Existence of Homoenergetic Affine Flows for the Boltzmann Equation

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Abstract

An existence theorem is proved for homoenergetic affine flows described by the Boltzmann equation. The result complements the analysis of TRUESDELL and of GALKIN on the moment equations for a gas of Maxwellian molecules. Existence of the distribution function is established here for a large class of molecular models (hard sphere and angular cut-off interactions). Some of the data lead to an implosion and infinite density in a finite time, in agreement with the physical picture of the associated flows; for the remaining set of data, global existence is shown to hold.

1. Introduction

At about the same time (1956), C. TRUESDELL [1] and V. S. GALKIN [2] independently investigated the steady homoenergetic flows of a gas of Maxwellian molecules according to the infinite system of moments associated with the Boltzmann equation. Later GALKIN [3–5] extended his analysis to some typical unsteady homogenergetic affine flows. The book by TRUESDELL & MUNCASTER [6] gives a unified discussion of all these works, and provides further calculations linking these flows to more current research in the kinetic theory.

While these analyses have the great advantage of leading to explicit solutions, which lend themselves to a detailed discussion of their properties, they suffer from two drawbacks:

1) They are restricted to Maxwellian molecules.
2) They provide solutions of the system of equations for moments, but no proof is given that a corresponding solution of the Boltzmann equation itself exists.

The aim of this paper is to complement the above-mentioned analyses by proving an existence theorem for the Boltzmann equation for a large class of molecular models (including Maxwellian molecules with cutoff), when the initial data are compatible with a homoenergetic affine flow. Some of the data lead to an implosion and infinite density in a finite time, in agreement with the physical picture of the associated flows; for the remaining set of data, global existence will be shown to hold. In connection with these solutions, it should be noted that NIKOL'SKII [7, 8] proved a general transformation which, for gases of inverse k^{th}-power molecules ($k \neq 7/3$) or hard spheres, generates a solution for dilatations directly from any given spatially homogeneous solution. If applied to KROOK & WU's solution [9, 10] this transformation produces a solution independently found by MUNCASTER [11]. The spatial dependence of the NIKOL'SKII solution [7] is a particular case of the one to be discussed here.

2. Homoenergetic affine flows

In this section we recall the basic ideas about homoenergetic affine flows. The defining properties are the following:
a) The body force (per unit mass) $\boldsymbol{X}$ acting on the molecules is constant:

$$\boldsymbol{X} = \text{const.} \tag{2.1}$$

b) The density ϱ, the internal energy per unit mass e, the stress tensor $\mathfrak{p}$ and the heat flux $\boldsymbol{q}$ may be functions of time but not of the space coordinates.
c) The bulk velocity $\boldsymbol{v}$ is an affine function of position $\boldsymbol{x}$:

$$\boldsymbol{v} = \mathfrak{K}(t)\,\boldsymbol{x} + \boldsymbol{v}_0(t). \tag{2.2}$$

This definition holds for a general material; for a gas described by the kinetic theory, a natural extension of property b) is immediate:
b′) The moments formed with the peculiar velocity

$$\boldsymbol{c} = \boldsymbol{\xi} - \boldsymbol{v} \tag{2.3}$$

may be functions of time but do not depend upon space coordinates. Here $\boldsymbol{\xi}$ is the molecular velocity with respect to an inertial frame.

The condition b′) holds for the solutions obtained by TRUESDELL [1] and GALKIN [2–5]. For analyses relating directly to the distribution function f, this condition is transformed into
b″) The variable $\boldsymbol{x}$ appears in f only through $\boldsymbol{v}$, given by Eq. (2.2), *i.e.*:

$$f = f(\boldsymbol{c}, t). \tag{2.4}$$

An analysis of the balance equations based on a), b) and c) immediately leads to the following restrictions on $\mathfrak{K}$ and $\boldsymbol{v}_0$:

$$\begin{aligned} \dot{\mathfrak{K}} + \mathfrak{K}^2 &= 0, \\ \dot{\boldsymbol{v}}_0 + \mathfrak{K}\boldsymbol{v}_0 &= \boldsymbol{X}. \end{aligned} \tag{2.5}$$

The general solution of this system is:

$$\begin{aligned} \mathfrak{K}(t) &= [\mathfrak{I} + t\mathfrak{K}(0)]^{-1}\,\mathfrak{K}(0), \\ v_0(t) &= [\mathfrak{I} + t\mathfrak{K}(0)]^{-1}\,[v_0(0) + tX + \tfrac{1}{2}t^2\mathfrak{K}(0)\,X], \end{aligned} \tag{2.6}$$

where $\mathfrak{I}$ is the 3×3 identity matrix. This solution exists globally for $t > 0$ if the eigenvalues of $\mathfrak{K}(0)$ are nonnegative; otherwise the solution ceases to exist for $t = t_0$, where $-t_0^{-1}$ is the largest, in absolute value, of the negative eigenvalues of $\mathfrak{K}(0)$.

In particular, if

$$[\mathfrak{K}(0)]^2 = 0, \tag{2.7}$$

then $[\mathfrak{I} + t\mathfrak{K}(0)]^{-1} = \mathfrak{I} - t\mathfrak{K}(0)$ and therefore $\mathfrak{K}(t)$ is independent of time, v is then steady if and only if

$$\mathfrak{K}(0)\,X = 0, \tag{2.8}$$

and if $v_0(0)$ is chosen in such a way that

$$\mathfrak{K}(0)\,v_0(0) = X. \tag{2.9}$$

In particular, this is always possible if $X = 0$.

Eq. (2.7) is satisfied if and only if a coordinate system exists for which the matrix representation of $\mathfrak{K}(0)$ is given by

$$((K_{ij})) = \begin{pmatrix} 0 & 0 & 0 \\ K & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{2.10}$$

For a simple proof of this, see the Appendix.

Eqs. (2.5) are certainly necessary for the existence of a solution satisfying conditions a), b″), c). They are derived [6] under the assumptions a), b), c), and, of course, b″) implies b).

In order to show that Eqs. (2.5) are also sufficient, we consider the Boltzmann equation

$$\frac{\partial f}{\partial t} + \xi\cdot\frac{\partial f}{\partial x} + X\cdot\frac{\partial f}{\partial \xi} = Q(f,f), \tag{2.11}$$

where $Q(f,f)$ is the collision operator [12, 13]. We choose c in place of ξ as an independent variable and use the same letter f for $f(x, \xi, t)$ and $f(x, c, t)$, although, of course, they are different functions of their arguments. Then we have to make the following replacements in Eq. (2.11):

$$\begin{aligned} \frac{\partial f}{\partial t} &\Rightarrow \frac{\partial f}{\partial t} - \frac{\partial f}{\partial c}\cdot\dot{\mathfrak{K}}x - \frac{\partial f}{\partial c}\dot{v}_0, \\ \frac{\partial f}{\partial x} &\Rightarrow -\frac{\partial f}{\partial c}\cdot\mathfrak{K}, \\ \frac{\partial f}{\partial \xi} &\Rightarrow \frac{\partial f}{\partial c}. \end{aligned} \tag{2.12}$$

Eq. (2.11) then becomes:

$$\frac{\partial f}{\partial t} - \frac{\partial f}{\partial \boldsymbol{c}} \cdot (\dot{\mathfrak{K}} + \mathfrak{K}^2)\, \boldsymbol{x} - \frac{\partial f}{\partial \boldsymbol{c}} \cdot (\dot{\boldsymbol{v}}_0 + \mathfrak{K}\boldsymbol{v}_0 - \boldsymbol{X}) - \frac{\partial f}{\partial \boldsymbol{c}} \cdot \mathfrak{K}\boldsymbol{c} = Q(f,f), \tag{2.13}$$

where $Q(f,f)$ is now expressed in terms of $\boldsymbol{c}$ rather than of $\boldsymbol{\xi}$.

In order to obtain a solution independent of $\boldsymbol{x}$, as required by Eq. (2.4), the first equation of system (2.5) must be satisfied. If the second is also satisfied, then Eq. (2.13) becomes:

$$\frac{\partial f}{\partial t} - \frac{\partial f}{\partial \boldsymbol{c}} \cdot \mathfrak{K}\boldsymbol{c} = Q(f,f), \tag{2.14}$$

and the space variable no longer appears explicitly. Note that the second equation of system (2.5) is also necessary for a solution satisfying Eq. (2.4) to exist, because if we multiply Eq. (2.13) by $\boldsymbol{c}$ and integrate, we obtain exactly Eq. (2.5), provided one recalls that, by definition,

$$\int \boldsymbol{c} f \, d\boldsymbol{c} = \boldsymbol{0}. \tag{2.15}$$

Thus the existence of homoenergetic affine flows is reduced to proving an existence theorem for Eq. (2.14). We note that the latter can be cast into integral form provided we determine the semigroup corresponding to collisionless flow.

To this end we consider the ordinary differential equation:

$$\frac{d\boldsymbol{c}}{dt} = -\mathfrak{K}\boldsymbol{c}, \qquad \boldsymbol{c}(0) = \boldsymbol{c}_0, \tag{2.16}$$

which can easily be solved if the expression of $\mathfrak{K}$ appearing in Eq. (2.6) is used. We obtain

$$\boldsymbol{c} = [\mathfrak{I} + t\mathfrak{K}(0)]^{-1}\, \boldsymbol{c}_0. \tag{2.17}$$

Hence, if we let

$$f^{\#}(\boldsymbol{c}, t) = f([\mathfrak{I} + t\mathfrak{K}(0)]^{-1}\, \boldsymbol{c}, t), \tag{2.18}$$

we obtain

$$\frac{\partial f^{\#}}{\partial t} = \left(\frac{\partial f}{\partial t} - \frac{\partial f}{\partial \boldsymbol{c}} \cdot \mathfrak{K}\boldsymbol{c}\right)^{\#} = [Q(f,f)]^{\#}. \tag{2.19}$$

Integration with respect to t yields the integral form

$$f^{\#}(\boldsymbol{c}, t) = f^{\#}(\boldsymbol{c}, 0) + \int_0^t [Q(f,f)]^{\#}\, (\boldsymbol{c}, s)\, ds. \tag{2.20}$$

Other integral forms are possible when $Q(f,f)$ can be split into two separate contributions (gain and loss terms), as is the case for hard sphere molecules and cut-off interactions.

3. A priori estimates of mass and energy densities

In order to obtain an existence theorem, we need *a priori* estimates on the mass and energy densities, *i.e.* on the moments

$$\varrho = \int f\,dc, \quad 2E = \int c^2 f\,dc = \operatorname{Tr}\mathfrak{p} \tag{3.1}$$

where Tr denotes the trace of a tensor. Eq. (2.14) yields the following two equations:

$$\frac{d\varrho}{dt} + \varrho \operatorname{Tr}\mathfrak{K} = 0, \tag{3.2}$$

$$\frac{dE}{dt} + \operatorname{Tr}(\mathfrak{K}\mathfrak{p}) + E\operatorname{Tr}\mathfrak{K} = 0. \tag{3.3}$$

Eq. (3.2) immediately yields:

$$\varrho(t) = \varrho(0)\exp\left[-\int_0^t \operatorname{Tr}\mathfrak{K}(s)\,ds\right], \tag{3.4}$$

and since $\mathfrak{K}(s)$ is explicitly known, the density is determined. Henceforth, in order to avoid trivialities, $\varrho(0)$ will be assumed to be different from zero.

Next let k denote the largest, in absolute value, of the elements of the matrix $\mathfrak{K}$. Then, if we use the inequality

$$|p_{ij}| \leqq \tfrac{1}{2}(p_{ii} + p_{jj}) \tag{3.5}$$

that applies to the stress tensor, Eq. (3.3) gives

$$\frac{dE}{dt} \leqq 6kE - E\operatorname{Tr}\mathfrak{K}, \tag{3.6}$$

and

$$E(t) \leqq E(0)\exp\left\{\int_0^t [6k(s) - \operatorname{Tr}\mathfrak{K}(s)]\,ds\right\}. \tag{3.7}$$

We remark that $|\operatorname{Tr}\mathfrak{K}|$ and k are bounded for $t \leqq T < t_0$, where $-t_0^{-1}$ is the largest among the negative eigenvalues of $\mathfrak{K}(0)$. If there are no negative eigenvalues, both k and $|\operatorname{Tr}\mathfrak{K}|$ are bounded for all positive t.

It is also important to note that the solution we find must satisfy Eq. (2.15). It is easy to see, however, that, if satisfied at $t = 0$, Eq. (2.15) holds at any time provided f satisfies Eq. (2.14). In fact, if

$$\boldsymbol{j} = \int cf\,dc, \tag{3.8}$$

Eq. (2.14) yields

$$\frac{d\boldsymbol{j}}{dt} + \boldsymbol{j}\operatorname{Tr}\mathfrak{K} + \mathfrak{K}\boldsymbol{j} = \boldsymbol{0}, \tag{3.9}$$

and this linear equation has $\boldsymbol{j} = \boldsymbol{0}$ as the unique solution taking $\boldsymbol{0}$ as an initial value.

4. The existence theorem

We prove the existence of a solution for cut-off collision operators in two steps. In the first we consider a collision term

$$Q(f,f) = \int\int\int B(\theta, |c - c_*|)\,(f'f'_* - ff_*)\,dc_*\,d\theta\,d\varepsilon, \tag{4.1}$$

where the kernel $B(\theta, |c - c_*|)$ is bounded:

$$B(\theta, |c - c_*|) \leqq a/\pi^2 \quad (a = \text{const.}). \tag{4.2}$$

In this case we establish existence and give a proof of the H-theorem. Unfortunately general molecules with a cut-off do not satisfy Eq. (4.2), but we show in step two how the general case can be handled as a limit of a series of problems in which Eq. (4.2) is valid.

We begin by noting that

$$J(f,f) = Q(f,f) + af\int f_*\,dc_* \tag{4.3}$$

is a positive functional of the distribution function f and increases when f increases. If $\phi(c)$, the initial value of f, is in L^1, let us consider the following iteration scheme, where $\varrho(t)$ is given by Eq. (3.4):

$$\begin{gathered}\frac{\partial}{\partial t} f^\#_{n+1} + a\varrho(t) f^\#_{n+1} = [J(f_n, f_n)]^\#, \quad f_{n+1}(c, 0) = \phi(c), \quad n \geqq 0 \\ f_0(c, t) = 0.\end{gathered} \tag{4.4}$$

Here t runs from 0 to T. If $\varrho(t)$ exists for any positive t, then T is arbitrary. If $\varrho(t)$ is finite only for $t < t_0$, then T is any positive number less than t_0. We find that $\{f_n\}$ is a monotone increasing sequence whose norm in L^1 is bounded by $\varrho(t)$; hence it tends to an L^1 function $f(c, t)$. We can now take the limits in Eqs. (4.4) $n \to \infty$. To this end we consider the explicit formula giving f_{n+1} in terms of f_n and ϕ, which has the advantage of containing a smooth monotone operator. Hence Eq. (4.4a) with f in place of f_n and f_{n+1} holds. As a consequence, f will have a density $\hat{\varrho}(t)$ such that

$$\begin{gathered}\frac{d\hat{\varrho}}{dt} + a\hat{\varrho}\,\mathrm{Tr}\,\mathfrak{K} = a\hat{\varrho}[\hat{\varrho} - \varrho(t)], \\ \hat{\varrho}(0) = \varrho(0).\end{gathered} \tag{4.5}$$

The unique solution of this initial value problem is

$$\hat{\varrho}(t) = \varrho(t). \tag{4.6}$$

Hence f will satisfy

$$\begin{gathered}\frac{\partial f^\#}{\partial t} = [Q(f,f)]^\#, \\ f(c, 0) = \phi(c).\end{gathered} \tag{4.7}$$

Hence an L^1 solution exists. This solution is the unique solution of the initial value problem (4.7); it is also unique among the solutions of the Boltzmann equation taking the same initial values and having bounded density in $0 \leqq t \leqq T$.

If we assume that the initial data have a finite second moment $E(0)$, then the inequality for $E(t)$ discussed in the previous section gives:

$$E(t) \leqq A_T \equiv E(0) \exp\left\{\int_0^T [6k(s) - \operatorname{Tr} \mathfrak{K}(s)]\, ds\right\}, \quad 0 \leqq t \leqq T. \tag{4.8}$$

We remark that the constant A_T depends only on the initial values (through $\mathfrak{K}(t)$ and $E(0)$) and T.

We can now prove a rigorous H-theorem for Eq. (4.7), under the assumption that the initial data have a finite H-functional, $H(0)$, where

$$H(t) = \int f \log f\, dc. \tag{4.9}$$

The form of the H-theorem for Eq. (4.7) follows formally from a simple calculation. By the identity

$$\operatorname{Det}[\mathfrak{I} + t\mathfrak{K}(0)] = e^{\int_0^t \operatorname{Tr}\mathfrak{K}(s)ds} \equiv D(t) \tag{4.10}$$

we have

$$\begin{aligned}\frac{d}{dt}[D(t)\, H(t)] &= \int (Q(f,f))^{\#} \log f^{\#}\, dc \\ &= D(t) \int Q(f,f) \log f\, dc \leqq 0,\end{aligned} \tag{4.11}$$

and hence

$$H(t) \leqq H(0)/D(t), \tag{4.12}$$

In order to obtain this inequality rigorously, we use the method of ARKERYD [14, 6]. One can write the Boltzmann equation in the form

$$\frac{\partial f^{\#}}{\partial t} + a\varrho(t) f^{\#} = [J(f,f)]^{\#}, \tag{4.13}$$

where for convenience we take $a\varrho(t) \geqq 1$ (this can be done by fixing the time interval and adjusting a in Eqs. (4.3) and (4.13)). We then define truncated initial data and collision terms by setting

$$J_p(f,f) = \min\,(J(f,f), p), \tag{4.14}$$

$$\phi_{n,p}(c) = \min\left(\phi(c) + \frac{1}{n} e^{-c^2}, p\right). \tag{4.15}$$

We also replace ϱ by ϱ_m where

$$\varrho_m(t) = \varrho(t) + \frac{\pi^{3/2}}{m} \exp\left[-\int_0^t \operatorname{Tr}\mathfrak{K}(s)\, ds\right]. \tag{4.16}$$

We can then approximate the solution to (4.13) by solutions of

$$\frac{\partial}{\partial t} f^{\#}_{n,m,p} + a\varrho_m(t) f^{\#}_{n,m,p} = [J_p(f_{n,m,p}, f_{n,m,p})]^{\#}, \quad f_{n,m,p}(0) = \phi_{n,p}. \tag{4.17}$$

The solution of the Boltzmann equation (4.13) with the initial value $\phi_{n,\infty}$ is precisely $f_{n,n,\infty}$. In fact $\phi_{n,n,\infty}$ is $\phi(c) + e^{-c^2}/n$ and $\varrho_{n,n,\infty}(0) = \varrho(0) + \pi^{3/2}/n$. The evolution equation for the density, Eq. (3.2) immediately gives $\varrho_{n,n,\infty}(t) = \varrho_n(t)$.

It is easy to see [14] that

$$f^{\#}_{n,m,p} \geqq f^{\#}_{n',m',p'} \quad p \geqq p', m \geqq m', n' \geqq n. \tag{4.18}$$

Moreover, since $a\varrho \geqq 1$, there are constants ε_T, C_T, and K_T such that ε_T and C_T are independent of $p \geqq 1$ and for finite m, n, and p

$$\varepsilon_T \exp(-C_T c^2) \leqq f^{\#}_{n,m,p}(c, t) \leqq K_T \leqq p. \tag{4.19}$$

Then for finite n and p we easily have $f^{\#}_{n,m,p} \log f^{\#}_{n,m,p} \in L^1$. Finally we proceed exactly as in [14, Theorem 2.1] to prove that

$$H^{\#} = \int f^{\#} \log f^{\#} \, dc = H(t)\, D(t) \tag{4.20}$$

is a nonincreasing function of t for $f = f_{n,m,p}$. In the last step of Eq. (4.20), as well as in Eq. (4.11), we have used the fact that if we let

$$c^{\#} = [\mathfrak{J} + t\mathfrak{K}(0)]^{-1} c, \tag{4.21}$$

then the Jacobian in this change of variables from c to $c^{\#}$ is $D(t)$. It is now easy though tedious to pass to the limit when m, n, p go to ∞, following again ARKERYD [14]. Eq. (4.11) is thus justified for any initial data ϕ having a finite H-functional and belonging to L^1.

As a consequence

$$H(t) \leqq H_T, \tag{4.22}$$

where again the constant H_T depends on the initial data and T. It is to be noted that Eq. (4.11) can be used to show that $H(t)$ decreases in time if $D(t)$ is nondecreasing.

Generally molecules with a cut-off do not satisfy the boundedness assumption stated in Eq. (4.2). In order to overcome this we consider the more general condition

$$B(\theta, |c - c_*|) \leqq \frac{b}{\pi^2}(1 + c^2 + c_*^2), \quad b = \text{const.} \tag{4.23}$$

To deal with this case we first replace B by a cutoff expression

$$B_m(\theta, |c - c_*|) = \min(B(\theta, |c - c_*|), m), \tag{4.24}$$

where m is a positive constant. Then

$$B_m(\theta, |c - c_*|) \leqq \frac{a_m}{\pi^2}, \quad a_m = \text{const.}, \tag{4.25}$$

and we can apply the previous result to conclude that there is a solution f_m of

$$\frac{\partial f_m^\#}{\partial t} = [Q(f_m, f_m)]^\#,$$
$$f_m(c, 0) = \phi(c). \tag{4.26}$$

In addition, if the initial value possesses a finite moment $E(0)$ and a finite H-functional, $H(0)$, then E_m and H_m will exist at any time $t \leqq T$ and will satisfy the inequalities

$$E_m(t) \leqq E_T, \tag{4.27}$$

$$H_m(t) \leqq H_T, \tag{4.28}$$

where the constants E_T and H_T do not depend on m. Finally, using the weak compactness criterion employed by ARKERYD [14] in his first proof and the equicontinuity in time of the sequence f_m, we arrive at the following

Existence Theorem. *There exists a solution f of Eq.* (4.7), *where the kernel $B(\theta, |c - c_*|)$ of the collision term $Q(f,f)$ satisfies Eq.* (4.23), *and the initial mass density, energy density, and H-functional are finite at time* 0. *These functionals remain bounded when $0 \leqq t \leqq T$. The time T has the meaning explained previously. It is arbitrary provided $\varrho(t)$ exists for an arbitrary time interval and the constants E_T and H_T are finite for any positive T; both conditions are satisfied if $\mathfrak{K}(0)$ has no negative eigenvalues. If $\mathfrak{K}(0)$ possesses negative eigenvalues and t_0^{-1} is their largest absolute value, then T must not be larger than t_0.*

If we add the assumption that the fourth moments of ϕ exist, then we can prove a uniqueness theorem. To this end we have to prove that the fourth moments remain finite:

$$Q(t) \equiv \int (1 + c^2)^2 f(c, t)\, dc \leqq C_T, \quad 0 \leqq t \leqq T. \tag{4.29}$$

This is easily done following ARKERYD [15] and using a special case of POVZNER'S inequality [16]

$$(1 + c'^2)^2 + (1 + c_*'^2)^2 - (1 + c^2)^2 - (1 + c_*^2)^2 \leqq 2(1 + c^2)(1 + c_*^2). \tag{4.30}$$

It can be shown that

$$\int (1 + c^2)^2 Q(f,f)\, dc \leqq 4b(E_T + R_T)\, Q(t), \tag{4.31}$$

where R_T is the maximum value of the density in $[0, T]$. Hence Eq. (4.7) gives

$$\frac{dQ}{dt} + Q \operatorname{Tr} \mathfrak{K} + 4 \int c\mathfrak{K}c(1 + c^2) f\, dc \leqq 4b(E_T + R_T)\, Q, \tag{4.32}$$

and Eq. (4.29) follows with

$$C_T = Q(0) \exp\{[4b(E_T + R_T) + 36k_T + M_T]\, T\}, \tag{4.33}$$

where M_T and K_T are the maximum values of $|\operatorname{Tr} \mathfrak{K}|$ and k in $[0, T]$.

Uniqueness Theorem. *Let f be the solution delivered by the Existence Theorem. If $Q(0)$ exists, so does $Q(t)$ in* $[0, T]$, *and then the solution of Eq.* (4.7) *is unique.*

Appendix

Here we give a simple proof that Eq. (2.7) is satisfied if and only if a coordinate system exists for which the components of the tensor $\mathfrak{K}(0)$ are given by Eq. (2.10). The "if" part is trivially proved by direct computation. We must show, then, that Eq. (2.7) implies the existence of a coordinate system for which Eq. (2.10) holds.

To this end we assume that $((K_{ij}))$ is not the zero matrix, since otherwise Eq. (2.10) is trivial. Then there is a vector $\boldsymbol{x}$ such that $\mathfrak{K}\boldsymbol{x} \neq \boldsymbol{0}$. Let us choose among the vectors having this property and also having unit (Euclidean) norm a particular vector $\boldsymbol{x}_1$ for which the norm of $\mathfrak{K}\boldsymbol{x}_1$ takes its maximum value, K. Let $\boldsymbol{x}_2$ denote $\mathfrak{K}\boldsymbol{x}_1/K$; then $\mathfrak{K}\boldsymbol{x}_2 = \boldsymbol{0}$ and $\boldsymbol{x}_2$ is a unit vector. We can prove that $\boldsymbol{x}_2$ is orthogonal to $\boldsymbol{x}_1$; otherwise, in fact, we could replace $\boldsymbol{x}_1$ by $(\boldsymbol{x}_1 - c\boldsymbol{x}_2)/(1 - c^2)^{1/2}$ where $c = \boldsymbol{x}_1 \cdot \boldsymbol{x}_2$ $(c < 1)$ and $\mathfrak{K}\boldsymbol{x}_1$ would equal $K/(1 - c^2)^{1/2} > K$. Let now $\boldsymbol{x}_3$ denote a third unit vector given by $\boldsymbol{x}_3 \wedge \boldsymbol{x}_2$, and hence orthogonal to both $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$. If we take now a coordinate system whose unit vectors are $\boldsymbol{x}_1$, $\boldsymbol{x}_2$ and $\boldsymbol{x}_3$, the elements of $((K_{ij}))$ are given by $\boldsymbol{x}_i \cdot \mathfrak{K}\boldsymbol{x}_j$ $(i, j = 1, 2, 3)$. We have

$$\begin{gathered} K_{i2} = \boldsymbol{x}_i \cdot \mathfrak{K}\boldsymbol{x}_2 = 0, \qquad K_{21} = \boldsymbol{x}_2 \cdot \mathfrak{K}\boldsymbol{x}_1 = K\boldsymbol{x}_2 \cdot \boldsymbol{x}_2 = K, \\ K_{11} = \boldsymbol{x}_1 \cdot \mathfrak{K}\boldsymbol{x}_1 = K\boldsymbol{x}_1 \cdot \boldsymbol{x}_2 = 0, \qquad K_{31} = \boldsymbol{x}_3 \cdot \mathfrak{K}\boldsymbol{x}_1 = K\boldsymbol{x}_3 \cdot \boldsymbol{x}_2 = 0. \end{gathered} \tag{A.1}$$

Five matrix elements are thus zero. Further let

$$\mathfrak{K}\boldsymbol{x}_3 = a\boldsymbol{x}_1 + b\boldsymbol{x}_2 + c\boldsymbol{x}_3; \tag{A.2}$$

then, applying $\mathfrak{K}$ on both sides, we obtain

$$\boldsymbol{0} = Ka\boldsymbol{x}_2 + c\mathfrak{K}\boldsymbol{x}_3. \tag{A.3}$$

This shows that either $\mathfrak{K}\boldsymbol{x}_3 = H\boldsymbol{x}_2$ for some constant H or $a = c = 0$; the latter case reduces to the former, however, due to Eq. (A.2). Hence

$$\begin{gathered} K_{33} = \boldsymbol{x}_3 \cdot \mathfrak{K}\boldsymbol{x}_3 = H\boldsymbol{x}_3 \cdot \boldsymbol{x}_2 = 0, \qquad K_{13} = \boldsymbol{x}_1 \cdot \mathfrak{K}\boldsymbol{x}_3 = H\boldsymbol{x}_1 \cdot \boldsymbol{x}_2 = 0, \\ K_{23} = \boldsymbol{x}_2 \cdot \mathfrak{K}\boldsymbol{x}_3 = H\boldsymbol{x}_2 \cdot \boldsymbol{x}_2 = H. \end{gathered} \tag{A.4}$$

H, however, must be zero because of the definition of K. In fact, otherwise, by letting

$$\bar{\boldsymbol{x}} = A\boldsymbol{x}_1 + B\boldsymbol{x}_3, \qquad A = K/N, \qquad B = H/N, \qquad N = (H^2 + K^2)^{1/2}, \tag{A.5}$$

we would find that the square of the norm of $\bar{\boldsymbol{x}}$ is given by $A^2 + B^2 = 1$ while that of $\mathfrak{K}\bar{\boldsymbol{x}}$ is $(AK + BH)^2 = K^2 + H^2 > K^2$. However this contradicts the fact that K was defined as the maximum value of $\mathfrak{K}\boldsymbol{x}$ over all vectors of unit norm.

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