The Initial Value Problem for the Navier-Stokes Equations with Data in L^p

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Introduction

We shall consider the initial value problem for the Navier-Stokes equations in the infinite cylinder $S_T = R^n \times [0, T)$. More precisely, given $g(x) = (g_1(x), \ldots, g_n(x))$ satisfying $\operatorname{div}(g)(x) = \sum_{j=1}^n \left(\frac{\partial}{\partial x_j}\right) g_j(x) = 0$, $x \in R^n$, we seek a solution vector $u(x,t) = (u_1(x,t), \ldots, u_n(x,t))$ and a pressure function $P(x,t)$ such that

(1) $\frac{\partial u_i}{\partial t}(x,t) - \sum_{j=1}^n \frac{\partial^2 u_i}{\partial x_j^2}(x,t) + \sum_{j=1}^n \frac{\partial u_i}{\partial x_j}(x,t)\, u_j(x,t) + \frac{\partial P}{\partial x_i}(x,t) = 0$ for $x \in R^n$, $t \in (0,T)$, and $i = 1, \ldots, n$.

(2) $\sum_j \frac{\partial}{\partial x_j} u_j(x,t) = 0, \quad x \in R^n,\ t \in (0,T).$

(3) $u(x,0) = g(x).$

If ∇u denotes the $n \times n$ matrix $(\partial u_i / \partial x_j)$, Δu the Laplacian of u, and ∇P the gradient of P, we abbreviate the first equation by

$$D_t u - \Delta u + (\nabla u)(u) + \nabla P = 0.$$

In studying the above problem we shall consider it in its weak form (see Section II). We shall show in Section II that solving the equation in weak form is *equivalent* to solving a certain non-linear integral equation. In Section III, with the use of a familiar imbedding theorem, we shall prove uniqueness for all values of T and existence for small values of T of solutions of the integral equation and, hence, of the differential equation, in the class of u for which

$$\sum_{j=1}^n \left[\int_0^T \Big(\int_{R^n} |u_j(x,t)|^p\Big)^{q/p}\right]^{1/q} \equiv \|u\|_{p,q} < \infty$$

where p, q satisfy the relations $\frac{n}{p} + \frac{2}{q} \leqq 1$ and $n < p < \infty$. We denote this class by $L^{p,q}(S_T)$. The data $g = (g_1, \ldots, g_n)$ is taken from the space $L^p(R^n)$ with $n < p < \infty$.

More precisely, if $g(x)=(g_1(x), \ldots, g_n(x))$ then

$$g\in L^p(R^n) \Leftrightarrow \|g\|_p=\sum_{i=1}^{n}\Big(\int_{R^n} |g_i(x)|^p\,dx\Big)^{1/p}<\infty .$$

We emphasize that no condition of integrability is assumed for any distribution derivative of g. (Of course, in the sense of distributions g will satisfy the condition $\operatorname{div}(g)=0$.)

In Section IV we also consider the problem of existence for all time. We show that when $g(x)\in L^p\cap L^{p'}(R^n)$, $p'<n<p$, has the property that the norm $\|g\|_{(L^p\cap L^{p'})(R^n)} = \|g\|_{L^p(R^n)}+\|g\|_{L^{p'}(R^n)}$ is small enough, then the solution u exists and is unique for all values of time.

Finally in Section V we consider the relation of the class $L^{p,q}(S_T)$ with the Hopf-Leray class of solutions u defined by the condition $\|u\|_{2,\infty}+\|\nabla u\|_{2,2}<\infty$. We prove that when $g\in L^2(R^n)\cap L^p(R^n)$, $2<n<p$, then (in the small) the solution $u\in L^{p,q}(S_T)$ also belongs to the Hopf-Leray class. Hence, using the results in [8] and [10], it follows that when $g\in L^2(R^n)\cap L^p(R^n)$, $2<n<p$, any two Hopf-Leray solutions must agree in a small time interval $(0, T_0)$.

I. Construction of a Divergence Free Fundamental Solution of the Heat Equation

In this section we shall construct an $n\times n$ (symmetric) matrix of functions $E(x,t)=(E_{ij}(x,t))$ defined for $x\in R^n$, $t>0$, such that

(*i*) $\Delta E_{ij}(x,t)-D_t E_{ij}(x,t)=0$ for $t>0$,

(*ii*) $\operatorname{div}(E_i)(x,t)=\sum_{j=1}^{n} D_{x_j}E_{ij}(x,t)=0,\ t>0,$

$E_i=(E_{i1}, E_{i2}, \ldots, E_{in})$,

(*iii*) if $g(x)\in L^p(R^n)$, $1\leq p<\infty$, with $\operatorname{div}(g)=0$ in the sense of distributions, then

$$\int_{R^n} E(x-y,t)\big(g(y)\big)\,dy\to g(x) \quad \text{in} \quad L^p(R^n) \quad \text{as} \quad t\to 0+ .$$

We shall now construct a formal solution of the above problem with the aid of the Fourier transform.

For $f\in L^1(R^n)$, $f=(f_1, \ldots, f_n)$, we put

$$\mathscr{F}_x(f_j)(x)=\mathscr{F}(f_j)(x)=\int_{R^n} f_j(y)\exp(i\langle x,y\rangle)\,dy, \qquad \mathscr{F}_x(f)=\mathscr{F}(f)=\big(\mathscr{F}(f_j)\big).$$

If $E(x,t)$ satisfies (*i*), $\mathscr{F}_x(E)$ should satisfy the differential equation

$$|x|^2\mathscr{F}_x(E)(x,t)=D_t\mathscr{F}_x(E)(x,t), \qquad t>0.$$

Hence

$$\mathscr{F}_x(E)(x,t)=\big(c_{ij}(x)\,e^{-|x|^2 t}\big),$$

and our object now is to determine the matrix $(c_{ij}(x))$.

Condition (*iii*) implies that for each x,

$$\mathscr{F}_x(E)(x,0)\mathscr{F}(g)(x)=\mathscr{F}(g)(x)$$

when

$$\langle \mathscr{F}(g)(x), x\rangle = \sum_{i=1}^{n} x_i \mathscr{F}(g_i)(x) = 0.$$

Since for x fixed we may consider $\mathscr{F}(g)(x)$ to be any vector v satisfying $\langle v, x\rangle = 0$, condition (*iii*) means that for each $x \neq 0$, $\mathscr{F}_x(E)(x, 0)$ is the identity matrix on the null space of the linear functional $v \to \langle v, x\rangle$. Hence $\mathscr{F}_x(E)(x, 0) = I + \langle \cdot, x\rangle v(x)$ where $v(x)$ is a fixed vector in R^n and I = identity matrix.

If we regard condition (*ii*) to be valid also for $t=0$, then $\mathscr{F}_x(E)(x, 0)(x) = 0$. This means that $0 = x + |x|^2 v(x)$, and hence for $x \neq 0$, $v(x) = -\frac{x}{|x|^2}$. Since the transformation $v \to \langle v, x\rangle \frac{-x}{|x|^2}$ is given by the matrix $\left(-\frac{x_i x_j}{|x|^2}\right)$, we conclude (formally) that

$$\mathscr{F}_x(E)(x, 0) = \left(\delta_{ij} - \frac{x_i x_j}{|x|^2}\right) = (c_{ij}(x)),$$

and therefore

$$E(x, t) = (\delta_{ij} \Gamma(x, t) - R_i R_j \Gamma(x, t)) \tag{1.1}$$

where

$$\Gamma(x, t) = \frac{e^{-|x|^2/4t}}{(4\pi t)^{n/2}}$$

and R_j is the j-th Riesz transform; that is, R_j is a singular integral operator on $L^p(R^n)$, $1 < p < \infty$, defined by

$$R_j(f)(x) = \lim_{\varepsilon \to 0} c_j \int_{|x-y|>\varepsilon} \frac{x_j - y_j}{|x-y|^{n+1}} f(y)\, dy,$$

the limit being taken in $L^p(R^n)$. See [1].

Working backwards, we now see from the definition of $E(x, t)$ that $E_{ij}(x, t) \in C^\infty(R^n x (0, \infty))$ and that indeed (*i*) and (*ii*) are satisfied. On the other hand, if $\Omega(x) = (4\pi)^{-n/2} \exp(-|x|^2/4)$,

$$E_{ij}(x, 1) = \Omega(x)\delta_{ij} - R_i R_j(\Omega)(x).$$

Therefore the continuity of R_i in $L^p(R^n)$ for $1 < p < \infty$ (see [1]) and the fact that $\Omega \in \bigcap_{1 \leq p \leq \infty} L^p(R^n)$ imply that $E_{ij}(\cdot, 1) \in \bigcap_{1<p<\infty} L^p(R^n)$. However $E_{ij}(\cdot, 1) \notin L^1(R^n)$ (its Fourier transform is discontinuous at the origin). Observe also that for $t > 0$

$$E_{ij}(x, t) = t^{-n/2} E_{ij}(x/t^{1/2}, 1);$$

hence for fixed $t > 0$

$$E_{ij}(t)(f)(x) = \int_{R^n} E_{ij}(x-y, t) f(y)\, dy$$

is a bounded mapping from $L^p(R^n)$ into $L^r(R^n)$ for $1 \leq p < \infty$, $p < r < \infty$.

When $g \in L^p(R^n)$, $1 \leq p < \infty$, is weakly divergence free (in the sense of (iii) above), we have

$$E_{ij}(t)(g)(x) = \int_{R^n} \Gamma(x-y, t) g(y)\, dy \quad \text{a.e.}, \tag{1.2}$$

and hence well-known properties of the Weierstrass kernel yield (*iii*). The following lemma proves identity (1.2).

Lemma (1.1). *Let* $g=(g_1, \dots, g_n)\in L^p(R^n)$, $1\leqq p<\infty$, *be weakly divergence free. Then*

$$\sum_{j=1}^{n} \int_{R^n} R_j(\Omega)(x-y)\, g_j(y)\, dy=0 \quad \text{a.e.}$$

Proof. Since $R_j(\Omega)\in \bigcap_{1<q<\infty} L^q(R^n)$, the operator

$$T(f)(x)=\sum_{j=1}^{n} \int_{R^n} R_j(\Omega)(x-y)\, f_j(y)\, dy$$

is bounded from $L^p(R^n)$ into $L^r(R^n)$ for $1\leqq p<\infty, p<r<\infty$.

Let $g\in L^p(R^n)$ be weakly divergence free. Chose $k\in C_0^\infty(R^n)$ (infinitely differentiable with compact support), such that $\int_{R^n} k(x)\,dx=1$ and set

$$g_\lambda(x)=\lambda^n \int_{R^n} k(\lambda y)\, g(x-y)\, dy.$$

Then (*a*) $g_\lambda\in C^\infty(R^n)\cap L^p(R^n)\cap L^\infty(R^n)$, (*b*) $\operatorname{div}(g_\lambda)=0$, and (*c*) $g_\lambda\to g$ in $L^p(R^n)$ as $\lambda\to\infty$.

Then using (*a*) and (*b*), we obtain

$$\mathscr{F}(T(g_\lambda))(x)=\frac{1}{|x|}\, e^{-|x|^2}\left(\sum_{i=1}^{n} x_i \mathscr{F}(g_{i,\lambda})\right)=0.$$

Hence $T(g_\lambda)=0$. The continuity of T and (*c*) imply that $T(g)=0$.

We can also see that for $t>0$

$$E_{ij}(x,t)=\delta_{ij}\Gamma(x,t)+\int_0^{1/t} D^2_{x_i x_j}\Omega(x s^{1/2})\, s^{\frac{n}{2}-1}\, ds, \tag{1.3}$$

the Fourier transform in x of both sides being equal, and

$$D_{x_k}E_{i,j}(x,t)=\delta_{ij}D_{x_k}\Gamma(x,t)+\int_0^{1/t} \frac{\partial^3\Omega}{\partial x_k\,\partial x_i\,\partial x_j}(x s^{1/2})\, s^{\frac{n}{2}-\frac{1}{2}}\, ds. \tag{1.4}$$

Formula (1.3) in the case $n=3$ was obtained by Oseen in [7].

Using the matrix $E(x,t)$, we now define an integral operator which, as we shall see in Section II, arises naturally in the study of the initial value problem. Given $u=(u_1, \dots, u_n)$, we let $\langle u(y,s), \nabla E(x-y,t-s)\rangle$ denote the $n\times n$ matrix

$$(\langle u(y,s),\ D_{x_k}E_i(x-y,t-s)\rangle)$$

where $E_i(x,t)$ is the i-th row of $E(x,t)$. We set

$$B(u,v)(x,t)=\int_0^t \int_{R^n} \langle u(y,s),\ \nabla E(x-y,t-s)\rangle\,(v(y,s))\,dy\,ds. \tag{1.5}$$

From formula (1.3) it is easy to see that $D_{x_k}E_{ij}(x,t)\in L^1(S_T)$. Hence if u and $v\in L^{p,q}(S_T)$ with $p\geqq 2$ and $q\geqq 2$, $B(u,v)\in L^{p/2,q/2}(S_T)$. The integral equation of particular interest in this work is

$$(1.6)\qquad u+B(u,u)=u_0, \quad \text{where } u_0(x,t)=\int_{R^n}\Gamma(x-y,t)\,(g(y))\,dy.$$

II. Equivalence of Weak Solutions of the Navier-Stokes Equation and Solutions of the Integral Equation $u+B(u,u)=u_0$

To study a class of solutions of the Navier-Stokes equation for which the pointwise equations (1), (2), (3) are meaningless, we shall make use of the notion of a weak or generalized solution.

Let $\mathscr{S}(R^n)$ denote the space of rapidly decreasing functions on R^n and $\mathscr{S}'(R^n)$ the space of tempered distributions.

We will denote by $\mathscr{D}_T$ those vector functions $\varphi(x,t)=(\varphi_1(x,t),\dots,\varphi_n(x,t))$ such that $\varphi_i(x,t)\in\mathscr{S}(R^{n+1})$, $\varphi_i(x,t)=0$ for $t\geqq T$, and $\operatorname{div}(\varphi)(x,t)=\sum_{i=1}^{n} D_{x_i}\varphi_i(x,t)=0$ for all t.

Definition. A function $u(x,t)=(u_1(x,t),\dots,u_n(x,t))$ is a weak solution of the Navier-Stokes equation with initial value g if the following conditions hold:

(a) $u(x,t)\in L^{p,q}(S_T)$ for some p, q with $p,q\geqq 2$.

(b) For $\varphi\in\mathscr{D}_T$

$$\int_0^T\int_{R^n}\langle u, D_t\varphi+\Delta\varphi+(\nabla\varphi)(u)\rangle\,dx\,dt=-\int_{R^n}\langle g(x),\varphi(x,0)\rangle\,dx.$$

(c) For almost every $t\in[0,T]$, $\operatorname{div}(u(\cdot,t))=0$ in the sense of distributions.

We assume of course that $g(x)$ is weakly divergence free.

Theorem (2.1). *Let $g\in L^r(R^n)$, $1\leqq r<\infty$, be weakly divergence free. Then $u(x,t)\in L^{p,q}(S_T)$, $p,q\geqq 2$, $p<\infty$, is a weak solution of the Navier-Stokes equation with initial value g if and only if u is a solution of the integral equation*

$$u+B(u,u)=\int_{R^n}\Gamma(x-y,t)\,g(y)\,dy.$$

Proof. Let us first consider the case where $u\in L^{p,q}(S_T)$ is a solution of the integral equation. Put $v=B(u,u)$ and let u_0 denote the right side of the integral equation. Note that we may assume q finite. Since g is weakly divergence free, $u_0(x,t)$ is weakly divergence free for each $t>0$. Let $w_m\in C_0^\infty(S_T)$ be a sequence such that $w_m\to u$ in $L^{p,q}(S_T)$. Set $v_m=B(w_m,w_m)$; then $v_m\to v$ in $L^{p/2,q/2}(S_T)$. In other words, $\|v_m(\cdot,t)-v(\cdot,t)\|_{L^p(R^n)}$ tends to zero in $L^q(0,T)$; hence a subsequence tends to zero for almost every t of $[0,T]$. Observe also that v_m is divergence free for every t. At this stage we are using property (*ii*) of our fundamental solution (E_{ij}) in the formula $v_m=B(w_m,w_m)$. Since limits of divergence free distributions are divergence free, $v(\cdot,t)$ is weakly divergence free for almost every t in $[0,T]$. Therefore $u=u_0-v$ is weakly divergence free almost everywhere in $[0,T]$.

To pass from the integral equation to the weak form of the differential equation, we will first consider $w\in L^{p,q}(S_T)$ such that $\dfrac{\partial w}{\partial x_j}\in L^\infty(S_T)$ and $\operatorname{div}(w(\cdot,t))=0$

for every $t \in [0, T]$. For such w

$$B(w, w)(x, t) = \int_0^t \int_{R^n} E(x-y, t-s)\big(\nabla w(w)\big)(y, s)\,dy\,ds$$
$$= \int_0^t \int_{R^n} \Gamma(x-y, t-s)\big(\nabla w(w)(y, s)\big)\,dy\,ds$$
$$- \int_0^t \int_{R^n} \Gamma(x-y, t-s)(R_i R_j)\big(\nabla w(w)\big)(y, s)\,dy\,ds.$$

Hence

$$(\Delta - D_t)\big(B(w, w)\big)(x, t) = -\nabla w(w)(x, t) + (R_i R_j)\big(\nabla w(w)\big)(x, t).$$

We can then conclude that for such w

$$\int_0^T \int_{R^n} \langle B(w, w), D_t\varphi + \Delta\varphi\rangle(x, t)\,dx\,dt$$
$$= -\int_0^T \int_{R^n} \langle \nabla w(w), \varphi\rangle\,dx\,dt + \int_0^T \int_{R^n} \langle (R_i R_j)\nabla w(w), \varphi\rangle\,dx\,dt.$$

Observe on the other hand that when $\varphi \in \mathscr{D}_T$, $(R_i R_j)(\varphi) = \big(R_j \sum_i R_i(\varphi_i)\big) = 0$, since

$$\mathscr{F}_x\Big(\sum_i R_i(\varphi_i)\Big) = \frac{1}{|x|}\sum_{i=1}^n x_i \mathscr{F}_x(\varphi_i) = \frac{c}{|x|}\mathscr{F}_x(\operatorname{div}\varphi) = 0.$$

Therefore,

$$\int_0^T \int_{R^n} \langle B(w, w), D_t\varphi + \Delta\varphi\rangle\,dx\,dt = \int_0^T \int_{R^n} \langle w, \nabla\varphi(w)\rangle\,dx\,dt.$$

Now let u be a solution of our integral equation and let $k(x, t)$ be an infinitely differentiable function with compact support in $R^n \times (0, \infty)$ such that

$$\int k(x, t)\,dx\,dt = 1.$$

Set

$$w_\lambda(x, t) = \lambda^{n+1} \int_0^t \int_{R^n} k\big(\lambda(x-y), \lambda(t-s)\big)\,u(y, s)\,dy\,ds.$$

Then

$$w_\lambda, \frac{\partial w_\lambda}{\partial x_j} \in L^{p,q}(S_T) \cap L^\infty(S_T) \quad \text{and} \quad \operatorname{div}\big(w_\lambda(\cdot, t)\big) = 0$$

for every $t \in [0, T]$; hence w_λ satisfies the above equality. But as $\lambda \to \infty$, $w_\lambda \to u$ in $L^{p,q}(S_T)$, and therefore $B(w_\lambda, w_\lambda)$ tends to $B(u, u)$ in $L^{p/2, q/2}(S_T)$; hence for the limit value u,

$$\int_0^T \int_{R^n} \langle B(u, u), D_t\varphi + \Delta\varphi\rangle\,dx\,dt = \int_0^T \int_{R^n} \langle u, \nabla\varphi(u)\rangle\,dx\,dt.$$

Finally

$$u + B(u, u) = u_0(x, t) = \int_{R^n} \Gamma(x-y, t)\,g(y)\,dy.$$

Then

$$\int_0^T \int_{R^n} \langle u, D_t\varphi + \Delta\varphi + \nabla\varphi(u)\rangle \, dx\, dt$$
$$= \int_0^T \int_{R^n} \langle u, D_t\varphi + \Delta\varphi\rangle \, dx\, dt + \int_0^T \int_{R^n} \langle u, \nabla\varphi(u)\rangle \, dx\, dt$$
$$= \int_0^T \int_{R^n} \langle u_0 - B(u,u), D_t\varphi + \Delta\varphi\rangle \, dx\, dt + \int_0^T \int_{R^n} \langle u, \nabla\varphi(u)\rangle \, dx\, dt$$
$$= \int_0^T \int_{R^n} \langle u_0, D_t\varphi + \Delta\varphi\rangle \, dx\, dt = -\int_{R^n} \langle g(x), \varphi(x,0)\rangle \, dx.$$

Hence solutions of the integral equation are weak solutions of the Navier-Stokes equations.

Assume now that u is a weak solution of the Navier-Stokes equations with initial data $g(x)$. If we could choose $\varphi_i(x,t) = (E_{ij}(x,t))_{j=1}^n$, $i = 1, \ldots, n$, as test functions in condition (b) of the definition of weak solutions, the theorem would follow immediately. Unfortunately $\varphi_i \notin \mathscr{D}_T$, and this fact complicates the argument. We get around the difficulty by regularizing E_{ij}.

Let $a \in C^\infty(R^n)$ such that $a(x) = 1$ when $|x| \geqq 2$ and $a(x) = 0$ when $|x| \leqq 1$. Let $\psi \in C^\infty(R)$ such that $\psi(t) = 1$ when $t \geqq 2$ and $\psi(t) = 0$ when $t \leqq 1$.

Set $a_\lambda(x) = a(\lambda x)$ and $\psi_\varepsilon(t) = \psi(t/\varepsilon)$. For $t > 0$ set $E_{ij}^{(\lambda)} = \mathscr{F}^{-1}(a_\lambda \mathscr{F}(E_{ij}))$, the Fourier transform $\mathscr{F}$ being taken in the x variable.

Now observe that $E_{ij}^{(\lambda)}(\cdot, t) \in \mathscr{S}(R^n)$ for t positive, and that

$$\sum_{j=1}^n \frac{\partial E_{ij}^{(\lambda)}}{\partial x_j}(x,t) = 0.$$

Moreover, since $b = 1 - a \in C_0^\infty(R^n)$, then $k = \mathscr{F}^{-1}(b) \in \mathscr{S}(R^n)$ and

$$\begin{aligned} E_{ij}^{(\lambda)}(x,t) &= \mathscr{F}^{-1}((1-b_\lambda)\mathscr{F}(E_{ij}))(x,t) \\ &= E_{ij}(x,t) - \lambda^{-n} \int_{R^n} k(\lambda^{-1}(x-y)) E_{ij}(y,t)\, dy. \end{aligned} \tag{2.1}$$

Observe that the second term on the right-hand side of the above identity tends to zero in $L^p(S_T)$, $1 < p < \infty$, as $\lambda \to \infty$, while any of its first spatial derivatives tend to zero also in $L^1(S_T)$.

We fix (x,t) and set

$$E_i^{(\lambda)} = (E_{ij}^{(\lambda)})_{j=1}^n, \qquad \varphi_{\varepsilon,\lambda}(y,s) = \psi(s+2)\,\psi_\varepsilon(t-s)\, E_i^{(\lambda)}(x-y, t-s).$$

For $t \leqq T$, $\varphi_{\varepsilon,\lambda} \in \mathscr{D}_T$; therefore

$$\int_0^T \int_{R^n} \langle u, D_s + \Delta_y(\varphi_{\varepsilon,y})\rangle \, dy\, ds + \int_0^T \int_{R^n} \langle u, \nabla_y(\varphi_{\varepsilon,\lambda})(u)\rangle \, dy\, ds$$
$$= -\int_{R^n} \langle g(y), \varphi_{\varepsilon,\lambda}(y,0)\rangle \, dy.$$

Since

$$(\Delta_x - D_t)(\psi_\varepsilon E_{ij}^{(\lambda)})(x,t) = -\frac{1}{\varepsilon} E_{ij}^{(\lambda)}(x,t)\, \psi'\left(\frac{t}{\varepsilon}\right),$$

the above identity shows that

$$-\frac{1}{\varepsilon}\int_{t-2\varepsilon}^{t-\varepsilon}\int_{R^n}\langle u(y,s), E_i^{(\lambda)}(x-y,t-s)\rangle\,\psi'((t-s)/\varepsilon)\,dy\,ds$$
$$-\int_0^{t-\varepsilon}\int_{R^n}\langle u(y,s), (\nabla_x E_i^{(\lambda)}(x-y,t-s))[u(y,s)]\rangle\,\psi_\varepsilon(t-s)\,dy\,ds$$
$$=-\psi_\varepsilon(t)\int_{R^n}\langle g(x), E_i^{(\lambda)}(x-y,t)\rangle\,dy.$$

Now we let λ tend to infinity. Using the properties of identity (2.1) we obtain

$$-\frac{1}{\varepsilon}\int_0^t\int_{R^n}\langle u(y,s), E_i(x-y,t-s)\rangle\,\psi'\left(\frac{t-s}{\varepsilon}\right)dy\,ds$$
$$-\int_0^t\int_{R^n}\langle u(y,s), (\nabla_x E_i(x-y,t-s))(u(y,s))\rangle\,\psi\left(\frac{t-s}{\varepsilon}\right)dy\,ds$$
$$=-\psi(t/\varepsilon)\int_{R^n}\langle g(y), E_i(x-y,t)\rangle\,dy.$$

Now $u(\cdot,s)\in L^p(R^n)$ and is weakly divergence free for a.e. $s\in(0,T)$. Using (1.2), we see that

$$\int_{R^n}\langle u(y,s), E_i(x-y,t-s)\rangle\,dy=\int_{R^n}\Gamma(x-y,t-s)\,u_i(y,s)\,dy.$$

Therefore,

$$-\frac{1}{\varepsilon}\int_{t-2\varepsilon}^{t-\varepsilon}\psi'\left(\frac{t-s}{\varepsilon}\right)\int_{R^n}\Gamma(x-y,t-s)\,u_i(y,s)\,dy\,ds$$
$$=-\frac{1}{\varepsilon}\int_{t-2\varepsilon}^{t-\varepsilon}\psi'\left(\frac{t-s}{\varepsilon}\right)u_i(x,s)\,ds$$
$$-\frac{1}{\varepsilon}\int_{t-2\varepsilon}^{t-\varepsilon}\psi'\left(\frac{t-s}{\varepsilon}\right)\int_{R^n}\Gamma(x-y,t-s)\,[u_i(y,s)-u_i(x,s)]\,dy\,ds$$
$$=-u_i(x,t)-\frac{1}{\varepsilon}\int_{t-2\varepsilon}^{t-\varepsilon}\psi'\left(\frac{t-s}{\varepsilon}\right)[u_i(x,s)-u_i(x,t)]\,ds$$
$$-\frac{1}{\varepsilon}\int_{t-2\varepsilon}^{t-\varepsilon}\int_{R^n}\Gamma(x-y,t-s)\,\psi'\left(\frac{t-s}{\varepsilon}\right)[u_i(y,s)-u_i(x,s)]\,dy\,ds.$$

Now letting $\varepsilon\to 0$ we see that the term

$$-\frac{1}{\varepsilon}\int_{t-2\varepsilon}^{t-\varepsilon}\int_{R^n}\langle u(y,s), E_i(x-y,t-s)\rangle\,\psi'\left(\frac{t-s}{\varepsilon}\right)dy\,ds$$

converges in $L^{p/2,q/2}(S_T)$, as $\varepsilon\to 0$, to the i-th coordinate function of the vector $u(x,t)$.

Since $g\in L^r(R^n)$, $1\leq r<\infty$, and is weakly divergence free,

$$\lim_{\varepsilon\to 0}\psi\left(\frac{t}{\varepsilon}\right)\int_{R^n}\langle g(y), E_i(x-y,t)\rangle\,dy=\int_{R^n}g_i(y)\,\Gamma(x-y,t)\,dy.$$

Hence u satisfies the integral equation $u+B(u,u)=\int_{R^n}\Gamma(x-y,t)(g(y))dy$.

III. Existence, Uniqueness, and Regularity of Solutions of the Integral Equation

We shall show in Theorem (3.1) below that the bilinear operator $B(u, v)$ is continuous from $L^{p,q} \times L^{p,q}(S_T) \to L^{p,q}(S_T)$ when $\frac{n}{p}+\frac{2}{q} \leqq 1$. The basic analytic tool is the following imbedding theorem, the proof of which can be found in [11].

Theorem (3.0) (Imbedding). *Suppose* $g \in L^{p_1}(R^d)$ *and set*

$$Tg(x)=\int_{R^d} \frac{g(y)}{|x-y|^{d-\alpha}} dy \quad \text{where} \quad 0<\alpha<d \quad (x \in R^d).$$

If $0<\frac{1}{p_1}-\frac{\alpha}{d}=\frac{1}{p}$, *then* T *is continuous from* $L^{p_1}(R^d)$ *into* $L^p(R^d)$.

Theorem (3.1). *For* $u, v \in L^{p,q}(S_T)$ *we have the following conclusions:*

(*i*) *If* $\frac{n}{p}+\frac{2}{q}=1$ *with* $n<p<\infty$, *then*

$$\|B(u, v)\|_{L^{p,q}(S_T)} \leqq C(n, p, q)\, \|u\|_{L^{p,q}(S_T)}\, \|v\|_{L^{p,q}(S_T)}.$$

(*ii*) *If* $\frac{n}{p}+\frac{2}{q}<1$ *with* $n<p \leqq \infty$, *then*

$$\|B(u, v)\|_{L^{p,q}(S_T)} \leqq C(n, p, q)\, T^{\frac{1}{2}\left(1-\frac{n}{p}-\frac{2}{q}\right)}\, \|u\|_{L^{p,q}(S_T)}\, \|v\|_{L^{p,q}(S_T)}.$$

Proof. Using the representation (1.4) of $D_{x_k} E_{ij}(x, t)$, it is not difficult to see that

$$|D_{x_k} E_{ij}(x, t)| \leqq \frac{C}{(|x|+t^{1/2})^{n+1}}$$

for each i, j, k. Hence

$$|B(u, v)(x, t)| \leqq C \int_0^t \int_{R^n} \frac{1}{[|x-y|+(t-s)^{1/2}]^{n+1}} |u(y, s)|\, |v(y, s)|\, dy\, ds.$$

We proceed to prove (*i*). For any θ, $0<\theta<1$,

$$\frac{1}{(|x|+t^{1/2})^{n+1}} \leqq \frac{C_\theta}{|x|^{(n+1)\theta}\, t^{((n+1)/2)(1-\theta)}}.$$

As a function of y, $|u(y, s)|\, |v(y, s)|$ belongs to $L^{p/2}(R^n)$ for almost every s. Hence by the imbedding theorem, if we choose θ such that

$$\frac{1}{p}=\frac{2}{p}-\frac{n-(n+1)\theta}{n} \qquad \left(0<\theta<\frac{n}{n+1}\right)$$

then

$$\|B(u, v)(\cdot, t)\|_{L^p(R^n)} \leqq C_{\theta, p, n} \int_0^t \frac{1}{(t-s)^{((n+1)/2)(1-\theta)}} \big\| |u(\cdot, s)|\, |v(\cdot, s)| \big\|_{L^{p/2}(R^n)}\, ds.$$

With the above choice of θ, if $n/p+2/q=1$ and $p>n$, then

$$\frac{1}{q}=\frac{2}{q}-\left[1-\frac{(n+1)}{2}(1-\theta)\right].$$

We again apply the imbedding theorem, observing that

$$\big\| |u(\cdot,s)|\,|v(\cdot,s)| \big\|_{L^{p/2}} \leqq \|u(\cdot,s)\|_{L^p(R^n)} \|v(\cdot,s)\|_{L^p(R^n)} \in L^{q/2}(0,T).$$

We obtain

$$\|B(u,v)\|_{L^{p,q}(S_T)} \leqq C(p,n)\, \|u\|_{L^{p,q}(S_T)} \|v\|_{L^{p,q}(S_T)}.$$

We proceed now to prove (*ii*). We shall have two cases, $p=\infty$ and $n<p<\infty$. In the first case

$$\|B(u,v)(\cdot,t)\|_{L^\infty(R^n)} \leqq C \int_0^t \frac{1}{(t-s)^{1/2}} \|u(\cdot,s)\|_{L^\infty(R^n)} \|v(\cdot,s)\|_{L^\infty(R^n)}\, ds.$$

If also $q=\infty$, then

$$\|B(u,v)\|_{L^\infty(S_T)} \leqq C T^{1/2} \|u\|_{L^\infty(S_T)} \|v\|_{L^\infty(S_T)}.$$

If $p=\infty$ and $q<\infty$, then by Theorem (3.0)

$$\|B(u,v)(\cdot,t)\|_{L^\infty(R^n)} \in L^r(0,T) \quad \text{when} \quad \frac{1}{r} = \frac{2}{q} - \frac{1}{2}.$$

Since $q>2$, we have $r>q$ and hence

$$\|B(u,v)\|_{L^{\infty,q}(S_T)} \leqq C T^{\frac{1}{q}-\frac{1}{r}} \|B(u,v)\|_{L^{\infty,r}(S_T)} \leqq C T^{\frac{1}{2}-\frac{1}{q}} \|u\|_{L^{\infty,q}(S_T)} \|v\|_{L^{\infty,q}(S_T)}.$$

Finally, when $n<p<\infty$, we choose q^* and r such that $q^* \leqq q \leqq r$ and

$$\frac{1}{r} = \frac{2}{q^*} + \frac{n}{2p} - \frac{1}{2} = \frac{1}{2}\left(\frac{4}{q^*} + \frac{n}{p} - 1\right).$$

One way of making the choice is to set $q=q^*$ when $\frac{4}{q}+\frac{n}{p}-1 \geqq 0$ and $r=\infty$ when $\frac{4}{q}+\frac{n}{p}-1 \leqq 0$. We proceed as before, setting

$$\frac{1}{p} = \frac{2}{p} - \left[\frac{n-(n+1)\theta}{n}\right].$$

Then $\frac{1}{r} = \frac{2}{q^*} - \left[1 - \left(\frac{n+1}{2}\right)(1-\theta)\right]$, and

$$\begin{aligned}
\|B(u,v)\|_{L^{p,q}(S_T)} &\leqq C T^{\left(\frac{1}{q}-\frac{1}{4}\right)} \|B(u,v)\|_{L^{p,r}(S_T)} \\
&\leqq C T^{\left(\frac{1}{q}-\frac{1}{r}\right)} \|u\|_{L^{p,q^*}(S_T)} \|v\|_{L^{p,q^*}(S_T)} \\
&\leqq C T^{\frac{1}{2}\left(1-\frac{n}{p}-\frac{2}{q}\right)} \|u\|_{L^{p,q}(S_T)} \|v\|_{L^{p,q}(S_T)}.
\end{aligned}$$

Theorem (3.2) (Existence). *Assume p and q satisfy the conditions of Theorem* (3.1). *Then there exists a constant* $C_0 = C_0(B,p,q,n)$ *such that when*

$$\|f\|_{L^{p,q}(S_T)} < C_0\, T^{-\frac{1}{2}\left(1-\frac{n}{p}-\frac{2}{q}\right)}$$

the integral equation $u + B(u,u) = f$ *has a solution* $u \in L^{p,q}(S_T)$.

Proof. To prove the theorem we use Theorem (3.1) and a very simple iterative technique. Set $v_0=f$, $v_m=-B(v_{m-1}, v_{m-1})+f$ for $m\geqq 1$. From Theorem (3.1) it follows that

$$\|v_m\|_{L^{p,q}(S_T)} \leqq C T^{\frac{1}{2}\left(1-\frac{n}{p}-\frac{2}{q}\right)} \|v_{m-1}\|^2_{L^{p,q}(S_T)} + \|f\|_{L^{p,q}(S_T)}$$

where $C=C(B,p,q,n)$. It follows that if $C_0=(4C)^{-1}$ and

$$\|f\|_{L^{p,q}(S_T)} < C_0\, T^{-\frac{1}{2}\left(1-\frac{n}{p}-\frac{2}{q}\right)},$$

then

$$\|v_m\|_{L^{p,q}(S_T)} \leqq \frac{\|f\|_{L^{p,q}(S_T)}}{1-2CT^{\frac{1}{2}\left(1-\frac{n}{p}-\frac{2}{q}\right)}\|f\|_{L^{p,q}(S_T)}} = A \qquad \text{for all } m.$$

(It suffices to show that if a non-negative sequence $\{a_m\}_{m=0}^{\infty}$ satisfies $a_m \leqq \lambda a_{m-1}^2 + a_0$ for $m\geqq 1$, then $a_m \leqq \dfrac{a_0}{1-2\lambda a_0}$ provided $4\lambda a_0<1$. This can be proved by induction.) Thus

$$\begin{aligned} v_{m+1}-v_m &= -B(v_m, v_m)+B(v_{m-1}, v_{m-1}) \\ &= -[B(v_m-v_{m-1}, v_{m-1})+B(v_m, v_m-v_{m-1})]. \end{aligned}$$

Hence

$$\|v_{m+1}-v_m\|_{L^{p,q}(S_T)} \leqq 2CT^{\frac{1}{2}\left(1-\frac{n}{p}-\frac{2}{q}\right)} A \|v_m-v_{m-1}\|_{L^{p,q}(S_T)}.$$

Now observe that if $0\leqq x<\frac{1}{2}$, then $\dfrac{x}{1-x}<1$. Since

$$2CT^{\frac{1}{2}\left(1-\frac{n}{p}-\frac{2}{q}\right)} \|f\|_{L^{p,q}(S_T)} < \tfrac{1}{2},$$

we see that

$$2ACT^{\frac{1}{2}\left(1-\frac{n}{p}-\frac{2}{q}\right)} < 1.$$

Thus $\lim_{m\to\infty} v_m = u$ exists in $L^{p,q}(S_T)$ and $u+B(u,u)=f$.

Theorem (3.3) (Uniqueness). *Suppose p, q satisfy the conditions of Theorem* (3.1). *In the class $L^{p,q}(S_T)$ there can exist at most one solution u of the equation $u+B(u,u)=f$ for $f\in L^{p,q}(S_T)$.*

Proof. Since $L^{p,\infty}\subset L^{p,r}$ for $1\leqq r\leqq\infty$, we may assume $q<\infty$. If $u+B(u,u)=f$ and $v+B(v,v)=f$ for u and $v\in L^{p,q}(S_T)$, then $u-v=-[B(u,u-v)+B(u-v,v)]$. Hence for $\delta\leqq T$,

$$\|u-v\|_{L^{p,q}(S_\delta)} \leqq C(\|u\|_{L^{p,q}(S_\delta)}+\|v\|_{L^{p,q}(S_\delta)})\,\|u-v\|_{L^{p,q}(S_\delta)}.$$

We now choose δ so small that

$$C(\|u\|_{L^{p,q}(R^n\times(a,b))}+\|v\|_{L^{p,q}(R^n\times(a,b))})<1 \qquad \text{for any } (a,b)\subset(0,T)$$

with $b-a=\delta$. We see that $u=v$ in $S_\delta=R^n\times(0,\delta)$. By a repetition of the argument $u=v$ also in $S_{2\delta}$. Continuing in the same way, we see that $u=v$ in S_T.

Theorem (3.4) (Regularity). *Let u be a solution of the equation $u+B(u,u)=f$, $u\in L^{p,q}(S_T)$, $\frac{2}{q}+\frac{n}{p}\leqq 1$. Let k be a positive integer such that $k+1<p,q<\infty$. If*

$$D_x^\alpha D_t^j f\in L^{p/(|\alpha|+2j+1),\,q/(|\alpha|+2j+1)}(S_T) \qquad \textit{whenever } |\alpha|+2j\leqq k,$$

then also

$$D_x^\alpha D_t^j u\in L^{p/(|\alpha|+2j+1),\,q/(|\alpha|+2j+1)}(S_T) \qquad \textit{for } |\alpha|+2j\leqq k.$$

Proof. Consider the case $k=1$. Then $f\in L^{p,q}(S_T)$ and $D_{x_i}f\in L^{p/2,q/2}(S_T)$. Now the i-th coordinate function $B(u,u)_i(x,t)$ equals

$$\sum_{l,k}\int_0^t\int_{R^n} u_l(y,s)\,[D_{x_k}\Gamma(x-y,t-s)]\,[\delta_{il}u_k(y,s)-R_iR_lu_k(y,s)]\,dy\,ds.$$

From the L^p-theory of singular integrals of elliptic and parabolic type (see [1], [5], or [9]) we see that $D_{x_j}B(u,u)\in L^{p/2,q/2}(S_T)$ provided $1<\frac{p}{2},\frac{q}{2}<\infty$. Since $u=B(u,u)+f$ we conclude that $D_{x_j}u\in L^{p/2,q/2}(S_T)$.

To obtain the general case we proceed by induction on k. Assume the theorem is true for the integer $k\geqq 1$ and consider the case when

$$D_t^j D_x^\alpha f\in L^{p/(2j+|\alpha|+1),\,q/(2j+|\alpha|+1)}(S_T) \qquad \text{for } 2j+|\alpha|\leqq k+1,\ p,q>k+2.$$

We want to show that $D_t^jD_x^\alpha B(u,u)\in L^{p/(2j+|\alpha|+1),\,q/(2j+|\alpha|+1)}(S_T)$ for $2j+|\alpha|\leqq k+1$. The induction hypothesis implies that

$$D_t^l D_x^\beta u\in L^{p/2l+|\beta|+1,\,q/2l+|\beta|+1}(S_T) \qquad \text{for } 2l+|\beta|\leqq k.$$

Hence the only case of interest is $2j+|\alpha|=k+1$.

If $j=0$ then $D_x^\alpha B(u,u)$ is a sum of terms of the form $D_{x_k}B(D_x^\beta u, D_x^\gamma u)$, where $|\beta|+|\gamma|=k$. Now

$$\|D_{x_k}B(D_x^\beta u, D_x^\gamma u)\|_{L^{p/k+2,q/k+2}}\leqq C\,\|D_x^\beta u\,D_x^\gamma u\|_{L^{p/k+2,\,q/k+2}(S_T)}.$$

Since $D_x^\beta u\in L^{\frac{p}{|\beta|+1},\frac{q}{|\beta|+1}}(S_T)$, $D_x^\gamma u\in L^{\frac{p}{|\gamma|+1},\frac{q}{|\gamma|+1}}(S_T)$, and $(|\beta|+|\gamma|+2)/(k+2)=1$, we see that

$$\|D_x^\alpha B(u,u)\|_{L^{p/k+2,\,q/k+2}(S_T)} \leqq C\sum_{|\beta|+|\gamma|=k}\|D_x^\beta u\|_{L^{\frac{p}{|\beta|+1},\frac{q}{|\beta|+1}}(S_T)}\,\|D_x^\alpha u\|_{L^{\frac{p}{|\gamma|+1},\frac{q}{|\gamma|+1}}(S_T)}.$$

If $j>0$ then

$$D_t^jD_x^\alpha B(u,u)=\sum_{|\beta|+|\gamma|=|\alpha|}C_{\beta,\gamma}D_t^jB(D_x^\beta u, D_x^\gamma u).$$

From the form of $B(u,v)_i$ it is not difficult to see that $D_t^jB(u,v)_i$ is a sum of terms of the form $D_t^{r-s}D_x^\nu u(x,t)R_jR_l(D_t^sD_x^\eta v)(x,t)$, $s\leqq r$, and $D_{x_k}B(D_x^{\beta'}u, D_x^{\gamma'}v)_l$ where $|\nu|+|\eta|+2r=2j-1$ and $|\beta'|+|\gamma'|=2j-1$. Replacing u by $D_x^\beta u$ and v by $D_x^\gamma u$

where $|\beta|+|\gamma|=|\alpha|$, we have

$$\|D_t^j D_x^\alpha B(u,u)\|_{L^{\frac{p}{k+2},\frac{q}{k+2}}(S_T)} \leqq C\sum \|(D_t^{r-s} D_x^\beta u)(D_t^s D_x^\gamma u)\|_{L^{p/k+2,\,q/k+2}(S_T)},$$

the summation being over $s\leqq r$, $|\beta|+|\gamma|+2r=k$. By induction

$$D_t^{r-s} D_x^\beta u \in L^{p/|\beta|+2r-2s+1,\,q/|\beta|+2r-2s+1}(S_T)$$

and

$$D_t^s D_x^\gamma u \in L^{p/|\gamma|+2s+1,\,q/|\gamma|+2s+1}(S_T).$$

Using Hölder's inequality, we see that the $L^{\frac{p}{k+2},\frac{q}{k+2}}(S_T)$ norm of the product $(D_t^{r-s}D_x^\beta u)(D_t^s D_x^\gamma u)$ is finite.

IV. Existence and Uniqueness Theorems for the Navier-Stokes Equation

In Section II we proved that any solution of the integral equation

$$u+B(u,u)=\int_{R^n} \Gamma(x-y,t)(g(y))\,dy$$

in the class $L^{p,q}(S_T)$, $p,q\geqq 2$, $p<\infty$, was indeed a weak solution of the initial-value problem for the Navier-Stokes equation, and, conversely, a weak solution u with initial value g was a solution of the integral equation. We shall now rephrase the results of Section III for the differential problem. Set

$$f(x,t)=\int_{R^n} \Gamma(x-y,t)(g(y))\,dy.$$

Suppose $g\in L^r(R^n)$, $1\leqq r<\infty$. Since

$$\|\Gamma(\cdot,t)\|_{L^s(R^n)}\leqq C\,t^{-\frac{n}{2}+\frac{n}{2s}},$$

if s is chosen so that $0<\frac{1}{p}=\frac{1}{s}+\frac{1}{r}-1$, then

$$\|f(\cdot,t)\|_{L^p(R^n)}\leqq C\,t^{-\frac{n}{2}+\frac{n}{2s}}\|g\|_{L^r(R^n)}.$$

If $q\left(1-\frac{1}{s}\right)<2/n$,

$$\|f\|_{L^{p,q}(S_T)}\leqq C\,T^{\frac{1}{q}-\frac{n}{2}\left(1-\frac{1}{s}\right)}\|g\|_{L^r(R^n)}.$$

Hence

$$\|f\|_{L^{p,q}(S_T)}\leqq C\,T^{\frac{1}{q}+\frac{n}{2p}-\frac{n}{2r}}\|g\|_{L^r(R^n)}, \qquad \frac{n}{p}+\frac{2}{q}>\frac{n}{r}.$$

As a consequence of Theorems (2.1), (3.2), and (3.3) we have the following existence and uniqueness theorems for the initial value problem for the Navier-Stokes equation.

Theorem (4.1) (Existence). *Assume* $\frac{n}{p}+\frac{2}{q}\leqq 1$ *with* $n<p<\infty$. *If* $g(x)$ *is weakly divergence free and belongs to* $L^r(R^n)$ *with* $\frac{n}{p}+\frac{2}{q}>\frac{n}{r}>0$, *then the Navier-Stokes* *equation with initial data $g(x)$ has a weak solution $u(x, t) \in L^{p,q}(S_T)$ at least for $0<T \leqq T_0$, $T_0=T_0(p, q, r, g)$.*

Theorem (4.2) (Uniqueness). *Again assume $\frac{n}{p}+\frac{2}{q} \leqq 1$ with $n<p<\infty$. There exists at most one weak solution $u(x, t) \in L^{p,q}(S_T)$ of the initial value problem for the Navier-Stokes equation.*

The main earlier results on uniqueness, for $n=3$, are due to LERAY [4] when $u \in C^{2,1} \cap L^{p,q}$ and PRODI [8] when $u \in L^{2,\infty} \cap L_1^{2,2} \cap L^{p,q}$. SERRIN [10] extended the result of PRODI to n-dimensions. (See also LIONS [6].) For a more extensive bibliography on uniqueness theorems, see [3], [6], [10].

Theorem (4.3). *Assume $\frac{n}{p}+\frac{2}{q}=1$ with $n<p<\infty$. Suppose $g(x)$ is weakly divergence free and belongs to $L^{r_1} \cap L^{r_2}(R^n)$ with*

$$\frac{n}{p}+\frac{2}{q}-\frac{n}{r_1}<0<\frac{n}{p}+\frac{2}{q}-\frac{n}{r_2}.$$

Set

$$\|g\|_{L^{r_1} \cap L^{r_2}(R^n)}=\|g\|_{L^{r_1}(R^n)}+\|g\|_{L^{r_2}(R^n)}.$$

If $\|g\|_{L^{r_1} \cap L^{r_2}(R^n)}$ is sufficiently small, then there exists a unique function $u(x, t)$ defined for almost all $t>0$ such that for each $T>0$, $u(x, t) \in L^{p,q}(S_T)$ and is a weak solution in S_T of the Navier-Stokes equation with initial value g.

Proof. Again set $f(x, t)=\int_{R^n} \Gamma(x-y, t)\, g(y)\, dy$. Since $\frac{n}{p}+\frac{2}{q}>\frac{n}{r_2}$,

$$\|f\|_{L^{p,q}(S_1)} \leqq C(p, q, r_2)\, \|g\|_{L^{r_2}(R^n)}.$$

Since $\frac{n}{p}+\frac{2}{q}<\frac{n}{r_1}$,

$$\|f\|_{L^{p,q}(R^n \times (1,\infty))} \leqq C(p, q, r_1)\, \|g\|_{L^{r_1}(R^n)}.$$

Hence

$$\|f\|_{L^{p,q}(R^n \times (0,\infty))} \leqq C(p, q, r_1, r_2)\, \|g\|_{L^{r_1} \cap L^{r_2}(R^n)}.$$

To complete the proof we use Theorems (2.1) and (3.1) to conclude that if $\|g\|_{L^{r_1} \cap L^{r_2}(R^n)}$ is small, then for each T there is a unique weak solution $u_T(x, t) \in L^{p,q}(S_T)$ of the initial value problem for the Navier-Stokes equation with initial data g. The uniqueness result of course implies that for $T_1<T_2$, $u_{T_1}=u_{T_2}$ in S_{T_1}.

We conclude Section IV with a discussion of existence and uniqueness of solutions of the following problem (4.4):

Given a weakly divergence free function $g(x) \in L^r(R^n)$, $1 \leqq r \leqq \infty$, and $f(x, t)=(f_1, \ldots, f_n)(x, t) \in L^{p_1,q_1}(S_T)$, $1 \leqq p_1, q_1$, find $u(x, t) \in L^{p,q}(S_T)$, $p, q \geqq 2$, such that

(1) For all $\varphi \in \mathscr{D}_T$

$$\int_0^T \int_{R^n} \langle u, D_t \varphi+\Delta \varphi+V(\varphi)(u)\rangle\, dx\, dt$$
$$=-\left[\int_0^T \int_{R^n} \langle f, \varphi\rangle\, dx\, dt+\int_{R^n} \langle g(x), \varphi(x, 0)\rangle\, dx\right].$$

(2) $u(x,t)$ is weakly divergence free for almost every $t\in(0,T)$.

The following theorems concerning the above problem are stated without proof; their proofs follow by straightforward modifications of those in the case $f=0$.

Theorem (4.4). *Let $g\in L^r(R^n)$, $1\leqq r<\infty$, be weakly divergence free and assume $f(x,t)\in L^{p_1,q_1}(S_T)$ with $1<p_1<\infty$ and $1\leqq q_1$. Then $u\in L^{p,q}(S_T)$, $p,q\geqq 2$, is a solution of the problem* (4.4) *if and only if $u(x,t)$ is a solution of the integral equation*

$$u(x,t)+B(u,u)(x,t)=\int_{R^n}\Gamma(x-y,t)(g(y))\,dy+\int_0^t\int_{R^n}E(x-y,t-s)(f(y,s))\,dy\,ds.$$

We observe that since $1<p_1<\infty$, then $(R_iR_j)(f)(x,t)\in L^{p_1,q_1}(S_T)$, and so the function

$$\int_0^t\int_{R^n}E(x-y,t-s)(f(y,s))\,dy\,ds=\int_0^t\int_{R^n}\Gamma(x-y,t-s)\big(f+(R_iR_j)(f)\big)(y,s)\,dy\,ds$$

also belongs to $L^{p_1,q_1}(S_T)$.

Theorem (4.5). *Assume $\frac{n}{p}+\frac{2}{q}\leqq 1$ with $n<p<\infty$. Suppose $g\in L^r(R^n)$ with $\frac{n}{p}+\frac{2}{q}>\frac{n}{r}>0$ and that g is weakly divergence free. Assume also that $f\in L^{p_1,q_1}(S_T)$, $1<p_1\leqq p$, $1<q_1<q$, and $\frac{1}{q_1}+\frac{n}{2p_1}\leqq\frac{1}{q}+\frac{n}{2p}+1$.*

Then problem (4.4) *has a solution $u(x,t)\in L^{p,q}(S_T)$ at least for $0<T<T_0=T_0(p,q,r,p_1,q_1)$.*

We should remark that the conditions imposed on p_1,q_1,p,q are sufficient to guarantee that

$$\int_0^t\int_{R^n}E(x-y,t-s)(f(y,s))\,dy\,ds\in L^{p,q}(S_T)$$

and that its $L^{p,q}$-norm over S_T is bounded by $C_T\|f\|_{L^{p_1,q_1}(S_T)}$ where $C_T=O(1)$ as $T\to 0$. We shall prove this remark, and the theorem will then follow from Theorem (3.2). Observe that $(R_iR_j)(f)(x,t)\in L^{p_1,q_1}(S_T)$; hence it suffices to show that the potential

$$w(x,t)=\int_0^t\int_{R^n}\Gamma(x-y,t-s)(f(y,s))\,dy\,ds\in L^{p,q}(S_T)$$

when $f\in L^{p_1,q_1}(S_T)$ and satisfies the desired norm inequality.

If $p_1=p$, then

$$\|w(\cdot,t)\|_{L^p(R^n)}\leqq C\int_0^t\|f(\cdot,s)\|_{L^p(R^n)}\,ds.$$

Hence in this case $w\in L^{p,\infty}(S_T)\subset L^{p,q_1}(S_T)$ for all $q_1\geqq 1$. If $1<p_1<p$, we set $1-\theta=\frac{1}{p_1}-\frac{1}{p}$ and observe that

$$|w(x,t)|\leqq C\int_0^t\int_{R^n}\frac{|f(y,s)|}{|x-y|^{n\theta}(t-s)^{(n/2)(1-\theta)}}\,dy\,ds.$$

Then

$$\|w(\cdot,t)\|_{L^p(R^n)} \leqq C\int_0^t \|f(\cdot,s)\|_{L^{p_1}(R^n)}(t-s)^{-(n/2)(1-\theta)}\,ds.$$

Now $\frac{n}{2}(1-\theta)\leqq 1+\frac{1}{q}-\frac{1}{q_1}<1$; therefore by the imbedding theorem we have $\|w\|_{L^{p,q}(S_T)}\leqq C\|f\|_{L^{p_1,q_1}(S_T)}$ provided that

$$\frac{1}{q}\geqq\frac{1}{q_1}-\left[1-\frac{n}{2}(1-\theta)\right],$$

that is, $\frac{1}{q_1}+\frac{n}{2p_1}\leqq\frac{1}{q}+\frac{n}{2p}+1$.

Theorem (4.6). *Again assume* $\frac{n}{p}+\frac{2}{q}\leqq 1$ *with* $n<p<\infty$. *There exists at most one solution* $u(x,t)$ *of the problem* (4.4) *in the class* $L^{p,q}(S_T)$.

V. Relation to the Hopf-Leray Class

J. LERAY [4], in dimension three, and E. HOPF [2], in the general case, have proved the following existence theorem.

Theorem (5.1) (HOPF-LERAY). *Suppose* $g(x)\in L^2(R^n)$ *and is weakly divergence free. Then in* S_T *(no restriction on T) there exists a weak solution of the initial value problem with the following properties:*

(*i*) $u(x,t)\in L^{2,\infty}(S_T)$.

(*ii*) $D_{x_i}u(x,t)\in L^{2,2}(S_T)$ *for* $i=1,\dots,n$.

(*iii*) $\sum_{i=1}^n \|u_i(\cdot,t)\|^2_{L^2(R^n)}+2\int_0^t \sum_{i,k=1}^n \|D_{x_k}u_i(\cdot,s)\|^2_{L^2(R^n)}\,ds\leqq\int_{R^n}|g(x)|^2\,dx.$

We shall call any weak solution $u(x,t)$ of the initial value problem in S_T satisfying (*i*) and (*ii*) a Hopf-Leray solution. In this section the initial data will be taken from the space $L^2(R^n)$.

With regard to the question of uniqueness in the class of Hopf-Leray solutions we state the following theorem.

Theorem (5.2). *Suppose u and v are Hopf-Leray solutions of the initial value problem for the Navier-Stokes equation with weakly divergence free data* $g(x)\in L^2(R^n)$. *Assume v satisfies the energy estimate* (*iii*) *in* (5.1). *If* $u\in L^{p,q}(S_T)$ *for a pair of exponents p and q satisfying* $\frac{n}{p}+\frac{2}{q}=1$ *with* $n<p<\infty$, *then* $u=v$ *in* S_T.

Theorem (5.2) is due to PRODI [8], when $n=3$, and to SERRIN [10, remarks to Theorem 6] in the general case.

In this section we restrict our attention to $n\geqq 2$, and we show that when the data $g(x)$ belongs to $L^p\cap L^2(R^n)$, $n<p<\infty$, any solution of the integral equation

$u+B(u,u)=\int_{R^n}\Gamma(x-y,t)\,g(y)\,dy$ in the class $L^{p,q}(S_T)$, $\frac{n}{p}+\frac{2}{q}=1$, is also a Hopf-Leray solution. More explicitly

Theorem (5.3). *Suppose* $g(x)\in L^2\cap L^p(R^n)$, $n<p<\infty$. *If* $u\in L^{p,q}(S_T)$, $\frac{n}{p}+\frac{2}{q}=1$, *is a solution of the integral equation*

$$u+B(u,u)=u_0 \quad \left(u_0(x,t)=\int_{R^n}\Gamma(x-y,t)\,g(y)\,dy\right),$$

then u is a Hopf-Leray solution.

As an immediate consequence of Theorems (5.2) and (5.3) we obtain

Theorem (5.4). *Suppose* $g(x)\in L^2\cap L^p(R^n)$, $n<p<\infty$, *and is weakly divergence free. Then there exists a number* $T_0=T_0(g,p)$ *such that if u and v are two Hopf-Leray solutions of the initial value problem for the Navier-Stokes equation with data g and if u and v satisfy the energy estimate,* 5.1 (*iii*), *then* $u\equiv v$ *in* S_{T_0}.

To prove Theorem (5.3) we shall make use of the following lemmas.

Lemma (5.1). *If* $g\in L^p(R^n)$, *then* $u_0(x,t)=\int_{R^n}\Gamma(x-y,t)\,g(y)\,dy$ *belongs to* $L^{r,s}(S_T)$ *for* $r=p$ *and* $s=\infty$ *and for* $r>p$ *and* $\frac{n}{p}<\frac{n}{r}+\frac{z}{s}$.

The lemma is an immediate consequence of Young's Inequality.

Lemma (5.2). *Let* $u\in L^{p,q}(S_T)$. *Then the following results hold:*

(*i*) *If* $p\geqq n$, $B(u,u)\in L^{p,q^*}(S_T)$ *where* $\frac{1}{q^*}=\frac{1}{q}+\frac{n}{2p}-\frac{1}{2}$.

(*ii*) *If* $q\geqq 2$, $B(u,u)\in L^{p/2,q}(S_T)$.

Proof. Observe that

$$\|D_{x_k}E_{ij}(\cdot,t)\|_{L^r(R^n)}=C\,t^{-\frac{1}{2}\left(n+1-\frac{n}{r}\right)}.$$

Hence using Young's Inequality $\left(\frac{1}{p}=\frac{2}{p}+\frac{1}{p'}-1,\ \frac{1}{p}+\frac{1}{p'}=1\right)$, we have

$$\|B(u,u)(\cdot,t)\|_{L^p(R^n)}\leqq C\int_0^t (t-s)^{-\frac{1}{2}\left(n+1-\frac{n}{p'}\right)}\|u(\cdot,s)\|^2_{L^p(R^n)}\,ds.$$

Using Young's Inequality again, we obtain (*i*). A similar argument (setting $r=1$) proves (*ii*).

Proof of Theorem (5.3). As a consequence of Lemma (5.1), if $g\in L^2(R^n)\cap L^p(R^n)$, $n<p<\infty$, then $u_0=\int_{R^n}\Gamma(x,y,t)\,g(y)\,dy$ belongs to $L^{2,\infty}(S_T)\cap L^{p,\infty}(S_T)$ and hence to any $L^{r,s}(S_T)$ when $2\leqq r\leqq p$, $s\geqq 1$. Therefore $u\in L^{r,s}(S_T)$, $2\leqq r\leqq p$, $1\leqq s$, if and only if $B(u,u)\in L^{r,s}(S_T)$. But then (*i*) of Lemma (5.2) implies that $u\in L^{p,\infty}(S_T)$, and (*ii*) implies now that u also belongs to $L^{2,\infty}(S_T)$.

To complete the proof we must verify that $D_x u \in L^{2,2}(S_T)$.

Observe that $\mathscr{F}(D_{x_k} u_0(\cdot, t))(x) = i x_k e^{-|x|^2 t} \mathscr{F}(g)(x)$. ($\mathscr{F}$ denotes once again the Fourier transform in the space variables.) Therefore, using Parseval's identity we find

$$\begin{aligned}\int_0^T \int_{R^n} |D_{x_k} u_0(x,t)|^2 \,dx\,dt &= \int_0^T \int_{R^n} x_k^2 e^{-2|x|^2 t} |\mathscr{F}(g)(x)|^2 \,dx\,dt \\ &= \int_{R^n} |\mathscr{F}(g)(x)|^2 \left\{ \int_0^T x_k^2 e^{-2|x|^2 t} \,dt \right\} dx \\ &\leqq 1/2 \int_{R^n} |\mathscr{F}(g)(x)|^2 \,dx = \tfrac{1}{2} \cdot \|g\|^2_{L^2(R^n)}.\end{aligned}$$

Hence $D_{x_k} u_0 \in L^{2,2}(S_T)$; therefore it suffices to show that $D_{x_k}(B(u,u)) \in L^{2,2}(S_T)$.

Extend $E_{ij}(x,t)$ to be zero for $t<0$. Then if $\mathscr{F}_{x,t}$ denotes the Fourier Transform in x and t, we have

$$\mathscr{F}_{x,t}(D_{x_k x_l} E_{ij})(x,t) = \frac{-x_k x_l}{|x|^2 - it} \left(\delta_{ij} - \frac{x_i x_j}{|x|^2} \right).$$

Hence $\mathscr{F}_{x,t}(D_{x_k x_l} E_{ij}) \in L^\infty(R^{n+1})$. Extending u to be zero outside S_T and using Parseval's identity in $L^2(R^{n+1})$, one obtains

$$\|D_{x_k} B(u,u)\|_{L^{2,2}(S_T)} \leqq C \|u\|^2_{L^{4,4}(S_T)}.$$

It suffices then to show that $u \in L^{4,4}(S_T)$ for some strip S_T. If $p \geqq 4$, the result follows ($u \in L^{r,s}(S_T)$ for $2 \leqq r \leqq p, s \geqq 1$). If $p<4$, by Lemma (5.1), $u_0 \in L^{n+2,n+2}(S_T)$ $\left(n \geqq 2,\ p < n+2,\ n/p < 1 = \frac{n}{n+2} + \frac{2}{n+2} \right)$. Hence for T small enough the sequence $\{v_n\}$ of Theorem (3.2) converges in $L^{n+2,n+2}(S_T)$ to our solution u (since the sequence depends only on u_0). Therefore $u \in L^{n+2,n+2}(S_T)$, $n+2 \geqq 4$; hence $u \in L^{4,4}(S_T)$ (since $u \in L^{p,p}(S_T)$) and the theorem follows.

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