Thermodynamics and Stability

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Contents

I. Introduction

The potential energy of an elastic body $\mathscr{B}$ in a configuration $\boldsymbol{x}$ has the form

$$U(\boldsymbol{x}) = U_e(\boldsymbol{x}) + \int_{\mathscr{B}} w(\boldsymbol{F})\, dm, \tag{P}$$

where $U_e(\boldsymbol{x})$ is the potential energy of the external loads, assumed conservative, w is the specific stored energy (strain energy), $\boldsymbol{F} = \nabla \boldsymbol{x}$, and dm is the mass measure on $\mathscr{B}$. The energy criterion for stability asserts that the stable configurations of $\mathscr{B}$ are exactly those that render U a minimum.[1] This criterion neglects thermodynamic influences and, in addition, is statical in nature. Thus, since many nonelastic materials exhibit elastic behavior in statical situations, the following questions seem appropriate:

(a) Can the energy criterion be given a precise dynamical significance?

(b) Does the energy criterion apply when thermal effects are present?

(c) Does the energy criterion apply when the material is not elastic?

(d) If the answer to (b) or (c) is yes, what interpretation should be given the stored energy?

[1] The minimum energy criterion is discussed at length in the treatise of KNOPS & WILKES (1973).

Recent studies,[1] all based, essentially, on two important papers of ERICKSEN (1966a, b), have shown that the answer to (a) and (b) is a qualified "yes"; and that the answer to (c) is also a qualified "yes" for materials with fading memory, for materials described by internal state variables, and, more generally, for any type of material that has, associated with it, an equilibrium free-energy. As a matter of fact, the work of COLEMAN & DILL (1973) and GURTIN (1973c) shows, in answer to (d), that the stored energy should, in all cases, be interpreted as the equilibrium free-energy evaluated at the environmental temperature.

With the exception of (1973c), each of the foregoing investigations is concerned with a single class of materials, and it is not immediately apparent which of the corresponding results are generic in nature, applicable to large classes of materials, and which are limited to the specific material under consideration. It seems reasonable to ask if there is, in fact, a general theory of stability which leads to appropriate generalizations of the above results, and which lays bare the simple concepts which must form the essence of the subject. The main purpose of this paper is to develop such a theory.

Chapter III begins with a fairly general treatment of the thermodynamics of continuous bodies. There we define state functions such as the total[2] energy E and entropy S, and we introduce the concept of an environmental temperature[3] θ_e. We show, as a consequence of the first two laws, that for certain types of environments

$$V = U_e + E - \theta_e S$$

is a natural Lyapunov function;[4] that is, V decreases with time on processes that are consistent with the environment. We then give the form V takes when the material is elastic, and, for this type of material, list its most important properties.

In Chapter IV we develop a general theory of dynamical systems. While our theory is motivated by the results of Chapter III, it is sufficiently general to describe the behavior of most materials now considered important in continuum mechanics. We define a dynamical system[5] to be a triplet $(\Sigma, \mathscr{P}, \text{Equil})$, where Σ is a set called the state space, $\mathscr{P}$ is a class of processes (functions of time with values in Σ), and Equil is an idempotent map from Σ into itself. The map Equil is a crucial ingredient in our theory, and its absence from other general theories of dynamical systems renders these theories virtually inapplicable to non-linear continuum mechanics. Our underlying physical assumption is that each possible state σ of the system has a unique equilibrium value Equil σ. For an elastic material a state is a triplet $(\boldsymbol{x}, \boldsymbol{v}, \theta)$, where $\boldsymbol{x}$ is a configuration, $\boldsymbol{v}$ a velocity field, and θ a temperature field, and

$$\text{Equil}(\boldsymbol{x}, \boldsymbol{v}, \theta) = (\boldsymbol{x}, \boldsymbol{0}, \theta_e),$$

where θ_e is the (constant) environmental temperature. For a material with memory a state involves the histories of $\boldsymbol{x}$ and θ as well as the present value of $\boldsymbol{v}$; the cor-

[1] KOITER (1969, 1971), GURTIN (1973a, b, c), COLEMAN & DILL (1973), KNOPS & WILKES (1973).

[2] Internal plus kinetic.

[3] It is not difficult to modify our theory to include a dependence of θ_e on time, provided $\dot{\theta}_e \geqq 0$. In this connection see GURTIN (1973c).

[4] In essence, this assertion is due to DUHEM (1911), Vol. 2, pp. 220–231.

[5] Our definition differs from the usual definition of a dynamical system (*cf.*, *e.g.*, ZUBOV (1964), HALE & INFANTE (1967), HALE (1969)) in two respects: (i) our processes are not generated by an operator semi-group; (ii) the introduction of the equilibrium map Equil. Also, in view of (ii), our definition differs from that of GILBERT & KNOPS (1967).

responding equilibrium state has $\boldsymbol{v}=\mathbf{0}$, the configuration history a constant equal to its present value, and the temperature history equal to the constant history with value θ_e.

We define a Lyapunov function V for the dynamical system to be a real-valued function of state that decreases with time on processes. In addition, we assume that the state function

$$T(\sigma)=V(\sigma)-V(\text{Equil}\,\sigma)$$

is non-negative on each process and vanishes on a process π only when π is a rest process. (This assumption is satisfied by all of the material bodies discussed previously.) We then define the potential energy U to be the restriction of V to equilibrium states; that is, to states of the form Equil σ.

The classical harmonic oscillator best illustrates the motivation behind these definitions. In this example a state is a pair (x, v), where x is the position coordinate and v the velocity of the oscillator, and

$$V(x, v)=\tfrac{1}{2}k\,x^2+\tfrac{1}{2}m\,v^2$$

with k the spring constant and m the mass. Equilibrium states[1] are then states of the form $(x, 0)$, so that $\text{Equil}(x, v)=(x, 0)$,

$$T(x, v)=\tfrac{1}{2}m\,v^2,$$

$$U(x, 0)=\tfrac{1}{2}k\,x^2.$$

Returning to the general theory, we show that the few simple axioms described above suffice to yield general forms of the results mentioned previously, as well as several new results. In particular, our main results can be stated roughly as follows:[2]

(i) When U has a local minimum at an equilibrium state σ_0, then there is at most one process starting at σ_0.

(ii) If an equilibrium state σ_0 is asymptotically stable with respect to orbits in the equilibrium space, then U must necessarily have a strict local minimum at σ_0.

(iii) If U has a global minimum at σ_0, then σ_0 is Lyapunov stable with respect to the Lyapunov topology; that is, with respect to the topology induced by the pseudo-metric $|V(\sigma)-V(\gamma)|$, $\sigma, \gamma\in\Sigma$.

(iv) If U has a strong local minimum at σ_0, then U is Lyapunov stable with respect to the combination of the Lyapunov topology and a certain topology induced on Σ by the equilibrium map.

The type of stability established in (iii) and (iv) is quite weak. But this is to be expected, since the underlying theory is aimed at large classes of materials. For individual materials one would expect stronger results. However, even in the simplest case of (non-linear) hyperelasticity non-trivial results concerning dynamical stability are lacking.

[1] Note that our notion of "equilibrium state" is much broader than the definition usually employed, under which only the state (0, 0) would qualify.

[2] For (i), (ii), and (iv) we need to assume that the space of equilibrium states is topological (indeed, for (i) and (ii), that it is a Hausdorff space); we need also certain mild continuity assumptions.

The chapter on dynamical systems is completely self contained; although motivated by – it does not make use of – specific continuum mechanical theories. For this reason it might be applicable, perhaps with minor modification, to areas outside of continuum mechanics.

In Chapter V we apply our results to specific materials. There we show that strengthened forms arise of the results established previously by KOITER (1969, 1971), GURTIN (1973a, b, c), COLEMAN & DILL (1973), and KNOPS & WILKES (1973). In particular, we show that for all of the specific materials considered here U has the form (P) with w the equilibrium free-energy at the environmental temperature.

II. Preliminary Definitions. Notation

1. Notation

Throughout this paper $\mathscr{E}$ denotes a three-dimensional Euclidean space with $\mathscr{V}$ the associated vector space. Lin is the space of linear transformations of $\mathscr{V}$ into itself;

$$\mathrm{Lin}^+ = \{\boldsymbol{F} \in \mathrm{Lin}: \det \boldsymbol{F} > 0\}.$$

We will consistently use the notation

$$\mathbb{R} = (-\infty, \infty), \qquad \mathbb{R}^+ = [0, \infty), \qquad \mathbb{R}^{++} = (0, \infty).$$

Let $\mathscr{B} \subset \mathscr{E}$ and let $\mathscr{W}$ denote either $\mathscr{E}$, $\mathscr{V}$, $\mathbb{R}$, or $\mathbb{R}^{++}$. We write $PC(\mathscr{B}, \mathscr{W})$ {respectively, $PC^1(\mathscr{B}, \mathscr{W})$} for the set of all piecewise continuous {respectively, continuous and piecewise smooth[1]} functions from $\mathscr{B}$ into $\mathscr{W}$. We will have frequent occasion to deal with functions $f(\boldsymbol{X}, t)$ of position $\boldsymbol{X} \in \mathscr{B}$ and time $t \in \mathbb{R}$. For such functions we write

$$\dot{f}(\boldsymbol{X}, t) = \frac{\partial}{\partial t} f(\boldsymbol{X}, t)$$

for the right-hand derivative with respect to time.

Given a function $f(a_1, a_2, \ldots, a_n)$ of several variables, we write

$$\partial_1 f = \frac{\partial f}{\partial a_1}, \ \partial_2 f = \frac{\partial f}{\partial a_2}, \ldots .$$

This notation is used also when certain of the a_i are vector or tensor variables, or even functions; the meaning will always be clear from the context.

2. Topological Considerations. Minima

Let Σ be a topological space with topology $\mathscr{S}$, and let $\sigma \in \Sigma$. The (open) neighborhood system of σ is the subfamily $\mathscr{S}_\sigma$ of $\mathscr{S}$ consisting of all open neighborhoods of σ. A **local base** B at σ is a subfamily B of $\mathscr{S}_\sigma$ such that every neighborhood of σ contains a member of B.

Let $\mathscr{T}$ denote a second topology for Σ. Then the **combination** of $\mathscr{S}$ and $\mathscr{T}$ is the smallest topology for Σ containing both $\mathscr{S}$ and $\mathscr{T}$. More precisely: the

[1] We use the term "smooth" as a synonym for "continuously differentiable".

combination of $\mathscr{S}$ and $\mathscr{T}$ is the family of all sets each of which is the union of a subfamily of the family

$$\{\Omega \cap \Lambda : \Omega \in \mathscr{S},\ \Lambda \in \mathscr{T}\}. \tag{2.1}$$

Proposition 2.1. *The family*

$$\{\Omega \cap \Lambda : \Omega \in \mathscr{S}_\sigma,\ \Lambda \in \mathscr{T}_\sigma\} \tag{2.2}$$

is a local base at σ with respect to the combination of the topologies $\mathscr{S}$ and $\mathscr{T}$.

Proof. Let $\mathscr{C}$ denote the combined topology. Trivially, (2.2) is a subfamily of $\mathscr{C}_\sigma$. Let Γ be an open neighborhood of σ. Then Γ is the union of a subfamily of (2.1), and, since $\sigma \in \Gamma$, σ must belong to at least one member Φ of this subfamily. Thus Φ belongs to (2.2) and $\Phi \subset \Gamma$. □

Let $F: \Sigma \to \mathbb{R}$, and let $\sigma \in \Sigma$. Then F has a **minimum at σ over $\Omega \subset \Sigma$** if

$$F(\sigma) \leqq F(\lambda) \quad \forall\, \lambda \in \Omega;$$

the minimum is **strict** if equality holds only for $\lambda = \sigma$. F has a **global minimum** at σ if F has a minimum at σ over Σ.

Let Σ be endowed with a topology. Then F has a **local minimum** {respectively, **strict local minimum**} at σ if F has a minimum {respectively, strict minimum} at σ over some neighborhood of Σ. A subset Ω of Σ is a **potential well** for F relative to σ if:

(a) Ω (with $\bar{\Omega} \subset \Sigma$) is an open neighborhood of σ;
(b) F has a minimum at σ over Ω;
(c) $\inf\{F(\lambda): \lambda \in \partial\Omega\} > F(\sigma)$.

F has a **strong local minimum** at σ if for some local base B at σ each $\Omega \in \mathsf{B}$ is a potential well for F relative to σ.

III. Motivation behind the General Theory

3. The Laws of Thermodynamics. The Lyapunov Function

Consider a body $\mathscr{B}$. In all of our applications $\mathscr{B}$ will be a continuous body; for the time being, however, it is not necessary to endow $\mathscr{B}$ with a mathematical structure. A **state** for $\mathscr{B}$ is a point σ in a set Σ, which we call the **state space.** The particular choice of Σ depends on the type of body under consideration; for example, the state of an elastic body is a triplet $(\boldsymbol{x}, \boldsymbol{v}, \theta)$, where $\boldsymbol{x}$ is a configuration. $\boldsymbol{v}$ is a velocity field, and θ is a temperature field.

A **process**[1] with values in Σ is a function

$$\pi: [0, d_\pi) \to \Sigma \quad (d_\pi > 0).$$

The number d_π, which may be ∞, is called the **duration** of π. For convenience, we write π_t for the value of π at *time* $t \in [0, d_\pi)$. Constant processes will be referred to as **rest processes.**

[1] *Cf.* NOLL (1972), p. 9.

A mapping

$$E: \Sigma \to \mathbb{R}$$

will be called a **state function.** Given a process π, we write $E(\pi)$ in place of $E \circ \pi$; that is, $E(\pi)$: $[0, d_\pi) \to \mathbb{R}$ is the function $t \mapsto E(\pi_t)$. Let $\mathscr{P}$ be a class of processes. Then E has a **trajectory derivative** over $\mathscr{P}$ if, for each $\pi \in \mathscr{P}$, $E(\pi)$ is differentiable on $[0, d_\pi)$; in this case we write $\dot{E}(\pi)$ for the function $t \mapsto \dfrac{d}{dt} E(\pi_t)$.

The thermodynamic behavior of $\mathscr{B}$ is governed by six state functions E, S, P, Q, J, and G, where for each state σ

$E(\sigma)$ is the (internal and kinetic) energy of $\mathscr{B}$ in σ,

$S(\sigma)$ is the entropy of $\mathscr{B}$ in σ,

$P(\sigma)$ is the power expended on $\mathscr{B}$ in σ,

$Q(\sigma)$ is the heat flow into $\mathscr{B}$ in σ,

$J(\sigma)$ is the entropy flow into $\mathscr{B}$ in σ,

$G(\sigma)$ is the entropy produced by $\mathscr{B}$ in σ.

We assume that for each $\sigma \in \Sigma$

$$G(\sigma) \geqq 0, \tag{3.1}$$

so that the entropy produced in each state is non-negative.

We consider now a given class $\mathscr{P}$ of processes over which E and S have trajectory derivatives. We assume that each $\pi \in \mathscr{P}$ is consistent with

balance of energy

$$\dot{E}(\pi) = Q(\pi) + P(\pi), \tag{3.2}$$

balance of entropy

$$\dot{S}(\pi) = J(\pi) + G(\pi). \tag{3.3}$$

We restrict our attention to situations in which the environment for $\mathscr{B}$ is described by a number $\theta_e > 0$ and three state functions U_e, P_e, and G_e, where

θ_e is the environmental temperature,

$U_e(\sigma)$ is the potential energy in σ of the *conservative* external loads,

$P_e(\sigma)$ is the power expended on $\mathscr{B}$ in σ by the *non-conservative* external loads,

$G_e(\sigma)$ is the entropy produced in σ at the interface between the body and the environment.

We assume that for each $\sigma \in \Sigma$

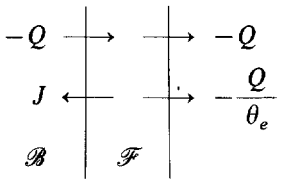
$$P_e(\sigma) \leqq 0, \qquad G_e(\sigma) \geqq 0. \tag{3.4}$$

The first of (3.4) asserts that the non-conservative loads are *dissipative*, since they result in an expenditure of power *by* $\mathscr{B}$; $(3.4)_2$ is simply the requirement that the interfacial entropy production be non-negative. When $P_e = 0$ {respectively, $G_e = 0$} we say that the environment is **conservative** {**thermally perfect**}.

We assume that each $\pi \in \mathscr{P}$ is consistent with the following **interface conditions** relating the environment and the body:[1]

$$\begin{aligned} P(\pi) &= -\dot{U}_e(\pi) + P_e(\pi), \\ J(\pi) &= \frac{1}{\theta_e} Q(\pi) + G_e(\pi). \end{aligned} \tag{3.5}$$

Equation $(3.5)_1$ is self-explanatory. To help motivate $(3.5)_2$ think of the interface between the environment and $\mathscr{B}$ as a thin film $\mathscr{F}$ across which the heat flux is continuous. Then $-Q(\pi)$ is the heat flow into $\mathscr{F}$ from $\mathscr{B}$ and hence into the environment from $\mathscr{F}$, so that $-\frac{1}{\theta_e} Q(\pi)$ is the entropy flow into the environment from $\mathscr{F}$. Therefore, since $J(\pi)$ is the entropy flow into $\mathscr{B}$ from $\mathscr{F}$, $J(\pi) - \frac{1}{\theta_e} Q(\pi)$ represents the net entropy flow out of $\mathscr{F}$. Thus $(3.5)_2$ is the requirement that the entropy produced in $\mathscr{F}$ be equal to the net entropy flow out of $\mathscr{F}$.



Let V and D be the state functions defined by

$$\begin{aligned} V(\sigma) &= U_e(\sigma) + E(\sigma) - \theta_e S(\sigma), \\ D(\sigma) &= \theta_e [G(\sigma) + G_e(\sigma)] - P_e(\sigma), \end{aligned} \tag{3.6}$$

so that, by (3.1) and (3.4),

$$D(\sigma) \geqq 0. \tag{3.7}$$

If we differentiate $V(\pi)$ with respect to time and use (3.2), (3.3), (3.5), and (3.6), we are led to the following important result:

Theorem 3.1.[2] *In each process $\pi \in \mathscr{P}$ the function $t \mapsto V(\pi_t)$ is monotone decreasing. In fact,*

$$\dot{V}(\pi) = -D(\pi). \tag{3.8}$$

This theorem shows that for thermodynamic systems of the type under consideration, which encompasses most of the material bodies studied in continuum mechanics under a variety of loading conditions, there is a natural *Lyapunov function:* namely, the state function V.

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[1] All of the early studies concerning stability were based on conservative external force fields. The first author to employ $(3.4)_1$, $(3.5)_1$ was GURTIN (1973c), although, as COLEMAN (1973) has remarked, ERICKSEN independently suggested an extension of this type during the Symposium on Nonlinear Elasticity held in Madison in April 1973.

[2] *Cf.* GURTIN (1973a), Corollary to Theorem 4.1. Related theorems were given by DUHEM (1911), Vol. 2, pp. 220–231, ERICKSEN (1966a, b), COLEMAN & DILL (1968), KOITER (1969, 1971), COLEMAN (1970), COLEMAN & DILL (1973), GURTIN (1973c), KNOPS & WILKES (1973).

4. *Continuous Bodies. Specific Environments*

For the applications considered here $\mathscr{B}$ will be a continuous body; for convenience, we identify $\mathscr{B}$ with the regular[1] region of Euclidean space it occupies in a fixed reference configuration. Points $X \in \mathscr{B}$ will be referred to as material points. A **configuration** of $\mathscr{B}$ is a one-to-one mapping $\boldsymbol{x} \in PC^1(\mathscr{B}, \mathscr{E})$ with $\boldsymbol{x}(\mathscr{B})$ a regular region and with $\det(\nabla \boldsymbol{x}) > 0$. A **velocity field** {respectively, **temperature field**} for $\mathscr{B}$ is a map $\boldsymbol{v} \in PC^1(\mathscr{B}, \mathscr{V})$ {respectively, $\theta \in PC(\mathscr{B}, \mathbb{R}^{++})$}. We use the following notation:

$$\text{Config} = \{\boldsymbol{x} : \boldsymbol{x} \text{ is a configuration of } \mathscr{B}\},$$

$$\text{Vel} = \{\boldsymbol{v} : \boldsymbol{v} \text{ is a velocity field for } \mathscr{B}\},$$

$$\text{Temp} = \{\theta : \theta \text{ is a temperature field for } \mathscr{B}\}.$$

Let $\mathscr{W}$ be a topological space. A **local state function** (with values in $\mathscr{W}$) is a map $\varphi : \Sigma \times \mathscr{B} \to \mathscr{W}$ such that for each state $\sigma \in \Sigma$ the map $\varphi_\sigma : \mathscr{B} \to \mathscr{W}$ defined by

$$\varphi_\sigma(X) = \varphi(\sigma, X)$$

is piecewise continuous.

In most continuum theories the behavior of $\mathscr{B}$ is governed by local state functions $\boldsymbol{x}$, $\boldsymbol{v}$, θ, ε, η, $\boldsymbol{S}$, $\boldsymbol{q}$, and γ, where for each state $\sigma \in \Sigma$:

$\boldsymbol{x}_\sigma \in \text{Config}$	is the configuration of $\mathscr{B}$ in σ,
$\boldsymbol{v}_\sigma \in \text{Vel}$	is the velocity field in σ,
$\theta_\sigma \in \text{Temp}$	is the temperature field in σ,
$\varepsilon_\sigma : \mathscr{B} \to \mathbb{R}$	is the specific internal energy in σ,
$\eta_\sigma : \mathscr{B} \to \mathbb{R}$	is the specific entropy in σ,
$\boldsymbol{S}_\sigma : \mathscr{B} \to \text{Lin}$	is the Piola-Kirchhoff stress in σ,
$\boldsymbol{q}_\sigma : \mathscr{B} \to \mathscr{V}$	is the heat flux in σ,
$\gamma_\sigma : \mathscr{B} \to \mathbb{R}^+$	is the specific entropy production in σ.

The local state functions determine the global state functions through the relations[2]

$$\begin{aligned} E(\sigma) &= \int_{\mathscr{B}} (\varepsilon_\sigma + \tfrac{1}{2}\boldsymbol{v}_\sigma^2)\, dm, & S(\sigma) &= \int_{\mathscr{B}} \eta_\sigma\, dm, \\ G(\sigma) &= \int_{\mathscr{B}} \gamma_\sigma\, dm, & P(\sigma) &= \int_{\partial\mathscr{B}} \boldsymbol{v}_\sigma \cdot \boldsymbol{S}_\sigma \boldsymbol{n}\, da, \\ Q(\sigma) &= -\int_{\partial\mathscr{B}} \boldsymbol{q}_\sigma \cdot \boldsymbol{n}\, da & J(\sigma) &= -\int_{\partial\mathscr{B}} \frac{1}{\theta_\sigma} \boldsymbol{q}_\sigma \cdot \boldsymbol{n}\, da, \end{aligned} \tag{4.1}$$

where dm is the element of mass on $\mathscr{B}$, while $\boldsymbol{n}$ is the outward unit normal to $\partial\mathscr{B}$. Thus the Lyapunov function V defined in $(3.6)_1$ is given by

$$V(\sigma) = U_e(\sigma) + \int_{\mathscr{B}} (\varepsilon_\sigma - \theta_e \eta_\sigma + \tfrac{1}{2}\boldsymbol{v}_\sigma^2)\, dm. \tag{4.2}$$

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[1] *Cf.*, *e.g.*, GURTIN (1972), p. 13. Roughly speaking, a regular region is a closed and bounded region whose boundary is piecewise smooth.

[2] It is clear from $(4.1)_{5,6}$ that heat supply due to long range influences, such as radiation, is assumed absent. Also, for convenience, we have restricted our attention to situations in which the body force $\boldsymbol{b}$ vanishes. Our results apply, without change, when $\boldsymbol{b}$ is the sum of conservative and dissipative force fields. Of course, in this case U_e includes the total potential energy of $\boldsymbol{b}$.

The configurations $\boldsymbol{x}_\sigma$, $\sigma \in \Sigma$, are called **admissible configurations**, and we write

$$\text{AdmissConfig} = \{\boldsymbol{x} \in \text{Config}: \boldsymbol{x} \text{ is admissible}\}.$$

We assume that AdmissConfig contains the **undeformed configuration**

$$\boldsymbol{x}(X) = X \qquad \forall X \in \mathscr{B}.$$

The point $\boldsymbol{x}_\sigma(X)$ is the position occupied by the material point X in the state σ, while $\mathscr{B}_\sigma = \boldsymbol{x}_\sigma(\mathscr{B})$ is the region of space occupied by $\mathscr{B}$ in σ. Let $\pi \in \mathscr{P}$. Then $\boldsymbol{x}_{\pi_t}(X)$ is the position occupied by X at time t in the process π, while $\boldsymbol{v}_{\pi_t}(X)$ is the velocity of X at time t in π. We assume that $\boldsymbol{x}$ and $\boldsymbol{v}$ are **compatible** in the following sense: in each $\pi \in \mathscr{P}$ and for all $X \in \mathscr{B}$ the map $t \mapsto \boldsymbol{x}_{\pi_t}(X)$ is differentiable and

$$\frac{\partial}{\partial t} \boldsymbol{x}_{\pi_t}(X) = \boldsymbol{v}_{\pi_t}(X). \tag{4.3}$$

The field $\boldsymbol{S}_\sigma \boldsymbol{n}$ on $\partial\mathscr{B}$ gives the surface force on the boundary per unit area of the reference configuration. The Cauchy stress $\boldsymbol{T}_\sigma: \mathscr{B} \to \text{Lin}$ is defined by

$$\boldsymbol{T}_\sigma = (\det \boldsymbol{F})^{-1} \boldsymbol{S}_\sigma \boldsymbol{F}^T, \qquad \boldsymbol{F} = \nabla \boldsymbol{x}_\sigma;$$

$\boldsymbol{T}_\sigma \boldsymbol{n}_\sigma$ gives the surface force per unit area of the actual configuration $\boldsymbol{x}_\sigma$. Here, of course, $\boldsymbol{n}_\sigma(X)$ is, for each $X \in \partial\mathscr{B}$, the outward unit normal to $\partial\mathscr{B}_\sigma$ at the point $\boldsymbol{x}_\sigma(X)$. Similarly, the heat flow across the boundary per unit area in $\boldsymbol{x}_\sigma$ is $\boldsymbol{h}_\sigma \cdot \boldsymbol{n}_\sigma$, where $\boldsymbol{h}_\sigma: \mathscr{B} \to \mathscr{V}$ is defined by

$$\boldsymbol{h}_\sigma = (\det \boldsymbol{F})^{-1} \boldsymbol{F} \boldsymbol{q}_\sigma.$$

We now give some examples in which the interface conditions (3.5) are satisfied. Examples (1)–(4) are concerned with the power relation $(3.5)_1$, while (5)–(7) are concerned with the entropy flux relation $(3.5)_2$.

i. Conservative Environments ($P_e = 0$).

(1) *Fixed boundary:* Each $\sigma \in \Sigma$ has

$$\boldsymbol{x}_\sigma(X) = X, \qquad \boldsymbol{v}_\sigma(X) = \boldsymbol{0}, \qquad \forall X \in \partial\mathscr{B},$$

so that $P = 0$. Thus $(3.5)_1$ is satisfied with $U_e = P_e = 0$.

(2) *Boundary held at constant pressure* p_0:[1] Each $\sigma \in \Sigma$ has

$$\boldsymbol{T}_\sigma \boldsymbol{n}_\sigma = -p_0 \boldsymbol{n}_\sigma \qquad \text{on } \partial\mathscr{B}.$$

In this instance $(3.5)_1$ is satisfied provided we take $P_e = 0$ and

$$U_e(\sigma) = p_0 \,\text{vol}(\mathscr{B}_\sigma),$$

where $\text{vol}(\mathscr{B}_\sigma)$ is the volume of $\mathscr{B}_\sigma$.

(3) *Dead loading:* Here there exists a function $\boldsymbol{s}: \partial\mathscr{B} \to \mathscr{V}$ such that

$$\boldsymbol{S}_\sigma \boldsymbol{n} = \boldsymbol{s} \qquad \text{on } \partial\mathscr{B}.$$

[1] *Cf.* COLEMAN (1970).

For this case we take $P_e=0$ and

$$U_e(\sigma)=-\int_{\partial\mathscr{B}} \boldsymbol{s}\cdot(\boldsymbol{x}_\sigma-\boldsymbol{z})\,da,$$

where $\boldsymbol{z}$ is an arbitrary fixed point of $\mathscr{E}$.

ii. A Non-conservative Environment.

(4) *Linear viscous damping on the boundary:* Each $\sigma\in\Sigma$ satisfies

$$\boldsymbol{S}_\sigma\boldsymbol{n}=-\boldsymbol{C}_\sigma\boldsymbol{v}_\sigma \quad \text{on } \partial\mathscr{B},$$

where, for each $\boldsymbol{X}\in\partial\mathscr{B}$, $\boldsymbol{C}_\sigma\in\text{Lin}$ is positive semi-definite. Here we take $U_e=0$ and

$$P_e(\sigma)=-\int_{\partial\mathscr{B}} \boldsymbol{v}_\sigma\cdot\boldsymbol{C}_\sigma\boldsymbol{v}_\sigma\,da.$$

iii. Thermally perfect environments ($G_e=0$).

(5) *Boundary held at constant temperature* θ_e: Each $\sigma\in\Sigma$ satisfies

$$\theta_\sigma=\theta_e \quad \text{on } \partial\mathscr{B},$$

and $(3.5)_2$ holds with $G_e=0$.

(6) *Insulated boundary*: Each $\sigma\in\Sigma$ has

$$\boldsymbol{q}_\sigma\cdot\boldsymbol{n}=0 \quad \text{on } \partial\mathscr{B}, \tag{4.4}$$

and again $(3.5)_2$ holds with $G_e=0$.

iv. An environment that is not thermally perfect.

(7) *$\mathscr{B}$ surrounded by a thin convective film with heat transfer coefficient k*: Here $k>0$ is a constant and each state $\sigma\in\Sigma$ satisfies

$$\boldsymbol{h}_\sigma\cdot\boldsymbol{n}_\sigma=k(\theta_\sigma-\theta_e) \quad \text{on } \partial\mathscr{B},$$

so that $(3.5)_2$ holds with

$$G_e(\sigma)=k\int_{\partial\mathscr{B}_\sigma}\frac{\theta_\sigma}{\theta_e}\left(1-\frac{\theta_e}{\theta_\sigma}\right)^2 da.$$

(More precisely, θ_σ should be replaced by $\theta_\sigma\circ\boldsymbol{x}_\sigma^{-1}$.)

Many other boundary conditions consistent with our underlying assumptions are easily arrived at, including those involving combinations of the above and those involving mixed boundary conditions.

In each of the examples (1)–(4) the potential energy U_e of the environment depends only on the configuration; that is,

$$U_e(\sigma_1)=U_e(\sigma_2) \quad \text{whenever} \quad \boldsymbol{x}_{\sigma_1}=\boldsymbol{x}_{\sigma_2}. \tag{4.5}$$

Henceforth we adopt this condition as a general hypothesis for all of the environments that we shall consider.

5. *Elastic Materials. The Equilibrium Map*

We now write $\hat{\boldsymbol{x}}$, $\hat{\boldsymbol{v}}$, and $\hat{\theta}$ for the local state functions $\boldsymbol{x}$, $\boldsymbol{v}$, and θ; the letter $\boldsymbol{x}$ will now stand for an arbitrary configuration of $\mathscr{B}$, $\boldsymbol{v}$ for an arbitrary velocity field, and θ for an arbitrary temperature field.

For an **elastic body:**

(E_1) The state space Σ is a subset of the set of all triplets of the form

$$\sigma=(\boldsymbol{x}, \boldsymbol{v}, \theta),$$

where $\boldsymbol{x}\in\text{Config}$, $\boldsymbol{v}\in\text{Vel}$, and $\theta\in\text{Temp}$.

(E_2) The local response functions $\hat{\boldsymbol{x}}$, $\hat{\boldsymbol{v}}$, and $\hat{\theta}$ are given by

$$\hat{\boldsymbol{x}}_\sigma=\boldsymbol{x}, \quad \hat{\boldsymbol{v}}_\sigma=\boldsymbol{v}, \quad \hat{\theta}_\sigma=\theta \tag{5.1}$$

for every $\sigma=(\boldsymbol{x}, \boldsymbol{v}, \theta)\in\Sigma$.

(E_3) The local response functions ε and η are given by constitutive equations of the form[1]

$$\begin{aligned}\varepsilon_\sigma(\boldsymbol{X})&=\hat{\varepsilon}\big(\boldsymbol{F}(\boldsymbol{X}), \theta(\boldsymbol{X}), \boldsymbol{X}\big),\\ \eta_\sigma(\boldsymbol{X})&=\hat{\eta}\big(\boldsymbol{F}(\boldsymbol{X}), \theta(\boldsymbol{X}), \boldsymbol{X}\big),\end{aligned} \tag{5.2}$$

for every $\boldsymbol{X}\in\mathscr{B}$ and $\sigma=(\boldsymbol{x}, \boldsymbol{v}, \theta)\in\Sigma$, where

$$\boldsymbol{F}=\nabla\boldsymbol{x}, \tag{5.3}$$

and where $\hat{\varepsilon}, \hat{\eta}$: $\text{Lin}^+\times\mathbb{R}^{++}\times\mathscr{B}\to\mathbb{R}$ are smooth.

The free-energy $\hat{\psi}$: $\text{Lin}^+\times\mathbb{R}^{++}\times\mathscr{B}\to\mathbb{R}$ is defined by

$$\hat{\psi}(\boldsymbol{F}, \theta)=\hat{\varepsilon}(\boldsymbol{F}, \theta)-\theta\hat{\eta}(\boldsymbol{F}, \theta), \tag{5.4}$$

where, for convenience, we have suppressed the argument $\boldsymbol{X}$. As is customary, we assume that $\hat{\psi}$ determines $\hat{\eta}$ through *entropy relation*

$$\hat{\eta}=-\partial_2\hat{\psi}. \tag{5.5}$$

We also assume that the specific heat $\partial_2\hat{\varepsilon}$ is strictly positive; or equivalently, by (5.4) and (5.5), that

$$\partial_2\partial_2\hat{\psi}<0. \tag{5.6}$$

An important consequence of these assumptions is

Ericksen's Lemma[2] *There exists a function* κ: $\text{Lin}^+\times\mathbb{R}^{++}\times\mathscr{B}\to\mathbb{R}$ *such that*

$$\hat{\varepsilon}(\boldsymbol{F}, \theta)-\theta_e\hat{\eta}(\boldsymbol{F}, \theta)=\hat{\psi}(\boldsymbol{F}, \theta_e)+\kappa(\boldsymbol{F}, \theta)(\theta-\theta_e)^2, \quad \kappa(\boldsymbol{F}, \theta)>0 \tag{5.7}$$

for all $\boldsymbol{F}\in\text{Lin}^+$ *and* $\theta\in\mathbb{R}^{++}$.

A proof of this lemma is given in the Appendix.

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[1] A similar constitutive equation holds for the stress, while the heat flux and specific entropy production depend, in addition, on the temperature gradient. These constitutive equations, however, are irrelevant to our treatment.

[2] ERICKSEN (1966a), eq. (30).

By (4.2), (5.1), and (5.2),

$$V(\sigma)=U_e(\sigma)+\int_{\mathscr{B}}[\hat{\varepsilon}(\boldsymbol{F},\theta)-\theta_e\hat{\eta}(\boldsymbol{F},\theta)+\tfrac{1}{2}\boldsymbol{v}^2]\,dm, \tag{5.8}$$

and thus (5.7) implies[1]

$$V(\sigma)=U_e(\sigma)+\int_{\mathscr{B}}[\hat{\psi}(\boldsymbol{F},\theta_e)+\kappa(\boldsymbol{F},\theta)(\theta-\theta_e)^2+\tfrac{1}{2}\boldsymbol{v}^2]\,dm. \tag{5.9}$$

Consider now the mapping

$$\text{Equil}\colon \Sigma\to\Sigma$$

which associates with every state $(\boldsymbol{x},\boldsymbol{v},\theta)\in\Sigma$ its equilibrium value $(\boldsymbol{x},\boldsymbol{0},\theta_e)$ (which we assume also belongs to Σ):

$$\text{Equil}(\boldsymbol{x},\boldsymbol{v},\theta)=(\boldsymbol{x},\boldsymbol{0},\theta_e). \tag{5.10}$$

We call

$$\Sigma_0=\text{Equil}\,\Sigma$$

the *equilibrium state space*, while elements $\sigma_0=(\boldsymbol{x},\boldsymbol{0},\theta_e)$ are *equilibrium states*. Since Equil is a projection,

$$\text{Equil } \textit{is idempotent}. \tag{5.11}$$[2]

Let $U\colon\Sigma_0\to\mathbb{R}$ denote the restriction of V to Σ_0,

$$U(\sigma_0)=V(\sigma_0)\qquad\forall\,\sigma_0\in\Sigma_0, \tag{5.12}$$

so that, by (5.9) and (5.10),

$$U(\sigma_0)=U_e(\sigma_0)+\int_{\mathscr{B}}\hat{\psi}(\boldsymbol{F},\theta_e)\,dm \tag{5.13}$$

for every equilibrium state σ_0. Thus $U(\sigma_0)$ is the total potential energy in σ_0; that is, $U(\sigma_0)$ equals the potential energy of the load system plus the free-energy of the body.

The integral

$$T(\sigma)=\int_{\mathscr{B}}[\kappa(\boldsymbol{F},\theta)(\theta-\theta_e)^2+\tfrac{1}{2}\boldsymbol{v}^2]\,dm, \tag{5.14}$$

measures the "distance" of a state σ from its equilibrium value. By (4.5) and (5.10),

$$U_e(\sigma)=U_e(\text{Equil}\,\sigma);$$

therefore (5.9), (5.13), and (5.14) imply that

$$V(\sigma)=U(\text{Equil}\,\sigma)+T(\sigma), \tag{5.15}$$

so that $V(\sigma)$ equals the potential energy of Equil σ plus the distance of σ from Equil σ.

---

[1] ERICKSEN (1966a), eq. $(32)_1$.

[2] A map $\varphi\colon\Sigma\to\Sigma$ is idempotent if $\varphi\circ\varphi=\varphi$.

Theorem 5.1. (Properties of the Lyapunov function)

(i) $T = V - V_\circ$ Equil.

(ii) $T \geqq 0$.

(iii) $T(\sigma) = 0$ *if and only if* σ *is an equilibrium state.*

(iv) *Given a process* $\pi \in \mathscr{P}$, *if* $T(\pi_t) = 0$ *for all* $t \in [0, d_\pi)$, *then* π *is a rest process.*

Proof. Assertion (i) follows from (5.12) and (5.15), while (ii) and (iii) are consequences of (5.14) and $(5.7)_2$. (Here we use the tacit assumption that $dm = \rho_0\, dv$, where ρ_0, the density in the reference configuration, is a strictly positive function on $\mathscr{B}$.)

Before proving (iv), note that each process $\pi \in \mathscr{P}$ can be identified with a triplet $(\boldsymbol{x}, \boldsymbol{v}, \theta)$, where $\boldsymbol{x}$, $\boldsymbol{v}$, and θ are functions on $\mathscr{B} \times [0, d_\pi)$ with $\boldsymbol{x}(\cdot, t) \in \text{Config}$, $\boldsymbol{v}(\cdot, t) \in \text{Vel}$, and $\theta(\cdot, t) \in \text{Temp}$ at each $t \in [0, d_\pi)$. Moreover, by (4.3),

$$\boldsymbol{v} = \dot{\boldsymbol{x}}. \tag{5.16}$$

To prove (iv) assume that $T(\pi) = 0$. Then, by (iii), π_t is an equilibrium state at each $t \in [0, d_\pi)$, so that $\theta(\boldsymbol{X}, t) \equiv \theta_e$ and $\boldsymbol{v}(\boldsymbol{X}, t) \equiv 0$. Thus we conclude from (5.16) that $\boldsymbol{x}(\boldsymbol{X}, t)$ is independent of t; hence π is a rest process. □

Theorem 5.1 will serve to motivate the general definition of a Lyapunov function given in the next section.

We close this section with a discussion of possible topologies for the underlying spaces. In applications the specific choice is dictated by the form of the free-energy function and by the actual environment under consideration.

Given any function f on $\mathscr{B}$ with values in a vector space, let

$$\|f\|_{L_p} = \Big[\int_{\mathscr{B}} |f|^p\, dv\Big]^{\frac{1}{p}}, \qquad \|f\|_{L_\infty} = \sup\{|f(\boldsymbol{X})| : \boldsymbol{X} \in \mathscr{B}\}.$$

Let $\boldsymbol{x}_1, \boldsymbol{x}_2 \in \text{Config}$, and let

$$\boldsymbol{F}_\alpha = \nabla \boldsymbol{x}_\alpha, \qquad \boldsymbol{C}_\alpha = \boldsymbol{F}_\alpha^T \boldsymbol{F}_\alpha \qquad (\alpha = 1, 2),$$

so that $\boldsymbol{C}_\alpha$ is the right Cauchy-Green strain tensor for $\boldsymbol{x}_\alpha$. Further, for p a positive integer or $p = \infty$, let

$$\begin{aligned} d_p(\boldsymbol{x}_1, \boldsymbol{x}_2) &= \|\boldsymbol{x}_1 - \boldsymbol{x}_2\|_{L_p} + \|\boldsymbol{F}_1 - \boldsymbol{F}_2\|_{L_p}, \\ c_p(\boldsymbol{x}_1, \boldsymbol{x}_2) &= \|\boldsymbol{C}_1 - \boldsymbol{C}_2\|_{L_p}. \end{aligned} \tag{5.17}$$

Then d_p is a metric for Config and therefore generates a Hausdorff topology $\mathscr{T}(d_p)$. On the other hand, c_p is a pseudo-metric and generates a non-Hausdorff topology $\mathscr{T}(c_p)$.

Most of our results require a topology only for the equilibrium space Σ_0. By (5.10), for an elastic material Σ_0 can be identified in a natural manner with the set AdmissConfig of all admissible configurations. If we consider situations in which the boundary of $\mathscr{B}$ is clamped, so that each $\boldsymbol{x} \in \text{AdmissConfig}$ satisfies

$$\boldsymbol{x}(\boldsymbol{X}) = \boldsymbol{X} \qquad \forall\, \boldsymbol{X} \in \partial\mathscr{B},$$

then c_p is a *metric* for AdmissConfig, and therefore $\mathscr{T}(c_p)$, as a topology for AdmissConfig, is a Hausdorff topology. The same assertion applies to situations

in which a portion $\mathscr{S}$ of $\partial\mathscr{B}$ is clamped, as long as this clamping precludes rigid deformations of $\mathscr{B}$.

The usual topology on Vel is that induced by the kinetic energy, while possible topologies on Temp are those induced by the L_p norms.

IV. General Theory of Dynamical Systems

In this chapter we shall develop a general theory of dynamical systems. While this theory is motivated by our previous results, it is sufficiently general to describe the behavior of most materials now considered important in continuum mechanics.

6. *Preliminary Results. Uniqueness*

Let $\pi\colon [0, d_\pi)\to\Sigma$ be a process[1] with values in a set Σ. The **orbit** of π is the set[2]

$$\text{Orbit}(\pi)=\{\pi_t\colon t\in[0, d_\pi)\}.$$

Each restriction of π of the form $\pi|_{[0,a)}$, $a\in(0, d_\pi]$, is called an **initial segment** of π, and, when $a<d_\pi$, π is a **continuation** of $\pi|_{[0,a)}$. Let $\mathscr{P}$ be a family of processes with values in Σ. Then $\mathscr{P}$ is a **process class** for Σ if each $\pi\in\mathscr{P}$ is maximal; that is, if each $\pi\in\mathscr{P}$ has no continuation in $\mathscr{P}$.

A **dynamical system** is a triplet $(\Sigma, \mathscr{P}, \text{Equil})$, where

(D_1) Σ is a set;

(D_2) $\mathscr{P}$ is a process class for Σ;

(D_3) Equil: $\Sigma\to\Sigma$ is idempotent.

We use the following terminology: Σ is the **state space;** elements $\sigma\in\Sigma$ are **states;**

$$\Sigma_0=\text{Equil}\ \Sigma$$

is the **equilibrium state space;** elements $\sigma_0\in\Sigma_0$ are **equilibrium states.** Given $\pi\in\mathscr{P}$, the set

$$\text{EquilOrbit}(\pi)=\{\text{Equil}\ \pi_t\colon t\in[0, d_\pi)\}$$

is the **equilibrium orbit** of π.

A **Lyapunov function** for the dynamical system $(\Sigma, \mathscr{P}, \text{Equil})$ is a mapping

$$V\colon \Sigma\to\mathbb{R}$$

with properties (L_1)–(L_3) listed below. In (L_2) and (L_3)

$$T=V-V\circ\text{Equil}. \tag{6.1}$$

(L_1) For each $\pi\in\mathscr{P}$ the function $t\mapsto V(\pi_t)$ is monotone decreasing.

(L_2) $T(\pi_t)\geqq 0$ for each $\pi\in\mathscr{P}$ and $t\in[0, d_\pi)$.

(L_3) If π is an initial segment of a process in $\mathscr{P}$, if $\pi_0\in\Sigma_0$, and if $T(\pi_t)=0$ for all $t\in[0, d_\pi)$, then π is a rest process.

Since Equil is idempotent,

$$\text{Equil}\ \sigma_0=\sigma_0 \qquad \forall\ \sigma_0\in\Sigma_0, \tag{6.2}$$

---

[1] The notions, process, duration, and rest process, are defined in the second paragraph of Section 3.

[2] This set, which is simply the range of π, is often called the positive orbit of π.

and therefore (6.1) yields

$$T(\sigma_0)=0 \qquad \forall\, \sigma_0\in\Sigma_0. \tag{6.3}$$

The function

$$U\colon \Sigma_0\to\mathbb{R}$$

defined by

$$U=V|_{\Sigma_0} \tag{6.4}$$

is called the **potential energy.** By (6.4), $V(\text{Equil}\,\sigma)=U(\text{Equil}\,\sigma)$; hence (6.1) implies that

$$V(\sigma)=U(\text{Equil}\,\sigma)+T(\sigma) \tag{6.5}$$

for every state σ. We therefore conclude from (L_1) that

$$\begin{aligned} &V(\pi_t)\leqq V(\pi_0),\\ &U(\text{Equil}\,\pi_t)+T(\pi_t)\leqq U(\text{Equil}\,\pi_0)+T(\pi_0). \end{aligned} \tag{6.6}$$

Let π be a process with values in Σ. Then π **starts from equilibrium** if $\pi_0\in\Sigma_0$; π **starts from** σ if $\pi_0=\sigma$; π **conserves potential energy** if $U(\text{Equil}\,\pi_t)=U(\text{Equil}\,\pi_0)$ for all $t\in[0, d_\pi)$. The first definition, $(6.6)_2$, (6.2), (6.3), and (L_2) yield

Theorem 6.1. *For any process $\pi\in\mathscr{P}$ starting from equilibrium*

$$U(\text{Equil}\,\pi_t)\leqq U(\pi_0)$$

for all $t\in[0, d_\pi)$.

Proposition 6.1.[1] *Let π be an initial segment of a process in $\mathscr{P}$ that starts from equilibrium. Then the following are equivalent:*

(i) *π is a rest process;*

(ii) $\text{Orbit}(\pi)\subset\Sigma_0$;

(iii) $\text{Equil}\,\pi_t=\pi_0$ *for all* $t\in[0, d_\pi)$;

(iv) *π conserves potential energy.*

Proof. Obviously, (i) implies (ii), (iii), and (iv), while (iii) implies (iv). Thus to complete the proof it suffices to show that (ii) and (iv) each imply (i), or, by (L_3), that (ii) and (iv) each imply

$$T(\pi_t)=0 \qquad \forall\, t\in[0, d_\pi). \tag{6.7}$$

That (6.7) follows from (ii) is a consequence of (6.3). On the other hand, if (iv) holds, then, since $T(\pi_0)=0$, $(6.6)_2$ implies $T(\pi_t)\leqq 0$ for all $t\in[0, d_\pi)$, and this result with (L_2) yields (6.7). □

The next two results[2] are uniqueness theorems appropriate to the initial-value problem for processes starting from equilibrium.

Theorem 6.2. *Let U have a global minimum at an equilibrium state σ_0. Then there is at most one process $\pi\in\mathscr{P}$ with initial-value σ_0. Moreover, π, if it exists, is a rest process.*

[1] *Cf.* GURTIN (1973c), eq. (3.7).

[2] *Cf.* GURTIN (1973a), Thm. 9.2.

Proof. Let $\pi \in \mathscr{P}$ have initial-value σ_0. By Theorem 6.1,

$$U(\text{Equil } \pi_t) \leqq U(\sigma_0);$$

but

$$U(\text{Equil } \pi_t) \geqq U(\sigma_0),$$

since U has a global minimum at σ_0. Thus π conserves potential energy, and we conclude from Proposition 6.1 that π is a rest process with value σ_0. That there is only one such rest process follows from the assumption that each $\pi \in \mathscr{P}$ is maximal. □

Theorem 6.3. *Assume that Σ_0 is a Hausdorff space. Let U have a local minimum at an equilibrium state σ_0. Then there is at most one process $\pi \in \mathscr{P}$ with initial-value σ_0 and with $t \mapsto$ Equil π_t continuous. Moreover, π, if it exists, is a rest process.*

Proof. By hypothesis, U has a minimum at σ_0 over an open neighborhood Ω_0 of σ_0 in Σ_0. Let $\pi \in \mathscr{P}$ start from σ_0, let

$$s = \sup\{\tau\colon \pi_t = \sigma_0 \ \forall\, t \in [0, \tau]\}, \tag{6.8}$$

and assume that $s < d_\pi$. Then, since $t \mapsto$ Equil π_t is continuous, Equil $\pi_s = \sigma_0$, and there exists an $h \in (s, d_\pi)$ such that Equil $\pi_t \in \Omega_0$ for all $t \in [s, h)$. Since this condition also holds on $[0, s]$, $U(\text{Equil } \pi_t) \geqq U(\sigma_0)$ for all $t \in [0, h)$, and we conclude from Theorem 6.1 that the initial segment $\pi|_{[0,h)}$ of π conserves potential energy. It therefore follows from Proposition 6.1 that $\pi|_{[0,h)}$ is a rest process, which clearly contradicts (6.8), since $s < h$; hence $s = d_\pi$ and π is a rest process. We have shown that if $\pi \in \mathscr{P}$ starts from σ_0, then π is a rest process with value σ_0. Since $\pi \in \mathscr{P}$ is maximal, there is at most one such rest process in $\mathscr{P}$. □

7. *Necessary Conditions for Stability*

In this section we give conditions which are necessary for the stability of a dynamical system $(\Sigma, \mathscr{P}, \text{Equil})$ with Lyapunov function V.

To avoid repeated hypotheses we assume, for the purpose of the next theorem and its two ensuing corollaries, that

(α) the equilibrium state space Σ_0 is a Hausdorff space;

(β) σ_0 is an equilibrium state at which U is continuous.

An equilibrium state λ_0 is **attracted to** σ_0 if there exists a process $\pi \in \mathscr{P}$ starting at λ_0 with

$$\text{Equil } \pi_t \to \sigma_0 \quad \text{as} \quad t \to d_\pi. \tag{7.1}$$

In view of this definition, Theorem 6.1 implies

Theorem 7.1.[1] *Let λ_0 be an equilibrium state and assume that λ_0 is attracted to σ_0. Then*

$$U(\sigma_0) \leqq U(\lambda_0).$$

Corollary 7.1. *Assume that each state in a neighborhood of σ_0 in Σ_0 is attracted to σ_0. Then U has a local minimum at σ_0.*

[1] GURTIN (1973 c), (4.1). *Cf.* GURTIN (1973 a), Thm. 9.5 and (1973 c) for an elastic material; COLEMAN & DILL (1973), Thm. 6.4 for a material with fading memory.

We say that σ_0 is **locally asymptotically stable with respect to equilibrium orbits** if there exist neighborhoods Ω_0, Λ_0 of σ_0 in Σ_0 with $\Omega_0 \subset \Lambda_0$ such that

(a) for each process $\pi \in \mathscr{P}$, $\pi_0 \in \Omega_0$ implies $\mathrm{EquilOrbit}(\pi) \subset \Lambda_0$;

(b) each $\lambda_0 \in \Lambda_0$ is attracted to σ_0.

Corollary 7.2.[1] *Let σ_0 be locally asymptotically stable with respect to equilibrium orbits. Then U has a strict local minimum at σ_0.*

Proof. Choose $\lambda_0 \in \Omega_0$, $\lambda_0 \neq \sigma_0$. Then λ_0 is attracted to σ_0, and there exists a process $\pi \in \mathscr{P}$ such that (7.1) holds. If π were a rest process, then π would have value λ_0, so that $\mathrm{Equil}\, \pi_t = \mathrm{Equil}\, \lambda_0 = \lambda_0$ and hence $\mathrm{Equil}\, \pi_t \to \lambda_0$. Thus since Σ_0 is a Hausdorff space, and since $\lambda_0 \neq \sigma_0$, π cannot be a rest process. Consequently, as $\pi_0 = \lambda_0 \in \Sigma_0$, we conclude from (L_2) and (L_3) that $T(\pi_0) = 0$ and $T(\pi_\tau) > 0$ at some $\tau \in (0, d_\pi)$; thus $(6.6)_2$ implies that

$$U(\mathrm{Equil}\, \pi_\tau) < U(\lambda_0).$$

Next, by (a), $\mathrm{Equil}\, \pi_\tau \in \Lambda_0$, and therefore (b) and Theorem 7.1 imply

$$U(\sigma_0) \leqq U(\mathrm{Equil}\, \pi_\tau).$$

The last two inequalities yield $U(\sigma_0) < U(\lambda_0)$, and the proof is complete. □

Remarks. (1) Theorem 7.1 and its two corollaries remain valid when (7.1) is replaced by the weaker requirement that there exist an increasing sequence $\{t_n\}$ in $[0, d_\pi)$, with $t_n \to d_\pi$ as $n \to \infty$, such that

$$\mathrm{Equil}\, \pi_{t_n} \to \sigma_0 \quad \text{as} \quad n \to \infty;$$

i.e., that σ_0 belong to the ω-limit set of the process $t \mapsto \mathrm{Equil}\, \pi_t$ in Σ_0.

(2) Theorem 7.1 and Corollary 7.1 remain valid when Σ_0 is not a Hausdorff space, provided (7.1) is replaced by the requirement that σ_0 be *a* limit of $\mathrm{Equil}\, \pi_t$ as $t \to d_\pi$.

A mapping

$$D: \Sigma \to \mathbb{R}^+$$

is a **dissipation function** consistent with V if given any $\pi \in \mathscr{P}$ the mapping $t \mapsto D(\pi_t)$ is integrable over every interval $[a, b] \subset [0, d_\pi)$ and

$$V(\pi_a) - V(\pi_b) \geqq \int_a^b D(\pi_t)\, dt. \tag{7.2}$$

If in the above definition (7.2) holds with $\geqq$ replaced by $=$, then D is a **maximal dissipation function.** Clearly, when D is maximal and each $\pi \in \mathscr{P}$ has $t \mapsto D(\pi_t)$ continuous, D is uniquely determined by V on the set of all states which are values of processes in $\mathscr{P}$.

Remark. It is not difficult to show that

$$D(\sigma) = 0$$

[1] This theorem generalizes Thm. 6.7 of COLEMAN & DILL (1973), whose result is applicable to materials with fading memory.

in each of the following two cases:

(i) σ is the value of a rest process in $\mathscr{P}$;

(ii) Σ is a Hausdorff space, V and D are continuous at σ, and $\pi_t \to \sigma$ as $t \to \infty$ for some $\pi \in \mathscr{P}$ with $d_\pi = \infty$.

We assume for the remainder of this section that

$$\Sigma \text{ is a Hausdorff space.}$$

A state σ is an ω-**limit point** of a process $\pi \in \mathscr{P}$ if there exists an increasing sequence $\{t_n\}$ in $[0, d_\pi)$, with $t_n \to d_\pi$, such that

$$\pi_{t_n} \to \sigma \quad \text{as} \quad n \to \infty. \tag{7.3}$$

The set $\omega(\pi)$ of all ω-limit points of π is the ω-**limit set** for π.

Theorem 7.2. *Let $\pi \in \mathscr{P}$ be a process with $\omega(\pi) \neq \emptyset$. Assume that V is continuous on $\omega(\pi)$, and that there exists a maximal dissipation function consistent with V. Then*

$$V|_{\omega(\pi)} \text{ is constant.} \tag{7.4}$$

Proof. Let $\{t_n\}$ be an appropriate sequence in the sense of (7.3). Then (7.2) (with $\leqq$ replaced by $=$) implies that

$$V(\pi_0) - V(\pi_{t_n}) = \int_0^{t_n} D(\pi_t)\, dt. \tag{7.5}$$

Since V is continuous at σ, the left-side of (7.5) tends to $V(\pi_0) - V(\sigma)$ as $n \to \infty$. Thus, since $D \geqq 0$, $t \mapsto D(\pi_t)$ must be integrable on $(0, d_\pi)$ and

$$V(\sigma) = V(\pi_0) - \int_0^{d_\pi} D(\pi_t)\, dt,$$

which yields (7.4). □

8. *Sufficient Conditions for Stability*

The results of the last section show that for an equilibrium state σ_0 to be stable U must necessarily have a minimum at σ_0. We now show that U a global minimum or U a strong local minimum is sufficient to yield certain types of Lyapunov stability.

Let $(\Sigma, \mathscr{P}, \text{Equil})$ be a dynamical system with Lyapunov function V, and let D be a dissipation function consistent with V.[1] We call

$$l(\lambda, \sigma) = |V(\lambda) - V(\sigma)| \tag{8.1}$$

the **Lyapunov distance** between the states λ and σ. Note that, by (6.1) and (L_2), in any process $\pi \in \mathscr{P}$ the number $T(\pi_t)$ represents the Lyapunov distance between π_t and its equilibrium value Equil π_t. The topology on Σ generated by the pseudo-metric l is called the **Lyapunov topology**. Given $\sigma \in \Sigma$ and $\varepsilon > 0$, we call

$$\Gamma_\varepsilon(\sigma) = \{\lambda \in \Sigma\colon\ l(\lambda, \sigma) < \varepsilon\} \tag{8.2}$$

[1] The assumption that there exist a dissipation function D consistent with V involves no loss in generality; indeed, $D(\sigma) \equiv 0$ is a dissipation function for any V.

the *l*-**ball** of radius ε. Also, when $\pi \in \mathscr{P}$ is a process and σ is a state, we write

$$l(\pi, \sigma) = \sup\{l(\pi_t, \sigma): t \in [0, d_\pi)\}.$$

Lemma 8.1. *Let σ_0 be an equilibrium state, and suppose that U has a minimum at σ_0 over $\Omega_0 \subset \Sigma_0$. Let $\pi \in \mathscr{P}$ have*

$$\text{EquilOrbit}(\pi) \subset \Omega_0. \tag{8.3}$$

Then $t \mapsto D(\pi_t)$ is integrable on $(0, d_\pi)$ and

$$l(\pi, \sigma_0), \int_0^{d_\pi} D(\pi_t)\, dt \leqq l(\pi_0, \sigma_0). \tag{8.4}$$

Proof. Since U has a minimum at σ_0 over Ω_0, we conclude from (8.3) that $U(\sigma_0) \leqq U(\text{Equil}\, \pi_t)$. Thus (L_2), (6.2), (6.3), and (6.5) imply that

$$V(\pi_t) - V(\sigma_0) \geqq 0;$$

hence

$$l(\pi_t, \sigma_0) = V(\pi_t) - V(\sigma_0), \tag{8.5}$$

and upon subtracting $V(\sigma_0)$ from both sides of $(6.6)_1$ we are lead at once to the conclusion that $l(\pi, \sigma_0) \leqq l(\pi_0, \sigma_0)$. Next, since $l(\pi_t, \sigma_0) \geqq 0$, the relations (7.2) and (8.5) imply that

$$l(\pi_0, \sigma_0) \geqq l(\pi_0, \sigma_0) - l(\pi_t, \sigma_0) \geqq \int_0^t D(\pi_t)\, d\tau,$$

and, as $D \geqq 0$, this clearly yields the desired conclusions. □

A set $\Omega \subset \Sigma$ is **invariant** if, given any process $\pi \in \mathscr{P}$,

$$\pi_0 \in \Omega \quad \text{implies} \quad \text{Orbit}(\pi) \subset \Omega.$$

A state σ is **Lyapunov stable** with respect to a topology $\mathscr{T}$ for Σ if given any $\mathscr{T}$-neighborhood Λ of σ there exists a $\mathscr{T}$-neighborhood Ω of σ such that

$$\pi_0 \in \Omega \quad \text{implies} \quad \text{Orbit}(\pi) \subset \Lambda. \tag{8.6}$$

A consequence of these definitions and Lemma 8.1 is

Theorem 8.1. *Let U have a global minimum at an equilibrium state σ_0. Then any l-ball at σ_0 is an invariant set, so that σ_0 is Lyapunov stable with respect to the Lyapunov topology.*

Corollary 8.1. *Let U have a global minimum at an equilibrium state σ_0. Then given any $\varepsilon > 0$ a process $\pi \in \mathscr{P}$ will have*

$$l(\pi, \sigma_0), \int_0^{d_\pi} D(\pi_t)\, dt < \varepsilon \tag{8.7}$$

as long as $l(\pi_0, \sigma_0) < \varepsilon$.

The inequality $(8.7)_2$ is most important when $d_\pi = \infty$, for then it takes the form

$$\int_0^{\infty} D(\pi_t)\, dt < \varepsilon.$$

The type of stability established in the above two results is quite weak. However, except for the Lyapunov topology, no underlying topology is required, and no assumptions of continuity are made concerning V, D, Equil, or the processes in $\mathscr{P}$. Thus, in particular, these results should be applicable to processes containing shock waves.

Our next step is to establish a stronger type of Lyapunov stability. With this in mind, we assume, for the remainder of this section, that

(α) the equilibrium state Σ_0 is a topological space;

(β) each $\pi \in \mathscr{P}$ has $t \mapsto \text{Equil}\, \pi_t$ continuous.

Given a subset Ω_0 of Σ_0, we call $\text{Equil}^{-1}\, \Omega_0$ the **equilibrium tube generated** by Ω_0; the topology for Σ consisting of all equilibrium tubes generated by open sets in Σ_0 is called the **equilibrium topology** on Σ.

Theorem 8.2. *Let $\Omega_0 \subset \Sigma_0$ be a potential well for U relative to an equilibrium state σ_0. Then there exists an $\varepsilon_0 > 0$ such that the intersection of the equilibrium tube $\text{Equil}^{-1}\, \Omega_0$ with any l-ball of radius $\leqq \varepsilon_0$ is an invariant set.*

Proof. Let Ω_0 be a potential well for U relative to σ_0, and let

$$\varepsilon_0 = \inf\{U(\lambda_0) \colon \lambda_0 \in \partial\Omega_0\} - U(\sigma_0), \tag{8.8}$$

so that $\varepsilon_0 > 0$. Let π be a process with

$$\text{Equil}\, \pi_0 \in \Omega_0, \qquad l(\pi_0, \sigma_0) < \varepsilon_0, \tag{8.9}$$

and let

$$s = \sup\{\tau \colon \text{Equil}\, \pi_t \in \Omega_0 \ \forall t \in [0, \tau]\}. \tag{8.10}$$

Assume that

$$s < d_\pi. \tag{8.11}$$

Since $t \mapsto \text{Equil}\, \pi_t$ is continuous and Ω_0 is open (in Σ_0), $(8.9)_1$ and (8.10) imply that $s > 0$ and $\text{Equil}\, \pi_s \in \partial\Omega_0$. Thus, by (8.8),

$$U(\text{Equil}\, \pi_s) - U(\sigma_0) \geqq \varepsilon_0.$$

On the other hand, (L_2), (6.3), (6.5), $(6.6)_1$, (8.1), and $(8.9)_2$ imply that

$$U(\text{Equil}\, \pi_t) - U(\sigma_0) \leqq V(\pi_t) - V(\sigma_0) \leqq V(\pi_0) - V(\sigma_0) \leqq l(\pi_0, \sigma_0) < \varepsilon_0$$

and we have a contradiction. Thus (8.11) cannot be valid, so that $s = d_\pi$ and $\text{EquilOrbit}(\pi) \subset \Omega_0$, or equivalently

$$\text{Orbit}(\pi) \subset \text{Equil}^{-1}\, \Omega_0. \tag{8.12}$$

Further, since U has a minimum at σ_0 over Ω_0, we conclude from Lemma 8.1 that (8.4) must hold. Thus (8.9) implies (8.4) and (8.12), which clearly leads to the desired conclusion. □

Corollary 8.2. *Let $\Omega_0 \subset \Sigma_0$ be a potential well for U relative to an equilibrium state σ_0. Then there exists an $\varepsilon_0 > 0$ such that given any $\varepsilon \in (0, \varepsilon_0)$ a process $\pi \in \mathscr{P}$ will have*

$$\text{EquilOrbit}(\pi) \subset \Omega_0, \qquad l(\pi, \sigma_0), \int_0^{d_\pi} D(\pi_t)\, dt < \varepsilon$$

provided

$$\text{Equil}\,\pi_0 \in \Omega_0, \quad l(\pi_0, \sigma_0) < \varepsilon.$$

This corollary is essentially a rewording of Theorem 8.2 together with an application of Lemma 8.1. A slightly less trivial consequence of Theorem 8.2 is

Corollary 8.3. *Let U have a strong local minimum at an equilibrium state σ_0. Then σ_0 is Lyapunov stable with respect to the combined Lyapunov and equilibrium topologies.*

Proof. Choose a neighborhood Λ of σ_0 with respect to the combined Lyapunov and equilibrium topologies. By Proposition 2.1, Λ must contain a set of the form (see (2.2)) $(\text{Equil}^{-1}\Lambda_0) \cap \Gamma_{\varepsilon_0}(\sigma_0)$, where Λ_0 is an open neighborhood of σ_0 in Σ_0. Since U has a strong local minimum at σ_0, σ_0 must have an open neighborhood Φ_0 in Σ_0 such that $\Phi_0 \subset \Lambda_0$ and Φ_0 is a potential well for U relative to σ_0. By Theorem 8.2 there exists an $\varepsilon \in (0, \varepsilon_0)$ such that $\Omega = (\text{Equil}^{-1}\Phi_0) \cap \Gamma_\varepsilon(\sigma_0)$ is an invariant set. Thus since $\Omega \subset \Lambda$, (8.6) must hold for any $\pi \in \mathscr{P}$. □

V. Applications of the General Theory

9. *Elastic Materials*

Consider now the elastic body discussed in Section 5. Recall that states are triplets of the form $(\boldsymbol{x}, \boldsymbol{v}, \theta)$, while equilibrium states have the form $(\boldsymbol{x}, \boldsymbol{0}, \theta_e)$. When convenient we shall identify the equilibrium state space Σ_0 with the set AdmissConfig of all admissible configurations. In particular, for the potential energy in an equilibrium state $(\boldsymbol{x}, \boldsymbol{0}, \theta_e)$ we shall write $U(\boldsymbol{x})$ rather than $U(\boldsymbol{x}, \boldsymbol{0}, \theta_e)$. Also, (4.5) justifies our writing $U_e(\boldsymbol{x})$ in place of $U_e(\boldsymbol{x}, \boldsymbol{v}, \theta)$, so that the domain of both U and U_e is here taken to be AdmissConfig.

We make no general assumptions concerning the class $\mathscr{P}$ of processes other than those made in Sections 3–6. (Thus, in particular, $\mathscr{P}$ is a process class.) Given a process $\pi = (\boldsymbol{x}, \boldsymbol{v}, \theta) \in \mathscr{P}$, we write

$$\boldsymbol{F} = \nabla \boldsymbol{x},$$

and we denote by $\boldsymbol{x}(t)$, $\boldsymbol{F}(t)$, $\boldsymbol{v}(t)$, and $\theta(t)$, respectively, the *fields* $\boldsymbol{x}(\cdot, t)$, $\boldsymbol{F}(\cdot, t)$, $\boldsymbol{v}(\cdot, t)$, and $\theta(\cdot, t)$. By (5.10), π starts from equilibrium provided

$$\boldsymbol{v}(0) = \dot{\boldsymbol{x}}(0) = \boldsymbol{0}, \quad \theta(0) = \theta_e \quad \text{on } \mathscr{B}.$$

By (5.11) and Theorem 5.1, $(\Sigma, \mathscr{P}, \text{Equil})$ is a dynamical system with V, defined by (5.8), an associated Lyapunov function. Thus all of the results of Chapter IV are applicable here. In particular, (5.13) and Theorem 6.1 yield

Theorem 9.1.[1] *Let $\pi = (\chi, v, \theta) \in \mathscr{P}$ be a process starting from equilibrium. Then*

$$\int_{\mathscr{B}} \hat{\psi}(\boldsymbol{F}(t), \theta_e)\, dm + U_e(\boldsymbol{x}(t)) \leqq \int_{\mathscr{B}} \hat{\psi}(\boldsymbol{F}(0), \theta_e)\, dm + U_e(\boldsymbol{x}(0)).$$

Theorem 9.1 asserts that the potential energy in the current configuration and at the environmental temperature θ_e is not greater than the potential energy in the initial configuration. This result is interesting, especially since $\theta(\boldsymbol{X}, t)$, for $t > 0$, is not required to coincide with θ_e.

[1] GURTIN (1973b), p. 25.

We say that $\hat{\psi}$ has a **global isothermal minimum** at $(\mathbf{1}, \theta)$ if

$$\hat{\psi}(\mathbf{1}, \theta) \leqq \hat{\psi}(\boldsymbol{F}, \theta) \tag{9.1}$$

for every $\boldsymbol{F} \in \mathrm{Lin}^+$. We say that $\hat{\psi}$ has a **local isothermal minimum** at $(\mathbf{1}, \theta)$ if for some $\varepsilon > 0$ (9.1) holds for every $\boldsymbol{F}$ with[1] $|\boldsymbol{C} - \mathbf{1}| < \varepsilon$ ($\boldsymbol{C} = \boldsymbol{F}^T\boldsymbol{F}$), where $\varepsilon > 0$ is independent of the material point $\boldsymbol{X}$ under consideration. (Recall our agreement to write $\hat{\psi}(\boldsymbol{F}, \theta)$ in place of $\hat{\psi}(\boldsymbol{F}, \theta, \boldsymbol{X})$.) These definitions imply

Lemma 9.1. *Let* Ψ: AdmissConfig $\to \mathbb{R}$ *be defined by*

$$\Psi(\boldsymbol{x}) = \int_{\mathscr{B}} \hat{\psi}(\boldsymbol{F}, \theta_e)\, dm.$$

(i) *If* $\hat{\psi}$ *has a global isothermal minimum at* $(\mathbf{1}, \theta_e)$, *then* Ψ *has a global minimum at the undeformed configuration.*

(ii) *If* $\hat{\psi}$ *has a local isothermal minimum at* $(\mathbf{1}, \theta_e)$, *then* Ψ *has a local minimum at the undeformed configuration with respect to the topology* $\mathscr{T}(c_\infty)$.

Here $\mathscr{T}(c_\infty)$ is the topology on AdmissConfig generated by the pseudo-metric c_∞ defined in $(5.17)_2$. It is clear that for a process $\pi = (\boldsymbol{x}, \boldsymbol{v}, \theta)$ to have $t \mapsto \mathrm{Equil}\,\pi_t$ continuous (with respect to $\mathscr{T}(c_\infty)$) it suffices to have $\boldsymbol{C}$ continuous on $\mathscr{B} \times [0, d_\pi)$, where $\boldsymbol{C} = \nabla\boldsymbol{x}^T \nabla\boldsymbol{x}$ is the right Cauchy-Green strain tensor in π. Thus, since $U = \Psi$ when $U_e = 0$ (see (5.13)), we have the following consequence of Lemma 9.1, Theorem 6.2, and Theorem 6.3:

Theorem 9.2.[2] *Assume that* $U_e = 0$. *Let* $\hat{\psi}$ *have a local isothermal minimum at* $(\mathbf{1}, \theta_e)$. *Then there is at most one process* $\pi = (\boldsymbol{x}, \boldsymbol{v}, \theta) \in \mathscr{P}$ *with* $\boldsymbol{C} = \nabla\boldsymbol{x}^T \nabla\boldsymbol{x}$ *continuous and*

$$\boldsymbol{x}(\boldsymbol{X}, 0) = \boldsymbol{X}, \quad \boldsymbol{v}(\boldsymbol{X}, 0) = \mathbf{0}, \quad \theta(\boldsymbol{X}, 0) = \theta_e \qquad \forall\, \boldsymbol{X} \in \mathscr{B}.$$

Moreover, this process, if it exists (in $\mathscr{P}$), *is the rest process*

$$\boldsymbol{x}(\boldsymbol{X}, t) = \boldsymbol{X}, \qquad \boldsymbol{v}(\boldsymbol{X}, t) = \mathbf{0}, \qquad \theta(\boldsymbol{X}, t) = \theta_e$$

for all $\boldsymbol{X} \in \mathscr{B}$ *and* $t \in [0, d_\pi)$. *If* $\hat{\psi}$ *has a global isothermal minimum at* $(\mathbf{1}, \theta_e)$, *then the requirement that* $\boldsymbol{C}$ *be continuous may be omitted.*

$U_e = 0$ corresponds to situations in which the conservative external loads vanish. Theorem 9.2 shows that, in such situations, when $\hat{\psi}$ has a local isothermal minimum at $(\mathbf{1}, \theta_e)$, the only possible process starting from the undeformed equilibrium state is the rest process.

Assume now that a Hausdorff topology has been chosen for the set AdmissConfig, and let Σ_0 be endowed with the corresponding induced topology. Let $\boldsymbol{x}_0, \boldsymbol{x}_1 \in$ AdmissConfig. We say that $\boldsymbol{x}_1$ **is attracted to** $\boldsymbol{x}_0$ if there exists a process $\pi = (\boldsymbol{x}, \boldsymbol{v}, \theta) \in \mathscr{P}$ starting from the equilibrium state $(\boldsymbol{x}_1, \mathbf{0}, \theta_e)$ such that

$$\boldsymbol{x}(t) \to \boldsymbol{x}_0 \quad \text{as} \quad t \to \infty$$

(the limit being in the topology chosen for AdmissConfig). Further, we say that $\boldsymbol{x}_0$ is **locally asymptotically stable** if there exist neighborhoods Ω_0, Λ_0 of $\boldsymbol{x}_0$ in AdmissConfig with $\Omega_0 \subset \Lambda_0$ such that

[1] Recall that the principle of material frame-indifference requires that, for fixed θ, the value of $\hat{\psi}$ is the same on any two $\boldsymbol{F}$ with equal right Cauchy-Green strain tensors $\boldsymbol{C} = \boldsymbol{F}^T\boldsymbol{F}$.

[2] *Cf.* GURTIN (1973a), Thm. 9.5, (1973b), p. 26.

(a) for each process $\pi=(\boldsymbol{x}, \boldsymbol{v}, \theta) \in \mathscr{P}$ starting from equilibrium, $\boldsymbol{x}(0)\in\Omega_0$ implies $\boldsymbol{x}(t)\in\Lambda_0$ for all $t\in[0, d_\pi)$;

(b) each $\boldsymbol{x}_1\in\Lambda_0$ is attracted to $\boldsymbol{x}_0$.

Note that in the above definitions nothing is said about the behavior of $\boldsymbol{v}(t)$ or $\theta(t)$ as $t\to d_\pi$.

Lemma 9.2. *Let $\boldsymbol{x}_0$ be an admissible configuration.*

(i) *An admissible configuration $\boldsymbol{x}_1$ is attracted to $\boldsymbol{x}_0$ if and only if $(\boldsymbol{x}_1, \boldsymbol{0}, \theta_e)$ is attracted to $(\boldsymbol{x}_0, \boldsymbol{0}, \theta_e)$ in the sense of the definition containing* (7.1).

(ii) *$\boldsymbol{x}_0$ is locally asymptotically stable if and only if $(\boldsymbol{x}_0, \boldsymbol{0}, \theta_e)$ is locally asymptotically stable with respect to equilibrium orbits.*

This lemma, Theorem 7.1, Corollary 7.1, and Corollary 7.2 yield

Theorem 9.3. *Assume that the potential energy*

$$U(\boldsymbol{x})=\int_{\mathscr{B}}\hat{\psi}(\boldsymbol{F}, \theta_e)dm+U_e(\boldsymbol{x}) \tag{9.2}$$

is continuous at an admissible configuration $\boldsymbol{x}_0$.

(i)[1] *If an admissible configuration $\boldsymbol{x}_1$ is attracted to $\boldsymbol{x}_0$, then*

$$U(\boldsymbol{x}_0)\leqq U(\boldsymbol{x}_1). \tag{9.3}$$

(ii) *If every configuration in a neighborhood of $\boldsymbol{x}_0$ in* AdmissConfig *is attracted to $\boldsymbol{x}_0$, then U has a local minimum at $\boldsymbol{x}_0$.*

(iii)[2] *If $\boldsymbol{x}_0$ is locally asymptotically stable, then U has a strict local minimum at $\boldsymbol{x}_0$.*

Note that in the case of dead loading (9.3) takes the form

$$\int_{\mathscr{B}} w(\boldsymbol{F}_0)dm\leqq\int_{\mathscr{B}} w(\boldsymbol{F}_1)dm-\int_{\partial\mathscr{B}}\boldsymbol{s}\cdot\boldsymbol{u}\,da, \tag{9.4}$$

where

$$w(\boldsymbol{F})=\psi(\boldsymbol{F}, \theta_e), \qquad \boldsymbol{u}=\boldsymbol{x}_1-\boldsymbol{x}_0.$$

The classical criterion for the mechanical stability of a given configuration $\boldsymbol{x}_0$ of an elastic body with stored energy function w is that (9.4) hold for every displacement $\boldsymbol{u}$ of $\mathscr{B}$ within a certain class. Theorem 9.3 shows that this classical criterion is, in a certain precise sense, necessary for dynamic stability, even under thermal influences, provided the stored energy is identified with the free-energy at the environmental temperature.

By (3.7) and (3.8), the state function D defined by $(3.6)_2$ is a maximal dissipation function for V, while (3.1) and (3.4) imply that $\theta_e G$, $\theta_e G_e$, and $-P_e$ are dissipation functions. Further, $\theta_e G$ is maximal when the environment is both conservative and thermally perfect.

Let us agree to write $\gamma(\pi_t)=\gamma_{\pi_t}$ for the value of the specific entropy production at time t in a process π; of course, $\gamma(\pi_t)$ is a field over $\mathscr{B}$ and, by $(4.1)_3$,

$$G(\pi_t)=\int_{\mathscr{B}}\gamma(\pi_t)dm.$$

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[1] *Cf.* GURTIN (1973a), Thm. 9.5, (1973b), p. 26.

[2] *Cf.* COLEMAN & DILL (1973), Thm. 6.7.

Since $\theta_e G$ is a dissipation function for V, Corollary 8.1, (8.1), (5.9), 5.12), and (5.13) yield the following result, in which it is not neccessary that AdmissConfig be endowed with a topology.

Theorem 9.4. *Assume that the potential energy* (9.2) *has a global minimum at an admissible configuration* $\boldsymbol{x}_0$. *Then given any* $\varepsilon>0$ *there exists a* $\delta>0$ *such that any process* $\pi=(\boldsymbol{x}, \boldsymbol{v}, \theta)\in\mathscr{P}$ *with*

$$\int_{\mathscr{B}} \boldsymbol{v}^2(0)\,dm<\delta, \qquad \int_{\mathscr{B}} \kappa(0)[\theta(0)-\theta_e]^2\,dm<\delta,$$
$$U(\boldsymbol{x}(0))-U(\boldsymbol{x}_0)<\delta, \tag{9.5}$$

satisfies

$$\int_{\mathscr{B}} \boldsymbol{v}^2(t)\,dm<\varepsilon, \qquad \int_{\mathscr{B}} \kappa(t)[\theta(t)-\theta_e]^2\,dm<\varepsilon,$$
$$U(\boldsymbol{x}(t))-U(\boldsymbol{x}_0)<\varepsilon, \qquad \int_0^{d_\pi}\int_{\mathscr{B}} \gamma(\pi_\tau)\,dm\,d\tau<\varepsilon \tag{9.6}$$

for all $t\in[0, d_\pi)$, *where* $\kappa(t)=\kappa(\boldsymbol{F}(t), \theta(t))$.

Remark. For an elastic material

$$\gamma(\sigma)=-\frac{1}{\rho\theta^2}\hat{\boldsymbol{q}}(\boldsymbol{F}, \theta, \boldsymbol{g})\cdot\boldsymbol{g}, \qquad \boldsymbol{g}=\nabla\theta. \tag{9.7}$$

Assume, for the moment, that Fourier's Law

$$\hat{\boldsymbol{q}}(\boldsymbol{F}, \theta, \boldsymbol{g})=-k(F, \theta)\boldsymbol{g}$$

holds, with k positive and bounded away from zero. Then there exists a constant $C>0$ such that

$$\int_{\mathscr{B}} \gamma(\sigma)\,dm=\int_{\mathscr{B}} k(F, \theta)|\nabla\ln\theta|^2\,dv\geqq C\,\|\nabla\ln\theta\|_{L_2}^2,$$

and $(9.6)_4$ implies that (for $d_\pi=\infty$)

$$C\int_0^\infty \|\nabla\ln\theta(t)\|_{L_2}^2\,dt<\varepsilon. \tag{9.8}$$

This type of inequality is the basis for proofs of asymptotic stability. Indeed, when (9.8) holds, and when $\dfrac{d}{dt}\|\nabla\ln\theta(t)\|_{L_2}^2$ is uniformly bounded in t, then $\|\nabla\ln\theta(t)\|_{L_2}\to 0$ as $t\to\infty$.

Corollary 9.1.[1] *Assume that* $U_e=0$ *and let* $\hat{\psi}$ *have a global isothermal minimum at* $(\mathbf{1}, \theta_e)$. *Then given any* $\varepsilon>0$ *there exists a* $\delta>0$ *such that any process* $\pi=(\boldsymbol{x}, \boldsymbol{v}, \theta)\in\mathscr{P}$ *with*

$$\int_{\mathscr{B}} \boldsymbol{v}^2(0)\,dm<\delta, \qquad \int_{\mathscr{B}} \kappa(0)[\theta(0)-\theta_e]^2\,dm<\delta$$
$$\int_{\mathscr{B}} [\hat{\psi}(\boldsymbol{F}(0), \theta_e)-\hat{\psi}(\mathbf{1}, \theta_e)]\,dm<\delta$$

[1] *Cf.* GURTIN (1973a), Theorem 9.3.

satisfies

$$\int_{\mathscr{B}} v^2(t)\,dm<\varepsilon, \qquad \int_{\mathscr{B}} \kappa(t)[\theta(t)-\theta_e]^2\,dm<\varepsilon,$$

$$\int_{\mathscr{B}} [\hat{\psi}(\boldsymbol{F}(t),\theta_e)-\hat{\psi}(\mathbf{1},\theta_e)]\,dm<\varepsilon, \qquad \int_0^{d_\pi}\int_{\mathscr{B}} \gamma(\pi_\tau)\,dm\,d\tau<\varepsilon$$

for all $t\in[0, d_\pi)$.

Assume now that the set AdmissConfig is endowed with a topology, and suppose that each $\pi=(\boldsymbol{x}, v, \theta)\in\mathscr{P}$ has $t\mapsto\boldsymbol{x}(t)$ continuous. Then Corollary 8.2, in the present theory, reads

Theorem 9.5. *Let* $\Omega_0\subset$ AdmissConfig *be a potential well for* U *relative to an admissible configuration* $\boldsymbol{x}_0$. *Then given any* $\varepsilon>0$ *there exists a* $\delta>0$ *such that any process* $\pi=(\boldsymbol{x}, v, \theta)\in\mathscr{P}$ *which is consistent with* (9.5) *and has* $\boldsymbol{x}(0)\in\Omega_0$ *satisfies* (9.6) *and has* $x(t)\in\Omega_0$ *for all* $t\in[0, d_\pi)$.

Further, when the topology on AdmissConfig is generated by a pseudo-metric d, then Theorem 9.5 and the definition of a strong local minimum imply

Theorem 9.6.[1] *Let* U *have a strong local minimum at an admissible configuration* $\boldsymbol{x}_0$. *Then given any* $\varepsilon>0$ *there exists a* $\delta>0$ *such that any process* $\pi=(x, v, \theta)\in\mathscr{P}$ *which is consistent with* (9.5) *and has*

$$d(\boldsymbol{x}(0), \boldsymbol{x}_0)<\delta$$

satisfies (9.6) *and has*

$$d(\boldsymbol{x}(t), \boldsymbol{x}_0)<\varepsilon$$

for all $t\in[0, d_\pi)$.

Remark. It is important to note that in deriving the results of this section the only constitutive relations used were (5.2) and (5.5). For an elastic material the stress is given by a constitutive relation identical in nature to (5.2) and obeys a stress relation similar to (5.5) The fact that these relations were not needed is far from trivial. Indeed, (5.2) and (5.5) are satisfied by large classes of *viscous* materials;[2] these include, of course, the classical linearly viscous fluid. Therefore the results of this section are valid, almost without change, for such materials. Of course, the corresponding specific entropy production γ will not have the simple form (9.7), but will generally involve the strain rate $\dot{\boldsymbol{F}}$. For a linearly viscous fluid γ is quadratic in the spatial velocity gradient. When the viscosity is constant and the velocity zero on the boundary, the Poincaré inequality can be used to derive, in conjunction with (9.8), an inequality of the form

$$C_1\int_0^\infty \|v\|_{L_2}^2\,dt<\varepsilon.$$

Remark. For an insulated boundary $(4.1)_{5,6}$ and (4.4) imply that Q and J are identically zero. We therefore conclude from $(3.5)_2$ that $G_e=0$ and $(3.4)_2$, $(3.5)_2$ are satisfied trivially *for any choice of environmental temperature* θ_e. There-

[1] *Cf.* KOITER (1969, 1971); GURTIN (1973a), Thm. 9.6. KOITER used an unnecessarily stringent definition of a strong local minimum.

[2] *Cf.* COLEMAN & NOLL (1963); COLEMAN & MIZEL (1964).

fore, for this type of environment there is a one-parameter family of equilibrium maps, one for each choice of θ_e, and for each such choice all of our results are valid.

10. Materials with Memory

For a material with memory the state involves not only the present configuration and temperature, but also the past histories of these quantities. With this in mind we define

$$\text{ConfigHist} = \text{the set of all maps } \boldsymbol{x}^*: (0, \infty) \to \text{Config},$$

$$\text{TempHist} = \text{the set of all maps } \theta^*: (0, \infty) \to \text{Temp}.$$

Each $\boldsymbol{x}^* \in \text{ConfigHist}$ represents a possible configuration history for $\mathscr{B}$; $\boldsymbol{x}^*(s)$, $s > 0$, gives the configuration occupied by $\mathscr{B}$, s time units into the past. A similar interpretation applies to each $\theta^* \in \text{TempHist}$. Let $\boldsymbol{x} \in \text{Config}$ and $\theta \in \text{Temp}$. Then $\boldsymbol{x}^c \in \text{ConfigHist}$ and $\theta^c \in \text{TempHist}$ are the *constant* histories defined by

$$\boldsymbol{x}^c(s) = \boldsymbol{x}, \quad \theta^c(s) = \theta \quad \forall\, s > 0.$$

A **body with memory** is a continuous body $\mathscr{B}$, consistent with the hypotheses of Sections 3 and 4, whose state space, local state functions, equilibrium map, and process class satisfy (M_1)–(M_5) listed below.

(M_1) The state space Σ is a subset of the set of all ordered arrays of the form

$$\sigma = (\boldsymbol{x}, \boldsymbol{v}, \theta, \boldsymbol{x}^*, \theta^*), \tag{10.1}$$

where

$\boldsymbol{x} \in \text{Config}$	is the current configuration,
$\boldsymbol{v} \in \text{Vel}$	is the current velocity,
$\theta \in \text{Temp}$	is the current temperature,
$\boldsymbol{x}^* \in \text{ConfigHist}$	is the past history of the configuration,
$\theta^* \in \text{TempHist}$	is the past history of the temperature.

We assume that $(\boldsymbol{x}, \boldsymbol{v}, \theta_e, \boldsymbol{x}^*, \theta^*)$, $(\boldsymbol{x}, \boldsymbol{v}, \theta, \boldsymbol{x}^c, \theta^c)$, and

$$\sigma_0 = (\boldsymbol{x}, \boldsymbol{0}, \theta_e, \boldsymbol{x}^c, \theta_e^c) \tag{10.2}$$

belong to Σ whenever (10.1) belongs to Σ.

(M_2) The local response functions $\hat{\boldsymbol{x}}$, $\hat{\boldsymbol{v}}$, and $\hat{\theta}$ are given by (5.1), (10.1).

(M_3) The equilibrium function is given by

$$\text{Equil}(\boldsymbol{x}, \boldsymbol{v}, \theta, \boldsymbol{x}^*, \theta^*) = (\boldsymbol{x}, \boldsymbol{0}, \theta_e, \boldsymbol{x}^c, \theta_e^c). \tag{10.3}$$

Thus an equilibrium state is one in which the current configuration is arbitrary, the current velocity vanishes, and the current temperature is the environmental temperature θ_e, and in which the configuration and temperature have, at all past times, been equal to their current values $\boldsymbol{x}$ and θ_e. In view of (10.3), the equilibrium state space Σ_0 can be identified, in a natural manner, with the set AdmissConfig of all admissible configurations. When we wish to make this identification explicit, we will refer to $(\boldsymbol{x}, \boldsymbol{0}, \theta_e, \boldsymbol{x}^c, \theta_e^c)$ as the **equilibrium state corresponding to $\boldsymbol{x}$.**

(M_4) The local response functions ε and η are given by constitutive equations of the form

$$\varepsilon_\sigma(X)=\tilde{\varepsilon}(f_\sigma(X), X),$$
$$\eta_\sigma(X)=\tilde{\eta}(f_\sigma(X), X),$$

where

$$f_\sigma=(F, \theta, F^*, \theta^*), \quad F=\nabla x, \quad F^*=\nabla x^*.$$

We assume that for each admissible choice of X, $F^*(X)$, and $\theta^*(X)$, $\tilde{\varepsilon}$ and $\tilde{\eta}$, considered as functions of $F(X)$ and $\theta(X)$, are smooth maps with domain $\mathrm{Lin}^+ \times \mathbb{R}^{++}$. The free-energy $\tilde{\psi}$ is defined by

$$\tilde{\psi}(f_\sigma)=\tilde{\varepsilon}(f_\sigma)-\theta\tilde{\eta}(f_\sigma), \tag{10.4}$$

where, for convenience, we have suppressed the argument X. We define the equilibrium free-energy $\hat{\psi}$ on $\mathrm{Lin}^+ \times \mathbb{R}^{++} \times \mathscr{B}$ by

$$\hat{\psi}(F, \theta)=\tilde{\psi}(F, \theta, F^c, \theta^c), \tag{10.5}$$

where, for $F \in \mathrm{Lin}^+$ and $\theta \in \mathbb{R}^{++}$, F^c and θ^c are the histories defined by $F^c(s)=F$ and $\theta^c(s)=\theta$ for all $s>0$. Thus $\tilde{\psi}(F, \theta)$ is the value of the free-energy when the deformation gradient and temperature have, at all past times, been equal to their current values F and θ. We assume that[1]

$$\tilde{\eta}=-\partial_2\tilde{\psi}, \quad \partial_2\partial_2\tilde{\psi}<0, \tag{10.6}$$[2]

and that, for each $\sigma \in \Sigma$,

$$\hat{\psi}(F, \theta) \leqq \tilde{\psi}(F, \theta, F^*, \theta^*). \tag{10.7}$$

(M_5) Each process

$$t \mapsto \pi_t=(x(t), v(t), \theta(t), x^t, \theta^t) \tag{10.8}$$

in $\mathscr{P}$ (where $x(t) \in \mathrm{Config}$, $v(t) \in \mathrm{Vel}$, $\theta(t) \in \mathrm{Temp}$, $x^t \in \mathrm{ConfigHist}$, $\theta^t \in \mathrm{TempHist}$) is maximal and satisfies the compatibility condition

$$x^t(s)=\begin{cases} x(t-s), & 0<s \leqq t \\ x^0(s-t), & s>t \end{cases} \qquad \theta^t(s)=\begin{cases} \theta(t-s), & 0<s \leqq t \\ \theta^0(s-t), & s>t \end{cases} \tag{10.9}$$

for all $t \in [0, d_\pi)$. Of course, (4.3) requires that

$$v=\dot{x}. \tag{10.10}$$

Note that, by (10.3), π starts from equilibrium if and only if

$$v(0)=0, \quad \theta(0)=\theta_e, \quad x^0=x(0)^c, \quad \theta^0=\theta_e^c. \tag{10.11}$$

Of future use is the following obvious generalization of Ericksen's Lemma.[3]

---

[1] COLEMAN (1964) (eqs. (7.8 b), (8.21)) has shown that for a material with fading memory $(10.6)_1$ and (10.7) are consequences of the second law of thermodynamics. The assumption $(10.6)_2$ is simply the requirement that the instantaneous specific heat $\partial_2\tilde{\varepsilon}$ be strictly positive.

[2] Here, of course, $\partial_2\tilde{\psi}$ is the derivative of $\tilde{\psi}(F, \theta, F^*, \theta^*, X)$ with respect to θ holding the remaining arguments fixed.

[3] COLEMAN & DILL (1973), Thm. 4.3.

Lemma 10.1. *There exists a function* $f_\sigma \mapsto \kappa(f_\sigma)$ *such that*

$$\tilde{\varepsilon}(f_\sigma)-\theta_e\,\tilde{\eta}(f_\sigma)=\tilde{\psi}(\boldsymbol{F},\theta_e,\boldsymbol{F}^*,\theta^*)+\kappa(f_\sigma)[\theta-\theta_e]^2, \qquad \kappa(f_\sigma)>0 \tag{10.12}$$

for every $\sigma\in\Sigma$.

The proof of this lemma is given in the Appendix.

By (4.2) and (10.12),

$$V(\sigma)=U_e(\boldsymbol{x})+\int_{\mathscr{B}}[\tilde{\psi}(\boldsymbol{F},\theta_e,\boldsymbol{F}^*,\theta^*)+\kappa(f_\sigma)(\theta-\theta_e)^2+\tfrac{1}{2}\boldsymbol{v}^2]\,dm,$$

where we have used (4.5) to justify writing $U_e(\boldsymbol{x})$ in place of $U_e(\sigma)$. In the equilibrium state (10.2), $f_{\sigma_0}=(\boldsymbol{F},\theta_e,\boldsymbol{F}^c,\theta_e^c)$; we therefore conclude from (6.1), (6.4), and (10.5) that

$$\begin{aligned} U(\boldsymbol{x})&=U_e(\boldsymbol{x})+\int_{\mathscr{B}}\hat{\psi}(\boldsymbol{F},\theta_e)\,dm,\\ T(\sigma)&=\int_{\mathscr{B}}[\tilde{\psi}(\boldsymbol{F},\theta_e,\boldsymbol{F}^*,\theta^*)-\hat{\psi}(\boldsymbol{F},\theta_e)+\kappa(f_\sigma)(\theta-\theta_e)^2+\tfrac{1}{2}\boldsymbol{v}^2]\,dm, \end{aligned} \tag{10.13}$$

where, for convenience, we have considered U as a function on AdmissConfig rather than as a function on Σ_0. It is important to note that *the potential energy U coincides with the potential energy of an elastic body provided we interpret the stored energy as the equilibrium free-energy $\hat{\psi}$.*

Proposition 10.1. $(\Sigma,\mathscr{P},\text{Equil})$ *is a dynamical system with V as an associated Lyapunov function.*

Proof. By definition $\mathscr{P}$ is a process class, while (M_3) insures that Equil is idempotent. Thus $(\Sigma,\mathscr{P},\text{Equil})$ is a dynamical system. By Theorem 3.1, V satisfies (L_1), while (10.7), $(10.12)_2$, and $(10.13)_2$ yield (L_2). To establish (L_3) let $\pi\in\mathscr{P}$ start from equilibrium and assume that $T(\pi_t)=0$ for all $t\in[0,d_\pi)$. Then (10.7), $(10.12)_2$, and $(10.13)_2$ imply that $\boldsymbol{v}(t)=0$ and $\theta(t)=\theta_e$ for all $t\in[0,d_\pi)$, so that $\boldsymbol{x}(t)=\boldsymbol{x}(0)$ for all $t\in[0,d_\pi)$. Since π starts from equilibrium, (10.11) holds; hence (10.9) yields the conclusion that $\boldsymbol{x}^t=\boldsymbol{x}(0)^c$ and $\theta^t=\theta_e^c$ for all $t\in[0,d_\pi)$. Thus π is a rest process. □

In view of Proposition 10.1 all of the results established in Chapter IV for general dynamical systems may be applied to materials with memory. In fact all of the results established in the previous section, for elastic materials, are valid here almost without change. We now list these results without proof.

Theorem 10.1.[1] *For a process in $\mathscr{P}$ starting from equilibrium*

$$\int_{\mathscr{B}}\hat{\psi}(\boldsymbol{F}(t),\theta_e)\,dm+U_e(\boldsymbol{x}(t))\leqq\int_{\mathscr{B}}\hat{\psi}(\boldsymbol{F}(0),\theta_e)\,dm+U_e(\boldsymbol{x}(0)).$$

Theorem 10.2. *Assume that $U_e=0$. Let $\hat{\psi}$ have a local isothermal minimum at $(\mathbf{1},\theta_e)$. Then there is at most one process in $\mathscr{P}$ with $\boldsymbol{C}=\nabla\boldsymbol{x}^T\nabla\boldsymbol{x}$ continuous,*

and

$$\boldsymbol{x}(\boldsymbol{X},0)=\boldsymbol{X}, \qquad \boldsymbol{v}(\boldsymbol{X},0)=\boldsymbol{0}, \qquad \theta(\boldsymbol{X},0)=\theta_e \qquad \forall\,\boldsymbol{X}\in\mathscr{B}$$

$$\boldsymbol{x}^0(s)(\boldsymbol{X})=\boldsymbol{X}, \qquad \theta^0(s)(\boldsymbol{X})=\theta_e \qquad \forall\,\boldsymbol{X}\in\mathscr{B},\ s>0$$

[1] *Cf.* GURTIN (1973c), p. 96.

Moreover, this process, if it exists (in $\mathscr{P}$), is the rest process with

$$\boldsymbol{x}(\boldsymbol{X}, t)=\boldsymbol{X}, \quad \boldsymbol{v}(\boldsymbol{X}, t)=\boldsymbol{0}, \quad \theta(\boldsymbol{X}, t)=\theta_e$$

for all $\boldsymbol{X}\in\mathscr{B}$ and $t\in[0, d_\pi)$. If $\hat{\psi}$ has a global isothermal minimum at $(\mathbf{1}, \theta_e)$, then the requirement that $\boldsymbol{C}$ be continuous may be omitted.

Assume now that a Hausdorff topology has been chosen for the set AdmissConfig. Let $\boldsymbol{x}_0, \boldsymbol{x}_1 \in$ AdmissConfig. We say that $\boldsymbol{x}_1$ **is attracted to** $\boldsymbol{x}_0$ if there exists a process in $\mathscr{P}$ starting from the equilibrium state corresponding to $\boldsymbol{x}_1$ such that

$$\boldsymbol{x}(t)\to\boldsymbol{x}_0 \quad \text{as} \quad t\to\infty$$

(the limit being in the topology chosen for AdmissConfig). Further, we say that $\boldsymbol{x}_0$ is **locally asymptotically stable** if there exist neighborhoods Ω_0, Λ_0 of $\boldsymbol{x}_0$ in AdmissConfig with $\Omega_0\subset\Lambda_0$ such that

(a) for each process in $\mathscr{P}$ starting from equilibrium, $\boldsymbol{x}(0)\in\Omega_0$ implies $\boldsymbol{x}(t)\in\Lambda_0$ for all $t\in[0, d_\pi)$:

(b) each $\boldsymbol{x}_1\in\Lambda_0$ is attracted to $\boldsymbol{x}_0$.

Theorem 10.3. *Assume that the potential energy*

$$U(\boldsymbol{x})=\int_{\mathscr{B}}\hat{\psi}(\boldsymbol{F}, \theta_e)\,dm+U_e(\boldsymbol{x}) \tag{10.14}$$

is continuous at an admissible configuration $\boldsymbol{x}_0$.

(i)[1] *If an admissible configuration $\boldsymbol{x}_1$ is attracted to $\boldsymbol{x}_0$, then*

$$U(\boldsymbol{x}_0)\leqq U(\boldsymbol{x}_1).$$

(ii)[2] *If every configuration in a neighborhood of $\boldsymbol{x}_0$ in* AdmissConfig *is attracted to $\boldsymbol{x}_0$, then U has a local minimum at $\boldsymbol{x}_0$.*

(iii)[3] *If $\boldsymbol{x}_0$ is locally asymptotically stable, then U has a strict local minimum at $\boldsymbol{x}_0$.*

Theorem 10.3 shows that the minimum energy criterion is, in a certain precise sense, necessary for the dynamic stability of a body with memory, provided the stored energy is identified with the *equilibrium* free-energy at the environmental temperature.

The next result does not require a topology for the set AdmissConfig.

Theorem 10.4.[4] *Assume that the potential energy* $(10.13)_1$ *has a global minimum at an admissible configuration $\boldsymbol{x}_0$. Then given any $\varepsilon>0$ there exists a $\delta>0$ such that any process in $\mathscr{P}$ with*

$$\begin{gathered}\int_{\mathscr{B}}\boldsymbol{v}^2(0)\,dm<\delta, \qquad \int_{\mathscr{B}}\kappa(0)[\theta(0)-\theta_e]^2\,dm<\delta,\\ \int_{\mathscr{B}}[\tilde{\psi}(\boldsymbol{F}(0), \theta_e, \boldsymbol{F}^0, \theta^0)-\hat{\psi}(\boldsymbol{F}(0), \theta_e)]\,dm<\delta,\\ U(\boldsymbol{x}(0))-U(\boldsymbol{x}_0)<\delta,\end{gathered} \tag{10.15}$$

[1] *Cf.* COLEMAN & DILL (1973), Thm. 6.4.
[2] *Cf.* COLEMAN & DILL (1973), Thm. 6.5.
[3] *Cf.* COLEMAN & DILL (1973), Thm. 6.7.
[4] A related theorem is Thm. 6.2 of COLEMAN & DILL (1973).

satisfies

$$\int_{\mathscr{B}} \boldsymbol{v}^2(t)\,dm<\varepsilon, \qquad \int_{\mathscr{B}} \kappa(t)[\theta(t)-\theta_e]^2\,dm<\varepsilon,$$
$$\int_{\mathscr{B}} [\tilde{\psi}(\boldsymbol{F}(t),\theta_e,F^t,\theta^t)-\hat{\psi}(\boldsymbol{F}(t),\theta_e)]\,dm<\varepsilon, \tag{10.16}$$
$$U(\boldsymbol{x}(t))-U(\boldsymbol{x}_0)<\varepsilon, \qquad \int_0^{d_\pi}\int_{\mathscr{B}} \gamma(\pi_\tau)\,dm\,d\tau<\varepsilon$$

for all $t\in[0,d_\pi)$, *where* $\kappa(t)=\kappa(\boldsymbol{f}_{\pi_t})$.

Corollary 10.1. *Assume that* $U_e=0$ *and let* $\hat{\psi}$ *have a global isothermal minimum at* $(\mathbf{1},\theta_e)$. *Then given any* $\varepsilon>0$ *there exists a* $\delta>0$ *such that any process in* $\mathscr{P}$ *with*

$$\int_{\mathscr{B}} \boldsymbol{v}^2(0)\,dm<\delta, \qquad \int_{\mathscr{B}} \kappa(0)[\theta(0)-\theta_e]^2\,dm<\delta$$
$$\int_{\mathscr{B}} [\tilde{\psi}(\boldsymbol{F}(0),\theta_e,\boldsymbol{F}^0,\theta^0)-\hat{\psi}(\mathbf{1},\theta_e)]\,dm<\delta$$

satisfies

$$\int_{\mathscr{B}} \boldsymbol{v}^2(t)\,dm<\varepsilon, \qquad \int_{\mathscr{B}} \kappa(t)[\theta(t)-\theta_e]^2\,dm<\varepsilon,$$
$$\int_{\mathscr{B}} [\tilde{\psi}(\boldsymbol{F}(t),\theta_e,\boldsymbol{F}^t,\theta^t)-\hat{\psi}(\mathbf{1},\theta_e)]\,dm<\varepsilon, \qquad \int_0^{d_\pi}\int_{\mathscr{B}} \gamma(\pi_\tau)\,dm\,d\tau<\varepsilon$$

for all $t\in[0,d_\pi)$.

Assume now that the set AdmissConfig is endowed with a topology, and suppose that each process in $\mathscr{P}$ has $t\mapsto\boldsymbol{x}(t)$ continuous.

Theorem 10.5. *Let* $\Omega_0\subset$AdmissConfig *be a potential well for* U *relative to an admissible configuration* $\boldsymbol{x}_0$. *Then given any* $\varepsilon>0$ *there exists a* $\delta>0$ *such that any process in* $\mathscr{P}$ *which is consistent with* (10.15) *and has* $\boldsymbol{x}(0)\in\Omega_0$ *satisfies* (10.16) *and has* $\boldsymbol{x}(t)\in\Omega_0$ *for all* $t\in[0,d_\pi)$.

Further, when the topology on AdmissConfig is generated by a pseudo-metric d, we have

Theorem 10.6.[1] *Let* U *have a strong local minimum at an admissible configuration* $\boldsymbol{x}_0$. *Then given any* $\varepsilon>0$ *there exists a* $\delta>0$ *such that any process in* $\mathscr{P}$ *which is consistent with* (10.15) *and has*

$$d(\boldsymbol{x}(0),\boldsymbol{x}_0)<\delta$$

satisfies (10.16) *and has*

$$d(\boldsymbol{x}(t),\boldsymbol{x}_0)<\varepsilon$$

for all $t\in[0,d_\pi)$.

11. Materials Described by Internal State Variables

For bodies of this type the internal state is described by an internal state vector $\boldsymbol{\alpha}=(\alpha_1,\alpha_2,\ldots,\alpha_n)\in\mathbb{R}^n$, whose value may vary from point to point. This

[1] *Cf.* COLEMAN & DILL (1973), Thm. 6.3.

vector enters all of the underlying constitutive equations of the material, and its evolution with time is governed by a constitutive equation of the form

$$\dot{\boldsymbol{\alpha}} = \boldsymbol{h}(\boldsymbol{F}, \theta, \boldsymbol{\alpha}), \tag{11.1}$$

where $\boldsymbol{F}$ is the deformation gradient. The equilibrium points of (11.1) are those for which

$$\boldsymbol{h}(\boldsymbol{F}, \theta, \boldsymbol{\alpha}) = 0. \tag{11.2}$$

We assume here that[1] for each $\boldsymbol{F}$ and θ there is a unique state vector

$$\boldsymbol{\alpha} = \hat{\boldsymbol{\alpha}}(\boldsymbol{F}, \theta)$$

such that (11.2) holds. These remarks should motivate the general theory presented below.

A **body described by internal state variables** is a continuous body $\mathscr{B}$ consistent with the hypotheses of Sections 3 and 4 and the hypotheses (S_1)–(S_5) below.

(S_1) The state space Σ is a subset of the set of all ordered arrays of the form

$$\sigma = (x, \boldsymbol{v}, \theta, \boldsymbol{\alpha}), \tag{11.3}$$

where $\boldsymbol{x} \in \text{Config}$, $\boldsymbol{v} \in \text{Vel}$, $\theta \in \text{Temp}$, and $\boldsymbol{\alpha} \in PC(\mathscr{B}, \mathscr{D})$ with $\mathscr{D}$ an open subset of $\mathbb{R}^n$.

(S_2) The local response function $\hat{\boldsymbol{x}}$, $\hat{\boldsymbol{v}}$, and $\hat{\theta}$ are given by (5.1), (11.3).

(S_3) The equilibrium function is defined by

$$\begin{aligned} &\text{Equil}(\boldsymbol{x}, \boldsymbol{v}, \theta, \boldsymbol{\alpha}) = (\boldsymbol{x}, \boldsymbol{0}, \theta_e, \boldsymbol{\alpha}_0), \\ &\boldsymbol{\alpha}_0(\boldsymbol{X}) = \hat{\boldsymbol{\alpha}}(\boldsymbol{F}(\boldsymbol{X}), \theta_e, \boldsymbol{X}), \qquad \boldsymbol{F} = \nabla \boldsymbol{x}, \end{aligned} \tag{11.4}$$

where $\hat{\alpha}$: $\text{Lin}^+ \times \mathbb{R}^{++} \times \mathscr{B} \to \mathscr{D}$. We assume, of course, that Equil has values in Σ.

(S_4) The local response functions ε and η are given by constitutive equations of the form

$$\begin{aligned} \varepsilon_\sigma(\boldsymbol{X}) &= \tilde{\varepsilon}(\boldsymbol{F}(\boldsymbol{X}), \theta(\boldsymbol{X}), \boldsymbol{\alpha}(\boldsymbol{X}), \boldsymbol{X}), \\ \eta_\sigma(\boldsymbol{X}) &= \tilde{\eta}(\boldsymbol{F}(\boldsymbol{X}), \theta(\boldsymbol{X}), \boldsymbol{\alpha}(\boldsymbol{X}), \boldsymbol{X}) \end{aligned} \tag{11.5}$$

with $\tilde{\varepsilon}$ and $\tilde{\eta}$ smooth functions on $\text{Lin}^+ \times \mathbb{R}^{++} \times \mathscr{D} \times \mathscr{B}$. We define the free-energy $\tilde{\psi}$ and the equilibrium free-energy $\hat{\psi}$ by

$$\begin{aligned} \tilde{\psi}(\boldsymbol{F}, \theta, \boldsymbol{\alpha}) &= \tilde{\varepsilon}(\boldsymbol{F}, \theta, \boldsymbol{\alpha}) - \theta \tilde{\eta}(\boldsymbol{F}, \theta, \boldsymbol{\alpha}), \\ \hat{\psi}(\boldsymbol{F}, \theta) &= \tilde{\psi}(\boldsymbol{F}, \theta, \hat{\boldsymbol{\alpha}}(\boldsymbol{F}, \theta)), \end{aligned} \tag{11.6}$$

and we assume that[2]

$$\tilde{\eta} = -\partial_2 \tilde{\psi}, \qquad \partial_2 \partial_2 \tilde{\psi} < 0, \tag{11.7}$$

and that[3]

$$\hat{\psi}(\boldsymbol{F}, \theta) \leqq \tilde{\psi}(\boldsymbol{F}, \theta, \boldsymbol{\alpha}) \tag{11.8}$$

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[1] *Cf.* COLEMAN & GURTIN (1967), Eq. (6.17).

[2] Equation $(11.7)_1$ is a consequence of the second law of thermodynamics (*cf.* COLEMAN & GURTIN (1967), Eq. (5.16)); $(11.7)_2$ is the assumption that the instantaneous specific heat be strictly positive.

[3] *Cf.* COLEMAN & GURTIN (1967), p. 607, where it is shown that (11.8) follows whenever each equilibrium point $(\boldsymbol{F}, \theta, \hat{\boldsymbol{\alpha}}(\boldsymbol{F}, \theta))$ of the evolution equation (11.1) is, in a precise sense, asymptotically stable in the large at constant strain and temperature.

for all $\boldsymbol{F}\in\mathrm{Lin}^+$, $\theta\in\mathbb{R}^{++}$, and $\boldsymbol{\alpha}\in\mathscr{D}$.

(S_5) Each process $\pi=(\boldsymbol{x},\boldsymbol{v},\theta,\boldsymbol{\alpha})\in\mathscr{P}$ is maximal. Note that, by (4.3),

$$\boldsymbol{v}=\dot{\boldsymbol{x}},$$

and that π starts from equilibrium if and only if

$$\boldsymbol{v}(0)=\boldsymbol{0},\qquad \theta(0)=\theta_e,\qquad \boldsymbol{\alpha}(0)=\hat{\boldsymbol{\alpha}}(\nabla\boldsymbol{x}(0),\theta_e).$$

Lemma 11.1. *There exists a function* $\kappa\colon \mathrm{Lin}^+\times\mathbb{R}^{++}\times\mathscr{D}\times\mathscr{B}\to\mathbb{R}$ *such that*

$$\tilde{\varepsilon}(\boldsymbol{F},\theta,\boldsymbol{\alpha})-\theta_e\,\tilde{\eta}(\boldsymbol{F},\theta,\boldsymbol{\alpha})=\tilde{\psi}(\boldsymbol{F},\theta,\boldsymbol{\alpha})+\kappa(\boldsymbol{F},\theta,\boldsymbol{\alpha})(\theta-\theta_e)^2,\qquad \kappa(\boldsymbol{F},\theta,\boldsymbol{\alpha})>0 \tag{11.10}$$

for all $\boldsymbol{F}\in\mathrm{Lin}^+$, $\theta\in\mathbb{R}^{++}$, *and* $\boldsymbol{\alpha}\in\mathscr{D}$.

The proof of this lemma is contained in the Appendix.

By (4.2) and $(11.10)_1$,

$$V(\sigma)=U_e(\boldsymbol{x})+\int_{\mathscr{B}}[\tilde{\psi}(\boldsymbol{F},\theta_e,\boldsymbol{\alpha})+\kappa(\boldsymbol{F},\theta,\boldsymbol{\alpha})(\theta-\theta_e)^2+\tfrac{1}{2}\boldsymbol{v}^2]\,dm,$$

and therefore (6.1), (6.4), (11.4), (11.5), and $(11.6)_2$ yield

$$U(\boldsymbol{x})=U_e(\boldsymbol{x})+\int_{\mathscr{B}}\hat{\psi}(\boldsymbol{F},\theta_e)\,dm,$$

$$T(\sigma)=\int_{\mathscr{B}}[\tilde{\psi}(\boldsymbol{F},\theta_e,\boldsymbol{\alpha})-\hat{\psi}(\boldsymbol{F},\theta_e)+\kappa(\boldsymbol{F},\theta,\boldsymbol{\alpha})(\theta-\theta_e)^2+\tfrac{1}{2}\boldsymbol{v}^2]\,dm,$$

where, as before, we have considered U and U_e as functions with AdmissConfig, rather than Σ_0, as domain. Again we note that *the potential energy U coincides with the potential energy of an elastic body provided we interpret the stored energy as the equilibrium free-energy $\hat{\psi}$.*

Proposition 11.1. $(\Sigma,\mathscr{P},\mathrm{Equil})$ *is a dynamical system with V an associated Lyapunov function.*

We omit the proof, which is almost identical to that of Proposition 10.1.

In view of this proposition, all of the results established in Chapter IV for general dynamical systems may be applied to bodies described by internal state variables. These results are completely analogous to those stated in Section 9 for elastic bodies and in Section 10 for bodies with fading memory and therefore will not be stated explicitly here.

Appendix

We here prove Ericksen's Lemma (5.7) as well as its two generalizations (10.12) and (11.2). To prove (5.7) we expand the free-energy (5.4) using Taylor's formula with remainder:

$$\hat{\psi}(\boldsymbol{F},\theta_e)=\hat{\psi}(\boldsymbol{F},\theta)+\partial_2\hat{\psi}(\boldsymbol{F},\theta)(\theta_e-\theta)+\tfrac{1}{2}\,\partial_2\partial_2\hat{\psi}(\boldsymbol{F},\theta_*)(\theta_e-\theta)^2,$$

where $\theta_* = \theta_*(\boldsymbol{F}, \theta)$ lies between θ_e and θ. Therefore, from the definition

$$\kappa(\boldsymbol{F}, \theta) = -\tfrac{1}{2}\,\partial_2\partial_2\hat{\psi}(\boldsymbol{F}, \theta_*),$$

(5.4), (5.5), and (5.6) yield (5.7).

In the above proof we used only the properties of the map $\theta \mapsto \hat{\psi}(\boldsymbol{F}, \theta)$. For a material with memory we apply Taylor's formula to the map $\theta \mapsto \tilde{\psi}(\boldsymbol{F}, \theta, \boldsymbol{F}^*, \theta^*)$ in exactly the same manner and use (10.4), (10.6).

Finally, for a material described by internal state variables we work with the map $\theta \mapsto \tilde{\psi}(\boldsymbol{F}, \theta, \boldsymbol{\alpha})$ and use $(11.6)_1$, (11.7).

Acknowledgment. I would like to thank C. DAFERMOS for stimulating discussions concerning the ω-limit set. This research was supported by the National Science Foundation.

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(Received June 20, 1975)