot{\theta}) - T\cdot\operatorname{grad} v - \underset{\sim}{\zeta}\cdot\underset{\sim}{v} - \beth\cdot\operatorname{grad}\underset{\sim}{v} - \frac{q\operatorname{grad}\theta}{\theta} \leqq 0. \tag{3.11}$$

Referential versions of (2.14), (3.3) and (3.11) can be easily given by the introduction of Piola-type stresses involving the use of the deformation gradient F and its determinant $\iota := \det F$, *i.e.*

$$P := \iota T(F^{-1}),^T \quad \underset{\sim}{\vartheta} := \iota\underset{\sim}{\zeta}, \quad \text{ר} := \iota\,\beth(F^{-1})^T, \tag{3.12}$$

and the operators Div and Grad on the reference placement, so that, in particular,

$$\iota\operatorname{div}\beth = \operatorname{Div}\text{ר}.$$

The equations are $(\varrho^* = \varrho\iota)$:

$$\varrho_*\left(\left(\frac{\partial\varkappa}{\partial\underset{\sim}{v}}\right)^{\cdot} - \frac{\partial\varkappa}{\partial\underset{\sim}{v}}\right) = \varrho_*\underset{\sim}{\beta} - \underset{\sim}{\vartheta} + \operatorname{Div}\beth, \tag{3.13}$$

$$\mathbf{e}\,(PF^T) = \aleph^T\underset{\sim}{\vartheta} + (\operatorname{Grad}\aleph)^t\,\beth, \tag{3.14}$$

$$\varrho_*(\dot{\psi} + \eta\dot{\theta}) - P\cdot\dot{F} - \underset{\sim}{\vartheta}\cdot\underset{\sim}{v} - \text{ר}\cdot\operatorname{Grad}\underset{\sim}{v} - \frac{(F^{-1}q)\cdot\operatorname{Grad}\theta}{\theta} \leqq 0. \tag{3.15}$$

Remark. In [4] the existence also of a thermal microstructure is conjectured. Here I keep to less shifting ground.

4. Latent microstructure

I say that the microstructure is latent when, though its effects are felt in the balance equations, all relevant quantities can be expressed in terms of geometric and kinematic quantities pertaining to apparent placements.

More precisely I suppose that hypotheses (i)–(iv) below are satisfied which are introduced in order of increasing severity so that consequences of corresponding specificity can be deduced. The first two hypotheses are:

(i) there is no inertia connected with the microstructure, *i.e.*, $\varkappa$ vanishes identically:

(ii) there are no external body actions on the microstructure, *i.e.*, $\underset{\sim}{\beta}$ vanishes identically;

These hypotheses alone have several consequences which I explore first; under (i) and (ii) the balance equation (2.14) reduces to

$$\underset{\sim}{\zeta} = \operatorname{div}\beth, \tag{4.1}$$

or, on a local chart,

$$\zeta^\alpha = \operatorname{div} t^\alpha,$$

and, in referential form,

$$\underset{\sim}{\vartheta} = \operatorname{Div}\text{ר}. \tag{4.2}$$

When constitutive prescriptions are assigned for $\underset{\sim}{\zeta}$ and $\beth$, eqn. (4.1) may take the analytical form of an evolution equation for the order parameters $\underset{\sim}{\nu}$.

As a consequence of (4.1) the balance equation of moment of momentum (3.3) becomes more compact

$$\mathbf{e}\, T = \operatorname{div}(\aleph^T \beth) \tag{4.3}$$

or

$$\mathbf{e}\, T = \operatorname{div}\left(\sum_1^q {}_\alpha\, r_\alpha \otimes t^\alpha\right),$$

and, in referential form,

$$\mathbf{e}\,(PF^T) = \operatorname{Div}(\aleph^T \text{ר}). \tag{4.4}$$

Formula (4.3) generalizes the Cosserats' moment equation obtained, within their context, under assumptions akin to (i) and (ii).

There is another interesting consequence of (4.1): the power density of internal actions due to the microstructure can be written as the divergence of a vector u:

$$\begin{aligned} \underset{\sim}{\zeta} \cdot \underset{\sim}{v} + \beth \cdot \operatorname{grad} \underset{\sim}{v} &= \operatorname{div} u, \\ u := \beth^T \underset{\sim}{v} &\left(= \sum_1^q {}_\alpha\, t^\alpha v_\alpha\right), \end{aligned} \tag{4.5}$$

so that (3.8) can be given the apparently standard form

$$\varrho \dot{\varepsilon} = T \cdot \operatorname{grad} v + \operatorname{div} q' + \varrho \chi, \tag{4.6}$$

where a modified flux vector q' appears

$$q' = q + u. \tag{4.7}$$

DUNN & SERRIN suggest, for their modified Neuman equation, the form (4.6), and call u the interstitial work flux. The introduction of different flux vectors, q' and q respectively, in the Neumann equation and in the Clausius-Duhem inequality has been suggested repeatedly by I. MÜLLER; it is, perhaps, worth remarking that, when only hypotheses (i) and (ii) are accepted, u and hence q' need not be objective.

If, instead, one assumes that:

(iii) the interstitial work flux is objective, then u must not change when $\underset{\sim}{v}$ is altered by the addition of a term of the form $\aleph w$, with w any vector; as a consequence (see $(4.5)_2$)

$$\beth^T \aleph = 0; \tag{4.8}$$

then condition (4.3) requires further that

$$T \in \mathrm{Sym}. \tag{4.9}$$

The last hypothesis is less precise; it is rather suggestive of many specific subcases, the simpler of which I study in the following Sections:

(iv) a set of frictionless holonomic or anholonomic constraints expresses either the order parameters $\underset{\sim}{\nu}$ in terms of displacement gradient F and, perhaps, its gradients or the generalized velocities $\underset{\sim}{v}$ in terms of $\operatorname{grad} v$ and, perhaps, higher gradients of v.

5. The elastic materials of Dunn & Serrin

I observe in this Section that the elastic materials studied by DUNN & SERRIN in [6] can be considered as special continua with latent microstructure when, on the one hand, hypothesis (iv) is rendered more specific as follows:
(iv)′ the internal parameters are constrained by a condition:

$$\underset{\sim}{v} = \hat{\underset{\sim}{v}}(F), \tag{5.1}$$

and, on the other hand, it is also presumed that
(v) free energy, entropy, heat flux, the stress and microstress are functions of F, Grad F, θ and grad θ.

The implications of (iv)′ and (v) are less cumbersome to obtain if Cartesian components (specified below, with latin letters for indices) are used (repeated indices are to be summed). From (5.1) it follows that

$$v_\alpha = \frac{\partial \hat{v}_\alpha}{\partial F_{aA}} v_{a,b} F_{bA} \tag{5.2}$$

and, in a rigid motion with angular velocity w,

$$v_\alpha^{(R)} = \mathbf{e}_{abc} \frac{\partial \hat{v}_\alpha}{\partial F_{aA}} F_{cA} w_b, \tag{5.3}$$

so that

$$\aleph_{\alpha a} = \mathbf{e}_{abc} \frac{\partial \hat{v}_\alpha}{\partial F_{cA}} F_{bA}. \tag{5.4}$$

Condition (4.8) now requires that the third-order tensor

$$\mathbf{s}_{ijk} := \sum_1^q{}_\alpha \beth_i^\alpha \frac{\partial \hat{v}_\alpha}{\partial F_{jA}} F_{kA} \tag{5.6}$$

be symmetric in the last two indices

$$\mathbf{s}_{ijk} = \mathbf{s}_{ikj}. \tag{5.7}$$

The condition that the constraint (5.1) is frictionless is interpreted in the usual sense, rendered precise in a general setting by GURTIN & PODIO-GUIDUGLI (see [7]):

(a) stress and microstresses are each the sum of two terms, active and reactive respectively,

$$T = \overset{a}{T} + \overset{r}{T}, \quad \underset{\sim}{\zeta} = \overset{a}{\underset{\sim}{\zeta}} + \overset{r}{\underset{\sim}{\zeta}}, \quad \beth = \overset{a}{\beth} + \overset{r}{\beth}; \tag{5.8}$$

(b) the power of the reactive parts of stress and microstress vanishes for all velocity fields allowed by the constraints; this condition is more conveniently expressed in terms of Piola-type stresses:

$$\overset{r}{P} \cdot \dot{F} + \overset{r}{\underset{\sim}{\vartheta}} \cdot \underset{\sim}{v} + \overset{r}{\daleth} \cdot \operatorname{Grad} \underset{\sim}{v} = 0, \tag{5.9}$$

for all $\dot{F}$ and all v given by (5.2).

It is easy to check that (5.9) implies

$$\overset{r}{P}_{iA} + \sum_1^q{}_{\alpha}\left(\overset{r}{\vartheta}{}^{\alpha}\frac{\partial \hat{v}_{\alpha}}{\partial F_{iA}} + \overset{r}{\daleth}{}^{\alpha}_{B}\left(\frac{\partial \hat{v}_{\alpha}}{\partial F_{iA}}\right)_{,B}\right) = 0, \tag{5.10}$$

$$\sum_1^q{}_{\alpha}\,\overset{r}{\daleth}{}^{\alpha}_{A}\frac{\partial \hat{v}_{\alpha}}{\partial F_{iB}} + \sum_1^q{}_{\alpha}\,\overset{r}{\daleth}{}^{\alpha}_{B}\frac{\partial \hat{v}_{\alpha}}{\partial F_{iA}} = 0. \tag{5.11}$$

On the other hand property (a) above and (5.11) imply

$$\sum_1^q{}_{\alpha}\,\overset{a}{\daleth}{}^{\alpha}_{A}\frac{\partial \hat{v}_{\alpha}}{\partial F_{iB}} - \sum_1^q{}_{\alpha}\,\overset{a}{\daleth}{}^{\alpha}_{B}\frac{\partial \hat{v}_{\alpha}}{\partial F_{iA}} = 0. \tag{5.12}$$

The interpretation *à la* COLEMAN-NOLL of the dissipation inequality and an ensuing well-known train of reasoning, together with hypothesis (v) and (5.12), exclude a dependence of ψ and η on grad θ, impose the usual relation between these two functions and, besides, lead to the relations

$$\overset{a}{P}_{iA} + \sum_1^q{}_{\alpha}\left(\overset{a}{\vartheta}{}^{\alpha}\frac{\partial \hat{v}_{\alpha}}{\partial F_{iA}} + \left(\overset{a}{\daleth}{}^{\alpha}_{B}\frac{\partial \hat{v}_{\alpha}}{\partial F_{iA}}\right)_{,B} - \overset{a}{\daleth}{}^{\alpha}_{B,B}\frac{\partial \hat{v}_{\alpha}}{\partial F_{iA}}\right) = \varrho_*\frac{\partial \psi}{\partial F_{iA}}, \tag{5.13}$$

$$\sum_1^q{}_{\alpha}\left(\overset{a}{\daleth}{}^{\alpha}_{B}\frac{\partial \hat{v}_{\alpha}}{\partial F_{iA}}\right) = \varrho_*\frac{\partial \psi}{\partial F_{iA,B}}; \tag{5.14}$$

observe that the tensor on the right-hand side of (5.14) is symmetric in the indices A, B.

Summing up term by term (5.10) and (5.13) and using (5.14) one arrives at the relation

$$P_{iA} = \varrho_*\frac{\partial \psi}{\partial F_{iA}} - \left(\varrho_*\frac{\partial \psi}{\partial F_{iA,B}}\right)_{,B} - \left(\sum_1^q{}_{\alpha}\,\overset{r}{\daleth}{}^{\alpha}_{B}\frac{\partial \hat{v}_{\alpha}}{\partial F_{iA}}\right)_{,B}. \tag{5.15}$$

To render explicit in terms of the derivatives of ψ the last term in (5.15) it suffices to exploit (5.7) in the manner shown by DUNN & SERRIN in Appendix A of their paper and so obtain

$$P_{iA} = \varrho_*\frac{\partial \psi}{\partial F_{iA}} - \left(\varrho_*\frac{\partial \psi}{\partial F_{iA,B}}\right)_{,B} - \left(\varrho_* F_{iB}\left(\frac{\partial \psi}{\partial F_{jB,C}}F^{-1}_{Aj} - \frac{\partial \psi}{\partial F_{jB,A}}F^{-1}_{Cj}\right)\right)_{,C}, \tag{5.16}$$

which is DUNN & SERRIN's formula (3.4).

As observed by DUNN & SERRIN, the symmetry of PF^T is ensured automatically because ψ is an objective scalar.

6. A subcase of the Korteweg fluid

The reduction of (5.16) to the special case when ψ depends (on the temperature θ,) on $\iota = \det F$ and its first gradient only has already been performed by DUNN & SERRIN and so needs not be reconsidered here. However, that case can be viewed also from a different angle, and therefore a few remarks are in order.

Suppose that the microstructure be such that $\underset{\sim}{v}$ (as happens when $\underset{\sim}{\nu}$ is determined through ι alone) be not affected by superposed rigid rotations ($\aleph = 0$); then the interstitial work flux is automatically objective and assertion (iii) becomes a theorem.

If, in addition, there is only one internal parameter (then the notation can reduce to the usual letters without superscript and wavy underlining: ν, ζ, t) and that parameter is constrained to coincide with ι, $\nu \equiv \iota$, then one does not even need referential arguments. One can take into account Euler's formula for $\dot{\iota}$

$$\dot{\iota} = \iota \operatorname{div} v$$

and its consequence

$$\operatorname{grad} \dot{\iota} = (\operatorname{grad} \iota)^{\cdot} + (\operatorname{grad} v)^T \operatorname{grad} \iota,$$

to put expression (3.2) for the power density into the form

$$\begin{aligned} A \cdot \operatorname{dev}(T + (\operatorname{grad} \iota) \otimes t) + \alpha\,(\operatorname{tr} T + 3\iota\zeta + (\operatorname{grad} \iota) \cdot t) \\ + t \cdot (\operatorname{grad} \iota)^{\cdot}, \end{aligned} \tag{6.1}$$

$$A := \operatorname{dev} \operatorname{grad} v, \qquad \alpha = \tfrac{1}{3} \operatorname{div} v, \tag{6.2}$$

to obtain directly, on the one hand,

$$\overset{r}{t} = 0, \quad \operatorname{dev} \overset{r}{T} = 0, \quad \operatorname{tr} \overset{r}{T} + 3\iota\overset{r}{\zeta} = 0 \tag{6.3}$$

and hence

$$\operatorname{tr} \overset{a}{T} - 3\iota\overset{a}{\zeta} = 0,$$

and, on the other hand,

$$\overset{a}{t} = \varrho \frac{\partial \psi}{\partial (\operatorname{grad} \iota)}, \tag{6.4}$$

$$\operatorname{div}(\overset{a}{T} + (\operatorname{grad} \iota) \otimes \overset{a}{t}) = 0,$$

$$\operatorname{tr} \overset{a}{T} + 3\iota\overset{a}{\zeta} + (\operatorname{grad} \iota) \cdot \overset{a}{t} = 3\iota\varrho \frac{\partial \psi}{\partial \iota},$$

together with (see (4.1))

$$\overset{a}{\zeta} + \overset{r}{\zeta} = \operatorname{div} \overset{a}{t}. \tag{6.5}$$

The final expression for T,

$$T = \left(\varrho\iota \frac{\partial \psi}{\partial \iota} - \iota \operatorname{div}\left(\varrho \frac{\partial \psi}{\partial (\operatorname{grad} \iota)}\right)\right) 1 - (\operatorname{grad} \iota) \otimes \varrho \frac{\partial \psi}{\partial (\operatorname{grad} \iota)}, \tag{6.6}$$

coincides with the expression (1.25) given by DUNN & SERRIN. Notice again that the symmetry of T is assured, because the dependence of ψ on grad ι may occur only through the modulus $|\operatorname{grad} \iota|$, as ψ is an objective scalar.

7. A more general elastic fluid of the Korteweg class

Suppose that $\mathcal{Q}$ coincides with $\mathcal{V}\times\mathbb{R}$ ($q=4$), that ν_1, ν_2, ν_3 are Cartesian components of a vector d and ν_4 is a scalar ν; then we can put

$$\underset{\sim}{\zeta} \equiv \begin{pmatrix} z \\ \hline \zeta \end{pmatrix} \quad \text{and} \quad \urcorner \equiv \begin{pmatrix} V \\ \hline t \end{pmatrix},$$

with $\zeta\in\mathbb{R}$, $z, t\in\mathcal{V}$ and $V\in\mathrm{Lin}$ and write the balance equation of moment of momentum in the form

$$z = \operatorname{div} V, \qquad \zeta = \operatorname{div} t. \tag{7.1}$$

Suppose further that

(iv)$''$ appropriate constraints bind d and ν to ι and $\operatorname{grad}\iota$; more precisely

$$d = \operatorname{grad}\varrho, \qquad \nu = \varrho. \tag{7.2}$$

Remark. Of course, ϱ is in one-to-one correspondence with ι and one could, perhaps more appropriately, accept an alternative direct definition of d and ν in terms of ι and $\operatorname{grad}\iota$. The choice (7.2) brings the following developments more strictly in line with those of DUNN & SERRIN.

When the notation (6.2) is used and account is taken of obvious consequences of the constraints (7.2)

$$\dot d = -3\operatorname{grad}(\varrho\alpha) - (A+\alpha 1)\operatorname{grad}\varrho, \qquad \dot\nu = -3\varrho\alpha,$$

the power density of internal actions

$$T\cdot\operatorname{grad} v + z\cdot\dot d + V\cdot\operatorname{grad}\dot d + t\cdot\operatorname{grad}\dot\nu + \zeta\dot\nu,$$

can be given the expression

$$\begin{aligned}
&\alpha(\operatorname{tr} T - (4z+3t)\cdot\operatorname{grad}\varrho - 4V\cdot\operatorname{grad}^2\varrho - 3\varrho\zeta)\\
&\quad - 3(\operatorname{grad}\alpha)\cdot(\varrho(z+t) + 3(\operatorname{sym} V)\operatorname{grad}\varrho)\\
&\quad - 3\varrho(\operatorname{grad}^2\alpha)\cdot V + (\operatorname{grad}^2 v)\cdot((\operatorname{grad}\varrho)\otimes V)\\
&\quad + A\cdot\operatorname{dev}(T - (\operatorname{grad}^2\varrho)V^T - (\operatorname{grad}\varrho)\otimes z).
\end{aligned}$$

It follows that the reactive components of stress and microstress satisfy the conditions

$$\overset{r}{T} = \varrho\overset{r}{\zeta}\,1 + (\operatorname{grad}\varrho)\otimes\overset{r}{z} - (\operatorname{grad}^2\varrho)\overset{r}{V},$$

$$\overset{r}{z} + \overset{r}{t} = 0, \qquad \overset{r}{V}\in\mathrm{Skw}. \tag{7.3}$$

Assume now that

(v)$'$ free energy, entropy, heat flux, stress and microstress are functions of ϱ, $\operatorname{grad}\varrho$, $\operatorname{grad}^2\varrho$, θ and $\operatorname{grad}\theta$.

Because

$$\begin{aligned}
(\operatorname{grad}^2\varrho)^{\cdot} = &-5\alpha\operatorname{grad}^2\varrho - 6\operatorname{sym}((\operatorname{grad}\varrho)\otimes(\operatorname{grad}\alpha))\\
&- 3\varrho\operatorname{grad}^2\alpha - 2\operatorname{sym}((\operatorname{grad}^2\varrho)A) - B,
\end{aligned}$$

with

$$B_{ij} = \varrho_{,h} v_{h,ij}, \qquad B \in \mathrm{Sym},$$

the total time derivative of ψ is given by

$$\dot{\psi} = \frac{\partial \psi}{\partial \theta}\dot{\theta} + \frac{\partial \psi}{\partial \theta_{,i}}(\theta_{,i})^{\cdot} - \left(3\varrho \frac{\partial \psi}{\partial \varrho} + 4\frac{\partial \psi}{\partial \varrho_{,i}}\varrho_{,i} + 5\frac{\partial \psi}{\partial \varrho_{,ij}}\varrho_{,ij}\right)\alpha$$

$$- \left(3\varrho \frac{\partial \psi}{\partial \varrho_{,i}} + 6\varrho_{,j}\frac{\partial \psi}{\partial \varrho_{,ij}}\right)\alpha_{,i} - 3\varrho \frac{\partial \psi}{\partial \varrho_{,ij}}\alpha_{,ij}$$

$$- \left(\frac{\partial \psi}{\partial \varrho_{,j}}\varrho_{,i} + \frac{\partial \psi}{\partial \varrho_{,kj}}\varrho_{,ik} + \frac{\partial \psi}{\partial \varrho_{,ik}}\varrho_{,jk}\right)A_{ij} - \frac{\partial \psi}{\partial \varrho_{,ij}}B_{ij}.$$

Then, again, the argument first introduced by COLEMAN & NOLL leads to the usual conclusion that ψ does not depend on θ and grad θ, and to the following relations for the active components of stress and microstress:

$$\overset{a}{V} = \varrho \frac{\partial \psi}{\partial\,(\mathrm{grad}^2 \varrho)}, \qquad \overset{a}{z} + \overset{a}{t} = \varrho \frac{\partial \psi}{\partial\,(\mathrm{grad}\,\varrho)},$$

$$\overset{a}{T} - (\mathrm{grad}\,\varrho) \otimes \overset{a}{z} - \varrho \overset{a}{\zeta}\, 1 = \tag{7.4}$$

$$-\varrho^2 \frac{\partial \psi}{\partial \varrho} 1 - \varrho\,(\mathrm{grad}\,\varrho) \otimes \frac{\partial \psi}{\partial\,(\mathrm{grad}\,\varrho)} - \varrho\,(\mathrm{grad}^2\,\varrho)\left(\frac{\partial \psi}{\partial\,(\mathrm{grad}\,^2\varrho)}\right).$$

Formulae (7.3), (7.4) together lead to

$$V = \varrho \frac{\partial \psi}{\partial\,(\mathrm{grad}^2 \varrho)} + \overset{r}{V}, \qquad \overset{r}{V} \in \mathrm{Skw},$$

$$z + t = \varrho \frac{\partial \psi}{\partial\,(\mathrm{grad}\,\varrho)}, \tag{7.5}$$

$$T = \varrho\left(\zeta - \varrho \frac{\partial \psi}{\partial \varrho}\right) 1 - (\mathrm{grad}\,\varrho) \otimes t - (\mathrm{grad}^2\,\varrho)\,V.$$

The balance equations (7.1) allow us to eliminate ζ and t from $(7.5)_3$

$$T = -\varrho\left(\varrho \frac{\partial \psi}{\partial \varrho} + \mathrm{div}\left(\mathrm{div}\left(\varrho \frac{\partial \psi}{\partial\,(\mathrm{grad}^2 \varrho)}\right) - \varrho \frac{\partial \psi}{\partial\,(\mathrm{grad}\,\varrho)}\right)\right) 1$$

$$+ (\mathrm{grad}\,\varrho) \otimes \mathrm{div}\,V - (\mathrm{grad}^2\,\varrho)\,V - \varrho\,(\mathrm{grad}\,\varrho) \otimes \frac{\partial \psi}{\partial\,(\mathrm{grad}\,\varrho)}. \tag{7.6}$$

Finally to express the reactive part of V in terms of the derivatives of ψ, we exploit the assumption of objectivity of the interstitial working. In a rigid motion with rotational speed w

$$\dot{\varrho} = 0, \qquad (\mathrm{grad}\,\varrho)^{\cdot} = (\mathrm{grad}\,\varrho) \times w,$$

so that the assumption mentioned is equivalent to the condition

$$((\mathrm{grad}\,\varrho) \otimes V)\,v \in \mathrm{Sym} \qquad \forall\, v \in \mathscr{V}. \tag{7.7}$$

Following now the argument, already cited, of DUNN & SERRIN, we obtain from (7.5) and (7.7)

$$\varrho_{,k}\overset{r}{V}_{ij} = \varrho\left(\varrho_{,i}\frac{\partial\psi}{\partial\varrho_{,jk}} - \varrho_{,j}\frac{\partial\psi}{\partial\varrho_{,ki}}\right),$$

with the conclusion that

$$\begin{aligned} T_{ij} = & -\varrho\left(\varrho\frac{\partial\psi}{\partial\varrho} + \left(\left(\varrho\frac{\partial\psi}{\partial\varrho_{,hk}}\right)_{,h} - \varrho\frac{\partial\psi}{\partial\varrho_{,k}}\right)_{,k}\delta_{ij}\right). \\ & + \varrho_{,i}\left(\left(\varrho\frac{\partial\psi}{\partial\varrho_{,jk}}\right)_{,k} - \varrho\frac{\partial\psi}{\partial\varrho_{,j}}\right) - \varrho_{,ik}\varrho\frac{\partial\psi}{\partial\varrho_{,kj}} \\ & + \varrho\left(\varrho_{,j}\frac{\partial\psi}{\partial\varrho_{,ki}} - \varrho_{,k}\frac{\partial\psi}{\partial\varrho_{,ij}}\right)_{,k}, \end{aligned} \tag{7.8}$$

a formula which generalizes (6.6) and may be the starting point for ampler analyses regarding the class of elastic fluids of the Korteweg type. The formula does not fall within the subclass considered by DUNN & SERRIN; one of their hypotheses on the interstitial working excludes a dependence of ψ on $\text{grad}^2\,\varrho$.

Acknowledgements. This work is part of a research program under the auspices of the Italian Ministry of Education. I thank also the Department of Mathematics of Heriot-Watt University in Edinburgh for support given me for a short visit, during which I had the leisure to write the final draft.

References

1. G. CAPRIZ & P. PODIO-GUIDUGLI, Materials with finite-dimensional structure, in A. P. S. SELVADURAI, ed., *Mechanics of Structured Media.* Elsevier (1981), 255–268.
2. G. CAPRIZ, Introductory remarks to the dynamics of continua with microstructure, im *Mathematical Methods and Models in Mechanics.* Banach Center Publ. **15** (1985), 85–109.
3. G. CAPRIZ & P. PODIO-GUIDUGLI, Structured continua from a Lagrangian point of view. *Ann. Mat. Pura Appl.* **135** (1983), 1–25.
4. G. CAPRIZ, Continui con microstruttura. *Boll. Un. Mat. Ital.* **3A** (1984), 181–195.
5. R. A. TOUPIN, Theories of elasticity with couple-stress. *Arch. Rational Mech. An.* **17** (1964), 85–112.
6. J. E. DUNN & J. SERRIN, On the thermomechanics of interstitial working. IMA Preprint 24 (1983).
7. G. A. MAUGIN, The method of virtual power in continuum mechanics. *Acta Mechanica* **35** (1980), 1–70.
8. M. GURTIN & P. PODIO-GUIDUGLI, The thermodynamics of constrained materials. *Arch. Rational Mech. An.* **51** (1973), 192–208.

Università di Pisa
Italy

(Received October 19, 1984)