Quasiconvexity and Uniqueness of Equilibrium Solutions in Nonlinear Elasticity

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Dedicated to J. L. Ericksen on his sixtieth birthday

1. Introduction

In this paper, we prove the uniqueness of the solution to certain simple displacement boundary value problems in the nonlinear theory of homogeneous hyperelasticity for a body occupying a star-shaped reference configuration $\Omega \subset \mathbb{R}^n$ whose boundary $\partial\Omega$ is subjected to an affine deformation i.e., there exists a constant $n \times n$ matrix F and a constant n vector b such that $x \mapsto Fx + b$ for all $x \in \partial\Omega$. We consider all smooth equilibrium configurations satisfying this boundary condition. Clearly, the homogeneous deformation $x \mapsto Fx + b$, for all $x \in \bar{\Omega}$, is one such solution. Our aim is to prove under suitable hypotheses that it is the only such solution.

In Section 4, we prove that when W is rank-one-convex, the homogeneous deformation has maximal energy amongst all possible smooth equilibrium solutions of the boundary value problem. When, in addition, W is strictly quasiconvex at F, it follows immediately that the homogeneous deformation is the only smooth equilibrium solution.

In Section 5, attention is restricted to *radial* deformations of a ball composed of *isotropic* material. We prove that *all* smooth radial solutions are of the form $x \to \lambda x$ for all $x \in \Omega$, where $\lambda > 0$, provided that *either* (i) W is rank-one-convex and W is strictly quasi-convex, *or* (ii) W is strictly rank-one-convex and W is quasi-convex. These results complement those of BALL [5] in his treatment of *weak* radial deformations of the sphere. He discusses a variety of conditions all roughly allied to strong-ellipticity but excluding the ones considered here.

Finally, in Section 6 the results of Section 4 are adapted to the corresponding boundary value problems for incompressible materials.

Sections 2 and 3 contain necessary definitions and other essential preliminaries

Uniqueness in nonlinear elastostatics has been previously studied by several authors including ERICKSEN & TOUPIN [8], HILL [12, 13], TRUESDELL & TOUPIN [20], F. JOHN [14] and more recently by GURTIN & SPECTOR [10], SPECTOR [17, 18], CHILLINGWORTH, MARSDEN & WAN [6, 7], MARSDEN & WAN [15] and WAN & MARSDEN [21]. These authors either deal with local uniqueness (in the sense of requiring the solution to be in a neighbourhood of a given deformation) or impose constitutive hypotheses closely linked with a notion of stability. In contrast, although our results do not cover such an extensive range of boundary problems, they are global in character and rely upon the comparatively modest requirements of rank-one-convexity and quasiconvexity of the stored-energy density. Of course, it is well-known that a convex stored-energy density, while ensuring uniqueness, implies other conditions unacceptable in elasticity, (*cp.* HILL [12], TRUESDELL & NOLL [19] and BALL [1]).

We are not concerned in this paper with existence of the solution. For a general discussion of this question, and in particular for the relation to the constitutive hypotheses adopted here, the interested reader should consult the papers by BALL [1, 2, 4].

Throughout, we adhere mainly to direct tensor notation.

2. Notation. The Displacement Boundary Value Problem

The inner product of two tensors A, B is given by $AB = \operatorname{tr} AB^t$, where B^t is the transpose of B and tr denotes the tensorial trace operator. By $\det A$ and $\operatorname{adj} A$ we shall mean the determinant and adjugate matrix of the matrix (tensor) A, respectively, so that we have the relation

$$A \operatorname{adj} A = I \det A,$$

where I is the identity. Thus, we see that $\operatorname{adj} A$ is the transposed matrix of co-factors of A. The gradient and divergence operators in $\mathbb{R}^n$ are written respectively as ∇ and div; hence, for the vector field v, ∇v is the tensor field with cartesian components $(\nabla v)_{ij} = \partial v_i / \partial x_j$, while for the tensor field T, $\operatorname{div} T$ is the vector field with cartesian components $\partial T_{ij}/\partial x_j = 0$.

Let M be the set of all $n \times n$ real matrices. We put

$$M_+ = \{F \in M : \det F > 0\}.$$

Thus, M can be identified with $\mathbb{R}^{n^2}$ with the usual euclidean norm while M_+ is an open subset of M.

For a function $W: M_+ \to \mathbb{R}$ which is continuously differentiable, we define $\partial W/\partial F$, $F \in M_+$, to be the tensor whose components are given by

$$(\partial W/\partial F)_{ij} = \partial W/\partial F_{ij}.$$

Let $\Omega \subset \mathbb{R}^n$ be an open bounded region of n-dimensional euclidean space whose boundary $\partial\Omega$ is piecewise continuously differentiable. Let $N: \partial\Omega \to \mathbb{R}^n$ denote the unit outward normal on $\partial\Omega$, so that N is defined almost everywhere on $\partial\Omega$.

We suppose a homogeneous elastic body in its reference configuration occupies the region Ω and is deformed by displacement of the boundary $\partial\Omega$ to a specified final position. Body-force is assumed to be zero.

We define the *deformation* to be the map $u: \bar{\Omega} \to \mathbb{R}^n$ such that $u \in C^1(\bar{\Omega}, \mathbb{R}^n)$ and the associated deformation gradient satisfies $\nabla u(x) \in M_+$ for all $x \in \bar{\Omega}$.

We assume that the elastic body possesses a function $W: M_+ \to \mathbb{R}$ defined on the deformation gradients in M_+ and which is taken to be the *stored-energy function* per unit volume of Ω. The requirement of "frame-indifference" means that W must satisfy the invariance condition

$$W(QF) = W(F) \quad \forall\, F \in M_+, \quad \forall\, Q \in \mathrm{SO}(n). \tag{2.1}$$

The *total energy* for the deformation u is given by

$$I(u) = \int_\Omega W(\nabla u(x))\, dx. \tag{2.3}$$

Let T denote the Piola-Kirchhoff stress tensor which for $W \in C^1(M_+, \mathbb{R})$ has the constitutive relation

$$T(\nabla u) = \frac{\partial W}{\partial F}(\nabla u). \tag{2.3}$$

Then we say that the deformation u is a *smooth equilibrium solution* for a material with $W \in C^1(M_+, \mathbb{R})$ provided $u \in C^2(\Omega, \mathbb{R}^n) \cap C^1(\bar{\Omega}, \mathbb{R}^n)$, $\nabla u(x) \in M_+$ for all $x \in \bar{\Omega}$ and the equilibrium equations are satisfied:

$$\operatorname{div} T(\nabla u(x)) = 0 \quad \forall\, x \in \Omega. \tag{2.4}$$

In the proof of uniqueness, we shall need the following results. The first is a conservation law due originally to GREEN.

Proposition 2.1 (GREEN [9]). *Let Ω be as defined above. Let u be a smooth equilibrium solution for a material with stored-energy density $W \in C^2(M_+, \mathbb{R})$. Then*

$$nI(u) = \int_{\partial\Omega} \left\{ (N \cdot x)\, W(\nabla u(x)) + \operatorname{tr} T(\nabla u(x)) \left[N \otimes \left(u(x) - r\frac{\partial u}{\partial r}(x) \right) \right] \right\} dS, \tag{2.5}$$

where $r\dfrac{\partial u}{\partial r}(x) = (x \cdot \nabla)\, u(x)$, *and* $a \otimes b$ *denotes the tensor product between the vectors* $a \in \mathbb{R}^n$, $b \in \mathbb{R}^n$.

Proof. The result follows directly by straightforward verification of the identity

$$nW(\nabla u) = \operatorname{div} \left[xW(\nabla u) + T^t(\nabla u) \left(u - r\frac{\partial u}{\partial r} \right) \right], \tag{2.6}$$

based upon the constitutive assumption (2.3) and the equilibrium equation (2.4). An application of the divergence theorem to (2.6) now leads to (2.5). □

Let us also note for later use that the divergence theorem applied to (2.4) immediately gives

$$\int_{\partial\Omega} TN\, dx = 0. \tag{2.7}$$

Lemma 2.1 (*cp.* HADAMARD [11]). *Let $u, v \in C^1(\bar{\Omega}, \mathbb{R}^n)$ with $u(x) = v(x)$ $\forall\, x \in \partial\Omega$.*

Then

$$\text{(i)}\quad \nabla(u-v)=\frac{\partial}{\partial N}(u-v)(x)\otimes N,\qquad x\in\partial\Omega,$$

$$\text{(ii)}\quad r\frac{\partial}{\partial r}(u-v)(x)=(N\cdot x)\frac{\partial}{\partial N}(u-v)(x),\qquad x\in\partial\Omega.$$

Proof. Fix $i\in(1,\dots,n)$ and set $f(x)=u_i(x)-v_i(x)$, where w_i denotes the i^{th} Cartesian component of the vector w. Then $f\in C^1(\overline{\Omega},\mathbb{R})$ and $f(x)\equiv 0$ on $\partial\Omega$. Hence, $\nabla f(x)=N(x)\dfrac{\partial f}{\partial N}(x)$, $\forall\, x\in\partial\Omega$, and so

$$\frac{\partial}{\partial x_j}(u_i(x)-v_i(x))=N_j\frac{\partial}{\partial N}(u_i-v_i)(x),$$

which proves (i).

Again,

$$r\frac{\partial f}{\partial r}(x)=(x\cdot\nabla)f(x)=(N(x)\cdot x)\frac{\partial f}{\partial N}(x),$$

and so (ii) holds. □

We observe that Part (i) of the previous proposition shows that $\nabla(u-v)$ is a rank-one matrix on $\partial\Omega$.

In the next section, we set out hypotheses which will be imposed on the stored-energy function, together with some related concepts.

3. The Stored-Energy Density

We recall here some selected definitions concerned with the general notion of convexity which will be used in our uniqueness theorem to restrict the stored-energy density. A complete discussion of these and related concepts may be found, for example, in the papers by BALL [1, 3, 4] and in the books by TRUESDELL & NOLL [19] and WANG & TRUESDELL [22].

Definition 3.1 (MORREY [16]). Let $W\in C(M_+,\mathbb{R})$. Then W is *quasi-convex at* $F\in M_+$ if and only if

$$\int_D W(F+\nabla\xi(x))\,dx\geqq\int_D W(F)\,dx=W(F)\,|D|,$$

for all non-empty open bounded subsets $D\subseteq\mathbb{R}^n$ and all Lipschitz continuous functions ξ which vanish on ∂D such that $F+\nabla\xi(x)\in M_+$, $\forall\, x\in D$. Here $|D|$ denotes the volume of D.

Definition 3.2. We say that W is *strictly quasi-convex at* $F\in M_+$ if and only if W is quasi-convex at F and equality holds only when $\xi=0$.

Definition 3.3. We say that W is (*strictly*) *quasi-convex* if and only if W is (strictly) quasi-convex at F for all (constant) $F \in M_+$.

Definition 3.4. Let $W \in C(M_+, \mathbb{R})$. Then W is rank-one-convex at F if and only if

$$W(F + \mu a \otimes b) \leqq \mu W(F + a \otimes b) + (1 - \mu)\, W(F), \tag{3.1}$$

for all $\mu \in [0, 1]$, $F \in M_+$, $a \in \mathbb{R}^n$, $b \in \mathbb{R}^n$, such that $F + ta \otimes b \in M_+$, for all $t \in [0, 1]$.

Definition 3.5. The function $W \in C(M_+, \mathbb{R})$ is *strictly rank-one-convex* at F if and only if W is rank-one-convex and equality holds in (3.1) only if $a \otimes b = 0$ or $\mu(1 - \mu) = 0$.

It immediately follows from (3.1) on taking the limit as $\mu \to 0$ and using the definition of a derivative that if $W \in C^1(M_+, \mathbb{R})$ is rank-one-convex at F then

$$W(F + a \otimes b) \geqq W(F) + \frac{\partial W}{\partial F}(F)\, a \otimes b \tag{3.2}$$

$$= W(F) + \operatorname{tr}\{T(F)\,(b \otimes a)\}, \tag{3.3}$$

for all $F \in M_+$, $a \in \mathbb{R}^n$, $b \in \mathbb{R}^n$ such that $F + a \otimes b \in M_+$. In (3.3), the Piola-Kirchhoff stress has been introduced from (2.3).

Definition 3.6. We say that W is (*strictly*) *rank-one-convex* if and only if W is (strictly) rank-one-convex at F for all $F \in M_+$.

Definition 3.7. Let $W \in C^2(M_+, \mathbb{R})$. Then W is *strongly-elliptic* if and only if

$$\frac{\partial^2 W(F)}{\partial F^2}(a \otimes b, a \otimes b) > 0$$

for all $F \in M_+$, $a \in \mathbb{R}^n$, $b \in \mathbb{R}^n$ such that a and b are not identically zero.

The above definitions are not independent but are related by the following implications:

(i) $W \in C(M_+, \mathbb{R})$ quasi-convex $\Rightarrow$ W rank-one-convex (*cp.* Morrey [16], Ball [1]).

(ii) $W \in C^2(M_+, \mathbb{R})$ strongly-elliptic $\Rightarrow$ W strictly rank-one-convex.

(iii) For $W \in C^2(M_+, \mathbb{R})$, then W rank-one-convex is equivalent to the *Legendre-Hadamard condition*:

$$\frac{\partial^2 W}{\partial F^2}(F)\,(a \otimes b, a \otimes b) \geqq 0 \tag{3.4}$$

for all

$$F \in M_+, \quad a \in \mathbb{R}^n, \quad b \in \mathbb{R}^n.$$

It is still a major open question whether the converse implication in (i) is valid, although the converse in (ii) is obviously false.

A sufficient condition for quasi-convexity is provided by the following pointwise condition. The function $W \in C(M_+, \mathbb{R})$ is said to be *polyconvex* (*cp.* MORREY [16], BALL [1]) if and only if there exists a function $G: M_+ \times M_+ \times (0, \infty) \to \mathbb{R}$ such that

$$W(F) = G(F, \operatorname{adj} F, \det F) \quad \forall\, F \in M_+$$

and G is convex on $M_+ \times M_+ \times (0, \infty)$. Furthermore, when G is strictly convex on $M_+ \times M_+ \times (0, \infty)$ at $(F, \operatorname{adj} F, \det F)$ then W is strictly quasi-convex at $F \in M_+$. However, in what follows, we make no direct use of this condition, which has been used by BALL [1] in his discussion of existence of solutions in nonlinear elastostatics.

We are now ready to state and prove our results concerning uniqueness.

4. Uniqueness in the Displacement Boundary Value Problem

Throughout this section, we suppose that $\Omega \subset \mathbb{R}^n$ is an open bounded domain of euclidean n-space which is *star-shaped* with respect to the point $x_0 \in \Omega$. That is, for all $y \in \mathbb{R}^n \setminus \{0\}$, the ray $\{x_0 + \lambda y : \lambda \geqq 0\}$ intersects the boundary $\partial\Omega$ in exactly one point. Furthermore, we also suppose that $\partial\Omega$ is piecewise continuously differentiable and we let the vector $N: \partial\Omega \to \mathbb{R}^n$ denote the outward unit normal field which is defined almost everywhere on $\partial\Omega$.

Since Ω is star-shaped with respect to x_0, it clearly follows that

$$N(x) \cdot (x - x_0) > 0 \quad \forall\, x \in \partial\Omega. \tag{4.1}$$

We now show that when the stored-energy density is rank-one-convex a simple inequality holds between equilibrium solutions having the same boundary values. This forms the content of Proposition 4.1. The uniqueness result is then derived as a Corollary under the additional assumption of quasi-convexity and affine displacement boundary data.

Proposition 4.1. *Let Ω be as above. Let $u: \bar{\Omega} \to \mathbb{R}^n$, $v: \bar{\Omega} \to \mathbb{R}^n$ be smooth equilibrium solutions for a material with stored-energy density $W \in C^2(M_+, \mathbb{R})$ such that $u(x) = v(x)$ for $x \in \partial\Omega$. When W is rank-one-convex we have*

$$n\{I(u) - I(v)\} \leqq \int_{\partial\Omega} \operatorname{tr}\, [T(\nabla u) - T(\nabla v)] \left[N \otimes \left(v - r\frac{\partial v}{\partial r}\right)\right] dS, \tag{4.2}$$

where $I(u)$ is the total energy defined by (2.2).

Proof. By a translation, without loss we may suppose that Ω is star-shaped with respect to $x_0 = 0$. Hence, $N(x) \cdot x > 0$ for all $x \in \partial\Omega$.

Now, by Proposition 2.1, we have

$$n\{I(u) - I(v)\} = \int_{\partial\Omega} (N\cdot x)\{W(\nabla u) - W(\nabla v)\}\,dS$$
$$+ \int_{\partial\Omega} \left(\operatorname{tr} T(\nabla u)\left[N \otimes \left(u - r\frac{\partial u}{\partial r}\right)\right]\right. \tag{4.3}$$
$$\left. - \operatorname{tr} T(\nabla v)\left[N \otimes \left(v - r\frac{\partial v}{\partial r}\right)\right]\right) dS.$$

However, we may use the fact that $u = v$ on $\partial\Omega$ and also Lemma 1.1(ii) to write,

$$\operatorname{tr} T(\nabla u)\left[N \otimes \left(u - r\frac{\partial u}{\partial r}\right)\right] - \operatorname{tr} T(\nabla v)\left[N \otimes \left(v - r\frac{\partial v}{\partial r}\right)\right]$$
$$= \operatorname{tr} T(\nabla u)\left[N \otimes r\frac{\partial(v-u)}{\partial r}\right] + \operatorname{tr}\,[T(\nabla u) - T(\nabla v)]\left[N \otimes \left(v - r\frac{\partial v}{\partial r}\right)\right] \tag{4.4}$$
$$= (N\cdot x)\operatorname{tr} T(\nabla u)\left[N \otimes \frac{\partial(v-u)}{\partial N}\right] + \operatorname{tr}\,[T(\nabla u) - T(\nabla v)]\left[N \otimes \left(v - r\frac{\partial v}{\partial r}\right)\right].$$

Furthermore, by Lemma 1.1(i), we know that on $\partial\Omega$

$$\nabla v = \nabla u + \nabla(v-u) = \nabla u + \frac{\partial(v-u)}{\partial N} \otimes N. \tag{4.5}$$

On putting the expressions (4.4) and (4.5) into (4.3) we are led to

$$n\{I(u) - I(v)\} = \int_{\partial\Omega} (N\cdot x)\left\{W(\nabla u) - W\left(\nabla u + \frac{\partial(v-u)}{\partial N} \otimes N\right)\right.$$
$$\left. + \operatorname{tr} T(\nabla u)\left[N \otimes \frac{\partial(v-u)}{\partial N}\right]\right\} dS + \int_{\partial\Omega} \operatorname{tr}\,[T(\nabla u) - T(\nabla v)]\left[N \otimes \left(v - r\frac{\partial v}{\partial r}\right)\right] dS,$$

and then the desired inequality follows by appealing to rank-one-convexity, (3.3) and the positivity of $N\cdot x$ on $\partial\Omega$. □

Remark 4.1. If W is strictly rank-one-convex then equality holds only if $\nabla u(x) \equiv \nabla v(x)$ for all $x \in \partial\Omega$.

Remark 4.2. It is clear from the proof that in Proposition 4.1, rank-one-convexity of W may be replaced by rank-one-convexity of W on the set of surface values of ∇u on $\partial\Omega$.

Corollary 4.2. *Let Ω be as above and let $F \in M_+$ be a constant matrix. Let $u: \bar{\Omega} \to \mathbb{R}^n$ be a smooth equilibrium solution for a material with $W \in C^2(M_+, \mathbb{R})$ which is rank-one-convex. Let b be a constant vector and let u satisfy the displacement boundary condition*

$$u(x) = Fx + b \quad \forall\, x \in \partial\Omega. \tag{4.6}$$

Then

$$I(u) \leqq I(v)$$

where $v(x) = Fx + b$ *for all* $x \in \bar{\Omega}$.

Proof. Clearly, $\nabla v(x) = F$ for all $x \in \bar{\Omega}$ and so v is a smooth equilibrium solution with $u(x) = v(x)$ for all $x \in \partial\Omega$. Also,

$$v - r\frac{\partial v}{\partial r} = b,$$

and hence by Proposition 4.1, we obtain

$$n(I(u) - I(v)) \leqq \operatorname{tr} \int_{\partial\Omega} \{T(\nabla u(x)) - T(\nabla v(x))\} N(x)\, dS \otimes b$$

so that the result now follows on recalling (2.7). □

Remark 4.3. The conclusion of Corollary 4.2 continues to hold without *any* restriction imposed on W (other than the smoothness already assumed), provided that instead u is postulated to satisfy the boundary condition (4.6) and also to be a smooth strong local minimizer *i.e.*, for all infinitely differentiable $w \in \mathbb{R}^n$ with $w - u$ of compact support on Ω and $\|w - u\|$ sufficiently small in the L_∞-norm, there holds

$$I(u) \leqq I(w).$$

For then, it follows from *e.g.*, BALL [1, Thm. 3.4] that W is rank-one-convex at $\nabla u(x)$ for all $x \in \bar{\Omega}$, and hence, in particular, at the boundary values of ∇u. Remark 4.2 together with Corollary 4.2 then yields the inequality

$$I(u) \leqq I(v), \quad v(x) = Fx + b \quad \forall\, x \in \bar{\Omega},$$

as required.

Corollary 4.3. *Let* Ω, F, u, v *be as in the previous corollary. Let* $W \in C^2(M_+, \mathbb{R})$ *besides being rank-one-convex, be in addition strictly quasi-convex at* F. *Then* $u(x) = Fx + b$ *for all* $x \in \bar{\Omega}$.

Proof. Let us suppose that $u(x) \not\equiv v(x) = Fx + b$. Then the strict quasi-convexity of W at F gives

$$I(u) > I(v),$$

which contradicts Corollary 4.2. Hence $u(x) \equiv v(x)$ for all $x \in \Omega$. □

Remark 4.4. When W is strictly rank-one-convex and quasi-convex at F then the above argument implies that

$$\nabla u(x) = \nabla v(x) = F \quad \forall\, x \in \partial\Omega.$$

Remark 4.5. We do not know whether the requirement that Ω be star-shaped is necessary. However, the well-known example due to F. JOHN [14] of nonuniqueness in the displacement boundary value problem for a spherical annulus shows that some restriction is essential on the geometrical or topological form of Ω.

This completes the discussion of general uniqueness. In the remaining two sections we turn our attention respectively to radial symmetric solutions and to incompressible material.

5. Radial Displacements

In this section, we restrict Ω to be the ball given by

$$\Omega = \{x \in \mathbb{R}^n : \|x\| < a\} \tag{5.1}$$

and we consider only *radial deformations* $u : \bar{\Omega} \to \mathbb{R}^n$ assumed of the form

$$u(x) = U(r)\, x/r \tag{5.2}$$

for all $x \in \bar{\Omega} \setminus \{0\}$, where $r = \|x\|$ and $U : (0, a] \to (0, \infty)$. We will show under hypotheses on the stored-energy density similar, but not identical, to those adopted in the previous section, that all such deformations are necessarily of the form

$$u(x) = \lambda x/r \tag{5.3}$$

for some positive constant λ. We now record for convenience some of the appropriate fundamental theory. (For further details, see, *e.g.*, BALL [5]).

For radial deformations, it follows that $U \in C^1((0, a))$ and that

$$\nabla u(x) = r^{-1}\, U(r)\, I + r^{-2}\,(U'(r) - U(r)/r)\, x \otimes x \tag{5.4}$$

for all $x \in \bar{\Omega} \setminus \{0\}$, where I is the identity matrix and a prime indicates differentiation with respect to the argument. Thus, we see that $\nabla u(x)$ is a symmetric matrix with $n - 1$ eigenvalues all equal to $r^{-1}\, U(r)$ with the remaining one equal to $U'(r)$. In particular,

$$\det \nabla u(x) = U'(r) \left(\frac{U(r)}{r}\right)^{n-1}, \tag{5.5}$$

and since by hypothesis $\nabla u(x) \in M_+$, we conclude that $U'(r) > 0$ for all $r \in (0, a)$.

Throughout the rest of this section, we suppose that the stored-energy density $W : M_+ \to \mathbb{R}$ is an isotropic function, which means that

$$W(FQ) = W(F) \tag{5.6}$$

for all $F \in M_+$ and all $Q \in SO(n)$. For a function $W : M_+ \to \mathbb{R}$ that is both frame-indifferent and isotropic there exists a symmetric function $\Phi : \mathbb{R}^n_{++} \to \mathbb{R}$ such that

$$W(F) = \Phi(v_1, v_2, \ldots, v_n),$$

where

$$\mathbb{R}^n_{++} = \{\xi \in \mathbb{R}^n : \xi_i > 0, \quad i = 1, 2, \ldots, n\},$$

and $v_i, \ldots, v_n$ are the eigenvalues of the matrix $(F^tF)^{\frac{1}{2}}$. BALL [5] has shown that $W \in C^k(M_+, \mathbb{R})$ if and only if $\Phi \in C^k(\mathbb{R}^n_{++}, \mathbb{R})$ for $k = 0, 1, 2,$ or ∞.

Thus, for a radial deformation $u(x)$ we may set $F = \nabla u(x)$ and then in the above notation

$$v_1 = U'(r), \quad v_2 = v_3 = \ldots = v_n = U(r)/r$$

so that

$$W(\nabla u(x)) = \Phi(U'(r), U(r)/r, \ldots, U(r)/r)$$

and the corresponding Piola-Kirchhoff stress tensor becomes

$$T(\nabla u(x)) = \Phi_2 I + r^{-2}(\Phi_1 - \Phi_2)\, x \otimes x,$$

where

$$\Phi_1 = \frac{\partial \Phi}{\partial v_1}(U'(r), U(r)/r, \ldots, U(r)/r)$$

and

$$\Phi_2 = \frac{\partial \Phi}{\partial v_k}(U'(r), U(r)/r, \ldots, U(r)/r), \quad k = 2, \ldots, n.$$

Thus we see that $T(\nabla u(x))$ is a symmetric matrix with eigenvalues Φ_1 (once) and Φ_2 (repeated $n-1$ times).

Now let us suppose that $u : \bar{\Omega} \to \mathbb{R}^n$ is a smooth radial equilibrium solution for an isotropic material with stored-energy density $W \in C^2(M_+, \mathbb{R})$. Then $U \in C^2((0, a))$ and U satisfies the differential equation

$$r\frac{\partial \Phi_1}{\partial r} + (n-1)(\Phi_1 - \Phi_2) = 0, \quad 0 < r < a$$

which equivalently may be written as

$$\Phi_{11}U'' + (n-1)\,\Phi_{12}(U/r)' + (n-1)(\Phi_1 - \Phi_2)/r = 0, \quad 0 < r < a, \tag{5.7}$$

where

$$\Phi_{11} = \frac{\partial^2 \Phi}{\partial v_1^2}(U'(r), U(r)/r, \ldots, U(r)/r)$$

and

$$\Phi_{12} = \frac{\partial^2 \Phi}{\partial v_k\, \partial v_1}(U'(r), U(r)/r, \ldots, U(r)/r) \quad k = 2, \ldots, n.$$

We shall appeal to equation (5.7) in our discussion of uniqueness.

Finally, let us note that strong-ellipticity of W implies $\Phi_{11} > 0$ on $\mathbb{R}^n_{++}$.

Proposition 5.1. *Let* $\Omega = \{x \in \mathbb{R}^n : \|x\| < a\}$ *and let* $\lambda > 0$. *Let* $u : \bar{\Omega} \to \mathbb{R}^n$ *be a smooth radial equilibrium solution for an isotropic elastic material with stored-*

energy density $W \in C^2(M_+, \mathbb{R})$. *Suppose that* W *is strictly rank-one convex and that* Φ_{11}, Φ_{12} *are Lipschitz continuous on* $\{t \in \mathbb{R}^n_{++} : t_k = t_2,\ 2 \leqq k \leqq n\}$. *Let* $u(x)$ *satisfy the displacement boundary condition* $u(x) = \lambda x$ *for all* $x \in \partial\Omega$. *Then the quasi-convexity of* W *at* λI *implies that* $u(x) = \lambda x$ *for all* $x \in \overline{\Omega}$.

Proof. Let $v(x) = \lambda x$ for all $x \in \overline{\Omega}$. Then v is a smooth radial equilibrium solution with $V(r) = \lambda r$, in an obvious notation.

By Remark 4.4, we have that $\nabla u(x) = \nabla v(x) = \lambda I$ for all $x \in \partial\Omega$. Thus $U(a) = V(a)$ and $U'(a) = V'(a)$. We may then use the known uniqueness of the Cauchy problem for equation (5.7) (with data specified at $r = a$) to conclude that $U(r) = V(r)$ for $r \in (0, a]$. Thus $u(x) = \lambda x$ for all $x \in \overline{\Omega}$. □

Remark 5.1. For $W \in C^2(M_+, \mathbb{R})$, we observe that strict rank-one-convexity of W is implied by strong-ellipticity of W.

Remark 5.2. Ball [5] has discussed several other conditions on W ensuring that the smooth radial equilibrium deformation must assume the form $u(x) = \lambda x$.

Remark 5.3. Corollary 4.3 together with Proposition 5.1 shows that uniqueness of smooth radial equilibrium solutions requires that we may take the strict form of either rank-one-convexity or quasi-convexity in the conditions for W.

Remark 5.4. Corollary 4.3 provides a partial answer to the question posed by Ball [5] concerning conditions to be imposed on W guaranteeing that the only sufficiently smooth (not necessarily radial) equilibrium solution with the boundary value $u(x) = \lambda x$, $x \in \partial\Omega$, is given by $u(x) = \lambda x$, $x \in \overline{\Omega}$.

Remark 5.5. Proposition 5.1 remains valid without any explicit mention of the boundary conditions, since on the boundary radial solutions assume the value $u(x) = U(a)\, x/a$ and we may therefore set $\lambda = U(a)/a$. In particular, for the traction boundary value problem, the hypotheses of the Proposition imply that smooth radial equilibrium solutions are given by $u(x) = \lambda x$, $x \in \overline{\Omega}$. Now, however, the constant λ is not uniquely determined by the load (*cp.* Ball [5]).

6. Incompressible Materials

In this final section, we modify the previous theory to incompressible elastic bodies in which, as is well-known, only isochoric deformations are possible. Thus, the stored-energy density is now defined on the set

$$M_1 = \{F \in M \colon \det F = 1\}. \tag{6.1}$$

However, given $W \colon M_1 \to \mathbb{R}$, it is possible to extend W to the whole of M_+ by the formula

$$\tilde{W}(F) = W(F/\det F) \quad \forall\, F \in M_+,$$

and then $W \in C^k(M_1, \mathbb{R})$ if and only if $\tilde{W} \in C^k(M_+, \mathbb{R})$. Furthermore, all the definitions in Section 3 may be easily adapted to incompressible materials by replacing M_+ by M_1.

The tensor $(\partial W/\partial F)$ is now called the *Piola-Kirchhoff extra stress* and we set

$$T(F) = \frac{\partial W}{\partial F}(F) - p(\operatorname{adj} F)^t \quad \forall\, F \in M_1, \tag{6.2}$$

where p is an arbitrary scalar function. Then $T(F)$ is called the *Piola-Kirchhoff stress.*

As before, we suppose that the incompressible material in its reference configuration occupies the open bounded region $\Omega \subset \mathbb{R}^n$ whose boundary $\partial\Omega$ is piecewise continuously differentiable. The unit outward normal field on $\partial\Omega$ is again denoted by N. We say that a function $u: \bar{\Omega} \to \mathbb{R}^n$ is an *incompressible deformation* if and only if $u \in C^1(\bar{\Omega}, \mathbb{R}^n)$ and $\nabla u(x) \in M_1$ for all $x \in \bar{\Omega}$. A pair (u, p) is a *smooth equilibrium solution for our incompressible elastic material* with stored-energy density $W \in C^2(M_1, \mathbb{R})$ if and only if u is an incompressible deformation such that $u \in C^2(\Omega, \mathbb{R}^n)$ and $p \in C^1(\Omega, \mathbb{R}) \cap C(\bar{\Omega}, \mathbb{R})$ satisfy the equilibrium equations with zero body-force,

$$\operatorname{div} T(\nabla u(x)) = 0 \tag{6.3}$$

for all $x \in \Omega$, where

$$T(\nabla u(x)) = \frac{\partial W}{\partial F}(\nabla u(x)) - p(x)\,(\operatorname{adj} \nabla u(x))^t. \tag{6.4}$$

The scalar function $p: \bar{\Omega} \to \mathbb{R}$ is now called a hydrostatic pressure corresponding to u.

Remark 6.1. Let (u, p) and (u, q) be two smooth equilibrium solutions for an incompressible material. Then on recalling the identity

$$\operatorname{div}\,[\operatorname{adj}\,(\nabla u(x))^t] = 0, \tag{6.5}$$

we see from (6.3) and (6.4) that the difference $p - q$ satisfies the equation

$$\nabla(p(x) - q(x)) \operatorname{adj}\,(\nabla u(x))^t = 0. \tag{6.6}$$

But $\det(\operatorname{adj} \nabla u(x))^t = 1$, and it then follows from (6.6) and the connectedness of Ω that $p - q$ is constant on Ω.

Thus, for a smooth equilibrium solution (u, p), the deformation determines the pressure p up to an arbitrary constant on Ω. Moreover, the equilibrium equations are equivalent to

$$\operatorname{div} \frac{\partial W}{\partial F}(\nabla u) - \nabla p\,(\operatorname{adj} \nabla u(x))^t = 0, \tag{6.7}$$

giving a formula for ∇p in terms of ∇u.

The proof of uniqueness for the incompressible displacement boundary value problem is closely patterned on that of Section 4 for the compressible case. Once again, basic to the argument is the following result analogous to Proposition 2.1.

Proposition 6.1. *Let Ω be as above. Let (u, p) be a smooth equilibrium solution for an incompressible material with stored-energy density $W \in C^2(M_1, \mathbb{R})$. Then*

$$nI(u) = \int_{\partial\Omega} \left\{(N \cdot x)\ W(\nabla u(x)) + \operatorname{tr} T(\nabla u(x)) \left[N \otimes \left(u(x) - r\frac{\partial u(x)}{\partial r}\right)\right]\right\} dS. \tag{6.8}$$

Furthermore, it follows that

$$\int_{\partial\Omega} \operatorname{tr} \operatorname{adj} (\nabla u(x))^t \left[N \otimes \left(u(x) - r\frac{\partial u(x)}{\partial r}\right)\right] dS = 0. \tag{6.9}$$

Proof. We begin by deducing (6.9) from (6.8). Let $\lambda \in \mathbb{R}$. Then $(u, p + \lambda)$ is also a smooth equilibrium solution and so by (6.8)

$$nI(u) = \int_{\partial\Omega} (N \cdot x)\ W(\nabla u(x))\, dS$$
$$+ \int_{\partial\Omega} \operatorname{tr} \left[\frac{\partial W}{\partial F}(\nabla u(x)) - (p + \lambda) \operatorname{adj} (\nabla u(x))^t\right] \left[N \otimes \left(u(x) - r\frac{\partial u(x)}{\partial r}\right)\right] dS$$

for all $\lambda \in \mathbb{R}$. Hence the result follows. (An alternative proof of (6.9) may be based on the divergence theorem applied to (6.9) together with use of the identity $F \operatorname{adj} F = I$.)

To prove (6.8), we will verify that the equation

$$nW(\nabla u) = \operatorname{div} \left[xW(\nabla u) + T^t(\nabla u)\left(u - r\frac{\partial u}{\partial r}\right)\right] \tag{6.10}$$

holds, as then the result follows immediately from the divergence theorem. To this end, let us first recall that when $\det F \neq 0$, we have

$$\frac{\partial}{\partial F} \det F = (\operatorname{adj} F^t) \tag{6.11}$$

so that

$$r\frac{\partial}{\partial r}(\det \nabla u) = \operatorname{tr} \operatorname{adj} (\nabla u)\, r\left(\frac{\partial \nabla u}{\partial r}\right). \tag{6.12}$$

But $\det \nabla u \equiv 1$, and so

$$\operatorname{tr} \operatorname{adj} (\nabla \mathrm{u})\, r\left(\frac{\partial \nabla u}{\partial r}\right) = 0. \tag{6.13}$$

Then, from elementary properties of the divergence operator, we get

$$\begin{aligned}
\operatorname{div}\left\{xW + T^t(\nabla u)\left(u - r\frac{\partial u}{\partial r}\right)\right\} &= nW + (x \cdot \nabla)\, W + \operatorname{div} T(\nabla u) - \operatorname{tr} T(\nabla u)\, r\left(\frac{\partial \nabla u}{\partial r}\right)^t \\
&= nW + \operatorname{tr} \frac{\partial W}{\partial F}(\nabla u)\, r\left(\frac{\partial \nabla u}{\partial r}\right)^t - \operatorname{tr} T(\nabla u)\, r\left(\frac{\partial \nabla u}{\partial r}\right)^t \\
&= nW + p \operatorname{tr} \operatorname{adj} (\nabla u)\, r\left(\frac{\partial \nabla u}{\partial r}\right) \\
&= nW,
\end{aligned}$$

in which (6.4), the equilibrium equations (6.3) and (6.13) have been used. Thus, (6.8) is proved.

The remaining steps in the uniqueness proof are given by the following sequence of results corresponding respectively to (3.3), Proposition 4.1 and Corollary 4.3 in the compressible problem:

Lemma 6.2. *Let the stored-energy density* $W \in C^1(M_1, \mathbb{R})$ *be rank-one-convex. Then*

$$W(F + a \otimes b) \geqq W(F) + \operatorname{tr} (T(F)\, b \otimes a\} \tag{6.14}$$

for all $F \in M_1$ *and* $a \in \mathbb{R}^n$, $b \in \mathbb{R}^n$ *such that* $F + a \otimes b \in M_1$, *where* $T(F)$ *is given by* (6.2).

Proof. We know that rank-one-convexity yields the inequality (cp. (3.2))

$$W(F + a \otimes b) \geqq W(F) + \operatorname{tr} \left\{ \frac{\partial W}{\partial F}(F)\, b \otimes a \right\}$$

$$= W(F) + \operatorname{tr} \{T(F)\, b \otimes a\} + p \operatorname{tr} \{(\operatorname{adj} F^t)\, b \otimes a\},$$

on using (6.2). On the other hand, from the definition of determinant, we have

$$\det (F + a \otimes b) = \det F + \operatorname{tr} \{(\operatorname{adj} F^t)\, b \otimes a\},$$

but $F \in M_1$, $F + a \otimes b \in M_1$, so it follows that

$$\operatorname{tr} \{(\operatorname{adj} F^t)\, b \otimes a\} = 0, \tag{6.15}$$

which proves the result. □

Proposition 6.3. *Let* Ω *be as in Section 4. Let* (u, p) *and* (v, q) *be two smooth equilibrium solutions for an incompressible elastic material with stored-energy density* $W \in C^2(M_1, \mathbb{R})$ *such that* $u(x) = v(x)$ *for all* $x \in \partial\Omega$. *If* W *is rank-one-convex, then*

$$n[I(u) - I(v)] \leqq \int_{\partial\Omega} \operatorname{tr} [T(\nabla u(x)) - T(\nabla v(x))] \left[N \otimes \left(v(x) - r \frac{\partial v(x)}{\partial r} \right) \right] dS, \tag{6.16}$$

where

$$T(\nabla u) = \frac{\partial W}{\partial F}(\nabla u) - p \operatorname{adj} (\nabla u)^t, \tag{6.4}$$

$$T(\nabla v) = \frac{\partial W}{\partial F}(\nabla v) - q \operatorname{adj} (\nabla v)^t. \tag{6.17}$$

Furthermore, it follows that

$$\int_{\partial\Omega} \operatorname{tr} [\operatorname{adj} (\nabla u)^t] \left[N \otimes \left(v - r \frac{\partial v}{\partial r} \right) \right] dS = 0. \tag{6.18}$$

Proof. By means of the equality (6.8) in Proposition 6.1, together with Lemma 6.2, the inequality (6.16) may be established in the same way as that in Proposition 4.1.

To obtain (6.18), we replace (u, p) by $(u, p + \lambda)$, where $\lambda \in \mathbb{R}$ is an arbitrary constant, in inequality (6.16). The result is then immediate. As an alternative proof, consider the identity

$$\int_{\partial\Omega} \operatorname{tr}\,[\operatorname{adj}(\nabla u)^t]\left[N \otimes \left(v - r\frac{\partial v}{\partial r}\right)\right] dS = \int_{\partial\Omega} \operatorname{tr}\,[\operatorname{adj}(\nabla u)^t]\left[N \otimes \left(u - r\frac{\partial u}{\partial r}\right)\right] dS$$
$$+ \int_{\partial\Omega} (N \cdot x) \operatorname{tr}\,[\operatorname{adj}(\nabla u)^t]\left[N \otimes \frac{\partial}{\partial N}(u - v)\right] dS,$$

where Lemma 1.1(ii) has been used. But we also have (4.5) for all $x \in \partial\Omega$ and since $\nabla u(x) \in M_1$, $\nabla v(x) \in M_1$ for $x \in \partial\Omega$, it follows from (6.15) that the second integral on the right of the last equation vanishes identically. The first integral vanishes by (6.9). □

Corollary 6.4. *Let Ω be as in Section 4 and let $F \in M_1$ be a constant matrix. Let (u, p) be a smooth equilibrium solution for an incompressible elastic material with stored-energy density $W \in C^2(M_1, \mathbb{R})$ which is rank-one-convex. Let $u(x)$ satisfy the displacement boundary condition*

$$u(x) = Fx + b, \tag{6.19}$$

for all $x \in \partial\Omega$, where $b \in \mathbb{R}^n$ is a constant vector. Then

$$I(u) \leqq I(v), \tag{6.20}$$

where $v(x) = Fx + b$ for all $x \in \bar{\Omega}$.

If W is also strictly quasi-convex at F, then $u(x) = Fx + b$ for all $x \in \bar{\Omega}$ and p is an arbitrary constant on $\bar{\Omega}$.

Proof. It is obvious that $(v, 0)$ is a smooth equilibrium solution and that $u(x) = v(x)$ for all $x \in \partial\Omega$. Also, $v - r\dfrac{\partial v}{\partial r} \equiv b$, and so by Proposition 6.3,

$$n[I(u) - I(v)] \leqq \operatorname{tr}\left\{\int_{\partial\Omega} [T(\nabla u) - T(\nabla v)]\, N\, dS\right\} \otimes b$$
$$= 0,$$

by the equilibrium equations (6.3). Hence (6.20) follows.

The strict quasi-convexity of W at $F = \nabla v$ next implies that $\nabla u(x) = \nabla v(x)$ for all $x \in \bar{\Omega}$, so that $u(x) = Fx + b$ for all $x \in \bar{\Omega}$. Finally, any pressure corresponding to $v(x)$ is constant on $\bar{\Omega}$. Thus, since $u(x) = v(x)$ for all $x \in \bar{\Omega}$, the pressure p is likewise constant on $\bar{\Omega}$. □

Remark 6.2. For an incompressible elastic material, the results analogous to those of Section 5 for radial deformations become trivial. In fact, on supposing that $u(x)$ is a radial incompressible deformation, we have as before that

$(ux) = U(r)\, x/r$ and $U(0) = 0$. However, $\det \nabla u = 1$ so that $U(r)$ satisfies the differential equation

$$U'(r)\,[U(r)/r]^{n-1} = 1.$$

An integration then gives

$$U(r) = r$$

since the constant of integration vanishes due to the condition $U(0) = 0$. Hence radial solutions always assume the form: $u(x) = x$.

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References

1. BALL, J. M., Convexity conditions and existence theorems in nonlinear elasticity. Arch. Rational Mech. Anal. **63**, 337–403, 1978.
2. BALL, J. M., Constitutive inequalities and existence theorems in nonlinear elastostatics. Nonlinear Analysis and Mechanics: Heriot-Watt Symposium Vol. I. pp. 187–238. Pitman. London 1977.
3. BALL, J. M., Strict convexity, strong ellipticity and regularity in the calculus of variations. Math. Proc. Camb. Phil. Soc. **87**, 501–503, 1980.
4. BALL, J. M., Existence of solutions in finite elasticity. Proc. IUTAM Symposium on Finite Elasticity. Lehigh University. August 1980 (Edited by D. E. Carlson and R. T. Shield). pp. 1–12.
5. BALL, J. M., Discontinuous equilibrium solutions and cavitation in nonlinear elasticity. Phil. Trans. Roy. Soc. Lond. A **306**, 557–611, 1982.
6. CHILLINGWORTH, D. R. J., J. E. MARSDEN & Y. H. WAN, Symmetry and bifurcation in three dimensional elasticity. Part I. Arch. Rational Mech. Anal. **80**, 295–331, 1982.
7. CHILLINGWORTH, D. R. J., J. E. MARSDEN & Y. H. WAN, Symmetry and bifurcation in three dimensional elasticity. Part II. Arch. Rational Mech. Anal. **83**, 362–395, 1983.
8. ERICKSEN, J. L., & R. A. TOUPIN, Implications of Hadamard's conditions for elastic stability with respect to uniqueness theorems. Canad. J. Math. **8**, 432–436, 1956.
9. GREEN, A. E., On some general formulae in finite elastostatics. Arch. Rational Mech. Anal. **50**, 73–80, 1973.
10. GURTIN, M. E., & S. J. SPECTOR, On stability and uniqueness in finite elasticity. Arch. Rational Mech. Anal. **70**, 152–165, 1970.
11. HADAMARD, J., Leçons sur la propagation des ondes. Paris, Hermann 1903 (Reprinted by Dover, 1952).
12. HILL, R., On uniqueness and stability in the theory of finite elastic strain. J. Mech. Phys. Sols. **5**, 229–241, 1957.
13. HILL, R., Bifurcation and uniqueness in non-linear mechanics of continua. Problems of Continuum Mechanics. Society for Industrial and Applied Mathematics. Philadelphia. 155–164, 1961.
14. JOHN, F., Uniqueness of non-linear elastic equilibrium for prescribed boundary displacements and sufficiently small strains. Comm. Pure Appl. Math. **25**, 617–634, 1972.

15. MARSDEN, J. E., & Y. H. WAN, Linearization stability and Signorini series for the traction problem in elastostatics. Proc. Roy. Soc. Edin. A **95**, 171–180, 1983.
16. MORREY, Jr., C. B., Quasi-convexity and the lower semi-continuity of multiple integrals, Pacific J. Math. **2**, 25–53, 1952.
17. SPECTOR, S., On uniqueness in finite elasticity with general loading. J. Elast. **10**, 149–161, 1980.
18. SPECTOR, S. J., On uniqueness for the traction problem in finite elasticity. J. Elast. **12**, 367–383, 1982.
19. TRUESDELL, C., & W. NOLL, The non-linear field theories of mechanics. Handbuch der Physik. Vol. III/3. Springer. Berlin. 1965.
20. TRUESDELL, C., & R. TOUPIN, Static grounds for inequalities in finite elastic strain. Arch. Rational Mech. Anal. **12**, 1–33, 1963.
21. WAN, Y. H., & J. E. MARSDEN, Symmetry and bifurcation in three-dimensional elasticity. Part III: Stressed reference configuration. Arch. Rational Mech. Anal. **84**, 203–233, 1984.
22. WANG, C.-C., & C. TRUESDELL, Introduction to Rational Elasticity. Noordhoff. Leyden. 1973.

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