Steady Solutions of the Boltzmann Equation for a Gas Flow Past an Obstacle, I. Existence

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1. Introduction

This is the first of two papers concerned with the exterior problem for the Boltzmann equation. In that problem the gas has prescribed constant velocity c at infinity and passes by an obstacle $\mathcal{O}$. It is shown that if c is small, then steady solutions exist and are stable in time. The existence theorem is presented in this paper while stability will be discussed in the companion paper [16], hereafter referred to as Part II. The assumption imposed on $\mathcal{O}$, the intermolecular potential and the law of reflection for gas molecules at the wall $\partial\mathcal{O}$ of $\mathcal{O}$ are physically plausible ones, stated after equation (1.12) of this section.

Until now, the exterior problem has been studied only for the Euler and Navier-Stokes equations. There is an extensive literature on existence and stability of flows of incompressible fluids, but little for compressible gases. We should mention the papers [3], [12] in which the Euler equation for compressible fluids is solved for small c for two-dimensional isentropic, irrotational steady flows, the stability of which remains an open question. We mention also [11], which solves over an infinite interval of time the Navier-Stokes equation for compressible fluids at rest at infinity.

In [1] and [14] we solved the Boltzmann equation for flows such that $c = 0$; steady solutions are then trivially given by Maxwellians (see (1.9) below). The aim of this paper is to deal with flows such that $c \neq 0$, and so nontrivial steady solutions appear.

We discuss gas flow in n-dimensional Euclidean space $\mathbb{R}^n$. Thus let $\mathcal{O}$ be a domain in $\mathbb{R}^n$ and denote its exterior $\mathbb{R}^n \setminus \bar{\mathcal{O}}$ by Ω. Then we have the following initial-boundary value problem:

$$\frac{\partial f}{\partial t} = -\xi \cdot \nabla_x f + Q[f,f], \quad (t, x, \xi) \in \bar{\mathbb{R}}_+ \times \Omega \times \mathbb{R}^n, \tag{1.1a}$$

$$\gamma^- f = M\gamma^+ f, \qquad (t, x, \xi) \in \bar{\mathbb{R}}_+ \times S^-, \tag{1.1b}$$

$$f \to g_c(\xi) \ (|x| \to \infty), \qquad (t, \xi) \in \bar{\mathbb{R}}_+ \times \mathbb{R}^n, \tag{1.1c}$$

$$f|_{t=0} = f_0, \qquad (x, \xi) \in \Omega \times \mathbb{R}^n. \tag{1.1d}$$

The notations are to be explained. The unknown is the (probability) density $f = f(t, x, \xi)$ of gas molecules having the position $x \in \bar{\Omega}$ and velocity $\xi \in \mathbb{R}^n$ at time $t \in \bar{\mathbb{R}}_+ = [0, \infty)$. Equation (1.1a) is the Boltzmann equation, in which ∇_x stands for the gradient with respect to x and $\cdot$ for the inner product in $\mathbb{R}^n$, while Q is a quadratic operator, called the collision operator, given by

$$Q[f,f] = \int_{\mathbb{R}^n \times S^{n-1}} q(v, \theta) \{f(\eta) f(\eta') - f(\xi) f(\xi')\} \, d\xi' \, d\omega, \tag{1.2}$$

where $f(\eta) = f(t, x, \eta)$, $v = |\xi - \xi'|$, $\omega \in S^{n-1}$, $\cos\theta = (\xi - \xi') \cdot \omega / v$, $\eta = \xi - (v \cos\theta)\,\omega$ and $\eta' = \xi' + (v \cos\theta)\,\omega$. $q(v, \theta)$ in (1.2) is a non-negative function of v and θ, called the collision cross-section and determined by the intermolecular potential. The two classical examples are the gas of hard balls, for which

$$q(v, \theta) = \sigma v \, |\cos\theta| \quad (\sigma > 0, \text{ constant}), \tag{1.3}$$

and the gas of molecules subject to an intermolecular potential varying $\propto r^{-s}$, $s > 1$, which gives rise to

$$q(v, \theta) = v^{-\gamma} q_0(\theta), \qquad \gamma = (s - 4)/s. \tag{1.4}$$

See [4] for the derivation of (1.1a) and [7] for discussion of $q(v, \theta)$.

Equation (1.1b) expresses the boundary condition on $\partial\Omega = \partial\mathcal{O}$. Let $n(x)$ be the unit outward normal to $\partial\Omega$ (inward to $\partial\mathcal{O}$) at $x \in \partial\Omega$, and define

$$S^{\pm} = \{(x, \xi) \in \partial\Omega \times \mathbb{R}^n \mid n(x) \cdot \xi \gtrless 0\} \text{ (same signs)}.$$

Then $\gamma^{\pm}$ are trace operators on $S^{\pm}$ defined by

$$\gamma^{\pm} f = f|_{S^{\pm}}$$

(for the precise definition, see § 4), and M is an operator which maps functions on S^+ onto functions on S^-. $\gamma^+ f$ is the density of gas molecules incident upon the wall $\partial\Omega$, $\gamma^- f$ is the density of molecules reflected by the wall. Hence M is determined by the law of reflection of the gas molecules at the wall. For example, if the reflection at $x \in \partial\Omega$ induces the deterministic change from the molecular velocity $\xi' = m(x, \xi)$ to ξ, then we are given a map $S^- \ni (x, \xi) \to (x, m(x, \xi)) \in S^+$, and

$$M\gamma^+ f = f(t, x, m(x, \xi)). \tag{1.5}$$

The specular reflection

$$m(x, \xi) = \xi - 2(n(x) \cdot \xi)\, n(x) \tag{1.6}$$

and the reverse reflection

$$m(x, \xi) = -\xi \tag{1.7}$$

are well-known; see [9]. Nondeterministic laws of reflection are also possible. In [6], the diffuse reflection

$$M\gamma^+ f = \int_{n(x)\cdot\xi' > 0} m(x, \xi, \xi') f(t, x, \xi') \, d\xi' \tag{1.8}$$

is treated, in which $m(x, \xi, \xi')$ is the probability that the reflection at $x \in \partial\Omega$ changes the velocity from ξ' to ξ.

The boundary condition at infinity is (1.1c), in which

$$g_c(\xi) = \exp\{-|\xi - c|^2/2\} \tag{1.9}$$

is a Maxwellian (Gaussian) density with appropriately normalized physical constants, which represents a gas in equilibrium, moving with the mean velocity $c \in \mathbb{R}^n$. Thus the gas is assumed to be in equilibrium at infinity.

The initial-boundary value problem (1.1) is the subject of Part II; here we discuss the steady problem

$$\begin{cases} -\xi \cdot \nabla_x f + Q[f, f] = 0, & (x, \xi) \in \Omega \times \mathbb{R}^n, \\ \gamma^- f - M\gamma^+ f = 0, & (x, \xi) \in S^-, \\ f \to g_c(\xi) \quad (|x| \to \infty), & \xi \in \mathbb{R}^n, \end{cases} \tag{1.10}$$

where the unknown is $f = f(x, \xi)$.

It is well known [4] that

$$Q[g_c, g_c] = 0$$

holds for any $c \in \mathbb{R}^n$. Thus $g_c(\xi)$ is a steady solution of (1.1a), but in general is not a solution of (1.10) unless $c = 0$, for otherwise the boundary condition is violated, as seen from (1.6) and (1.7); the gas flow is disturbed by the obstacle $\mathcal{O}$. In this paper, assuming that the disturbance is small, we shall seek solutions of (1.10) in the form $f = g_c + g_0^{\frac{1}{2}} u$, where $g_0 = g_{c=0}$.

Define

$$\begin{aligned} L_c u &= 2 g_0^{-\frac{1}{2}} Q[g_c, g_0^{\frac{1}{2}} u], \\ \Gamma_0[u, v] &= g_0^{-\frac{1}{2}} Q[g_0^{\frac{1}{2}} u, g_0^{\frac{1}{2}} v], \end{aligned} \tag{1.11}$$

where $Q[\,,\,]$ is the bilinear symmetric operator induced by the quadratic operator Q of (1.2). Put

$$\begin{aligned} M_0 \gamma^+ u &= (\gamma^- g_0^{-\frac{1}{2}})\, M\gamma^+(g_0^{\frac{1}{2}} u), \\ \tilde{M}_0 &= M_0 \gamma^+ - \gamma^-, \\ h_c &= (\gamma^- g_0^{-\frac{1}{2}})\, (\gamma^- g_c - M\gamma^+ g_c). \end{aligned}$$

Then (1.10) reduces to

$$\begin{aligned} & -\xi \cdot \nabla_x u + L_c u + \Gamma_0[u, u] = 0, && (x, \xi) \in \Omega \times \mathbb{R}^n, \\ & \tilde{M}_0 u = h_c, && (x, \xi) \in S^-, \\ & u \to 0 \quad (|x| \to \infty), && \xi \in \mathbb{R}^n. \end{aligned} \tag{1.12}$$

The precise definition of the solution $u = u(x, \xi)$ of (1.12) will be stated in § 4, after we have established a trace theorem.

Now we shall state the assumptions under which (1.12) is to be solved. For the domain $\mathcal{O}$, we assume

[$\mathcal{O}$] *$\mathcal{O}$ is a bounded convex domain of $\mathbb{R}^n$ with a piecewise smooth boundary $\partial\mathcal{O} = \partial\Omega$.* The restrictive requirement of convexity will be removed in a future paper.

The intermolecular potential employed here is the cutoff hard potential of GRAD [7]:

[q] (i) *$q(v, \theta)$ is a nonnegative continuous function of $v > 0$ and $\theta \in [0, \pi]$.*
(ii) *There are constants $q_1, q_2 > 0$ and $\delta \in [0, 1)$ such that*

$$q(v, \theta) \leqq q_1 \, |\cos\theta| \, (v + v^{-\delta}), \qquad \int_{S^{n-1}} q(v, \theta) \, d\omega \geqq q_2 v/(v + 1).$$

It is clear that [q] is satisfied by (1.3) and by (1.4) if $s \geqq 4$ and if $q_0(\theta)/|\cos\theta|$ is bounded and bounded away from zero (the assumption of an angular cutoff [7]).

As for the boundary condition on $\partial\Omega$, different assumptions are required according to the type of the operator M. When M is of the form (1.5), the function $m = m(x, \xi)$ is assumed to satisfy

$[M]_1$ (i) *$m : S^- \to \mathbb{R}^n$ is a piecewise smooth map such that $(x, m(x, \xi)) \in S^+$ whenever $(x, \xi) \in S^-$.*
(ii) *$|m(x, \xi)| = |\xi|$ for all $(x, \xi) \in S^-$.*
(iii) *Let $m_\xi = (\partial m_j/\partial \xi_k)$ denote the Jacobian matrix of m with respect to ξ with x fixed. There is a constant $m_0 > 0$ such that*

$$|n(x) \cdot \xi| = |n(x) \cdot m(x, \xi)| \, |\det m_\xi(x, \xi)|,$$
$$|\det m_\xi(x, \xi)| \geqq m_0,$$

for $(x, \xi) \in S^-$.
(iv) *$m(x, t\xi) = tm(x, \xi)$ for all $(x, \xi) \in S^-$ and $t > 0$.*

This statement called the regular reflection law (see [9]) includes the specular reflection law (1.6) and the reverse law (1.7).

Let $p, q \in [1, \infty]$ and $\beta \in \mathbb{R}$. We define the space $Y^{p,q,\pm}_\beta$ of functions $u(x, \xi)$ on $S^\pm$ (same signs) by

$$Y^{p,q,\pm}_\beta \ni u \Leftrightarrow \int_{\partial\Omega} \left(\int_{n(x)\cdot\xi \gtrless 0} |(1 + |\xi|)^\beta u(x, \xi)|^q \varrho(x, \xi) \, d\xi \right)^{p/q} d\sigma_x < \infty$$

when $p, q < \infty$, and with integrals replaced by the supremum when $p = \infty$ and/or $q = \infty$. Here $d\sigma$ is the measure on $\partial\Omega$ and

$$\varrho(x, \xi) = |n(x) \cdot \xi|.$$

To simplify the notation, we put

$$Y^{p,\pm}_\beta = Y^{p,p,\pm}_\beta, \qquad Y^{p,\pm} = Y^{p,\pm}_0.$$

All of these spaces are Banach spaces with obvious norms. In the sequel, $\mathbb{B}(X, Y)$ will denote the Banach space of all bounded linear operators from a Banach space X into another Banach space Y. The assumption on M_0 for the case (1.8) is as follows:

$[M]_2$ (i) *$M_0 \in \mathbb{B}(Y^{2,+}, Y^{2,-})$ with the operator norm $\leqq 1$.*
(ii) *$\tilde{M}_0 g_0^{\frac{1}{2}} = 0$.*

(iii) $M_0 \in \mathbb{B}(Y_\beta^{\infty,+}, Y_\beta^{\infty,-})$ *for all* $\beta \in \mathbb{R}$.
(iv) $M_0 \in \mathbb{B}(Y_\beta^{\infty,p_0,+}, Y_\beta^{\infty,-})$ *for all* $\beta \in \mathbb{R}$ *with some (sufficiently large)* $p_0 < \infty$.

An example of a kernel $m(x, \xi, \xi')$ satisfying $[M]_2$ is found in [6]. Obviously $[M]_2$ covers a wider class of M's than those given by (1.8). For example, it holds for M given by

$$M\gamma^+ u = \int_{S^+} m(x, x', \xi, \xi')\, u(x', \xi')\, d\sigma_{x'}\, d\xi'.$$

Remark 1.1. Suppose M be given by (1.5), and suppose that $[M]_1$ (i) (ii) (iii) be satisfied. Then it can be verified easily that $M = M_0$ and that M_0 satisfies $[M]_2$ (i) (ii) (iii). However $[M]_1$ (iv) does not lead to $[M]_2$ (iv). These two assumptions will not be used explicitly before § 7 of Part II (see the remark after Proposition 7.5 of this paper).

Remark 1.2. A boundary condition satisfying $[M]_2$ (i) is said to be dissipative. If in addition $[M]_2$ (ii) is satisfied, then the norm of M_0 is unity. Thus our boundary condition is conservative. Note that (ii) gives $M\gamma^+ g_c = \gamma^- g_c$ for $c = 0$, but not for $c \neq 0$.

Remark 1.3. For our purpose, $[M]_2$ (iii) (iv) has to be satisfied only for $\beta = 0$ and $\beta = \beta_0$, with some $\beta_0 > n + 1$.

Remark 1.4. All the results below hold if M_0 is a convex linear combination of M_0's subject to $[M]_1$ or $[M]_2$.

The plan of this paper is as follows. The notations and function spaces used in the sequel are introduced in § 2, and properties of operators L_c and Γ_0 defined in (1.11) are summarized in § 3. In § 4 a trace theorem is established under which $\gamma^\pm$ make sense. The definition of "solution" of (1.12) is also stated. Sections 5, 6 and 7 are devoted to the study of the linear operator

$$B_c = -\xi \cdot \nabla_x + L_c, \tag{1.13}$$

which arises from the linear part of (1.12) and is called the linearized Boltzmann operator. Since x is a mere parameter in L_c, the operator B_c can be defined for all $x \in \mathbb{R}$ as well as for $x \in \Omega$. In the former case, B_c will be denoted by B_c^∞ and will be studied in § 5 and 6; the latter case will be discussed in § 7 together with the homogeneous boundary condition $\tilde{M}_0 u = 0$ on S^-. The main aim is to study the resolvent of B_c and, in particular, to prove the existence of the inverse B_c^{-1}. Our proof relies on a method of perturbation in which the unperturbed operator is B_c^∞. The perturbation by $\mathcal{O}$ is seen to be compact in a certain sense. In § 8 we solve the linear inhomogeneous boundary-value problem

$$\begin{aligned} -\xi \cdot \nabla_x \phi + L_c \phi &= 0, \qquad && (x, \xi) \in \Omega \times \mathbb{R}^n, \\ \tilde{M}_0 \phi &= h_c, && (x, \xi) \in S^-, \\ \phi \to 0 \quad (|x| &\to \infty), && \xi \in \mathbb{R}^n, \end{aligned} \tag{1.14}$$

which comes from (1.12) by ignoring the nonlinear term $\Gamma_0[u, u]$. Writing the solution to (1.14) as $\phi = \phi_c$, we see that (1.12) is equivalent to

$$u = -B_c^{-1}\Gamma_0[u, u] + \phi_c. \tag{1.15}$$

This equation will be solved in § 9 by use of the classical contraction-mapping principle.

The main results of our two papers were announced in [15]. However there is a difference in the proofs presented here. The estimates of B_c^{-1} and ϕ_c used in [15] were uniform in the parameter c, which permitted us to solve (1.15) when $n \geqq 4$ by means of the contraction-mapping principle. At the same time, this procedure required us to use the Nash-Moser-Nirenberg scheme for the physically important case $n = 3$, supplemented by decay estimates of ϕ_c for large $|x|$. In this paper we derive non-uniform estimates which make it possible to avoid the Nash-Moser-Nirenberg scheme (see Remark 9.4). Most of the techniques employed here were developed in [1] and [14] for the special case $c = 0$, but we reproduce them here with emphasis on the role of c.

2. Notations and Function Spaces

Let X be a Banach Space. The norm of X will be denoted by $\| \ \|_X$. Let Y be another Banach space. $\mathbb{B}(X, Y)$ will denote the space of bounded linear operators from X into Y, and $\mathbb{C}(X, Y)$ will denote its subset consisting of compact operators. They are Banach spaces with operator norm. We write $\mathbb{B}(X) = \mathbb{B}(X, X)$, $\mathbb{C}(X) = \mathbb{C}(X, X)$. In what follows, we shall often encounter operators defined formally, and we shall study their realizations in various spaces. To simplify notations, therefore, any realization of a formal operator A will be denoted by the same symbol A. Thus, in particular, the statement $A \in \mathbb{B}(X, Y)$ will be understood to mean that a formal operator A has a unique realization belonging to $\mathbb{B}(X, Y)$.

Let A be a (not necessarily bounded) linear operator defined in X with the range also in X. $D(A)$ will denote the domain of A, while $\varrho(A)$ and $\sigma(A)$ will denote the resolvent set and spectrum of A. The essential spectrum (in the sense of [8], p. 243) and discrete spectrum (the set of isolated eigenvalues of finite multiplicity) will be expressed as $\sigma_e(A)$ and $\sigma_d(A)$ respectively, and the resolvent $(\lambda I - A)^{-1}$, I being the identity, will be written simply as $(\lambda - A)^{-1}$.

Let D be a domain in $\mathbb{R}^n$. $L^p(D; X)$ $p \in [1, \infty]$, will denote the space of L^p-functions defined on D with values in a Banach space X, and $L^p_\beta(\mathbb{R}^n; X)$, $\beta \in \mathbb{R}$, the space of functions $u(y)$, $y \in \mathbb{R}^n$, such that $(1 + |y|)^\beta u(y) \in L^p(\mathbb{R}^n; X)$. They are Banach spaces with obvious norms. Let $\mathscr{E}^0(D; X)$ denote the set of strongly continuous functions on D with values in X, and write $\mathscr{B}^0(D; X) = \mathscr{E}^0(D; X) \cap L^\infty(D; X)$. As usual, X will be suppressed in the above notations when $X = \mathbb{C}$, $\mathbb{C}$ being the set of complex numbers.

Let Ω be as in § 1 and put $Q = \Omega_x \times \mathbb{R}^n_\xi$. We define the spaces

$$L^{p,q}_\beta(Q) = L^p(\Omega_x; L^q_\beta(\mathbb{R}^n_\xi)),$$

$$\tilde{L}^{p,q}_\beta(Q) = L^q_\beta(\mathbb{R}^n_\xi; L^p(\Omega_x)),$$

for $p, q \in [1, \infty]$ and $\beta \in \mathbb{R}$, with β dropped in subscript when $\beta = 0$. Put

$$L^p_\beta(Q) = L^{p,p}_\beta(Q).$$

Also put $Q^\infty = \mathbb{R}^n_x \times \mathbb{R}^n_\xi$ and define $L^{p,q}_\beta(Q^\infty)$, $\tilde{L}^{p,q}_\beta(Q^\infty)$ and $L^p_\beta(Q^\infty)$ similarly. These are Banach spaces, and for both Q and Q^∞,

$$\text{(2.1)} \qquad \begin{gathered} L^p_\beta = L^{p,p}_\beta = \tilde{L}^{p,p}_\beta, \\ L^{p,q}_\beta \subset \tilde{L}^{p,q}_\beta, \quad \tilde{L}^{q,p}_\beta \subset L^{q,p}_\beta \quad \text{if} \quad p \leqq q, \end{gathered}$$

with continuous injection [2].

Given a function $u = u(x, \xi)$, let $\hat{u} = \mathscr{F}_x u$ denote its Fourier transform with respect to x;

$$\hat{u}(k, \xi) = \mathscr{F}_x u(k, \xi) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-ik\cdot x} u(x, \xi)\, dx, \qquad k \in \mathbb{R}^n, \qquad i = \sqrt{-1}.$$

Put $\hat{Q}^\infty = \mathbb{R}^n_k \times \mathbb{R}^n_\xi$ and define $L^{p,q}_\beta(\hat{Q}^\infty)$ and $\tilde{L}^{p,q}_\beta(\hat{Q}^\infty)$ as before. It is useful to note that for all $p \in [1, 2]$, $\frac{1}{p} + \frac{1}{q} = 1$, $r \in [1, \infty]$ and $\beta \in \mathbb{R}$, the following relations hold:

$$\text{(2.2)} \qquad \begin{aligned} &\text{(i)} \quad \mathscr{F}_x \in \mathbb{B}(L^{p,2}_\beta(Q^\infty), L^{q,2}_\beta(\hat{Q}^\infty)), \\ &\text{(ii)} \quad \mathscr{F}_x \in \mathbb{B}(\tilde{L}^{p,r}_\beta(Q^\infty), \tilde{L}^{q,r}_\beta(\hat{Q}^\infty)). \end{aligned}$$

In fact, the definition of $\mathscr{F}_x$ makes (2.2) obvious when $p = 1$, and the Parseval theorem shows that it is true for $p = 2$. Hence the case $p \in (1, 2)$ follows by interpolation [2]. Similarly, the inverse Fourier transformation $\mathscr{F}_x^{-1}$ satisfies

$$\text{(2.3)} \qquad \begin{aligned} &\text{(i)} \quad \mathscr{F}_x^{-1} \in \mathbb{B}(L^{p,2}_\beta(\hat{Q}^\infty), L^{q,2}_\beta(Q^\infty)), \\ &\text{(ii)} \quad \mathscr{F}_x^{-1} \in \mathbb{B}(\tilde{L}^{p,r}_\beta(\hat{Q}^\infty), \tilde{L}^{q,r}_\beta(Q^\infty)), \end{aligned}$$

where p, q, r and β are the same as in (2.2).

Let $\sigma \in \mathbb{R}$ and define the open half-planes

$$\mathbb{C}_\pm(\sigma) = \{\lambda \in \mathbb{C} \mid \operatorname{Re} \lambda \gtrless \sigma\} \qquad \text{(associated signs)}$$

of the complex plane $\mathbb{C}$, and let $\overline{\mathbb{C}}_\pm(\sigma)$ denote its closure. Let $B[a]$ denote the closed ball

$$B[a] = \{y \in \mathbb{R}^n \mid |y| \leqq a\}$$

in $\mathbb{R}^n_y$, y being ξ, k or c in the sequel, and $a > 0$.

$\chi(|\xi| < a)$ will denote any smooth function such that

$$\chi(|\xi| < a) = 1 \text{ for } \xi \in B[a], = 0 \text{ for } \xi \in \mathbb{R}^n \setminus B[a+1],$$

or the characteristic function for $B[a]$; again $a > 0$. In either case, χ induces multiplication operators in the function spaces on $\mathbb{R}^n$, Q, Q^∞ and $\hat{Q}^\infty$ as well, all of which will be denoted again by $\chi(|\xi| < a)$. The symbols $\chi(|\xi| > a)$, $\chi(|k| < a)$ and $\chi(|k| > a)$ are to be understood similarly, but the latter two will be used, in addition, to express the operators $\mathscr{F}_x^{-1} \chi(|k| \lessgtr a) \mathscr{F}_x$.

3. Collision Operators

The operator L_c defined in (1.11) has been investigated thoroughly in [7] for the case in which $c = 0$. Using the results given there, we will deduce its properties for $c \neq 0$.

Let τ_c, $c \in \mathbb{R}^n$, denote translation in ξ; $\tau_c u(\xi) = u(\xi + c)$. By a simple change of variables in (1.2), we see that the quadratic operator Q commutes with τ_c; $\tau_c Q[f, f] = Q[\tau_c f, \tau_c f]$. As a consequence, L_c can be written formally as

$$L_c = \theta_c^{-1} L_0 \theta_c, \tag{3.1}$$

where $L_0 = L_{c=0}$ and

$$\theta_c u(\xi) = \tau_c(e^{-\xi \cdot c/2} u)(\xi).$$

Suppose [q] of § 1 be satisfied. Then it is known [7], [13] that L_0 has the decomposition

$$L_0 u(\xi) = -\nu(\xi) u(\xi) + \int_{\mathbb{R}^n} K(\xi, \xi') u(\xi') \, d\xi',$$

where the functions $\nu(\xi)$ and $K(\xi, \xi')$ have the following properties:

(3.2) (i) $\nu(\xi)$ is real and continuous.

(ii) $\nu_0 \leqq \nu(\xi) \leqq \nu_1(1 + |\xi|)$ holds

with some positive constants ν_0 and ν_1.

(3.3) (i) For $\xi \neq \xi'$, $K(\xi, \xi')$ is real, continuous and symmetric.

(ii) With some constants $k_0 \geqq 0$ and $\delta \in [0, 1)$, there holds

$$|K(\xi, \xi')| \leqq$$

$$k_0 \left[(v + v^{-\delta}) \exp\left\{ -\frac{1}{4}(|\xi|^2 + |\xi'|^2) \right\} + (v^{-1} + v^{-(n-2)}) \exp\left\{ -\frac{1}{8}(v^2 + \zeta^2) \right\} \right]$$

where $v = |\xi - \xi'|$ and $\zeta = (|\xi|^2 - |\xi'|^2)/v$. This has been proven in [7] when $n = 3$ and in [13] when $n \geqq 4$.

In view of (3.1), L_c also has the form

$$L_c u(\xi) = -\nu_c(\xi) u(\xi) + \int_{\mathbb{R}^n} K_c(\xi, \xi') u(\xi') \, d\xi'$$

where $\nu_c(\xi) = \nu(\xi - c)$ and $K_c(\xi, \xi') = K(\xi - c, \xi' - c) \exp\{\frac{1}{2}(\xi - \xi') \cdot c\}$. From (3.2) it follows that

(3.4) $\nu_c(\xi)$ enjoys (3.2) for each $c \in \mathbb{R}^n$ with the same constant

$$\nu_0 = \inf_\xi \nu_c(\xi) = \inf_\xi \nu(\xi).$$

Furthermore ν_c is continuous in both c and ξ.

As for $K_c(\xi, \xi')$, we note the

Lemma 3.1. *Let* $n \geqq 3$, $1 \leqq p < n/(n-2)$, $\alpha \in \mathbb{R}$, $\beta \in \mathbb{R}$ *and* $c \in \mathbb{R}^n$. *There is a constant* $C \geqq 0$ *such that for all* $\xi \in \mathbb{R}^n$,

$$\int_{\mathbb{R}^n} |K_c(\xi, \xi')|^p e^{\alpha|\xi'|} (1 + |\xi'|)^{-\beta} d\xi' \leqq C e^{\alpha|\xi|} (1 + |\xi|)^{-\beta-1}. \tag{3.5}$$

Here the constant C is locally bounded as a function of c, and (3.5) *is valid also for* $K_c(\xi', \xi)$.

Proof. In (3.5), replace ξ by $\xi + c$ and ξ' by $\xi' + c$, and note that

$$(1 + |c|)^{-1} (1 + |\xi|)^{-1} \leqq (1 + |\xi \pm c|)^{-1} \leqq (1 + |c|) (1 + |\xi|)^{-1}.$$

Then it suffices to show

$$\int_{\mathbb{R}^n} |G(\xi, \xi')|^p (1 + |\xi'|)^{-\beta} d\xi' \leqq C(1 + |\xi|)^{-\beta-1}, \tag{3.6}$$

where $G = K(\xi, \xi') \exp\left\{|c|\,|\xi - \xi'| + \dfrac{\alpha}{p}(|\xi'| - |\xi|)\right\}$. For $n = 3$, $c = 0$ and $\alpha = 0$, this statement has been proven in [7]. The proof given there makes use only of the estimate (3.3) (ii) and remains valid when $n \geqq 3$ if $p \in (0, n/(n-2))$. Moreover, in that proof the factors $\frac{1}{4}$ and $\frac{1}{8}$ in (3.3) (ii) can be replaced by any positive numbers. Note that for any $\varepsilon > 0$,

$$|\xi'| - |\xi| \leqq |\xi - \xi'| \leqq \varepsilon\,|\xi - \xi'|^2 + \frac{1}{4\varepsilon} \leqq 2\varepsilon(|\xi|^2 + |\xi'|^2) + \frac{1}{4\varepsilon},$$

which shows that (3.3) (ii) is valid for G if the factors k_0, $\frac{1}{2}$ and $\frac{1}{8}$ are modified. Hence (3.6) and consequently (3.5) are true.

Define the integral operator K_c by

$$K_c u = \int_{\mathbb{R}^n} K_c(\xi, \xi')\, u(\xi')\, d\xi'.$$

Let $L^q_{\beta,\alpha}$ be the space defined by

$$L^q_{\beta,\alpha} \ni u(\xi) \Leftrightarrow e^{\alpha|\xi|}\, u(\xi) \in L^q_\beta(\mathbb{R}^n_\xi).$$

Lemma 3.2. *Let* $1 \leqq q \leqq r \leqq \infty$ *and* $\alpha, \beta \in \mathbb{R}$. *Put* $\gamma_0 = 1 - \dfrac{1}{q} + \dfrac{1}{r}$. *Then*

$$K_c \in \mathbb{B}(L^q_{\beta,\alpha}, L^r_{\beta+\gamma,\alpha}) \quad \textit{if} \quad \frac{1}{q} - \frac{1}{r} < \frac{2}{n} \quad \textit{and} \quad \gamma \leqq \gamma_0.$$

The operator norm is locally bounded in $c \in \mathbb{R}^n$.

Proof. The two special cases $q = 1$ and $r = \infty$ follow readily from (3.5) with $p = r$ and $p = q/(q-1)$ respectively, by the aid of the Hölder inequality. Then the interpolation [2] leads to the general case.

Lemma 3.3. *Under the same conditions assumed in Lemma 3.2,*

$$K_c \in \mathscr{E}^0(\mathbb{R}^n_c; \mathbb{B}(L^q_{\beta,\alpha}, L^r_{\beta+\gamma,\alpha})) \quad \textit{for any} \quad \gamma < \gamma_0.$$

Proof. Lemma 3.2 indicates that if $\delta > 0$, then

$$\|\chi(|\xi| > a)\, K_c\|,\ \|K_c \chi(|\xi| > a)\| \leqq C(1+a)^{-\delta} \tag{3.7}$$

holds in $\mathbb{B}(L^q_{\beta,\alpha}, L^r_{\beta+\gamma,\alpha})$ with $\gamma \leqq \gamma_0 - \delta$. Put $K'(\xi, \xi') = \chi(|\xi| < a) \times \chi(|\xi'| < a)\, \chi(|\xi - \xi'| < \varepsilon)\, K_c(\xi, \xi')$ and let K' be the integral operator induced thereby. It can be easily seen from (3.3) (ii) that

$$\|K'\| \to 0 \text{ as } \varepsilon \to 0$$

in the norm of (3.7), locally uniformly for c. In view of this fact and (3.7), therefore, it suffices to prove the lemma for the integral operator K'' induced by the kernel $K''(\xi, \xi') = \chi(|\xi| < a) \times \chi(|\xi'| < a)\, \chi(|\xi - \xi'| > \varepsilon)\, K_c(\xi, \xi')$, for each $a, \varepsilon > 0$. Such is the case since by (3.3) (i) $K''(\xi, \xi')$ is continuous in c as well as in ξ and ξ' and is of compact support in ξ and ξ'.

The last argument implies that $K'' \in \mathbb{C}(L^q_{\beta,\alpha}, L^q_{\beta+\gamma,\alpha})$ for each fixed a, $\varepsilon > 0$. Therefore the above proof, together with [10], Theorem III.4.7, proves the

Lemma 3.4. *Under the assumptions of Lemma 3.3,*

$$K_c \in \mathbb{C}(L^q_{\beta,\alpha}, L^q_{\beta+\gamma,\alpha}) \quad \textit{for each} \quad c \in \mathbb{R}^n.$$

Since K_c acts only on the variable ξ, it can be regarded as an operator also in function spaces on Q and Q^∞. Then Lemmas 3.2 and 3.3 lead to the

Proposition 3.5. *Let* $p, q, r \in [1, \infty]$, $q \leqq r$ *and* $\beta \in \mathbb{R}$. *Put* $\gamma_0 = 1 - \dfrac{1}{q} + \dfrac{1}{r}$ *and suppose*

$$\frac{1}{q} - \frac{1}{r} < \frac{2}{n}.$$

Then

(i) $K_c \in \mathbb{B}(L^{p,q}_\beta(Q), L^{p,r}_{\beta+\gamma}(Q))$ *for each* $c \in \mathbb{R}^n$ *if* $\gamma \leqq \gamma_0$.

(ii) $K_c \in \mathscr{E}^0(\mathbb{R}^n_c; \mathbb{B}(L^{p,q}_\beta(Q), L^{p,r}_{\beta+\gamma}(Q))$ *if* $\gamma < \gamma_0$.

Here the space L *can be replaced by* $\tilde{L}$ *and* Q *by* Q^∞.

Let us study the operator L_c. First we define the multiplication operator

$$\Lambda_c = \nu_c(\xi) \times .$$

In the sequel we write $L^q_\beta = L^q_\beta(\mathbb{R}^n_\xi)$. The maximal domain of Λ_c in L^q_β is given by

$$D(\Lambda_c) = \{u \in L^q_\beta \mid \nu_c(\xi)\, u(\xi) \in L^q_\beta\}.$$

By (3.4) it follows that $L^q_{\beta+1} \subset D(\Lambda_c)$ and that

$$\sigma(-\Lambda_c) = \sigma_e(-\Lambda_c) = \overline{\{-\nu_c(\xi) \mid \xi \in \mathbb{R}^n\}} \subset (-\infty, -\nu_0].$$

In particular $\sigma(-\Lambda_c)$ is invariant in c.

Since $K_c \in \mathbb{B}(L^q_\beta)$ by Lemma 3.2, L_c can be defined in L^q_β as

$$L_c = -\Lambda_c + K_c, \quad D(L_c) = D(\Lambda_c),$$

and by Lemma 3.4 and according to [10], Theorem IV.3.35,

$$\text{(3.8)}\qquad \begin{aligned} &\text{(i)}\quad \sigma_e(L_c) = \sigma_e(-A_c) \subset (-\infty, -\nu_0],\\ &\text{(ii)}\quad \sigma(L_c) \cap \varrho(-A_c) = \sigma_d(L_c), \end{aligned}$$

for each c. Thus $\sigma_e(L_c)$ is invariant in c. We prove also

Proposition 3.6. *$\sigma_d(L_c)$ is invariant in c as well as in q, β.*

Proof. Let $\lambda \in \sigma_d(L_c)$ in L^q_β and let $u \in D(L_c)$ be a corresponding eigenfunction. Put $h = h(\xi) = (\lambda + \nu_c(\xi))^{-1}$ and $H = hK_c$. By (3.8), $h \in L^\infty(\mathbb{R}^n_\xi)$. Since $\lambda u = L_c u$, then $u = Hu$. Put

$$H_1 = \chi(|\xi| > a)\, H, \qquad H_2 = \chi(|\xi| < a)\, H.$$

By (3.7) it follows that for each fixed c and if $a > 0$ is large enough,

$$\|H_1\| \leqq \frac{1}{2} \quad \text{in} \quad \mathbb{B}(L^q_{\beta,\alpha}),$$

and so the Neumann series converges and $(I - H_1)^{-1} \in \mathbb{B}(L^q_{\beta,\alpha})$ exists. Since $\chi(|\xi| < a) \in \mathbb{B}(L^q_\beta, L^q_{\beta,\alpha})$ for any $\alpha \in \mathbb{R}$, so is H_2 by Lemma 3.2 and because $h \in L^\infty(\mathbb{R}^n_\xi)$. Hence so is $H_3 \equiv (I - H_1)^{-1} H_2$. Rewriting the equation $u = Hu$ as $u = H_3 u$, we then see that $u \in L(L^q_{\beta,\alpha})$ for all $\alpha \in \mathbb{R}$. Let θ_c be that of (3.1). Apparently

$$\theta_c, \theta_c^{-1} \in \mathbb{B}(L^q_{\beta,\alpha}, L^q_{\beta+1}),$$

provided $\alpha > |c|$. Hence $v \equiv \theta_c u \in L^q_{\beta+1} \subset D(L_0)$, and in virtue of (3.1), $\lambda v = L_0 v$ holds. This means that $\lambda \in \sigma_d(L_0)$ because obviously $v \neq 0$. The converse can be proven similarly, and the invariance in c follows. Let u be as before. Then by iteration we get $u = H^l u$ for any $l \in \mathbb{N}^+$. By repeated use of Lemma 3.2 it follows that for any $\gamma \geqq 0$, there is an $l \in \mathbb{N}_+$ such that $H^l \in \mathbb{B}(L^q_\beta, L^\infty_{\beta+\gamma})$. Hence $u \in L^\infty_{\beta+\gamma}$ and in particular $u \in L^\infty_{\beta+n+2} \subset L^1_{\beta+1}$ or $u \in D(L_c)$ in L^1_β. This shows the invariance in q and β.

It is well known *cf.* [4], that in the space $L^2 = L^2_0 = L^2(\mathbb{R}^n_\xi)$, L_0 is nonpositive and selfadjoint with $0 \in \sigma_d(L_0)$, the eigenspace of which is spanned by the functions

$$\text{(3.9)}\qquad \phi_0 = g_0^{\frac{1}{2}}, \quad \phi_j = \xi_j g_0^{\frac{1}{2}} \quad (1 \leqq j \leqq n), \quad \phi_{n+1} = |\xi^2|\, g_0^{\frac{1}{2}}.$$

On account of (3.8) and Proposition 3.6, therefore, $\sigma(L_c) \subset (-\infty, 0]$, $\sigma(L_c) \cap (-\nu_0, 0] \subset \sigma_d(L_c)$ and

$$0 \in \sigma_d(L_c) \text{ with multiplicity } n + 2,$$

for any $c \in \mathbb{R}^n$. Denote by P_c the eigenprojection for $0 \in \sigma_d(L_c)$ ([10], p. 180). A significant feature of P_c is the

Proposition 3.7. *The null space of P_c is invariant in c.*

Proof. Let $\{\psi_j\}$ $(0 \leqq j \leqq n+1)$ be an orthonormal set in L^2 constructed by the Schmidt orthogonalization procedure applied to (3.9) and denote by $(\cdot, \cdot)$ the inner product in L^2. Then

$$P_0 = \sum_{j=0}^{n+1} \psi_j(\cdot, \psi_j)$$

in L^2. From (3.1) we see that $P_c = \theta_c^{-1} P_0 \theta_c$ formally, and hence

$$P_c = \sum_{j=0}^{n+1} \theta_c^{-1}\psi_j\,(\cdot, \theta_c^* \psi_j), \tag{3.10}$$

where $\theta_c^* = \exp(-\xi \cdot c/2)\,\tau_{-c}$ is an adjoint to θ_c. Denote by sp $\{\phi_j\}$ the linear span of the functions (3.9), and similarly for sp $\{\theta_c^*\phi_j\}$. It is not hard to see that sp $\{\theta_c^*\phi_j\}$ = sp $\{\phi_j\}$, and thereby sp $\{\theta_c^*\psi_j\}$ = sp $\{\psi_j\}$ and is invariant in c. Since $P_c u = 0$ if and only if u is orthogonal to sp $\{\theta_c^*\psi_j\}$ due to (3.10), this proves the proposition.

Because of the properties of the functions (3.9),

$$\theta_c^{-1}\psi_j,\ \theta_c^*\psi_j \in \mathscr{E}^0(\mathbb{R}_c^n; L_\beta^q), \qquad 0 \leqq j \leqq n+1,$$

for any $q \in [1, \infty]$ and $\beta \in \mathbb{R}$. Consequently (3.10) provides an exact expression of P_c in L_β^q and proves the

Lemma 3.8. *For any p, q, $r \in [1, \infty]$ and β, $\gamma \in \mathbb{R}$,*

(i) $P_c \in \mathscr{E}^0(\mathbb{R}_c^n; \mathbb{B}(L_\beta^q, L_\gamma^r))$,

(ii) $P_c \in \mathscr{E}^0(\mathbb{R}_c^n; \mathbb{B}(L_\beta^{p,q}(Q), L_\gamma^{p,r}(Q)))$.

In (ii) the space L can be replaced by $\tilde{L}$ and Q by Q^∞.

Finally we state some properties of the bilinear symmetric operator Γ_0 defined in (1.11).

Lemma 3.9. *For any $c \in \mathbb{R}^n$,*

(i) $P_c \Gamma_0[u, v] = 0$ *for any u, v.*

(ii) *For any p, r, $s \in [1, \infty]$ with $\frac{1}{p} = \frac{1}{r} + \frac{1}{s}$ and $\beta \geqq 0$,*

$$\|\Lambda_c^{-1}\Gamma_0[u, v]\|_{\tilde{L}_\beta^{p,\infty}} \leqq C\,\|u\|_{\tilde{L}_\beta^{r,\infty}}\,\|v\|_{\tilde{L}_\beta^{s,\infty}}$$

where $\tilde{L}_\beta^{p,\infty} = \tilde{L}_\beta^{p,\infty}(Q)$ etc. *The constant $C \geqq 0$ is independent of u, v and locally bounded in c.*

Proof. (i) has been shown in [4] and [8] for the case in which $c = 0$; hence because of Proposition 3.7 (i) holds also if $c \neq 0$. In [8], the estimate in (ii) has been derived with $\tilde{L}_\beta^{p,\infty}$ replaced by $L_\beta^\infty(\mathbb{R}_\xi^n)$, *etc.* when $c = 0$. Because of (3.4) the proof given there remains valid if $c \neq 0$. Repeat the proof, noting that Γ_0 acts only on the variable ξ and that $u\,v \in L^p(\Omega)$ whenever $u \in L^r(\Omega)$, $v \in L^s(\Omega)$ with $\frac{1}{p} = \frac{1}{r} + \frac{1}{s}$ (Hölder inequality). Then (ii) follows.

4. Trace Theorem

Throughout this section $h = h(\xi)$ is a function of ξ alone and such that $h \in L^\infty_{\text{loc}}(R^n_\xi)$ and $h_0 \equiv \inf_\xi \operatorname{Re} h(\xi) > 0$. The function spaces in which our trace theorem is to be established are

$$W_h^{p,\pm}(Q) = \{u \in L^p(Q) \mid (\xi \cdot \nabla_x \pm h(\xi))\, u \in L^p(Q)\} \text{ (same signs)},$$

where $Q = \Omega_x \times \mathbb{R}^n_\xi$ as in § 2, and $\xi \cdot \nabla_x$ is understood in the sense of distributions. Endowed with the norms

$$\|u\|_{W_h^{p,\pm}(Q)} = \|u\|_{L^p(Q)} + \|(\xi \cdot \nabla_x \pm h(\xi))\, u\|_{L^p(Q)},$$

these spaces are Banach spaces if $1 \leqq p \leqq \infty$. As usual, $C_0^\infty(Q^\infty)$ denotes the set of C^∞-functions on $Q^\infty = \mathbb{R}^n_x \times \mathbb{R}^n_\xi$ with compact support. We put $C_0^\infty(\overline{Q}) = \{u|_Q \mid u \in C_0^\infty(Q^\infty)\}$ and recall the definition of $Y^{p,\pm}$ of § 1.

Theorem 4.1. *Under the assumption* [$\mathcal{O}$] *of* § 1, *if* $1 \leqq p < \infty$, *there are unique trace operators* $\gamma^\pm$ *such that*

(i) $\gamma^\pm \in B(W_h^{p,\pm}(Q), Y^{p,\pm})$,

(ii) $\gamma^\pm u = u|_{S^\pm}$ *whenever* $u \in C_0^\infty(\overline{Q})$.

Proof. For each $\xi \in \mathbb{R}^n$, define

$$\text{(4.1)} \qquad \begin{aligned} \partial\Omega^\pm(\xi) &= \{X \in \partial\Omega \mid n(X) \cdot \xi \gtrless 0\}, \\ \Omega^\pm(\xi) &= \{x \in \Omega \mid x = X \mp t\xi,\ X \in \partial\Omega^\pm(\xi),\ t > 0\}. \end{aligned}$$

Because of [$\mathcal{O}$], $\Omega^\pm(\xi) \subset \Omega$ and the maps $\partial\Omega^\pm(\xi) \times (0, \infty) \ni (X, t) \to x = X \mp t\xi \in \Omega^\pm(\xi)$ are bijective with the Jacobian $\varrho(X, \xi) = |n(X) \cdot \xi|$, so

$$\text{(4.2)}_\pm \qquad \int_{\Omega^\pm(\xi)} w(x)\, dx = \int_{\partial\Omega^\pm(\xi)} \int_0^\infty w(X \mp t\xi)\, \varrho(X, \xi)\, dt\, d\sigma_X$$

holds for $w \in L^1$, where $d\sigma_X$ is the measure on $\partial\Omega$. Define

$$S^\pm = \bigcup_\xi \partial\Omega^\pm(\xi) \times \{\xi\}, \qquad Q^\pm = \bigcup_\xi \Omega^\pm(\xi) \times \{\xi\}.$$

The definition of $S^\pm$ given here and that in § 1 are identical. From (4.2) we deduce

$$\text{(4.3)}_\pm \qquad \int_{Q^\pm} w(x, \xi)\, dx\, d\xi = \int_{S^\pm} \int_0^\infty w(X \mp t\xi, \xi)\, \varrho(X, \xi)\, dt\, d\sigma_X\, d\xi.$$

Let $u \in C_0^\infty(\overline{Q})$ and put $v^\pm = (\xi \cdot \nabla_x \pm h(\xi))\, u$. Then

$$\frac{\partial}{\partial t}(e^{-h(\xi)t} u\,(X \mp t\xi, \xi)) = \mp\, e^{-h(\xi)t} v^\pm(X \mp t\xi, \xi).$$

Integration yields for each $(X, \xi) \in S^{\pm}$,

$$(4.4)_{\pm} \qquad u(X, \xi) = \mp \int_0^{\infty} e^{-h(\xi)t} v^{\pm}(X \mp t\xi, \xi)\, dt.$$

The Hölder inequality with $h_0 > 0$ shows that

$$|u(X, \xi)|^p \leqq (qh_0)^{-p/q} \int_0^{\infty} |v^{\pm}(X \mp t\xi, \xi)|^p\, dt,$$

where $\frac{1}{p} + \frac{1}{q} = 1$. Multiply both sides by $\varrho(X, \xi)$ and integrate over $S^{\pm}$. Writing $\gamma^{\pm} u = u|_{S^{\pm}}$ and using (4.3), we conclude for $u \in C_0^{\infty}(\overline{Q})$ that

$$(4.5)_{\pm} \quad \|\gamma^{\pm} u\|_{Y^{p,\pm}} \leqq (qh_0)^{-1/q} \|(\xi \cdot \nabla_x \pm h(\xi))\, u\|_{L^p(Q^{\pm})} \leqq (qh_0)^{-1/q} \|u\|_{W_h^{p,\pm}(Q)}.$$

Now we claim that $C_0^{\infty}(\overline{Q})$ is dense in $W_h^{p,\pm}(Q)$ for $p \in [1, \infty)$. This can be proved by a classical procedure using mollifiers and the homothetic transformations $x \to \alpha x$, $\alpha > 0$. The details are omitted. From (4.5) it then follows that $\gamma^{\pm}$ possess unique continuous extensions in $\mathbb{B}(W_h^{p,\pm}(Q), Y^{p,\pm})$ if $p \in [1, \infty)$. The extensions thus obtained and denoted again by $\gamma^{\pm}$ are the desired trace operators.

Remark 4.2. Roughly speaking, $(\gamma^{\pm} u)(X, \xi)$, $X \in \partial\Omega^{\pm}(\xi)$, are the limits of $u(X \mp t\xi, \xi)$ as $t \downarrow 0$.

Remark 4.3. Let $u \in W_h^{p,+}(Q)$. Our theorem ensures the existence only of $\gamma^+ u$, not of $\gamma^- u$. However $\gamma^- u$ exists in the following sense. Since $\chi(|\xi| < a)$ commutes with $\xi \cdot \nabla_x$ for each $a > 0$, and since $h \in L_{\text{loc}}^{\infty}(R_\xi^n)$, $\chi(|\xi| < a)\, u \in W_h^{p,-}(Q)$; by Theorem 4.1 it follows that $\gamma^-(\chi(|\xi| < a)\, u) \in Y^{p,-}$ exists. Since $\chi(|\xi| < a)$ commutes also with γ^-, and since a is arbitrary, this means that $\gamma^- u \in Y_{\text{loc}}^{p,-}$ exists for $u \in W_h^{p,+}(Q)$. Similarly, $u \in W_h^{p,-}(Q)$ has the trace $\gamma^+ u \in Y_{\text{loc}}^{p,+}$.

The above results remain valid if Q is replaced by Q^{∞}. To be precise, define the space $W_h^{p,\pm}(Q^{\infty})$ similarly. If $u \in W_h^{p,\pm}(Q^{\infty})$, then $u|_Q \in W_h^{p,\pm}(Q)$, and so Theorem 4.1 and Remark 4.3 apply to $u|_Q$. To simplify notation, write $\gamma^{\pm} u$ for $\gamma^{\pm}(u|_Q)$. A significant difference between Q and Q^{∞} is shown by the following strengthened version of Remark 4.3.

Proposition 4.4. *If* $u \in W_h^{p,+}(Q^{\infty})$, *then* $\gamma^- u \in Y^{p,-}$ *as well as* $\gamma^+ u \in Y^{p,+}$, *and similarly for* $u \in W_h^{p,-}(Q^{\infty})$.

Proof. Put $\tilde{\Omega}^{\pm}(\xi) = \Omega^{\pm}(\xi) \cup \mathcal{O}$ (see (4.1)). As in (4.2),

$$(4.6)_{\pm} \qquad \int_{\tilde{\Omega}^{\pm}(\xi)} w(x)\, dx = \int_{\partial\Omega^{\mp}(\xi)} \int_0^{\infty} w(X \mp t\xi)\, \varrho(X, \xi)\, dt\, d\sigma_X.$$

Put $\tilde{Q}^{\pm} = Q^{\pm} \cup (\mathcal{O} \times \mathbb{R}^n)$. Let $u \in C_0^\infty(Q^\infty)$ and put $v^{\pm} = (\xi \cdot \nabla_x \pm h)u$. Proceed as in (4.4) to deduce for $(X, \xi) \in S^{\pm}$,

$$u(X, \xi) = \pm \int_0^\infty e^{-h(\xi)t} v^{\mp}(X \pm t\xi, \xi)\, dt, \tag{4.7$_\pm$}$$

which is never possible if $u \in C_0^\infty(\bar{Q})$. Then instead of (4.5),

$$\|\gamma^{\pm} u\|_{Y^{p,\pm}} \leqq (q h_0)^{-1/q} \|v^{\mp}\|_{L^p(\tilde{Q}^{\mp})}. \tag{4.8$_\pm$}$$

The rest of the proof is similar to that of Theorem 4.1.

These traces $\gamma^{\pm} u$ are the limits of $u(X \pm t\xi, \xi)$, $(X, \xi) \in S^{\pm}$ as $t \downarrow 0$ (compare Remark 4.2). Another difference is the

Proposition 4.5. *Assume* $\operatorname{Im} h(\xi) \in L^\infty(\mathbb{R}^n)$. *Then*

$$W_h^{2,+}(Q^\infty) = W_h^{2,-}(Q^\infty) = \{u \in L^2(Q^\infty) \mid \xi \times \nabla_x u,\ h(\xi)\, u \in L^2(Q^\infty)\}.$$

Proof. Since $\mathcal{F}_x(\xi \cdot \nabla_x u)\,(k, \xi) = ik \cdot \xi \hat{u}(k, \xi)$, then the Parseval relation shows that

$$\|(\xi \cdot \nabla_x \pm h(\xi))\, u\|_{L^2(Q^\infty)} = \|(ik \cdot \xi + h(\xi))\, \hat{u}\|_{L^2(\hat{Q}^\infty)}$$

holds for $u \in W_h^{2,\pm}(Q^\infty)$. Since $|ik \cdot \xi \pm h(\xi)|^2 = |\operatorname{Re} h(\xi)|^2 + |k \cdot \xi + \operatorname{Im} h(\xi)|^2$, Parseval's relation indicates that $\operatorname{Re} h(\xi)\, u$ and $(\xi \cdot \nabla_x \pm \operatorname{Im} h(\xi))\, u$ are in $L^2(Q^\infty)$, whence the proposition follows.

For later purposes, we state Green's formula:

Lemma 4.6. *Let* $u \in W_h^{2,+}(Q)$ *and suppose* $\gamma^- u \in Y^{2,-}$ *exists. Then*

$$h_0 \|u\|^2 \leqq \operatorname{Re}((\xi \cdot \nabla_x + h(\xi))\, u, u) + \frac{1}{2}\{\|\gamma^- u\|_-^2 - \|\gamma^+ u\|_+^2\}, \tag{4.9}$$

where $\|\ \|$ *and* $(\ ,\ )$ *denote the norm and the inner product of* $L^2(Q)$ *while* $\|\ \|_\pm$ *the norms of* $Y^{2,\pm}$.

Proof. Let $u \in C_0^\infty(\bar{Q})$. Then by the divergence theorem or by integration by parts, it readily follows that

$$2 \operatorname{Re}(\xi \cdot \nabla_x u, u) = \|\gamma^+ u\|_+^2 - \|\gamma^- u\|_-^2.$$

Since $h(\xi)\, u \in L^2(Q)$ if $u \in C_0^\infty(\bar{Q})$, this proves (4.9) for such u. Next, let $u \in W_h^{2,+}(Q)$ and put $u_a = \chi(|\xi| < a)\, u$. For any fixed $a > 0$, $u_a \in W_h^{2,\pm}(Q)$ with $\gamma^{\pm} u \in Y^{2,\pm}$ by Remark 4.3, and since $C_0^\infty(\bar{Q})$ is dense in $W_h^{2,\pm}(Q)$, there is a sequence $\{u^n\} \subset C_0^\infty(\bar{Q})$ such that

$$u^n \to u_a \text{ in } W_h^{2,\pm}(Q),\ h(\xi)\, u^n \to h(\xi)\, u_a \text{ in } L^2(Q),$$

$$\gamma^{\pm} u^n \to \gamma^{\pm} u_a \text{ in } Y^{2,\pm},$$

strongly as $n \to \infty$. Therefore (4.9) is valid with u_a substituted for u. Finally since $u_a \to u$ strongly in $W_h^{2,+}(Q)$ as $a \to \infty$, then $\gamma^+ u_a \to \gamma^+ u$ in $Y^{2,+}$ due to Theorem 4.1(i), and if in addition $\gamma^- u \in Y^{2,-}$ exists, then $\gamma^- u_a \to \gamma^- u$ in $Y^{2,-}$ strongly because $\gamma^- u_a = \gamma^- u$ for $\xi \in B[a]$ owing to Remark 4.3. The proof of the lemma is complete.

All the results of this section will be applied in the sequel with $h(\xi) = \lambda + \nu_c(\xi)$, $\lambda \in \mathbb{C}$. We should assume $\lambda \in \mathbb{C}_+(-\nu_0)$ in order that $h_0 > 0$, due to (3.4). Then $W^{p,\pm}_{\lambda+\nu_c}(Q) = W^{p,\pm}_{\nu_c}(Q)$ with equivalent norms. Of course $\lambda = 0$ when (1.12) is concerned. Let $u \in W^{p,+}_{\nu_c}(Q)$. Then $(-\xi \cdot \nabla_x + L_c)\, u \in L^p(Q)$ in view of Proposition 3.5, and $\gamma^+ u \in Y^{p,+}$ and $\gamma^- u \in Y^{p,-}_{\mathrm{loc}}$ exist. By an interpolation [2] between $[\mathrm{M}]_2$(i) and (iii) for $\beta = 0$ (see also Remark 1.1), $M_0 \in \mathrm{B}(Y^{p,-}, Y^{p,+})$ for $p \geqq 2$. Then $\gamma^- u \in Y^{p,-}$ necessarily exists if u satisfies the boundary condition $\tilde{M}_0 u = 0$ or $\gamma^- u = M_0 \gamma^+ u$. Furthermore, if $p < \infty$, then $u \in L^p(Q)$ can be said to satisfy the condition $u \to 0(|x| \to \infty)$ in a generalized sense. Thus we arrive at the

Definition 4.7. *Let* $p \in [2, \infty)$. *A function* $u = u(x, \xi)$ *is said to satisfy* (1.12) *in the* L^p*-sense or to be an* L^p*-solution to* (1.12) *if*

(i) $u \in W^{p,+}_{\nu_c}(Q)$,

(ii) $\Gamma_0[u, u] \in L^p(Q)$ *and* $(-\xi \cdot \nabla_x + L_c)\, u = -\Gamma_0[u, u]$ *in* $L^p(Q)$,

(iii) $\gamma^- u \in Y^{p,-}$ *with* $\tilde{M}_0 u = 0$ *in* $Y^{p,-}$.

L^p-solutions ϕ_c to (1.14) can be defined similarly.

5. The Operator A_c^∞

In this section we study the operator B_c of (1.13) for the simplest case: $\Omega = \mathbb{R}^n$ and $K_c = 0$. To avoid confusion, we write this operator as A_c^∞;

$$A_c^\infty = -\xi \cdot \nabla_x - \nu_c(\xi) \times, \quad (x, \xi) \in Q^\infty.$$

First let us consider this operator in the space $L^2(Q^\infty)$. Then its maximal domain $D(A_c^\infty)$ is $W^{2,+}_{\nu_c}(Q^\infty)$, but by virtue of (3.4) and Proposition 4.5 that domain can be specified:

$$D(A_c^\infty) = W^{2,+}_{\nu_c}(Q^\infty) = W^{2,-}_{\nu_c}(Q^\infty) \tag{5.1}$$
$$= \{u \in L^2(Q^\infty) \mid \xi \cdot \nabla_x u, \nu_c(\xi)\, u \in L^2(Q^\infty)\},$$

and for $u \in D(A_c^\infty)$,

$$\mathscr{F}_x A_c^\infty u(k, \xi) = -(ik \cdot \xi + \nu_c(\xi))\, \hat{u}(k, \xi). \tag{5.2}$$

From this relation we readily see that

(5.3)
(i) A_c^∞ is maximally dissipative in $L^2(Q^\infty)$,
(ii) $\sigma(A_c^\infty) = \sigma_e(A_c^\infty) = \overline{\{-ik \cdot \xi - \nu_c(\xi) \,|\, k, \xi \in \mathbb{R}^n\}} \subset \bar{C}_-(-\nu_0)$,
$\varrho(A_c^\infty) \supset C_+(-\nu_0)$,

where $\nu_0 > 0$ is the constant in (3.4). (5.3(i) implies that A_c^∞ generates a semigroup $F_c^\infty(t) = \exp tA_c^\infty$ in $L^2(Q^\infty)$, and by (5.2),

$$(5.4) \qquad \begin{aligned} &\text{(i)} \quad \mathscr{F}_x F_c^\infty(t)\, u = \exp\{-(ik\cdot\xi + \nu_c(\xi))\}\, \hat{u}(k,\xi), \\ &\text{(ii)} \quad \mathscr{F}_x(\lambda - A_c^\infty)^{-1} u = (\lambda + ik\cdot\xi + \nu_c(\xi))^{-1}\hat{u}(k,\xi). \end{aligned}$$

Taking the inverse Fourier transform, we obtain

$$(5.5) \qquad \begin{aligned} &\text{(i)} \quad F_c^\infty(t)\, u = \exp\{-\nu_c(\xi)\, t\}\, u(x - t\xi, \xi), \\ &\text{(ii)} \quad (\lambda - A_c^\infty)^{-1} u = \int_0^\infty \exp\{-(\lambda + \nu_c(\xi))\}\, u(x - t\xi, \xi)\, dt. \end{aligned}$$

The main aim of this section is to study (5.5) in various spaces other than $L^2(Q^\infty)$. In other words, we shall regard (5.5) as formal definitions of $F_c^\infty(t)$ and $(\lambda - A_c^\infty)^{-1}$ and investigate their realizations. To simplify the notation, we write

$$L_\beta^{p,q} = L_\beta^{p,q}(Q^\infty), \qquad \tilde{L}_\beta^{p,q} = \tilde{L}_\beta^{p,q}(Q^\infty).$$

All the results in this section are valid for any $p, q \in [1,\infty]$, $\beta \in \mathbb{R}$ and $c \in \mathbb{R}^n$ unless otherwise stated.

Lemma 5.1. *For any* $t > 0$,

(i) $\|F_c^\infty(t)\| \leqq \exp(-\nu_0 t)$ *in* $\mathbb{B}(\tilde{L}_\beta^{p,q})$,

(ii) $\|F_c^\infty(t)\| \leqq t^{-n\alpha}\exp(-\nu_0 t)$ *in* $\mathbb{B}(L_\beta^{q,p}, L_\beta^{p,q})$ *if* $q \leqq p$, *with* $\alpha = \dfrac{1}{q} - \dfrac{1}{p}$.

Proof. Denote the right-hand member of (5.5)(i) by $v = v(t, x, \xi)$. Then by the change of variable $x \to y = x - t\xi$,

$$(5.6) \qquad \|v(t,\cdot,\xi)\|_{L^p(\mathbb{R}_x^n)} \leqq e^{-\nu_c(\xi)t}\, \|u(\cdot,\xi)\|_{L^p(\mathbb{R}_x^n)}$$

for each $\xi \in \mathbb{R}^n$, which on account of (3.4) proves (i). Another change of variable $\xi \to y = x - t\xi$ leads to

$$\|v(t,x,\cdot)\|_{L^q(\mathbb{R}_\xi^n)} \leqq t^{-n/q} e^{-\nu_0 t}\, \|u(y,(y-x)/t)\|_{L^q(\mathbb{R}_y^n)}.$$

Write the last norm as $w(t,x)$. Put $r = p/q$ and assume $r \geqq 1$. Then

$$\|w(t,\cdot)^q\|_{L^r(\mathbb{R}_x^n)} \leqq \int_{\mathbb{R}^n} \|\,|u(y,(y-x)/t)|^q\|_{L^r(\mathbb{R}_x^n)}\, dy.$$

Change the variable: $x \to \xi = (y - x)/t$. Then the last integral is found to be equal to $t^{n/r}\|u\|_{L^{q,p}}^q$. Combining these conclusions proves (ii).

Lemma 5.2. *Let* $\lambda \in \mathbb{C}_+(-\nu_0)$.

(i) $\|(\lambda - A_c^\infty)^{-1}\| \leqq (\operatorname{Re}\lambda + \nu_0)^{-1}$ *in* $\mathbb{B}(\tilde{L}_\beta^{p,q})$,

(ii) $\|(\lambda - A_c^\infty)^{-1}\| \leqq \Gamma(\gamma)(\operatorname{Re}\lambda + \nu_0)^{-\gamma}$ *in* $\mathbb{B}(L_\beta^{q,p}, L_\beta^{p,q})$,

if $0 \leqq \frac{1}{q} - \frac{1}{p} < \frac{1}{n}$, *where* $\gamma = 1 - n\left(\frac{1}{q} - \frac{1}{p}\right)$ *and* Γ *is the gamma function.*
Let $\mathbb{B}$ *stand for either of the spaces mentioned in* (i) *and* (ii).
(iii) $(\lambda - A_c^\infty)^{-1}$ *is analytic in* $\lambda \in \mathbb{C}_+(-\nu_0)$ *in* $\mathbb{B}$ *for each fixed c.*
Let δ, $c_0 > 0$ *and define the set* Σ_0 *of points* (λ, c) *by*

$$\Sigma_0 = \overline{\mathbb{C}}_+(-\nu_0 + \delta) \times B[c_0].$$

(iv) *For each* a, δ, $c_0 > 0$,

$$\chi(|\xi| < a)\,(\lambda - A_c^\infty)^{-1} = (\lambda - A_c^\infty)^{-1}\,\chi(|\xi| < a) \in \mathscr{B}^0(\Sigma_0; \mathbb{B}).$$

Proof. From (5.5)(ii), we have

$$\|(\lambda - A_c)^{-1}\| \leqq \int_0^\infty e^{-\mathrm{Re}\lambda t}\,\|F_c^\infty(t)\|\,dt. \tag{5.7}$$

Evaluate the last integral using Lemma 5.1 to get (i), (ii). The restriction on p, q in (ii) serves to imply that $\gamma > 0$. According to a theorem of Lebesgue, Lemma 5.1 permits us to differentiate the right-hand side of (5.5)(ii) with respect to λ under the integral sign and so to prove (iii). In view of (3.4), the function $\Sigma_0 \ni (\lambda, c) \to \chi(|\xi| < a) \exp\{-(\lambda + \nu_c(\xi))\,t\} \in L^\infty(\mathbb{R}_+ \times \mathbb{R}_\xi^n)$ is continuous for each fixed $a > 0$, so the Lesbesgue dominated convergence theorem can be applied to (5.5)(ii) in virtue of Lemma 5.1. Hence (iv) follows.

Define the multiplication operator $A_c^\alpha = \nu_c(\xi)^\alpha \times$. By (3.4), $A_c^\alpha \in \mathbb{B}(\tilde{L}_{\beta-\alpha}^{p,q}, \tilde{L}_\beta^{p,q})$ for $\alpha \in \mathbb{R}$. Therefore the following is a strengthened version of Lemma 5.2(i), which is essentially due to GRAD [8].

Lemma 5.3. *Write* $\mathbb{B} = \mathbb{B}(\tilde{L}_\beta^{p,q})$ *and let* $\lambda \in \mathbb{C}_+(-\nu_0)$, $\alpha \leqq 1$.
(i) $\|(\lambda - A_c^\infty)^{-1} A_c^\alpha\|_{\mathbb{B}} \leqq \eta_0\,(\mathrm{Re}\,\lambda + \nu_0)^{\alpha - 1}$ *where* $\eta_0 = \max(1, \nu_0^\alpha(\mathrm{Re}\,\lambda + \nu_0)^{-\alpha})$.
(ii) *Let* Σ_0 *be as in Lemma 5.2. For any* a, δ, $c_0 > 0$,

$$\chi(|\xi| < a)\,(\lambda - A_c^\infty)^{-1} A_c^\alpha \in \mathscr{B}^0(\Sigma_0; \mathbb{B}).$$

Proof. Let $\|\ \|$ denote the norm of $L^p(\mathbb{R}_x^n)$. By (5.6), (5.7),

$$\int_0^\infty e^{-\mathrm{Re}\lambda t}\,\|A_c^\alpha v(t, \cdot, \xi)\|\,dt \leqq \int_0^\infty e^{-(\mathrm{Re}\lambda + \nu_c(\xi))t}\nu_c(\xi)^\alpha\,dt\,\|u(\cdot, \xi)\|.$$

The last integral is equal to $\nu_c(\xi)^\alpha/(\mathrm{Re}\,\lambda + \nu_c(\xi))$, which is majorized by η_0 if $\alpha \leqq 1$. Hence (i) readily follows, and (ii) can be proven in the same way as Lemma 5.2(iv).

Lemma 5.2(ii), in which the space L cannot be replaced by the space $\tilde{L}$, is essential in the proof of

Lemma 5.4. *Let* p, r, $s \in [1, \infty]$ *with* $p \leqq r$, *and let* $\beta \in \mathbb{R}$. *Then for any positive* δ *and* c_0 *there is an* $l \in \mathbb{N}^+$ *such that*

$$((\lambda - A_c^\infty)^{-1} K_c)^l \in \mathscr{B}^0(\Sigma_0; \mathbb{B}(L^p, \tilde{L}_\beta^{r,s})),$$

where $L^p = L^p(Q^\infty)$.

Proof. Let $q \geqq p$ be such that $\frac{1}{p} - \frac{1}{q} < \frac{1}{n}$ and write $L_0^{p,q} = L^{p,q}$. Then by Proposition 3.5,

$$K_c \in \mathscr{B}^0(\Sigma_0; \mathbb{B}(L^p, L^{p,q}) \cap \mathbb{B}(L^{q,p}, L^q)),$$

while by Lemma 5.2(ii) and for each $a > 0$,

$$(\lambda - A_c^\infty)^{-1} \chi(|\xi| < a) \in \mathscr{B}^0(\Sigma_0; \mathbb{B}(L^{p,q}, L^{q,p}) \cap \mathbb{B}(L^q)).$$

Put $H_1 = (\lambda - A_c^\infty)^{-1} \chi(|\xi| < a) K_c$. Then $H_1^2 \in \mathscr{B}^0(\Sigma_0; \mathbb{B}(L^p, L^q))$, and by iteration

$$H_1^{l'} \in \mathscr{B}^0(\Sigma_0; \mathbb{B}(L^p, L^r))$$

for any $r \geqq p$ with some $l' \in \mathbb{N}_+$. Similarly, by Proposition 3.5 for the space $\tilde{L}$ and Lemma 5.2(i) (iv),

$$H_1^{l''} \in \mathscr{B}^0(\Sigma_0; \mathbb{B}(\tilde{L}^{r,r}, \tilde{L}^{r,\infty}_{\beta+n+1}))$$

with some $l'' \in \mathbb{N}_+$ depending on β, but not on $r \geqq 1$. Since $\tilde{L}^{r,r} = L^r$ and since $\tilde{L}^{r,\infty}_{\beta+\gamma} \subset L^{r,s}_\beta$ is a continuous embedding if $\gamma > n/s$, then H_1^l, $l = l' + l''$, satisfies the lemma for each fixed $a > 0$. Now put $H = (\lambda - A_c^\infty)^{-1} K_c$. It is easy to see that the above proof applies also to H if the space $\mathscr{B}^0$ is replaced by L^∞. Thus

$$H^l \in L^\infty(\Sigma_0; \mathbb{B}(L^p, \tilde{L}^{r,s}_\beta)),$$

and hence so is $H^l - H_1^l$. Furthermore the same proof, combined with (3.7), shows that $H^l - H_1^l \to 0$ as $a \to \infty$ in the norm of this space uniformly for $(\lambda, c) \in \Sigma_0$. Then the lemma follows.

Let $\gamma^\pm$ be the trace operators of § 4. Owing to (5.1) and Proposition 4.4, $\gamma^\pm(\lambda - A_c^\infty)^{-1} \in \mathbb{B}(L^2, Y^{2,\pm})$ exist if $\lambda \in \varrho(A_c^\infty)$. We need their realizations in other spaces. To this end we shall introduce the space $\tilde{Y}^{p,q,\pm}_\beta$ of functions $u(X, \xi)$ on $S^\pm$ satisfying

$$\int_{\mathbb{R}^n} \left((1 + |\xi|)^\beta \left(\int_{\partial\Omega^\pm(\xi)} |u(X, \xi)|^p \varrho(X, \xi)\, d\sigma_X\right)^{1/p}\right)^q d\xi < \infty,$$

where $\partial\Omega^\pm(\xi)$ was defined in (4.1). Compare with the space $Y^{p,q,\pm}_\beta$ of § 1. Let η_0 be the constant of Lemma 5.3(i).

Lemma 5.5. *Let* $\lambda \in \mathbb{C}_+(-\nu_0)$ *and* $\alpha \leqq 1$.

(i) $\|\gamma^\pm(\lambda - A_c^\infty)^{-1} A_c^\alpha u\|_{\tilde{Y}^{p,q,\pm}_\beta} \leqq \eta_0^{(p-1)/p} \|A_c^{\alpha/p} u\|_{\tilde{L}^{p,q}_\beta}$.

(ii) *For any* a, δ, $c_0 > 0$,

$$\chi(|\xi| < a)\, \gamma^\pm(\lambda - A_c^\infty)^{-1} A_c^\alpha \in \mathscr{B}^0(\Sigma_0; \mathbb{B}(\tilde{L}^{p,q}_\beta, \tilde{Y}^{p,q,\pm}_\beta)).$$

Proof. In $(4.4)_+$ and $(4.7)_-$, put $h(\xi) = \lambda + \nu_c(\xi)$ and replace u by $w = (\lambda - A_c^\infty)^{-1} A_c^\alpha u = A_c^\alpha(\lambda - A_c^\infty)^{-1} u$. Then by the Hölder inequality and by $(4.2)_+$, $(4.6)_+$,

$$\int_{\partial\Omega^+(\xi)} |w(X, \xi)|^p \varrho(X, \xi)\, d\sigma_X \leqq \eta_0^{p/p'} \|A_c^{\alpha/p} u(\cdot, \xi)\|_{L^p(\Omega^+(\xi))}$$

for each $\xi \in \mathbb{R}^n$, and similarly with $\partial\Omega^+$, Ω^+ replaced by $\partial\Omega^-$, $\tilde{\Omega}^+$. Here $(\xi \cdot \nabla_x + h(\xi))\, w = \Lambda_c^\alpha u$ was used. Then (i) readily follows, and (ii) can be proven like Lemma 5.2(iv).

In the above, in contrast with Lemma 5.3(i) $\Lambda_c^{\alpha/p} u$ cannot be replaced by u.

Lemma 5.6. *$\gamma^\pm(\lambda - A_c^\infty)^{+1}$ is analytic in $\lambda \in \mathbb{C}_+(-\nu_0)$ in $\mathbb{B}(\tilde{L}_\beta^{p,q}, \tilde{Y}_\beta^{p,q,\pm})$.*

Proof. Put $R' = (\lambda - A_c^\infty)^{-1}$. As a property of the resolvent,

$$\partial R'/\partial\lambda = -R'^2$$

holds for $\lambda \in \varrho(A_c^\infty)$ in $L^2(Q^\infty)$. To conclude the lemma, apply Lemmas 5.2(i) and 5.5(i) with $\alpha = 0$ to $\gamma^\pm R'^2 = \{\gamma^\pm R'\}\, R'$.

6. The Operator $\mathbf{B}_c^\infty$

We continue the study of B_c in Q^∞, now including $K_c \neq 0$. We define the operator B_c^∞ as follows:

$$B_c^\infty = -\xi \cdot \nabla_x + L_c = -\xi \cdot \nabla_x - \Lambda_c + K_c. \quad (x, \xi) \in Q^\infty.$$

Since $K_c \in \mathbb{B}(L^2(Q^\infty))$ (Proposition 3.5), it can be defined in $L^2(Q^\infty)$ by

(6.1) $$B_c^\infty = A_c^\infty + K_c, \quad D(B_c^\infty) = D(A_c^\infty),$$

with $D(A_c^\infty)$ given by (5.1), and it generates a semigroup, denoted in Part II as $E_c^\infty(t)$, in $L^2(Q_c^\infty)$, due to (5.3)(i) and [10] Theorem IX.2.1. Here we discuss the resolvent of B_c^∞.

Write $L^2 = L^2(\mathbb{R}_\xi^n)$ and introduce a family of operators $\hat{A}_c(k)$, $k \in \mathbb{R}^n$ in L^2 by

(6.2) $$D(\hat{A}_c(k)) = \{u \in L^2 \mid k \cdot \xi u, \nu_c(\xi)\, u \in L^2\},$$
$$\hat{A}_c(k) = -(ik \cdot \xi + \nu_c(\xi)) \times .$$

This was suggested by (5.1), (5.2). We conclude statements similar to (5.3):

(6.3)
(i) $\hat{A}_c(k)$ is maximally dissipative in L^2.
(ii) $\sigma(\hat{A}_c(k)) = \sigma_e(\hat{A}_c(k)) = \overline{\{-(ik \cdot \xi + \nu_c(\xi) \mid \xi \in \mathbb{R}^n\}} \subset \overline{\mathbb{C}}^-(-\nu_0)$,
$$\varrho(\hat{A}_c(k)) \supset \mathbb{C}_+(-\nu_0).$$

Next, noting that $K_c \in \mathbb{B}(L^2)$, we define

(6.4) $$\hat{B}_c(k) = \hat{A}_c(k) + K_c, \quad D(\hat{B}_c(k)) = D(\hat{A}_c(k))$$

(again *cf.* (6.1)). It generates a semigroup in L^2. As for the spectrum, we observe that for all $c, k \in \mathbb{R}^n$,

(6.5) $$\sigma_e(\hat{B}_c(k)) = \sigma_e(\hat{A}_c(k)) \subset \overline{\mathbb{C}}_-(-\nu_0),$$
$$\sigma(\hat{B}_c(k)) \cap \varrho(\hat{A}_c(k)) \subset \sigma_d(\hat{B}_c(k)),$$

since $K_c \in \mathbb{C}(L^2)$ (Lemma 3.3) and by [10], Theorem IV.5.35. Further, following the argument in the proof of Lemma 3.6, this time with $h(\xi) = (\lambda + ik \cdot \xi + \nu_c(\xi))^{-1}$, and noting that $\theta_c^{-1}(k \cdot \xi)\, \theta_c = k \cdot \xi - k \cdot c$ where θ_c is defined in (3.1), we can claim that

(6.6) if $\lambda \in \mathbb{C}_+(-\nu_0)$, then $\lambda \in \sigma_d(\hat{B}_c(k))$ if and only if $\lambda - ik \cdot c \in \sigma_d(\hat{B}_0(k))$,

for all $k, c \in \mathbb{R}^n$.

Now we state two theorems, keys to later developments, which concern a part of $\sigma_d(\hat{B}_c(k))$ near the imaginary axis. They have been proven for the special case $c = 0$ in [5] (see also [1], [14]) and hence can be concluded readily for $c \neq 0$ by the aid of (6.5) and (6.6).

Theorem 6.1. *For any $\varkappa > 0$, there is a positive number $\sigma_1 = \sigma_1(\varkappa)$ such that $\sigma_1 < \nu_0$, and*

$$\varrho(\hat{B}_c(k)) \subset \mathbb{C}_+(-\sigma_1(\varkappa))$$

for all $k \in \mathbb{R}^n$, $|k| > \varkappa$, and for all $c \in \mathbb{R}^n$.

Here $\sigma_1(\varkappa) \to 0$ as $\varkappa \to 0$, as seen from

Theorem 6.2. *There are positive numbers $\varkappa_1$, σ_2 ($\sigma_2 < \nu_0$), a positive integer m ($m \leqq n + 2$) and scalar functions $\lambda_j(\varkappa)$, $j = 1, 2, \ldots, m$ defined on the interval $[-\varkappa_1, \varkappa_1]$ such that the following hold for any $k \in B[\varkappa_1]$ and $c \in \mathbb{R}^n$.*

(i) *Put $\mu_j(k, c) = \lambda_j(|k|) + ik \cdot c$. Then*

$$\sigma_d(\hat{B}_c(k)) \cap \overline{\mathbb{C}}_+(-\sigma_2) = \{\mu_j(k, c)\}_{j=1}^m .$$

(ii) *$\lambda_j(\varkappa)$ is a C^∞-function of $\varkappa \in [-\varkappa_1, \varkappa_1]$ with*

$$\lambda_j(\varkappa) = i\alpha_j\varkappa - \beta_j\varkappa^2 + O(|\varkappa|^3) \quad (|\varkappa| \to 0)$$

where $\alpha_j \in \mathbb{R}$ and $\beta_j > 0$ are constants.

(iii) *Denote by $P_j(k, c)$ the eigenprojection corresponding to the eigenvalue $\mu_j(k, c) \in \sigma_d(\hat{B}_c(k))$ (see* [10], p. 180). *Let P_c be the projection in Lemma* 3.8. *Then*

$$\sum_{j=0}^{m} P_j(0, c) = P_c .$$

Moreover,

$$P_j(k, c) = P_j(0, c) + |k|\, P_j'(k, c)$$

with some $P_j'(k, c) \in \mathbb{B}(L^2)$.

(iv) *For $\lambda \in \overline{\mathbb{C}}_+(-\sigma_2)$, the Laurent expansion*

$$(\lambda - \hat{B}_c(k))^{-1} = \sum_{j=0}^{m} (\lambda - \mu_j(k, c))^{-1} P_j(k, c) + S(\lambda, k, c)$$

holds, where

$$S(\lambda, k, c) = (\lambda - \hat{B}_c(k))^{-1}\left(I - \sum_{j=0}^{m} P_j(k, c)\right) \in \mathbb{B}(L^2)$$

is analytic in λ, *and* $SP_j = P_jS = 0, \; 1 \leqq j \leqq m.$

For our purpose it is further necessary that the operators appearing above be continuous in the parameters λ, k and c. Define for each $\varkappa$, $c_0 > 0$,

$$\Sigma_1 = \overline{\mathbb{C}}_+(-\sigma_1(\varkappa)) \times (\mathbb{R}^n \setminus B[\varkappa]) \times B[c_0],$$
$$\Sigma_2 = B[\varkappa_1] \times B[c_0],$$
$$\Sigma_3 = \overline{\mathbb{C}}_+(-\sigma_2) \times \Sigma_2.$$

In the sequel, Σ_1 and Σ_3 will be considered as sets of points (λ, k, c) and Σ_2 as a set of points (k, c).

Proposition 6.3. *For any* $\varkappa$, c_0, $a > 0$, *the following hold.*

(i) $(\lambda - \hat{B}_c(k))^{-1} \in L^\infty(\Sigma_1; \mathbb{B}(L^2))$; $\quad (\lambda - \hat{B}_c(k))^{-1}\chi(|\xi| < a) \in \mathscr{B}^0(\Sigma_1; \mathbb{B}(L^2))$.
(ii) $P_j(k, c), P_j'(k, c)\,(I - P_c) \in \mathscr{B}^0(\Sigma_2; \mathbb{B}(L^2, L^r_\beta)), \; 1 \leqq j \leqq m,$
for all $r \in [2, \infty]$ *and* $\beta \in \mathbb{R}$, *where* $L^r_\beta = L^r_\beta(\mathbb{R}^n_\xi)$.
(iii) $S(\lambda, k, c) \in L^\infty(\Sigma_3; \mathbb{B}(L^2))$; $S(\lambda, k, c)\,\chi(|\xi| < a) \in \mathscr{B}^0(\Sigma_3; \mathbb{B}(L^2))$.

We relegate the proof to the end of this section, and we return to the study of the operator B_c^∞ of (6.1). Choose $\varkappa_0 > 0$ $(\varkappa_0 \leqq \varkappa_1)$ so small that there are positive numbers β and δ such that

$$\operatorname{Re}\lambda_j(\varkappa) \leqq -\beta\,|\varkappa|^2, \quad |\operatorname{Im}\lambda_j(\varkappa)| \leqq \delta\,|\varkappa| \tag{6.6}$$

for $1 \leqq j \leqq m$ and $|\varkappa| \leqq \varkappa_0$. That is possible by virtue of Theorem 6.2(ii). Let $c_0 > 0$ and put

$$\begin{aligned} \sigma_0 &= \min(\sigma_1(\varkappa_0), \sigma_2), \\ a_0 &= \beta\{4(\delta + c_0 + 1)\}^{-2}. \end{aligned} \tag{6.7}$$

Define a closed set $\Sigma(a_0, \sigma_0)$ in $\mathbb{C}$ by

$$\Sigma(a_0, \sigma_0) = \{\lambda \in \overline{\mathbb{C}}_+(-\sigma_0) \mid -\operatorname{Re}\lambda \leqq a_0\,|\operatorname{Im}\lambda|^2\}. \tag{6.8}$$

By virtue of Theorems 6.1 and 6.2(i) (ii), it then follows that

$$\begin{aligned} &\varrho(\hat{B}_c(k)) \subset \Sigma(a_0, \sigma_0) \text{ for all } k \in \mathbb{R}^n \setminus B[\varkappa_0], \\ &\mu_j(k, c) \notin \Sigma(a_0, \sigma_0) \text{ for all } k \in B[\varkappa_0] \setminus \{0\}, \; 1 \leqq j \leqq m, \end{aligned} \tag{6.9}$$

for any $c \in B[c_0]$. Obviously $\mathscr{F}_x(\lambda - B_c^\infty)^{-1}u = (\lambda - \hat{B}_c(k))^{-1}\hat{u}$, so (6.9) suggests

$$\varrho(B_c^\infty) \subset \Sigma(a_0, \sigma_0) \setminus \{0\}, \quad 0 \in \sigma(B_c^\infty), \tag{6.10}$$

and Theorem 6.2(iv) suggests the decomposition of the resolvent

$$(6.11) \qquad (\lambda - B_c^\infty)^{-1} = \sum_{j=0}^{m} U_j(\lambda, c),$$

$$U_0(\lambda, c) = \mathscr{F}_x^{-1}\{\chi(|k| > \varkappa_0)(\lambda - \hat{B}_c(k))^{-1} + \chi(|k| < \varkappa_0) S(\lambda, k, c)\} \mathscr{F}_x,$$

$$U_j(\lambda, c) = \mathscr{F}_x^{-1}\chi(|k| < \varkappa_0)(\lambda - \mu_j(k, c))^{-1} P_j(k, c) \mathscr{F}_x, \quad 1 \leqq j \leqq m.$$

This is an orthogonal decomposition, $U_j U_k = 0$ $(j \neq k)$ in $L^2 = L^2(Q^\infty)$.
Define the sets of points $(\lambda. c)$,

$$\Sigma_4 = \overline{\mathbb{C}}_+(-\sigma_0) \times B[c_0],$$

$$\Sigma_5 = \Sigma(a_0, \sigma_0) \times B[c_0] \subset \Sigma_4, \quad \Sigma_5' = \{(\lambda, c) \in \Sigma_5 \mid |\lambda| \geqq r_0\}.$$

In what follows, all results are valid for any fixed c_0, $r_0 > 0$. The following gives a justification of (6.10) and (6.11).

Proposition 6.4. *Write* $\mathbb{B} = \mathbb{B}(L^2(Q^\infty))$ *and let* $a > 0$.

(i) $U_0(\lambda, c) \in L^\infty(\Sigma_4; \mathbb{B})$, $\quad U_0(\lambda, c)\, \chi(|\xi| < a) \in \mathscr{B}^0(\Sigma_4; \mathbb{B})$.

(ii) $U_j(\lambda, c) \in \mathscr{B}^0(\Sigma_5'; \mathbb{B})$, $\quad 1 \leqq j \leqq m$.

Proof. In view of the Parseval relation, (i) follows from Proposition 6.3(i) (iii) while (ii) follows from Proposition 6.3(ii) because $|\lambda - \mu_j(k, c)| \geqq \varepsilon$ for $(\lambda, c) \in \Sigma_5'$, $k \in B[\varkappa_0]$ with some $\varepsilon > 0$ depending only on $r_0 > 0$ (*cf.* (6.9)(ii)).

To study the case $r_0 = 0$, we first introduce the integrals $I = I(\lambda, c, l, \alpha)$ and $J = J(\lambda, \lambda', c, c', l, \alpha)$ by

$$I = \max_{1 \leqq j \leqq m} \int_{B[\varkappa_0]} |\lambda - \mu_j(k, c)|^{-l} |k|^\alpha \, dk,$$

$$J = \max_{1 \leqq j \leqq m} \int_{B[\varkappa_0]} |(\lambda - \mu_j(k, c))^{-1} - (\lambda' - \mu_j(k, c'))^{-1}|^l |k|^\alpha \, dk,$$

Lemma 6.5. $\forall l > 0$, $\forall \alpha > 0$, $\forall \theta \in [0, 1]$, $\exists C \geqq 0$, $\forall (\lambda, c)$, $(\lambda', c') \in \Sigma_5$,

(i) $I \leqq C |c|^{-\theta} |\mathrm{Im}\, \lambda|^{\min(0, \gamma)}$,

(ii) $J \leqq C(|c|^{-\theta} + |c'|^{-\theta})(|\lambda - \lambda'|^\delta + |c - c'|^\delta)$ *if* $\gamma \geqq \delta > 0$,

where $\gamma = n + \alpha + \theta - 2l$.

The proof of this technical lemma will be given in the Appendix. The following is a substitute of Proposition 6.4(ii) for $r_0 = 0$, which is a key to the sequel. With β dropped if $\beta = 0$, write

$$\tilde{L}_\beta^{p,q} = \tilde{L}_\beta^{p,q}(Q^\infty), \quad \tilde{L}_\beta^p = \tilde{L}_\beta^{p,p}.$$

Proposition 6.6. *Let* $1 \leqq q \leqq 2 \leqq p$, $r \leqq \infty$, $\beta \geqq 0$, $\theta \in [0, 1)$, $l \geqq 0$ *and* $\alpha = 0, 1$. *Put* $\gamma = \frac{1}{q} - \frac{1}{p}$, *and* $\gamma' = (n+\theta)\gamma + \alpha - 2l$. *Write* $\mathbb{B} = \mathbb{B}(\tilde{L}^{q,2}, \tilde{L}^{p,r}_\beta)$. *The following hold for all* j, $1 \leqq j \leqq m$.

(i) *There is a constant* $C \geqq 0$ *and for any* $(\lambda, c) \in \Sigma_5$,

$$\| U_j(\lambda, c)^l (I - P_c)^\alpha \|_{\mathbb{B}} \leqq C\,|c|^{-\theta} |\,\mathrm{Im}\,\lambda|^{\min(0,\gamma')},$$

where P_c *is the projection occurring in Proposition 3.7.*

(ii) *Let* $\Sigma_5'' = \Sigma(a_0, \sigma_0) \times (B[c_0] \setminus \{0\})$. *If* $\gamma > (2-\alpha)/(n+\theta)$, *then*

$$U_j(\lambda, c)\,(I - P_c)^\alpha \in \mathscr{B}^0(\Sigma_5''; \mathbb{B}).$$

If $\theta = 0$, Σ_5'' *can be replaced by* Σ_5.

Proof. Let $u \in \tilde{L}^{q,2}$ and put $v = U_j(\lambda, c)^l u$. By definition (6.11),

$$\hat{v}(k, \xi) = (\lambda - \mu_j(k, c))^{-l} P_j(k, c)\,\hat{u}(k, \xi), \qquad k \in B[\varkappa_0],$$

and $= 0$ for $k \in \mathbb{R}^n \setminus B[\varkappa_0]$. Therefore

$$\begin{aligned}
\|v\|_{\tilde{L}^{p,r}_\beta} &\leqq C\,\|\hat{v}\|_{\tilde{L}^{s,r}_\beta(\hat{Q}^\infty)} \quad \left(\frac{1}{p} + \frac{1}{s} = 1, \ \text{by (2.3)}\right)\\
&\leqq C\,\|\hat{v}\|_{L^{s,r}_\beta(\hat{Q}^\infty)} \quad \text{(by (2.1))}\\
&\leqq C\left(\int_{B[\varkappa_0]} |\lambda - \mu_j(k, c)|^{-ls}\,\|\hat{u}(k, \cdot)\|^s_{L^2(\mathbb{R}^n_\xi)} dk\right)^{1/s} \quad \text{(by Proposition 6.3(ii))}\\
&\leqq C I(\lambda, c, l/\gamma, 0)^\gamma\,\|\hat{u}\|_{L^{t,2}(\hat{Q}^\infty)} \quad \left(\frac{1}{t} + \frac{1}{q} = 1, \ \text{by Hölder inequality}\right).
\end{aligned}$$

By (2.1) and (2.2)

$$\|\hat{u}\|_{L^{t,2}(\hat{Q}^\infty)} \leqq \|\hat{u}\|_{\tilde{L}^{t,2}(\hat{Q}^\infty)} \leqq C\,\|u\|_{\tilde{L}^{q,2}}.$$

Consequently, one gets

$$\|v\|_{\tilde{L}^{p,r}_\beta} \leqq C I(\lambda, c, l/\gamma, 0)^\gamma\,\|u\|_{\tilde{L}^{q,2}}. \tag{6.12}$$

This and Lemma 6.5(i) yield a desired estimate in (i) for $\alpha = 0$. Note from Theorem 6.2(iii) that $P_j(k, c)(I - P_c) = |k|\,P_j'(k, c)(I - P_c)$. If $P_c u = 0$, therefore, (6.12) holds with $I(\lambda, c, l/\gamma, 1/\gamma)$ instead of with $I(\lambda, c, l/\gamma, 0)$ (see Proposition 6.3(ii)). This proves (i) for $\alpha = 1$. To prove (ii), one has only to put $v = (U_j(\lambda, c) - U_j(\lambda', c'))\,u$ and repeat the proof of (6.12) with the aid of Lemma 6.5(ii).

Combining above two propositions, we can now deduce the main result of this section. Recall $\Lambda_c^\alpha = \nu_c(\xi)^\alpha \times$.

Theorem 6.7. *Let* $1 \leqq q \leqq 2 \leqq p$, $r \leqq \infty$, $\beta \geqq 0$, $\alpha \in [0, 1]$, $\theta \in [0, 1)$ *and* $m = 0, 1$. *Put* $\gamma = \frac{1}{q} - \frac{1}{p}$ *and suppose* $\gamma > (2-m)/(n+\theta)$.

(i) *There is a constant* $C \geqq 0$ *and for all* $(\lambda, c) \in \Sigma_5$,

(6.13) $\|(\lambda - B_c^\infty)^{-1}(I - P_c)^m \Lambda_c^\alpha u\|_{\tilde{L}_\beta^{p,r}} \leqq C(\|u\|_{\tilde{L}_\beta^{p,r}} + \|\Lambda_c^\alpha u\|_{L^2} + |c|^{-\theta\gamma}\|\Lambda_c^\alpha u\|_{\tilde{L}^{q,2}})$.

(ii) *Let* Σ_5'' *be that of Proposition 6.6(ii) and put*

$$X = \tilde{L}_\beta^{p,r} \cap L^2 \cap \tilde{L}^{q,2}.$$

Then for any $a > 0$,

$$(\lambda - B_c^\infty)^{-1}(I - P_c)^m \Lambda_c^\alpha \chi(|\xi| < a) \in \mathscr{B}^0(\Sigma_5''; \mathrm{B}(X, \tilde{L}^{p,r})).$$

If $\theta = 0$, Σ_5'' *may be replaced by* Σ_5.

Proof. To simplify the notation, put

$$R = (\lambda - B_c^\infty)^{-1}, \quad R' = (\lambda - A_c^\infty)^{-1}, \quad G = (\lambda - A_c^\infty)^{-1} K_c.$$

Then the second resolvent equation of (6.1) can be written as $R = R' + GR$. Iterate this to obtain

$$R = \sum_{h=0}^{l-1} G^h R' + G^l R, \quad l \in \mathbb{N}_+. \tag{6.14}$$

The proof of Lemma 5.4 shows that

$$G^h \in \mathscr{B}^0(\Sigma_5; \mathrm{B}(\tilde{L}_\beta^{p,r})), \quad h \in \mathbb{N}_+, \tag{6.15}$$

while by (3.4) and Lemma 3.8(ii),

$$P_c \Lambda_c^\alpha \in \mathscr{B}^0(\Sigma_5, \mathrm{B}(\tilde{L}_\beta^{p,r})), \tag{6.16}$$

both of which are valid for all $p, r \in [1, \infty]$ and $\beta \in \mathbb{R}$. Combine (6.15), (6.16) with Lemma 5.3(i) to evaluate the sum for $0 \leqq h \leqq l - 1$ of (6.14) multiplied by $(I - P_c)^m \Lambda_c^\alpha$. This gives the first term on the right side of (6.13). Decompose the last term $G^l R$ of (6.14) according to the decomposition (6.11), and take l of Lemma 5.4 for $p = 2$. Then $G^l U_0(\lambda, c)$ can be evaluated by Proposition 6.4, and $G^l U_j(\lambda, c)$, $1 \leqq j \leqq m$ by (6.15) and Proposition 6.6, from which follow the second and last terms of the right side of (6.13), respectively. This argument completes the proof of (i) of the theorem. Assertion (ii) can be concluded from this proof if the statements of continuity in the propositions and lemmas used are taken into account.

Remark 6.8. (6.13) provides no estimates when $c = 0$ unless also $\theta = 0$. The argument in [15] relies only on (6.13) with $\theta = 0$, for which $\gamma = \frac{1}{q} - \frac{1}{p}$ should take a larger value than for $\theta \neq 0$. See also Remark 9.4.

We shall discuss also the operators $\gamma^\pm(\lambda - B_c^\infty)^{-1}$ which exist in the sense stated for $\gamma^\pm(\lambda - A_c^\infty)^{-1}$ in § 5.

Theorem 6.9. *Let* $1 \leqq q \leqq 2 \leqq p$, $r \leqq \infty$, $\beta \geqq 0$, $\theta \in [0, 1)$, $\alpha \in [0, 1]$ *and* $m = 0, 1$. *Suppose* $1/q > (2 - m)/(n + \theta)$.

(i) *There is a constant* $C \geqq 0$ *such that for all* $(\lambda, c) \in \Sigma_5$,

$$(6.17)\quad \|\gamma^{\pm}(\lambda - B_c^{\infty})^{-1}(I - P_c)^m \Lambda_c^{\alpha} u\|_{\tilde{Y}_{\beta}^{p,r,\pm}} \leqq C(\|\Lambda_c^{\alpha/p} u\|_{\tilde{L}_{\beta}^{p,r}} + \|\Lambda_c^{\alpha} u\|_{L^2} + |c|^{-\theta/q} \|\Lambda_c^{\alpha} u\|_{L^{q,2}}).$$

(ii) *Let* Σ_5'' *and* X *be as in Theorem 6.7(ii). For any* $a > 0$,

$$\gamma^{\pm}(\lambda - B_c^{\infty})^{-1}(I - P_c)^m \Lambda_c^{\alpha} \chi(|\xi| < a) \in \mathscr{E}^0(\Sigma_5''; \mathbb{B}(X, \tilde{Y}_{\beta}^{p,r,\pm})).$$

Proof. Write $G' = K_c R'$, $\Xi = (I - P_0)^m \Lambda_c^{\alpha}$, and deduce from (6.14)

$$(6.18)\qquad \gamma^{\pm} R \Xi = \gamma^{\pm} R' \sum_{h=0}^{l-1} (G')^h \Xi + \gamma^{\pm} G^l R \Xi.$$

Evaluate the sum for $0 \leqq h \leqq l - 1$ by the aid of Lemma 5.5 and (6.16), noting that (6.15) remains valid if G is replaced by G'. Then the first term on the right side of (6.17) follows. Since $\varrho(X, \xi) \leqq |\xi|$,

$$\|\gamma^{\pm} v\|_{\tilde{Y}_{\delta}^{r,s,\pm}} \leqq C \|v\|_{L_{\beta}^{\infty}},$$

with $\beta > \delta + \dfrac{1}{r} + \dfrac{n}{s}$. Put $v = G^l U_j(\lambda, c)\, \Xi u$ and take l large enough for Lemma 5.4 to hold with $p = 2$, $r = s = \infty$. Then the last term of (6.18) can be evaluated by means of Propositions 6.4(i) and 6.6 if (6.11), (6.15) and (6.16) are taken into account, giving the remaining terms in the right side of (6.17). (ii) can be proven like Theorem 6.7(ii).

It remains to prove Proposition 6.3. First, note from (6.2) that the adjoint $\hat{A}_c(k)^*$ to $\hat{A}_c(k)$ is given as

$$\hat{A}_c(k)^* = (ik \cdot \xi - \nu_c(\xi)) \times, \qquad D(\hat{A}_c(k)^*) = D(\hat{A}_c(k)).$$

Hence $\hat{A}_c(k)^* = \hat{A}_c(-k)$. This implies that $\hat{B}_c(k)^* = \hat{A}_c(-k) + K_c^*$. Since $K_0^* = K_0$ is selfadjoint in $L^2 = L^2(\mathbb{R}_{\xi}^n)$ (see § 3), we then have

Proposition 6.10. $B_c(k)^*$ *enjoys all the conclusions of Theorems 6.1 and 6.2 with obvious modifications.*

Furthermore, it should be mentioned that the eigenprojection $P_j^*(k, c)$ for the eigenvalue $\overline{\mu_j(k, c)} \in \sigma_d(\hat{B}_c(k)^*)$ is the adjoint to $P_j(k, c)$ ([10], p .184).

In what follows, for simplicity, we put

$$\hat{R} = (\lambda - \hat{B}_c(k))^{-1}, \hat{R}' = (\lambda - \hat{A}_c(k))^{-1}, \hat{G} = (\lambda - \hat{A}_c(k))^{-1} K_c$$

and we define for each δ, $c_0 > 0$,

$$\Sigma_6 = \mathbb{C}_+(-\nu_0) \times \mathbb{R}^n \times B[c_0],$$

$$\Sigma_7 = \overline{\mathbb{C}}_+(-\nu_0 + \delta) \times \mathbb{R}^n \times B[c_0] \subset \Sigma_6,$$

which are sets of points (λ, k, c). Also we write $L^p_\beta = L^p_\beta(\mathbb{R}^n_\xi)$ and $\mathbb{B} = \mathbb{B}(L^2)$. Again from (5.4) we note

$$(6.19)\qquad \begin{aligned} &\text{(i)} \quad \|\hat{R}'\|_{\mathbb{B}} \leqq (\operatorname{Re}\lambda + \nu_0)^{-1} \text{ in } \Sigma_6, \\ &\text{(ii)} \quad \hat{R}'\chi(|\xi| < a) \in \mathscr{B}^0(\Sigma_7; \mathbb{B}) \text{ for each } a > 0. \end{aligned}$$

Lemma 6.11. *(i)* $\forall\, \delta > 0,\ \forall\, c_0 > 0,\ \exists\, C \geqq 0,\ \|\hat{G}\|_{\mathbb{B}} \leqq C(\operatorname{Re}\lambda + \nu_0)^{-1}$ *in* Σ_6. *(ii)* $\hat{G} \in \mathbb{C}(L^2)$ *in* Σ_6. *(iii)* $\hat{G}$ *is analytic in* $\mathbb{B}$ *in* $\lambda \in \mathbb{C}_+(-\nu_0)$. *(iv)* $\forall\, p \in [2, \infty]$, $\forall\, \beta \geqq 0,\ \exists\, l \geqq 0,\ \forall\, \delta > 0,\ \forall\, c_0 > 0,$

$$\hat{G}^l \in \mathscr{B}^0(\Sigma_7; \mathbb{B}(L^2, L^p_\beta)).$$

Here $l = [\beta] + 1$ *if* $p = 2$.

Proof. In view of (6.19)(i), (i) follows from Lemma 3.2 and (ii) from Lemma 3.4. (iii) is evident from (6.3)(ii). Put

$$\hat{G}_1 = \chi(|\xi| < a)\,\hat{G}, \qquad \hat{G}_2 = \hat{G} - \hat{G}_1.$$

Lemma 3.3 and (6.19)(ii) show that $\hat{G}_1 \in \mathscr{B}^0(\Sigma_7; \mathbb{B})$ for each $a > 0$ while by (3.7) and (6.19)(i),

$$(6.20)\qquad \|\hat{G}_2\|_{\mathbb{B}} \leqq C(1 + a)^{-1}(\operatorname{Re}\lambda + \nu_0)^{-1} \text{ in } \Sigma_6$$

for each $a > 0$. This proves (iv) for $p = 2$, $\beta = 0$, and similarly for more general cases since Lemma 5.4 is also valid for $\hat{G}$ with obvious modifications.

We shall need the asymptotic behavior of $\hat{G}$ for large λ and large k.

Lemma 6.12. $\forall\, \delta > 0,\ \forall\, c_0 > 0,\ \exists\, C \geqq 0,\ \exists\, \gamma > 0,\ \forall\, (\lambda, k, c) \in \Sigma_7,\ \forall\, \varkappa > 0,$

(i) $\|\hat{G}\|_{\mathbb{B}} \leqq C\varkappa(1 + |\operatorname{Im}\lambda|)^{-\gamma}$ *if* $k \in B[\varkappa]$,
(ii) $\|\hat{G}\|_{\mathbb{B}} \leqq C(1 + |k|)^{-\gamma}$.

Proof. By Lemma 3.2 and Hölder inequality,

$$\|\hat{G}_1 u\|_{L^2} \leqq I\,\|K_c u\|_{L^{2p}} \leqq CI\,\|u\|_{L^2}$$

whenever $1 \leqq p \leqq n/(n-4)$, where

$$I = I(\lambda, k, c) = \left(\int_{B[a]} |\lambda + ik\cdot\xi + \nu_c(\xi)|^{-2q}\, d\xi\right)^{\frac{1}{2}q}, \quad \frac{1}{p} + \frac{1}{q} = 1.$$

We can choose $p > 1$ so that $q < \infty$. Write $\lambda = \sigma + i\tau$ and define

$$\Xi_1 = \{\xi \in B[a] \mid |\tau + k\cdot\xi| \geqq |k|\,\varepsilon\}, \qquad \Xi_2 = B[a] \setminus \Xi_1$$

with $\varepsilon > 0$. It is not hard to see that

$$\operatorname{mes}\Xi_1 \leqq Ca^n, \operatorname{mes}\Xi_2 \leqq Ca^{n-1}\varepsilon,$$

with some constant $C \geqq 0$ independent of k, a, τ and ε. Hence

$$I^{2q} = \int_{B[a]} = \int_{\Xi_1} + \int_{\Xi_2} \leqq Ca^n \{(|k|\varepsilon)^{-2q} + a^{-1}\, \varepsilon(\sigma + \nu_0)^{-2q}\}.$$

Choose $\varepsilon = (1 + |k|)^{-2q/(2q+1)}$ to get

$$\|\hat{G}_1\|_{\mathbb{B}} \leqq CI \leqq Ca^{n/q}(1 + |k|^{-1/(2q+1)}).$$

Choose $a = (1 + |k|)^{-1/(2q+1)(n+1)}$ in the above and in (6.20) to prove (ii). Suppose $k \in B[\varkappa]$ and $|\tau| \geqq 2a\varkappa$. Then $|\tau + k \cdot \xi| \geqq |\tau|/2$ for $\xi \in B[a]$, so $I \leqq Ca^{n/2q}\,|\tau|^{-1}$. Choose $a = |\tau|^{2q/(2q+n)}\,\varkappa^{-1}$ here and in (6.20) to get (i) for $|\tau| \geqq 2a\varkappa$ or $|\tau| \geqq 2^{1+2q/n}$.

If $\delta > 0$ and $c_0 \geqq 0$, Lemmas 6.11(i) and 6.12 deliver positive numbers r_0 and $\varkappa_2$ such that

$$\|\hat{G}\|_{\mathbb{B}} \leqq \frac{1}{2} \text{ for all } (\lambda, k, c) \in \Sigma_8$$

where

$$\Sigma_8 = \Sigma_7 \setminus \{(\lambda, k, c) \in \Sigma_7 \mid |\lambda| \leqq r_0, \quad k \in B[\varkappa_2]\}.$$

Then the Neumann series converges and $\|(I - \hat{G})^{-1}\|_{\mathbb{B}} \leqq 2$. Moreover, in view of [10, Theorem IV.1.6] and Lemma 6.11(iv),

(6.21) $(I - \hat{G})^{-1}$ is continuous in $\mathbb{B}$ in a neighborhood of a point (λ, k, c) if $1 \in \varrho(\hat{G}(\lambda, k, c))$.

Thus we have proven

Lemma 6.13. $\forall\, \delta > 0,\ \forall\, c_0 > 0,\ \exists\, r_0 > 0,\ \exists\, \varkappa_2 > 0,$

(i) $1 \in \varrho(\hat{G})$ *in* Σ_8,

(ii) $(I - \hat{G})^{-1} \in \mathscr{B}^0(\Sigma_8, \mathbb{B})$.

Proof of Proposition 6.3(i). Write the second resolvent equation $\hat{R} = \hat{R}' + \hat{G}\hat{R}$ as

(6.22) $$\hat{R} = (I - \hat{G})^{-1}\, \hat{R}'$$

and conclude that if $\lambda \in \varrho(\hat{A}_c(k))$, then

(6.23) $$\lambda \in \varrho(\hat{B}_c(k)) \text{ if and only if } 1 \in \varrho(\hat{G}(\lambda, k, c)).$$

Apply (6.19) and Lemma 6.13 to (6.22). Then

(6.24) $$\hat{R} \in L^\infty(\Sigma_8; \mathbb{B}), \qquad \hat{R}\chi(|\xi| < a) \in \mathscr{B}^0(\Sigma_8; \mathbb{B}).$$

Combine (6.21) and (6.23) with Theorem 6.1. Then $(I - \hat{G})^{-1} \in \mathscr{E}^0(\Sigma_1; \mathbb{B})$, so that $(I - \hat{G})^{-1} \in \mathscr{B}^0(\Sigma_1 \setminus \Sigma_8; \mathbb{B})$ since $\overline{\Sigma_1 \setminus \Sigma_8}$ is compact. Consequently (6.24) is also valid if Σ_8 is replaced by $\Sigma_1 \setminus \Sigma_8$, which, together with (6.24), completes the proof.

Proof of Proposition 6.3(ii). Let j, $1 \leqq j \leqq m$, be arbitrarily fixed and let $P = P(k, c)$ denote either $P_j(k, c)$ or $P_j'(k, c)\,(I - P_c)$. Put $H = \hat{G}(\mu_j(k, c), k, c)$.

By Theorem 6.2, $(\mu_j(k, c) - \hat{B}_c(k))\, P = 0$, or equivalently $P = HP$, so $P = H^l P$ by iteration. In view of Lemma 6.11(iv), therefore, Proposition 6.3(ii) will follow if

$$P(k, c) \in \mathscr{B}^0(\Sigma_2; \mathbb{B}). \tag{6.25}$$

The proof of (6.6) implies

$$P(k, c) = \theta_c^{-1}\, P(k, 0)\, \theta_c,$$

where θ_c is that of (3.1). Let $L^p_{\beta,\alpha} = L^p_{\beta,\alpha}(\mathbb{R}^n_\xi)$ be as in Lemma 3.2. Obviously, θ^c and θ_c^{-1} are continuous in $c \in B[c_0]$ in the strong topologies of both $\mathbb{B}(L^2_{0,\alpha}, L^2)$ and $\mathbb{B}(L^2, L^2_{0,-\alpha})$ if $\alpha > c_0$. Consequently, (6.25) will follow ([10], Lemma III.3.9) if

$$P(k, 0) \in \mathscr{B}^0(B[\varkappa_0];\, \mathbb{C}(L^2_{0,-\alpha}, L^2_{0,\alpha})), \qquad \alpha > c_0. \tag{6.26}$$

This has been proven for $\alpha = 0$ in [4]. Note that P is of finite rank (Theorem 6.2(iii) and so it is compact if bounded. Repeat the proof of Lemma 3.6 with $H_1 = \chi(|\xi| > a)\, H$ and $H_2 = H - H_1$, and combine (6.26) for $\alpha = 0$ to see that

$$P(k, 0) \in \mathscr{B}^0(B[\varkappa_0];\, \mathbb{C}(L^2, L^2_{0,\alpha}))$$

holds for any $\alpha \in \mathbb{R}$. This is also true for $P^*(k, c)$ (Proposition 6.10), so that passing to the adjoint and noting the remark after Proposition 6.10, we see that

$$P(k, 0) \in \mathscr{B}^0(B[\varkappa_0];\, \mathbb{C}(L^2_{0,-\alpha}, L^2))$$

for any $\alpha \in \mathbb{R}$. Now (6.26) follows by interpolation [2].

Proof of Proposition 6.3(iii). By virtue of (6.24) and Proposition 6.3(ii), $S = \hat{R}\left(I - \sum_j P_j\right)$ also satisfies (6.24), and so it is sufficient for S to be continuous in any compact set Σ_9 of Σ_3. Recall the integral formula ([10], p. 179)

$$S(\lambda, k, c) = \frac{1}{2\pi i} \int_\Gamma \hat{R}(\zeta, k, c)\,(\zeta - \lambda)^{-1}\, d\zeta,$$

where Γ is any simple closed rectifiable curve in $\overline{\mathbb{C}}_+(-\sigma_2)$ enclosing points λ and $\mu_j(k, c)$, $1 \leqq j \leqq m$, but not other points of $\sigma(\hat{B}_c(k))$. In view of Theorem 6.2, Γ can be chosen independently of λ, k, c as far as $(\lambda, k, c) \in \Sigma_9$. Since $\Gamma \subset \varrho(\hat{B}_c(k))$, $\hat{R}(\lambda, k, c)$ enjoys (6.24) with $\Gamma \times \Sigma_2$ substituted for Σ_8. Now we arrive at the desired result, evaluating the integral in the above.

7. The Operator B_c

In this section the operator B_c of (1.13) is discussed in $Q = \Omega_x \times R^n$ under the boundary condition $\gamma^- u = M_0 \gamma^+ u$. The main aim is to study the behavior of its resolvent near $\lambda = 0$. As in [1] and [14], the essential point is to derive its explicit expression. First of all, we need to introduce the solution operator $R_c(\lambda)$ of the boundary value problem

$$\begin{cases} (\lambda + \xi \cdot \nabla_x + \nu_c(\xi))\, u = 0 & \text{in } Q, \\ \gamma^- u = h & \text{on } S^-. \end{cases} \tag{7.1}$$

Proposition 7.1. *For each* $(\lambda, c) \in C_+(-\nu_0) \times \mathbb{R}^n$ *and* $h \in Y^{2,-}$, (7.1) *possesses a unique solution* $u = u(x, \xi) \in W_\nu^{2,+}(Q)$. *The operator* $R_c(\lambda)$ *defined by* $u = R_c(\lambda)\, h$ *is in* $\mathbb{B}(Y^{2,-}, W_{\nu_c}^{2,+}(Q))$ *and satisfies*

(i) $R_c(\lambda) \in B(Y^{2,-}, L^2(Q))$ *with the norm* $\leqq \{2\,(\operatorname{Re}\lambda + \nu_0)\}^{-\frac{1}{2}}$,

(ii) $\gamma^+ R_c(\lambda) = 0, \quad \gamma^- R_c(\lambda) = I.$

Proof. We begin with uniqueness. Let $u \in W_{\nu_c}^{2,+}(Q)$ be a solution to (7.1). By $(4.5)_+$ with $h(\xi) = \lambda + \nu_c(\xi)$ and by the first equation of (7.1), it follows that $\gamma^+ u = 0$. Then (4.9) gives

$$h_0\, \|u\|_{L^2(Q)} \leqq \frac{1}{2}\, \|h\|_{Y^{2,-}} \tag{7.2}$$

where $h_0 = \inf h(\xi) = \operatorname{Re}\lambda + \nu_0 > 0$, proving uniqueness. Note that this also proves (i) and (ii) if $R_c(\lambda)$ exists. In order to prove existence, define for each $(x, \xi) \in \overline{Q^-}$,

$$t^-(x, \xi) = \inf\{t \geqq 0 \mid x - t\xi \in \partial\Omega\},$$

and note that

(7.3) (i) $t^-(X, \xi) = 0$ if $(X, \xi) \in S^-$,

(ii) $x - t^-(x, \xi)\, \xi \in \partial\Omega^-(\xi)$ for all $(x, \xi) \in Q^-$,

(iii) $t^-(x + t\xi, \xi) = t^-(x, \xi) + t$ for all $t \geqq 0$, $(x, \xi) \in Q^-$,

(iv) $t^-(x, \xi) = t^-(x, \tilde{\xi})/|\xi|$, $\tilde{\xi} = \xi/|\xi|$.

Define $u = u(x, \xi)$ by

$$u(x, \xi) = e^{-(\lambda + \nu_c(\xi))t^-(x,\xi)}\, h(x - t^-(x, \xi)\, \xi, \xi) \tag{7.4}$$

for $(x, \xi) \in \overline{Q^-}$ and $u = 0$ for $(x, \xi) \in Q \setminus \overline{Q^-}$. By (7.3)(i), $u = h$ on S^-, while by (7.3)(iii),

$$(\xi \cdot \nabla_x u)\,(x + t\xi, \xi) = \frac{\partial}{\partial t} u(x + t\xi, \xi) = -\{(\lambda + \nu_c(\xi))\, u\}\,(x + t\xi, \xi)$$

for all $t \geqq 0$ and $(x, \xi) \in Q$, which, when $t = 0$, is just the same as the first equation of (7.1). Hence this u satisfies (7.1), though formally. We shall prove that $u \in W^{2,+}_{\nu_c}(Q)$. Put $w = |u|^2$ in $(4.3)_-$ and use (7.3)(i)(ii)(iii) to obtain

$$\|u\|^2_{L^2(Q)} = \int_{S^-} \int_0^\infty e^{-2(\mathrm{Re}\lambda + \nu_c(\xi))t} |h(X, \xi)|^2 \, dt \, d\sigma_X \, d\xi, \tag{7.5}$$

which leads again to (7.2). Hence if $h \in Y^{2,-}$, then $u \in L^2(Q)$ and so is $(\xi \cdot \nabla_x + \nu_c(\xi)) u = -\lambda u$ by the first equation of (7.1). This completes the proof of the proposition.

The operator $R_c(\lambda)$ will be used in various spaces. In what follows we write

$$\tilde{L}^{p,q}_\beta = \tilde{L}^{p,q}_\beta(Q), \quad L^p_\beta = \tilde{L}^{p,p}_\beta(Q).$$

The subscript β will be supressed when $\beta = 0$. Also we recall the definitions of the spaces $Y^{p,\pm}_\beta$ of § 1 and $\tilde{Y}^{p,q,\pm}_\beta$ of § 4, and put

$$\Sigma_0 = \overline{\mathbb{C}}_+(-\nu_0 + \delta) \times B[c_0].$$

The following results will be valid for all δ, $c_0 > 0$.

Lemma 7.2. *Let* $1 \leqq r \leqq p \leqq \infty$, $1 \leqq s \leqq q \leqq \infty$, $\beta \in \mathbb{R}$ *and* $\gamma \geqq 0$. *Put* $\mathbb{B} = \mathbb{B}(\tilde{Y}^{p,q,-}_{\beta+\gamma}, \tilde{L}^{r,s}_\beta)$ *and* $\gamma_0 = \dfrac{1}{r} - \dfrac{1}{p} + n\left(\dfrac{1}{s} - \dfrac{1}{q}\right)$.

(i) *If* $\gamma > \gamma_0$ ($\gamma \geqq \gamma_0$ *when* $q = s$),

$R_c(\lambda) \in L^\infty(\Sigma_0; \mathbb{B})$,

$\chi(|\xi| < a) R_c(\lambda) = R_c(\lambda) \chi(|\xi| < a) \in \mathscr{B}^0(\Sigma_0; \mathbb{B})$,

for any $a > 0$.

(ii) *If* $\gamma > \gamma_0 - 1$,

$K_c R_c(\lambda) \in \mathscr{B}^0(\Sigma_0; \mathbb{B})$.

(iii) *For* γ *of* (i), $R_c(\lambda)$ *is analytic in* $\lambda \in \mathbb{C}_+(-\nu_0)$ *in* $\mathbb{B}$ *for each fixed* c.

Proof. Let u be given by (7.4) and put $w = |u|^r$ in $(4.2)_-$. Proceeding as in (7.5) yields

$$\|u(\cdot, \xi)\|^r_{L^r(\mathbb{R}^n_x)} \leqq (r(\mathrm{Re}\,\lambda + \nu_0))^{-1} \int_{\partial\Omega^-(\xi)} |h(X, \xi)|^r \varrho(X. \xi) \, d\sigma_X$$

for each $\xi \in \mathbb{R}^n$, whence follows readily the first assertion of (i) for the special case $p = r$, $q = s$, $\gamma = 0$. The general case is a direct consequence of this and of the fact that since $\varrho(X, \xi) \leqq |\xi|$ and by Hölder,

$$\tilde{Y}^{p,q,-}_{\beta+\gamma} \subset \tilde{Y}^{r,s,-}_\beta \text{ is a continuous embedding} \tag{7.6}$$

for γ specified in (i). Then the second assertion of (i) can be proven like Lemma 5.2(iv) by appeal to the continuity of the function

$$\Sigma_0 \ni (\lambda, c) \to \chi(|\xi| < a) \exp\{-(\lambda + \nu_c(\xi)) t^-(x, \xi)\} \in L^\infty(Q^-).$$

In view of (i) and Proposition 3.5, it is clear that $K_c\chi(|\xi| < a)\,R_c(\lambda)$ enjoys (ii) of the lemma for each fixed $a > 0$, while by (3.7), $K_c\chi(|\xi| > a)\,R_c(\lambda) \to 0$ as $a \to 0$ in the norm of $\mathbb{B}$ uniformly in Σ_0. This proves (ii). Finally, differentiation of (7.4) with respect to λ gives rise to

$$\frac{\partial u}{\partial \lambda} = -t^- R_c(\lambda)\,h,$$

formally, where $t^- = t^-(x, \xi)$. Evaluate the norm of this right member just as in the proof of (i). Then it is seen that $t^- R_c(\lambda)$ also enjoys (i). This verifies (iii).

Recall (6.8) and define the set of points (λ, c),

$$\Sigma^0 = \Sigma(a_0, \sigma_0) \times B[c_0] \quad (= \Sigma_5 \text{ of } \S\,6).$$

We shall study the operator

$$T_c(\lambda) = \tilde{M}_0(\lambda - B_c^\infty)^{-1}\, eK_cR_c(\lambda),$$

where $\tilde{M}_0 = M_0\gamma^+ - \gamma^-$ (see (1.11)) and e stands for the extension,

$$eu = u \text{ in } Q, \quad eu = 0 \text{ in } Q^\infty \setminus Q, \tag{7.7}$$

u being a function on Q.

Lemma 7.3. *Let* $n \geqq 3$, $p \in [2, \infty]$, $\beta \geqq n\left(\frac{1}{2} - \frac{1}{p}\right)$.

(i) $T_c(\lambda) \in \mathscr{B}^0(\Sigma^0; \mathbb{B}(\check{Y}_\beta^{p,-}))$.

(ii) $T_c(\lambda)$ *is analytic in* $\mathbb{B}(Y^{2,-})$ *in* $\lambda \in \Sigma(a_0, \sigma_0) \setminus \{0\}$.

Proof. In (6.17), put $u = eK_cR_c(\lambda)\,h$ and set $p = r$, $q = 1$, $\alpha = \theta = m = 0$, for which $1/q = 1 > 2/n$ is satisfied if $n \geqq 3$. Then

$$T_c(\lambda) \in L^\infty(\Sigma^0; \mathbb{B}(Y_\beta^{p,-})),$$

by Lemma 7.2(ii) used thrice with $p = q = r = s$, $\gamma = 0$, with $p = q$, $r = s = 2$, $\beta = 0$ and with $p = q$, $r = 1$, $s = 2$, $\beta = 0$. This proof, combined with (3.7), shows that $\tilde{M}_0(\lambda - B_c^\infty)^{-1}\, e\chi(|\xi| > a)\,K_cR_c(\lambda) \to 0$ in $\mathbb{B}(Y_\beta^{p,-})$ as $a \to 0$ uniformly in Σ^0, while Theorem 6.9(ii) and Lemma 7.2(ii) show that $\tilde{M}_0(\lambda - B_c^\infty)^{-1}$ $e\chi(|\xi| < a)\,K_cR_c(\lambda)$ satisfies (i) of the lemma. Hence (i) follows. From (6.18) for $l = 1$,

$$\gamma^\pm(\lambda - B_c^\infty)^{-1} = \gamma^\pm(\lambda - A_c^\infty)^{-1}\{I + K_c(\lambda - B_c^\infty)^{-1}\},$$

which is analytic in λ due to Lemma 5.6 ($p = q = 2$) and (6.10). Then Lemma 7.2(iii) ($p = q = r = s = 2$) and $[\mathrm{M}]_2$(i) ensure that (ii) holds.

Proposition 7.4. *Under the assumptions of Lemma 7.3, there are constants* $C \geqq 0$ *and* $\gamma > 0$ (γ *being independent of* p, β) *such that for all* $(\lambda, c) \in \Sigma^0$,

(i) $\|T_c(\lambda)\| \leqq C(1 + |\lambda|)^{-\gamma/p}$ *in* $\mathbb{B}(Y_\beta^{p,-})$,

(ii) $\|T_c(\lambda)^3\| \leqq C(1 + |\lambda|)^{-\gamma}$ *in* $\mathbb{B}(Y_\beta^{\infty,-})$.

This proposition plays an essential role in § 6 of Part II as well as in the following proposition. Its lengthy proof will be effected in § 7 of Part II.

Proposition 7.5. *Replace* a_0, σ_0 *in* (6.8) *by* $\bar{a}_0$, $\bar{\sigma}_0$ *and define*

$$\bar{\Sigma} = \Sigma(\bar{a}_0, \bar{\sigma}_0) \times B[\bar{c}_0].$$

Under the assumptions of Lemma 7.3, there are constants $\bar{a}_0$, $\bar{\sigma}_0$, $\bar{c}_0 > 0$ *such that*

(i) $1 \in \varrho(T_c(\lambda))$ *in* $\mathbb{B}(Y_\beta^{p,-})$ *for all* $(\lambda, c) \in \bar{\Sigma}$.

(ii) $(I - T_c(\lambda))^{-1} \in \mathscr{B}^0(\bar{\Sigma}; \mathbb{B}(Y_\beta^{p,-}))$.

(iii) $(I - T_c(\lambda))^{-1}$ *is analytic in* $\mathbb{B}(Y^{2,-})$ *in* $\lambda \in \Sigma(\bar{a}_0, \bar{\sigma}_0) \setminus \{0\}$. *Of course,* $\bar{a}_0, \bar{\sigma}_0, \bar{c}_0$ *are independent of p and* β.

Proof. According to [1] and [14], there are constants $\bar{a}_0$, $\bar{\sigma}_0 > 0$ such that the proposition holds when $c = 0$. On the other hand, Lemma 7.3(i) permits us to restate (6.21) for $T_c(\lambda)$ and, moreover, Lemma 7.3(ii) permits us to replace "continuous in (λ, c)" in (6.21) by "analytic in $\mathbb{B}(Y^{2,-})$ in λ". It then follows that for any $r_0 > 0$, there is a constant $\bar{c}_0 > 0$ such that the proposition is true if $\bar{\Sigma}$ is replaced by $\{(\lambda, c) \in \bar{\Sigma} \mid |\lambda| \leqq r_0\}$. Note that this $\bar{c}_0$ can be chosen independently of p, because in Lemma 7.3(i), $T_c(\lambda)$ is equicontinuous with respect to $p \in [2, \infty]$ as seen by the interpolation between the cases $p = 2$ and $p = \infty$. Moreover, it can be assumed without loss of generality that $\bar{a}_0 \leqq a_0$, $\bar{\sigma}_0 \leqq \sigma_0$. By interpolation between Proposition 7.4(i) for $p = 2$ and (ii) it is seen that Proposition 7.4(ii) remains valid with $Y_\beta^{p,-}, p \in [2, \infty]$ substituted for $Y_\beta^{\infty,-}$. Then there is an $r_0 > 0$ such that

$$\|T_c(\lambda)^3\| \leqq \frac{1}{2} \quad \text{in} \quad \mathbb{B}(Y_\beta^{p,-})$$

for all $(\lambda, c) \in \Sigma^0$, $|\lambda| \geqq r_0$. Then the Neumann series converges and $(I - T_c(\lambda))^{-1} \in \mathbb{B}(Y_\beta^{p,-})$ exists for (λ, c) having norm $\leqq 2$. This and (6.21) modified as before prove the proposition for such (λ, c). Thus the proposition is proven.

The proof in [1] and [14] of the above proposition when $c = 0$ is based on the fact that $T_0(\lambda)$ is compact on $Y_\beta^{p,-}$ for $p \in [2, \infty)$ and so is $T_0(\lambda)^3$ for $p = \infty$, which is true also when $c \neq 0$. It is for the proofs of the compactness of $T_c(\lambda)^3$ for $p = \infty$ and of Proposition 7.4(ii) that the assumptions $[\mathrm{M}]_1$(iv) and $[\mathrm{M}]_2$(iv) are required. All the results for $p \in [2, \infty)$ are valid only under $[\mathrm{M}]_2$(i)(ii)(iii) (see also Remark 1.1).

Now we define the operator A_c in $L^2(Q)$ by

(7.8.) $$D(A_c) = \{u \in W_{\nu_c}^{2,+}(Q) \mid \gamma^- u \in Y^{2,-}, \ \tilde{M}_0 u = 0\},$$

$$A_c u = -(\xi \cdot \nabla_x + \nu_c(\xi))\, u.$$

Cf. Definition 4.7. Since $C_0^\infty(Q) \subset D(A_c)$, it is densely defined.

Proposition 7.6. *The following hold for each* $c \in \mathbb{R}^n$.

(i) *A_c is maximally dissipative in* $L^2 = L^2(Q)$.

(ii) $\varrho(A_c) \subset \mathbb{C}_+(-\nu_0)$.

(iii) *For each* $\lambda \in \mathbb{C}_+(-\nu_0)$,

$$(7.9) \qquad (\lambda - A_c)^{-1} = r(\lambda - A_c^\infty)^{-1}\, e + R_c(\lambda)\, \tilde{M}_0(\lambda - A_c^\infty)^{-1}\, e,$$

where e is as in (7.7) *and r is the restriction* $ru = u|_Q$ *to functions u on* Q^∞.

Proof. By (7.8), (4.9) can be applied to $u \in D(A_c)$ with $h(\xi) = \nu_c(\xi)$. Substitute into it the boundary condition $\tilde{M}_0 u = 0$ or $\gamma^- u = M_0 \gamma^+ u$. By $[\mathrm{M}]_2$(i), the last term $\frac{1}{2}\{\ldots\}$ of (4.9) $\leqq 0$, and so

$$\operatorname{Re}(A_c u, u) \leqq -\nu_0 \|u\|^2 \text{ in } L^2.$$

This inequality shows that A_c is dissipative and, moreover, that $\lambda - A_c$ is one-to-one in L^2 if $\lambda \in \mathbb{C}_+(-\nu_0)$. Denote the first and second terms on the right-hand side of (7.9) by R_1, R_2, respectively, and put $R = R_1 + R_2$. Let $\lambda \in \mathbb{C}_+(-\nu_0)$. By (5.1) and (5.3)(ii), $R_1 \in \mathbb{B}(L^2, W^{2,+}_{\nu_c}(Q))$, and by the remark given above Lemma 5.5 and by Proposition 7.1 and $[\mathrm{M}]_2$(i), so does R_2 with $\tilde{M}_0 R_2 = -\tilde{M}_0 R_1$. Hence R has a range in $D(A_c)$ and on account of (7.1), $(\lambda - A_c)\, R = I$ on L^2. Hence $\lambda - A_c$ is surjective and (7.9) now holds. Since $R \in \mathbb{B}(L^2)$, we can conclude that $\lambda - A_c$ is a closed operator in L^2. Now the proof of the proposition is complete.

We are ready to study B_c. Since $K_c \in \mathbb{B}(L^2)$ (Proposition 3.5), B_c can be defined in L^2 as

$$(7.10) \qquad B_c = A_c + K_c, \quad D(B_c) = D(A_c).$$

In view of Proposition 7.6(i) and [10], Theorem IX.2.1, B_c generates a semigroup on L^2. This semigroup, denoted by $E_c(t)$, will be studied in Part II. An expression similar to (7.9) may be obtained for B_c. Let $\bar{\Sigma}$ be as in Proposition 7.5 and put $\Sigma' = \Sigma(\bar{a}_0, \bar{\sigma}_0) \setminus \{0\}$.

Theorem 7.7. *For all* $c \in B[\bar{c}_0]$, *the following hold in* L^2:

(i) $\varrho(B_c) \subset \Sigma', 0 \in \sigma(B_c)$.

(ii) *Put* $S_c(\lambda) = R_c(\lambda) + r(\lambda - B_c^\infty)^{-1}\, e K_c R_c(\lambda)$. *For any* $\lambda \in \Sigma'$,

$$(7.11) \quad (\lambda - B_c)^{-1} = r(\lambda - B_c^\infty)^{-1}\, e + S_c(\lambda)\,(I - T_c(\lambda))^{-1}\, \tilde{M}_0(\lambda - B_c^\infty)^{-1}\, e.$$

Proof. Put $\Sigma'' = \mathbb{C}_+(-\nu_0 + \|K_c\|) \subset \Sigma'$, the norm being that of $\mathbb{B}(L^2)$. Let $\lambda \in \Sigma''$. By Proposition 7.6(ii), $\lambda \in \varrho(B_c)$ and $(\lambda - B_c)^{-1} \in \mathbb{B}(L^2)$ exists ([10], Theorem IX.2.1). Put $R_1 = r(\lambda - B_c^\infty)^{-1}\, e$, $R_2 = R_c(\lambda)\, \tilde{M}_0 R_1$ and $R = R_1 + R_2$. Proceeding as in the proof of Proposition 7.6(iii), we find that $R \in \mathbb{B}(L^2)$ with a range in $D(B_c)$ and satisfies

$$(7.12) \qquad (\lambda - B_c)\, R = I - K_c R_2 \equiv I - UV,$$

where $U = K_c R_c(\lambda) \in \mathbb{B}(Y^{2,-}, L^2)$ and $V = \tilde{M}_0 R_1 \in \mathbb{B}(L^2, Y^{2,-})$. Note that

$$(7.13) \qquad \text{(i)} \qquad \varrho(UV) \setminus \{0\} = \varrho(VU) \setminus \{0\},$$

$$\text{(ii)} \qquad (\mu - UV)^{-1} = \frac{1}{\mu}(I + U(\mu - VU)^{-1} V),$$

$$V(\mu - UV)^{-1} = (\mu - VU)^{-1} V, \quad \mu \in \varrho(VU) \setminus \{0\}.$$

The proof is easy and is omitted. Since $VU = T_c(\lambda)$, (7.13)(i) and Proposition 7.5(i) state that $(I - UV)^{-1} \in \mathbb{B}(L^2)$ exists. Then (7.12) yields

$$(\lambda - B_c)^{-1} = R(I - UV)^{-1} = R_1(I - UV)^{-1} + R_c(\lambda)\, V(I - UV)^{-1}.$$

Rewrite the last member by the aid of (7.13)(ii) with $\mu = 1$ to deduce (7.11) for $\lambda \in \Sigma''$. By Proposition 7.5(ii) and by Lemma 7.3(ii) and its proof, the right member of (7.11) exists for all $\lambda \in \Sigma'$ and is analytic in λ, both in $\mathbb{B}(L^2)$. Now the proof of the theorem is complete.

Define $\bar{\Sigma}' = \Sigma(\bar{a}_0, \bar{\sigma}_0) \times (B[\bar{c}_0] \setminus \{0\})$, and

$$X^p_\beta = \tilde{L}^{p,\infty}_{\beta-1/p} \cap L^\infty_\beta, \qquad Z^q = L^2 \cap \tilde{L}^{q,2}.$$

The main result of this section is as follows.

Theorem 7.8. *Let* $n \geqq 3$, $1 \leqq q \leqq 2 \leqq p \leqq \infty$, $\beta > \frac{n}{2}$, $\theta \in [0, 1)$, $\alpha \in [0, 1]$ *and* $m = 0, 1$. *Put* $\gamma = 1 + \frac{1}{q} - \frac{1}{p}$ *and suppose*

$$(7.13) \qquad \frac{1}{q} - \frac{1}{p} > \frac{2 - m}{n + \theta}, \qquad \frac{1}{p} < 1 - \frac{2}{n + \theta}.$$

(i) *There is a constant* $C \geqq 0$ *such that*

$$\|(\lambda - B_c)^{-1} (I - P_c)^m \Lambda^\alpha_c u\|_{\tilde{L}^{p,\infty}_{\beta-1/p}} \leqq C\,|c|^{-\theta\gamma} (\|u\|_{X^p_\beta} + \|\Lambda^\alpha_c u\|_{Z^q})$$

for all $(\lambda, c) \in \bar{\Sigma}$.

(ii) *For each* $a > 0$,

$$\chi(|\xi| < a)\,(\lambda - B_c)^{-1} (I - P_c)^m \Lambda^\alpha_c \chi(|\xi| < a) \in \mathscr{B}^0(\bar{\Sigma}'; \mathbb{B}(X^p_\beta \cap Z^q, \tilde{L}^{p,\infty}_{\beta-1/p})).$$

Here $\bar{\Sigma}'$ *can be replaced by* $\bar{\Sigma}$ *if* $\theta = 0$.

Proof. (i) will follow if each term on the right-hand side of (7.11) is evaluated. The first term was evaluated in Theorem 6.7(i), which delivers the first condition of (7.13). Combine Theorem 6.7(i) for $r = \infty$, $q = 1$ and $m = \alpha = 0$ with Lemma 7.2(i) and (ii). Then if the second condition of (7.13) is satisfied,

$$\|S_c(\lambda)\, h\|_{\tilde{L}^{p,\infty}_{\beta-1/p}} \leqq C(\|h\|_{\tilde{Y}^{p,\infty,-}_{\beta-1/p}} + |c|^{-\theta\gamma'} \|h\|_{Y^{2,-}}),$$

where $\gamma' = 1 - p^{-1}$. Put $h = (I - T_c(\lambda))^{-1} h'$. By Proposition 7.5 for $p = \infty$ and $p = 2$, and by (7.6),

$$\|h\|_{\tilde{Y}^{p,\infty,-}_{\beta-1/p}} \leqq C\,\|h\|_{Y^{\infty,-}_{\beta}} \leqq C\,\|h'\|_{Y^{\infty,-}_{\beta}},$$

$$\|h\|_{Y^{2,-}} \leqq C\,\|h'\|_{Y^{2,-}}.$$

Now put $h' = \tilde{M}_0(\lambda - B_c)^{-1}\, e(I - P_c)^m\, \Lambda_c^{\alpha} u$, and use Theorem 6.9(i) twice for $p = r = \infty$ and for $p = r = 2$, $\beta = 0$ to obtain estimates of h' in $Y^{\infty,-}_{\beta}$ and $Y^{2,-}$ respectively. The restriction for q in Theorem 6.9 is absorbed in the first of (7.13). Now we can conclude (i), and by taking into account the statements regarding continuity in theorems and assumptions, then we also conclude (ii).

As a corollary, we state the

Proposition 7.9. *Let* $u = u(c)$ *be a function of* c *such that*

$$u(c) \in L^{\infty}(B[\bar{c}_0];\, X^p_{\beta}) \cap \mathscr{B}^0(B[\bar{c}_0];\, X^p_{\beta-\varepsilon}), \qquad \Lambda_c^{\alpha} u(c) \in \mathscr{B}^0(B[\bar{c}_0];\, Z^q).$$

Under the conditions of Theorem 7.8, suppose that $\delta > \theta\gamma$ *and put*

$$v(\lambda, c) = |c|^{\delta}\, (\lambda - B_c)^{-1}\, (I - P_c)^m\, \Lambda_c^{\alpha} u(c),$$

for $c \neq 0$, *and put* $v(\lambda, 0) = 0$. *Then for any* $\varepsilon > 0$,

$$v(\lambda, c) \in L^{\infty}(\bar{\Sigma};\, \tilde{L}^{p,\infty}_{\beta-1/p}) \cap \mathscr{B}^0(\bar{\Sigma};\, \tilde{L}^{p,\infty}_{\beta-1/p-\varepsilon}).$$

Proof. Since $\delta > \theta\gamma$, Theorem 7.8(i) leads to

$$v(\lambda, c) \in L^{\infty}(\bar{\Sigma};\, \tilde{L}^{p,\infty}_{\beta-1/p}),$$

$$v(\lambda, c) \to 0 \ (c \to 0) \text{ strongly in } \tilde{L}^{p,\infty}_{\beta-1/p}, \text{ uniformly for } \lambda \in \Sigma(\bar{a}_0, \bar{\sigma}_0).$$

The first of these shows that

$$\|\chi(|\xi| > a)\, v(\lambda, c)\|_{\tilde{L}^{p,\infty}_{\beta-1/p-\varepsilon}} \leqq (1 + a)^{-\varepsilon}\, \|v(\lambda, c)\|_{\tilde{L}^{p,\infty}_{\beta-1/p}} \to 0 \qquad (a \to \infty),$$

uniformly for $(\lambda, c) \in \bar{\Sigma}$. Similarly, if $v'(\lambda, c)$ denotes the function $v(\lambda, c)$ with $\chi(|\xi| > a)\, u(c)$ substituted for $u(c)$, then Theorem 7.8(i) yields

$$\|v'(\lambda, c)\|_{\tilde{L}^{p,\infty}_{\beta-1/p-\varepsilon}} \leqq C((1 + a)^{-\varepsilon}\, \|u(c)\|_{X^p_{\beta}} + \|\Lambda_c^{\alpha} \chi(|\xi| > a)\, u(c)\|_{Z^q}),$$

whence $v'(\lambda, c) \to 0$ $(a \to \infty)$ in $\tilde{L}^{p,\infty}_{\beta-1/p-\varepsilon}$ uniformly in $\bar{\Sigma}$. Now Theorem 7.8(ii) completes the proof of the proposition.

This proposition indicates that at the point $\lambda = 0 \in \sigma(B_c)$ (see Theorem 7.7(i)), B_c enjoys what is called a limiting absorption principle, familiar in scattering theory. To be precise, put

$$v(\lambda) = (\lambda - B_c)^{-1}\, u, \qquad u \in L^2.$$

In the sequel $c \in B[\bar{c}_0]$ is fixed. In view of Theorem 7.7(i), $v(\lambda)$ is analytic and hence continuous in L^2 in $\lambda \in \Sigma(\bar{a}_0, \bar{\sigma}_0) \setminus \{0\}$, but it is not at $\lambda = 0$. However, Proposition 7.9 with $m = \alpha = 0$ states that if $u \in X^p_\beta \cap Z^q \subset L^2$, there is a unique limit $v(0) \in \tilde{L}^{p,\infty}_{\beta-1/p}$ such that

$$v(\lambda) \to v(0)\ (\lambda \to 0, \lambda \in \Sigma(\bar{a}_0, \bar{\sigma}_0)) \text{ in } \tilde{L}^{p,\infty}_{\beta-1/p-\varepsilon}. \tag{7.14}$$

This fact ensures that B_c has an inverse in the following sense. Define

$$D = \{v(0) \mid u \in X^p_\beta \cap Z^q\}.$$

Proposition 7.10. *Let p, q, β be as in Theorem 7.8 for* $m = \alpha = 0$.

(i) $D \subset \{v \in W^{p,+}_{\nu_c}(Q) \mid \gamma^- v \in Y^{p,-}, \tilde{M}_0 v = 0\}$.

(ii) *The operator B_c of* (1.13) *is bijective as a map*

$$B_c;\, D \to X^p_\beta \cap Z^q,$$

with $B_c^{-1} \in \mathbb{B}(X^p_\beta \cap Z^q, \tilde{L}^{p,\infty}_{\beta-1/p})$.

Proof. Let $X = X^p_\beta \cap Z^q$ and let $v(\lambda)$ be as above with $u \in X$. Put $w(\lambda) = -\lambda v(\lambda) + K_c v(\lambda) + u$. If $\lambda \neq 0$, then $v(\lambda) \in D(B_c)$ and

$$(\xi \cdot \nabla_x + \nu_c(\xi))\, v(\lambda) = w(\lambda) \text{ in } Q, \tag{7.15a}$$

$$\gamma^- v(\lambda) = M_0 \gamma^+ v(\lambda) \text{ on } S^- \tag{7.15b}$$

hold in the L^2-sense. This is also the case in the L^p-sense. For, since $\tilde{L}^{p,\infty}_{\beta-1/p} \subset L^p$ for $\beta > (n+1)/p$, then $u \in L^p$ and also $v(\lambda) \in L^p$ by Theorem 7.8(i). Consequently so is $w(\lambda)$, and thereby (7.15a) implies that $v(\lambda) \in W^{p,+}_{\nu_c}(Q)$ and is satisfied in L^p. By Theorem 4.1, then, $\gamma^+ v(\lambda) \in Y^{p,+}$ exists and (7.15b) holds in $Y^{p,-}$ since

$$M_0 \in \mathbb{B}(Y^{p,+}, Y^{p,-}), \quad p \in [2, \infty]$$

(interpolation between $[\mathrm{M}]_2$(i) and (iii), [2]). Thus $v(\lambda)$ is a solution to (7.15) in the L^p-sense. Next, note from (7.14) that $v(\lambda) \to v(0)$ in L^p and thereby $w(\lambda) \to w(0) = -K_c v(0) + u$ in L^p as $\lambda \to 0$. Take the limit as $\lambda \to 0$ in (7.15). On account of the completeness of $W^{p,+}_{\nu_c}(Q)$ and of Theorem 4.1(i), statement (i) of the theorem follows, and, moreover, it is found that $B_c v(0) = -u$ holds in the L^p-sense. The latter means that the map B_c; $D \to X$ is surjective. Suppose $u = 0$. Then $v(\lambda) = 0$ for $\lambda \neq 0$, so $v(0) = 0$ by (7.14), that is, B_c is one-to-one. Now the proof of (ii) is completed by Theorem 7.8.

8. Solutions of the Boundary-Value Problem (1.14)

Let $\phi = \phi_c$ be a solution to (1.14) and put $\psi_1 = R_c(0)\, h_c$, $\psi_2 = \phi_c - \psi_1$, where h_c is defined in (1.11). In view of (7.1) and Proposition 7.1(ii), (1.14) reduces to

$$\begin{aligned} (-\xi \cdot \nabla_x + L_c)\, \psi_2 &= -K_c \psi_1 \quad \text{in } Q, \\ \tilde{M}_0 \psi_2 &= 0 \text{ on } S^-. \end{aligned} \tag{8.1}$$

Formally, this can be written as $B_c\psi_2 = -K_c\psi_1$, and so

$$\phi_c = R_c(0)\, h_c - B_c^{-1} K_c R_c(0)\, h_c \tag{8.2}$$

is expected to solve (1.14). To verify that it does, we need the

Lemma 8.1. $\forall\, p, r \in [1, \infty]$, $\forall\, \beta \in \mathbb{R}$,

$$h_c \in \mathscr{E}^0(\mathbb{R}^n_c;\, \tilde{Y}^{p,r,-}_\beta), \qquad \|h_c\| = O(|c|) \qquad (c \to 0).$$

Proof. Since $Y^{\infty,-}_\gamma \subset \tilde{Y}^{p,r,-}_\beta$ with continuous injection if γ is large enough, one has only to prove the lemma for $p = r = \infty$. In view of $[\mathrm{M}]_2$(ii) and Remark 1.1, h_c can be written as $h_c = -\tilde{M}_0 \tilde{g}_c$, where $\tilde{g}_c = g_0^{-\frac{1}{2}} g_c - g_0^{\frac{1}{2}}$. Observe that

$$\gamma^\pm \tilde{g}_c \in \mathscr{E}^0(\mathbb{R}^n_c;\, Y^{\infty,\pm}_\beta),\ \|\gamma^\pm h g_c\| = O(|c|) \qquad (c \to 0),$$

the last of which is obtained using $\tilde{g}_c(\xi) = c \cdot (\xi - \mu c)\, g_0^{-\frac{1}{2}}\, g_{\mu c}$, $0 < \mu < 1$, (the mean-value theorem). This and $[\mathrm{M}]_2$(iii) yield the lemma.

Theorem 8.2. *Let ϕ_c be given by (8.2). Let $n \geqq 3$, $p \in [2, \infty]$, $\beta \geqq 0$ and $\theta \in [0, 1)$. Put $\gamma = 2 - 1/p$ and suppose*

$$\frac{1}{p} < 1 - \frac{2}{n + \theta}. \tag{8.3}$$

Let $\bar{c}_0 > 0$ be that of Proposition 7.5.

(i) $\phi_c \in \mathscr{B}^0(B[\bar{c}_0];\, \tilde{L}^{p,\infty}_\beta)$, $\quad \|\phi_c\| = O(|c|^{1-\theta\gamma})$.

(ii) *For each $c \in B[\bar{c}_0]$, ϕ_c is in $W^{p,+}_{v_c}(Q)$ with $\gamma^- \phi_c \in Y^{p,-}$ and solves (1.14) uniquely in the L^p-sense.*

Proof. (i) will follow if (i) is true for ψ_j, $j = 1, 2$. For $j = 1$, one has only to combine lemmas 7.2(ii), 8.1 with the fact that $\chi(|\xi| > a)\, \psi_1 \to 0$ $(a \to \infty)$ in $\tilde{L}^{p,\infty}_\beta$ locally uniformly in $c \in \mathbb{R}^n$ (*cf.* the proof of Proposition 7.9). Note that p and β can be arbitrary here. Put $u(c) = K_c \psi_1$. Clearly, it satisfies the first condition for $u(c)$ of Proposition 7.9 for any p, β, and thereby also the second, because $\tilde{L}^{p,\infty}_\beta \subset \tilde{L}^{p,2}_1$ if $\beta > (n+1)/2$ and because $\Lambda^\alpha_c \in \mathrm{B}(\tilde{L}^{p,r}_{\beta+\alpha}, \tilde{L}^{p,r}_\beta)$ by (3.4). One can choose $q = 1$. Then Proposition 7.9 proves (i) for ψ_2, in which (7.13) becomes identical with (8.3). Now, one can note from Proposition 7.10 that ψ_2 is a unique L^p-solution to (8.1), and proceeding as in the proof of Proposition 7.10, that ψ_1 is a unique L^p-solution to (7.1) with $h = h_c$, whence (ii) follows.

9. Construction of Steady Solutions

First we shall show that the suitable space in which (1.15) is to be solved is

$$X^p_\beta = L^\infty_\beta(Q) \cap \tilde{L}^{p,\infty}_{\beta-1/p}(Q),$$

which was introduced in § 7.

Proposition 9.1. *Let* $n \geqq 3$ *and let* $\bar{c}_0$ *be as in Theorem 7.8. Suppose that* $p \in [2, 4]$ *and* $\theta \in [0, 1)$ *be such that*

$$\frac{n+\theta}{n+\theta-2} < p < n+\theta, \tag{9.1}$$

and let $\beta > \frac{n}{2} + 1$. *Then there is a constant* $C \geqq 0$, *independent of* c, u, v, *such that for all* $c \in B[\bar{c}_0]$,

$$\| B_c^{-1} \Gamma_0[u, v] \|_{X_\beta^p} \leqq C\, |c|^{-\theta(1+2/p)} \| u \|_{X_\beta^p} \| v \|_{X_\beta^p}. \tag{9.2}$$

Proof. Put $w = \Lambda_c^{-1} \Gamma_0[u, v]$ and note that for any $r \geqq s/2$,

$$\| u \|_{\tilde{L}_\beta^{2r,\infty}} \leqq \| u \|_{\tilde{L}_\beta^\infty}^{(2r-s)/2r} \| u \|_{\tilde{L}_\beta^{s,\infty}}^{s/2r}.$$

Then Lemma 3.9(ii) for $r = s$ and (3.4) yield for $q \in [1, 2]$,

$$\| w \|_{X_\gamma^p} \leqq C \| u \|_{X_\gamma^p} \| v \|_{X_\gamma^p}, \tag{9.3}$$

$$\| \Lambda_c w \|_{Z^q} \leqq C(\| w \|_{\tilde{L}_\alpha^{2,\infty}} + \| w \|_{\tilde{L}_\alpha^{q,\infty}}) \leqq C \| u \|_{X_\alpha^{2q}} \| v \|_{X_\alpha^{2q}}, \tag{9.4}$$

with $\alpha > \frac{n}{2} + 1$. Recall Lemma 3.9(i) and use Theorem 7.8 for w with $m = \alpha = 1$. In virtue of (9.3) and (9.4), one then gets

$$\| B_c^{-1} \Lambda_c w \|_{\tilde{L}_\gamma^{p,\infty}} \leqq C\, |c|^{-\theta(1+1/q)} (\| u \|_{X_{\gamma+1/p}^p} \| v \|_{X_{\gamma+1/p}^p} + \| u \|_{X_\alpha^{2q}} \| v \|_{X_\alpha^{2q}}). \tag{9.5}$$

Put $q = p/2$. Then (7.13) turns to (9.1) with $p \in [2, 4]$. Use (9.5) twice with $p = \infty$, $\alpha = \gamma = \beta$, and with $p = p$, $\alpha = \beta$, $\gamma = \beta - 1/p$, to deduce (9.2).

If $\theta \neq 0$, (9.2) becomes meaningless as $c \to 0$. Nevertheless, it can be used to solve (1.15). Choose an α such that

$$\theta\left(1 + \frac{2}{p}\right) < \alpha < 1 - \theta\left(2 - \frac{1}{p}\right), \tag{9.6}$$

which is possible for all $p \geqq 2$ if $\theta \in [0, 2/7]$. Put $u = |c|^\alpha v$ in (1.15) and note that Γ_0 is bilinear. Then v should solve

$$v = |c|^{-\alpha} \phi_c - |c|^\alpha B_c^{-1} \Gamma_0[v, v] \tag{9.7}$$

when $c \neq 0$. Write this right-hand side as $H(v, c)$ and put $H(v, 0) = 0$. If $v = v(c)$ is a function of c, so is $H(v(c), c)$. Define the nonlinear map $\tilde{H}$ by $\tilde{H}[v](c) = H(v(c), c)$. Let $\| \ \|$ denote the norm of X_β^p, and for each $a, \bar{c}, \varepsilon > 0$, define the space $V = V(a, \bar{c}, \varepsilon, p, \beta)$ by

$$V = \left\{ v(c) \in L^\infty(B[\bar{c}]; X_\beta^p) \cap \mathscr{B}^0(B[\bar{c}]; X_{\beta-\varepsilon}^p) \;\middle|\; \sup_{|c| \leqq \bar{c}} \| v(c) \| \leqq a \right\},$$

which is a complete metric space with the natural metric

$$d(u, v) = \sup_{|c| \leqq \bar{c}} \| u(c) - v(c) \|.$$

Lemma 9.2. *Under the assumptions of Proposition 9.1, suppose* $\theta \in [0, 2/7)$. *Then there are positive constants* a *and* $\bar{c}$ *such that for any* $\varepsilon > 0$, *the map* $\tilde{H}$; $V \to V$ *is a contraction.*

Proof. Put $\sigma = 1 - \theta\left(2 - \frac{1}{p}\right) - \alpha$, $\tau = \alpha - \theta\left(1 + \frac{2}{p}\right)$. By virtue of Theorem 8.2(i) and Proposition 9.1, there are constants $C_1, C_2 > 0$ such that

$$(9.8)\qquad \begin{aligned} \|H(v, c)\| &\leq C_1 |c|^\sigma + C_2 |c|^\tau \|v\|^2, \\ \|H(v, c) - H(v', c)\| &\leqq C_2 |c|^\tau \|v + v'\| \|v - v'\|. \end{aligned}$$

for all $c \in B[\bar{c}_0]$. Since $\sigma, \tau > 0$ by (9.6), we can choose a $\bar{c}$ such that $0 < \bar{c} < \min(\bar{c}_0, (4C_1C_2)^{1/(\sigma+\tau)})$. Put $\mu = 1 - (1 - 4C_1C_2\bar{c}^{\sigma+\tau})^{\frac{1}{2}}$ and $a = \mu/(2C_2\bar{c}^\tau)$. It now follows from (9.8) that if v, $v' \in V$ and if $c \in B[\bar{c}]$, then

$$\|\tilde{H}[v](c)\| \leqq C_1\bar{c}^\sigma + C_2\bar{c}^\tau a^2 = a,$$

$$\|\tilde{H}[v](c) - \tilde{H}[v'](c)\| \leqq \mu \|v(c) - v'(c)\|.$$

Note that $\mu \in (0, 1)$. Moreover, it can easily be checked by the aid of (9.3) and (9.4) that Proposition 7.9 applies to $u(c) = \Lambda_c^{-1}\Gamma_0[v(c), v(c)]$. Combining these conclusions yields the lemma.

This lemma indicates that $\tilde{H}$ has a unique fixed point $v = v(c)$ in V. Obviously this $v(c)$ solves (9.7), so $u(c) = |c|^\alpha v(c)$ is a solution of (1.15). Note that $u(c) = |c|^\alpha H(v(c), c)$ and use Proposition 7.10 and Theorem 8.2(ii) to see that $u(c) \in W^{p,+}_{v_c}(Q)$ with $\gamma^- u(c) \in Y^{p,-}$ and that $u(c)$ satisfies (1.12) in the L^p-sense. Thus we have proven the main result of this paper stated as

Theorem 9.3. *Suppose* $[\Theta]$, $[q]$, $[M]$ *of* § 1 *be satisfied. Let* $n \geqq 3$, $\theta \in [0, 2/7)$ *and* $\beta > \frac{n}{2} + 1$, *and choose a* $p \in [2, 4]$ *satisfying* (9.1) *and an* α *satisfying* (9.6). *Then there is a positive number* $\bar{c}$ *such that for each* $c \in B[\bar{c}]$, (1.12) *admits a unique* L^p*-solution* $u(c)$ *satisfying*

$$u(c) \in \mathscr{B}^0(B[\bar{c}]; X^p_\beta), \qquad \|u(c)\| = O(|c|^\alpha).$$

Remark 9.4. If $n \geqq 4$, we may take $\theta = 0$ in (9.1), and Theorem 9.3 is valid for $p \in (2, 4)$ if $n = 4$ and $p \in [2, 4]$ if $n \geqq 5$. When $n = 3$, however, (9.1) becomes vacuous for $\theta = 0$ and hence $\tilde{H}$ cannot be shown to be a contraction. In [15] we were able to derive only for $\theta = 0$ the estimates obtained so far, including (9.2), and so the physically important case $n = 3$ was handled by use of Nash's implicit function theorem supplemented by decay estimates for ϕ_c for large x. The estimates, non-uniform in c, which are derived in this paper permit us to use a much simpler contraction mapping principle.

Appendix

In this appendix we prove Lemma 6.5. Define the integral

$$I_0 = I_0(\lambda, a, b, l, m) = \int_0^a |\lambda - ib\varkappa + \varkappa^2|^{-l} \varkappa^m \, d\varkappa.$$

Write $\lambda = \sigma + i\tau$ and for each $b_0 > 0$, put

$$\Sigma_0 = \{\lambda \in \mathbb{C} \mid -\sigma \leqq |\tau|^2/\{2(b_0+1)\}^2\}.$$

Lemma A.1. *Let* $a, b_0, m \geqq 0, l \geqq 1, \theta \in [0, 1)$. *Put* $\gamma = m + 1 + \theta - 2l$ *and suppose* $\gamma \neq 0$. *There is a constant* $C \geqq 0$ *and for all* $b \in [-b_0, b_0]$ *and* $\lambda \in \Sigma_0$,

$$I_0 \leqq C\,|b|^{-\theta}\,|\tau|^{\min(0,\gamma)}.$$

Proof. Put $\varkappa = |\tau|\, y$ and $\zeta = \sigma/|\tau|^2$. Then

$$I_0 \leqq 2\,|\tau|^{m+1-l} \int_0^{a/|\tau|} (|\tau|\,|\zeta + y^2| + |1 - by|)^{-l}\, y^m \, dy.$$

Write the last integral as I_1 and split it into two:

$$I_1 = \int_{J_2} + \int_{J_3} \equiv I_2 + I_3$$

with $J_2 = [0, b_1]$, $J_3 = [b_1. a/|\tau|]$, where $b_1 = 1/(b_0 + 1)$. If $y \in J_2$, then $|1 - by| \geqq 1 - |b|\, b_1 \geqq b_1$ and hence

$$I_2 \leqq C b_1^{m+1-\theta}.$$

Let $y \in J_3$. Then $|\zeta + y^2| \geqq y^2 - b_1^2/2 \geqq y^2/2$ for $\lambda \in \Sigma_0$, and since $(|\tau|\, y^2/2 + |1 - by|)^l \geqq (|\tau|\, y^2/2)^{l-\theta}\, |1 - by|^\theta$,

$$I_3 \leqq C\,|\tau|^{-l+\theta} \int_{J_3} y^{m-2(l-\theta)}\, |1 - by|^{-\theta}\, dy.$$

Denote the last integral as I_4 and split it as follows:

$$I_4 = \int_{J_5} + \int_{J_6} \equiv I_5 + I_6.$$

where $J_5 = J_3 \cap [2/(3\,|b|), 2/|b|]$, $J_6 = J_3 \setminus J_5$. If $a/|\tau| < 2/(3\,|b|)$, then $J_5 = \varphi$ and $I_5 = 0$. If not, then

$$I_5 \leqq C\,|b|^{-(m+2\theta-2l)} \int_{J_5} |1 - by|^{-\theta}\, dy \leqq C\,|b|^{-\theta-\gamma} \leqq C\,|b|^{-\theta}\,|\tau|^{-\gamma'},$$

where $\gamma' = \max(0, \gamma)$. Finally if $y \in J_6$, then $|1 - by| \geqq |b|\, y/2$, so

$$I_6 \leqq C\,|b|^{-\theta} \int_{J_4} y^{\gamma-1}\, dy \leqq C\,|b|^{-\theta}\,|\tau|^{-\gamma'}$$

with the same γ' as above. Combining all these estimates completes the proof of the lemma.

Let $\mu(k, c)$ denote any one of the $\mu_j(k, c)$ of Theorem 6.2(i), and α, β corresponding α_j, β_j of Theorem 6.2(ii). Note that $\alpha \in \mathbb{R}$, $\beta > 0$ and that $\alpha = 0$ occurs for some j's (see [4]). Put

$$\mu_0(k, c) = -\beta\,|k|^2 + i\alpha\,|k| + ik \cdot c.$$

Note from Theorem 6.2(ii) that

$$\mu(k, c) = \mu_0(k, c) + O(|k|^3) \quad (|k| \to 0). \tag{A.1}$$

Let $\varkappa_0$, a_0, σ_0 be as in (6.6) and (6.7), and define

$$\Sigma = \Sigma(a_0, \sigma_0) \times B[\varkappa_0] \times B[c_0].$$

Lemma A.2. $\forall\,(\lambda, k, c) \in \Sigma$, $\;|\lambda - \mu(k, c)| \geqq \frac{1}{2}\,|\lambda - \mu_0(k, c)|$.

Proof. First we prove that

$$|\lambda - \mu_0(k, c)| \geqq \eta_0\,|k|^2 \tag{A.2}$$

holds in Σ with some constant $\eta_0 > 0$. Put $\lambda = \sigma + i\tau$. When $\sigma \geqq 0$, (A.2) is obvious because $\beta > 0$. Thus we suppose $\sigma < 0$. If $|\sigma| \leqq \beta\,|k|^2/2$, then $|\sigma + \beta\,|k|^2| \geqq \beta\,|k|^2/2$ and (A.2) follows. If not, then $|\tau| \geqq (|\sigma|/a_0)^{\frac{1}{2}} \geqq \eta_1\,|k|$ for $\lambda \in \Sigma(a_0, \sigma_0)$ with $\eta_1 = \delta + c_0 + 1$ (see (6.7)). Therefore $|\tau - \alpha\,|k| - k \cdot c| \geqq |\tau| - (|\alpha| + c_0)\,|k| \geqq |k|^2/\varkappa_0$ in Σ, since we may assume without loss of generality that $|\alpha| \leqq \delta$ in (6.6). This completes the proof of (A.2). In view of (A.1) we may also assume that $\varkappa_0$ is so chosen that $|\mu(k, c) - \mu_0(k, c)| \leqq \eta_0\,|k|^2/2$. This and (A.2) then prove the lemma.

Proof of Lemma 6.5. Let $I = I(\lambda, c, l, m)$ be the integral of Lemma 6.5(i). By Lemma A.2,

$$I \leqq 2^{-l} \int_{B[\varkappa_0]} |\lambda - \mu_0(k, c)|^{-\theta}\,|k|^m\,dk.$$

Denote the last integral by I_7, and put $\varkappa = \sqrt{\beta}\,|k|$, $t = k \cdot c/(|k|\,|c|)$. Then

$$I_7 \leqq C\beta^{-(m+n)/2} \int_{-1}^{1} I_0\left(\lambda, \sqrt{\beta}\,\varkappa_0, \frac{\alpha + |c|\,t}{\sqrt{\beta}}, l, m + n - 1\right)(1 - t^2)^{(n-3)/2}\,dt.$$

Use Lemma A.1 and note that for $\theta \in [0, 1)$,

$$\int_{-1}^{1} |\alpha + |c|\,|t||^{-\theta}\,(1 - t^2)^{(n-3)/2}\,dt \leqq C \times \begin{cases} 1, & \alpha \neq 0, \\ |c|^{-\theta}, & \alpha = 0. \end{cases}$$

Then Lemma 6.5(i) follows. Next, observe the inequality

$$|a^{-1} - b^{-1}| \leqq |a - b|^{\varepsilon}\,(|a|^{-\gamma} + |b|^{-\gamma}), \gamma = 1 + \varepsilon, \tag{A.3}$$

for $a, b \in \mathbb{C}$ and $\varepsilon > 0$. Put $a = \lambda - \mu(k, c)$, $b = \lambda' - \mu(k, c')$ and $\varepsilon = \delta/l$, and let J be the integral of Lemma 6.5(ii). Then by (A.3),

$$J \leqq (|\lambda - \lambda'| + \varkappa_0\,|c - c'|)^{\delta}\,\{I(\lambda, c, l + \delta, m) + I(\lambda', c', l + \delta, m)\}.$$

Combine this and Lemma 6.5(i) to conclude Lemma 6.5(ii).

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