Solutions of Pseudo-Heat Equations in the Whole Space

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1. Introduction

Various physical problems [3, 4, 18, 32] are governed by equations of the form,

$$L[u]\equiv(A-I)u_t+Bu=0 \tag{1.1}$$

for an unknown function $u(x_1, \dots, x_m, t)\equiv u(x, t)$, where

$$Au\equiv\sum_{i,j=1}^{m} A_{ij}D_{ij}u, \quad D_{ij}u\equiv\partial^2 u/\partial x_i\partial x_j, \qquad u_t=\partial u/\partial t, \tag{1.1'}$$

$$Bu\equiv\sum_{i,j=1}^{m} b_{ij}D_{ij}u+\sum_{i=1}^{m} b_i D_i u-bu, \qquad D_i u\equiv\partial u/\partial x_1. \tag{1.1''}$$

The real constant matrices (a_{ij}) and (b_{ij}) will be assumed to be symmetric and positive definite, and the constant b will be assumed to be positive. Let $\Lambda(A)$, $\lambda(A)$, $\Lambda(B)$ and $\lambda(B)$ be, respectively, the maximum and the minimum eigenvalues of the matrices (a_{ij}) and (b_{ij}). Then for every real vector $\xi=(\xi_1, \dots, \xi_m)$, we have

$$\lambda(A)|\xi|^2\leqq\sum_{i,j=1}^{m} A_{ij}\xi_i\xi_j\leqq\Lambda(A)|\xi|^2,$$

$$\lambda(B)|\xi|^2\leqq\sum_{i,j=1}^{m} b_{ij}\xi_i\xi_j\leqq\Lambda(B)|\xi|^2.$$

A brief survey of equations similar to (1.1) was made in [30], where mixed problems for pseudo-parabolic equations had been studied. We refer to the references in [30] for earlier results on equations of this type. Much progress on mixed problems in a cylindrical domain has recently been made [19, 27–31]. The objective of this work is to study the solutions of (1.1) in the whole space $R^m\times R$ under the initial condition

$$y(x, 0)=u_0(x) \tag{1.2}$$

in an appropriate function space.

Because of the lack of L_p and Schauder estimates for solutions of elliptic equations in the whole space, we have restricted our attention to equations with real constant coefficients. On the other hand, the constancy of the coefficients insures that the problem considered here is truly in the whole space R^m.

By means of Fourier transforms, the solution can be informally expressed in closed form; see equation (2.4). This formula is elegant and it furnishes a method for numerical computations for problems of this type. The question is how to insure that this formula will be valid under various initial conditions. In fact, it is not even obvious that it defines a real-valued function for real $u_0(x)$ and accordingly it is necessary to demonstrate this fact. When the Cauchy data u_0 is rapidly decreasing, the justification can be carried out in an elementary manner; see §2. When $u_0(x)$ belongs to a Sobolev space $W^{k,p}(R^m)$, we have appealed to the Fourier multiplier theorem for integrals; see §3. For pointwise solutions, we use the principal fundamental solution; see §4. Although this involves some hard analysis, nevertheless it gives the finest results; moreover, it is through the principal fundamental solution that formula (2.4) is justified under the broadest possible initial conditions; see §5. Under various initial conditions, we have established the existence, uniqueness and asymptotic behavior of the solution, with particular attention to the regularity question. In short, the solution is just as regular as its initial data, but no more.

2. Rapidly Decreasing Solutions

Throughout this section we shall assume that the Cauchy data $u_0(x)$ is a real-valued function in the space S of rapidly decreasing functions [36]. We wish to construct a solution $u(x, t)$ of the problem (1.1) and (1.2), which stays in the space S for all time t in R, R being the whole real line. To this end, we assume for the moment that such a solution exists and that its time derivative also belongs to S for all t in R. By taking the Fourier transform of (1.1) and (1.2), we find that the Fourier transform $\hat{u}(\xi, t)$ of the solution $u(x, t)$ satisfies the equations

$$(2.1)\qquad (1+A(\xi))\,\hat{u}_t(\xi, t) = -(B(\xi)+b(\xi)+b)\,\hat{u}(\xi, t) \quad \text{in } R^m \times R,$$

$$(2.2)\qquad \hat{u}(\xi, 0) = \hat{u}_0(\xi) \quad \text{in } R^m$$

where

$$A(\xi) \equiv (2\pi)^2 \sum_{i,j=1}^{m} a_{ij}\,\xi_i\,\xi_j, \qquad B(\xi) \equiv (2\pi)^2 \sum_{i,j=1}^{m} b_{ij}\,\xi_i\,\xi_j,$$

$$b(\xi) \equiv -2\pi i \sum_{i=1}^{m} b_j\,\xi_j.$$

Consequently, for all values of ξ and t, $\hat{u}(\xi, t)$ is explicitly given by the expression

$$(2.3)\qquad \hat{u}(\xi, t) = \exp\left\{ -\frac{B(\xi)+b(\xi)+b}{A(\xi)+1}\, t \right\} \hat{u}_0(\xi) \equiv F(\xi, t)\,\hat{u}_0(\xi).$$

Now define the function $u(x, t)$ in $R^m \times R$ by the formula

$$(2.4)\qquad u(x, t) \equiv \int_{R^m} F(\xi, t)\,\hat{u}_0(\xi)\, e^{2\pi i \langle \xi, x \rangle}\, d\xi$$

where $\langle \xi, x\rangle$ denotes the inner product of the vectors ξ and x. We proceed to show that the function $u(x, t)$ defined in (2.4), together with its time derivatives of all orders, is in S for all t in R and that it is the solution of the Cauchy problem (1.1) and (1.2). To show this, we first consider the functions

$$(2.5)\qquad F(\xi, t)\equiv\exp\left\{-\frac{B(\xi)+b(\xi)+b}{A(\xi)+1}t\right\},\quad f(\xi)\equiv-\frac{B(\xi)+b(\xi)+b}{A(\xi)+1}$$

Let α be an m-vector with non-negative integer components, and let

$$D_i^{\alpha_i}\equiv\partial^{\alpha_i}/\partial\xi_i^{\alpha_i},\qquad D^\alpha\equiv D_1^{\alpha_1}D_2^{\alpha_2}\dots D_m^{\alpha_m},\qquad |\alpha|=\alpha_1+\dots+\alpha_m.$$

An application of induction shows that for all α, $|\alpha|\geqq 0$,

$$(2.6)\qquad D^\alpha F(\xi, t)=\frac{P_\alpha(\xi, t)}{[A(\xi)+1]^{2|\alpha|}}F(\xi, t)\quad\text{in } R^m\times R,$$

where $P_\alpha(\xi, t)$ is a polynomial in ξ and t whose degree in $\xi_1, \dots, \xi_m$ is less than that of the denominator by at least $|\alpha|$. Accordingly, for every α there is a constant $M_\alpha(t)$ depending on α and t such that

$$(2.7)\qquad |\xi|^{|\alpha|}|D^\alpha F(\xi,t)|\leqq M_\alpha(t)$$

uniformly in R^m. Thus $F(\xi, t)$ is infinitely differentiable, and its derivatives in ξ of order $|\alpha|$ are of the order $O(|\xi|^{-|\alpha|})$ as $|\xi|\to\infty$.

Since the Fourier transform maps the space S onto itself continuously [35, 36], it follows from the estimates in (2.7) that the function $F(\xi, t)\,\hat{u}_0(\xi)$ lies in S for all t in R and that the function $u(x, t)$ *defined in* (2.4) *belongs to S for all t in R.*

Clearly, for all integers $n>0$, we have

$$\partial^n u(x, t)/\partial t^n=\int_{R^m}[f(\xi)]^n F(\xi, t)\,\hat{u}_0(\xi)\,e^{2\pi i\langle\xi, x\rangle}d\xi$$

in $R^m\times R$, by virtue of the absolute and uniform convergence of the integral. Moreover, in analogy with (2.6) we have

$$(2.6')\qquad D^\alpha[(f(\xi))^n F(\xi, t)]=\frac{Q_{n\alpha}(\xi, t)}{[A(\xi)+1]^{2n|\alpha|}}F(\xi, t)\quad\text{in } R^m\times R$$

where $Q_{n\alpha}(\xi, t)$ is a polynomial in ξ and t whose degree in ξ is less than that of the denominator by at least $|\alpha|$. Thus, for all α and n, there is a constant $M_{n\alpha}$ depending on n, α and t such that

$$(2.7')\qquad |\xi|^\alpha\left|D^\alpha([f(\xi)]^n F(\xi, t))\right|\leqq M_{n\alpha}(t)$$

uniformly in ξ. Hence *the time derivatives of $u(x, t)$ of all orders are in S for all t in R.*

Since all the derivatives of $u(x, t)$ in (2.4) can be calculated by differentiation under the integral sign, it follows from (2.4) that $u(x, t)$ *satisfies equation* (1.1) *pointwise* in $R^m\times R$. Moreover, by letting $t\to 0$ in (2.4), the Fourier inversion theorem [37] insures that (1.2) is also pointwise satisfied in R^m.

Consider now the asymptotic behavior of the solution as $t\to\infty$. It follows directly from (2.4) that

$$|u(x,t)|\leqq \sup|F(\xi,t)|\int_{R^m}|\hat{u}_0(\xi)|\,d\xi. \tag{2.8}$$

Moreover, the positivity of the quadratic forms $A(\xi)$ and $B(\xi)$ insures that, for all t in R,

$$|F(\xi,t)|\leqq \exp\left\{-\frac{4\pi^2\lambda(B)|\xi|^2+b}{4\pi^2\Lambda(A)|\xi|^2+1}\,t\right\}.$$

Using the expression on the right-hand side of the above inequality, we conclude that

$$F(\xi,t)\leqq\begin{cases}\exp\{-b\,t\} & \text{if } \lambda(B)\leqq\Lambda(A)\,b,\\ \exp\{-t\lambda(B)/\Lambda(A)\} & \text{if } \lambda(B)\geqq\Lambda(A)\,b.\end{cases} \tag{2.9}$$

Thus we have from (2.8) and (2.9)

$$|u(x,t)|\leqq\exp\{-k\,t\}\int_{R^m}|\hat{u}_0(\xi)|\,d\xi$$

where $k=b$ or $\lambda(B)/\Lambda(A)$. That is, $u(x,t)$ *decays expontially in* t *as* $t\to\infty$. *Similar reasoning shows that the time derivatives of* $u(x,t)$ *of all orders also decay to zero expontially in* t *as* $t\to\infty$. We summarize the preceeding results in

Theorem 2.1. *If* $u_0(x)$ *belongs to the space* S *of rapidly decreasing functions, then the Cauchy problem* (1.1) *and* (1.2) *has a unique solution* $u(x,t)$ *which, together with its time derivatives of all orders, belongs to* S *for all* t *in* R. *Moreover, both* $u(x,t)$ *and its derivatives decay to zero expontially in* t *as* $t\to\infty$.

Remark 2.2. As we shall see in §5, the proof of the uniqueness of the solution in the space S (for all t in R) is precisely the same as that given in [30].

Remark 2.3. Theorem 2.1 assures us that if $u(x,t)$ is a pointwise solution of (1.1) and if its restriction to any instant t belongs to S, then it lies in S for all times t.

Remark 2.4. It is not obvious that if the Cauchy data $u_0(x)$ is real-valued, then so is the solution $u(x,t)$. That this is so will be proved in §5.

3. Solutions in Sobolev Spaces

We have seen that if the Cauchy data $u_0(x)\in S$, then the solution $u(x,t)$ is in S for all t in R and is given explicitly by formula (2.4). We now wish to consider whether the formula (2.4) still holds if the initial values $u_0(x)$ belong to $W^{k,p}(R^m)$, the Sobolev space of functions whose distribution derivatives up to and including the order k, $k\geqq 2$, belong to $L^p(R^m)$, $1<p<\infty$. To this end, we note that the concepts of strong and weak (distribution) derivatives are identical. Also, for $1<p<2$, the Fourier transform carries $W^{k,p}(R^m)$ into $W^{k,q}(R^m)$ with $1/p+1/q=1$. If u is in $L^p(R^m)$, $p>2$, then its Fourier transform is in general a temperate distribution.

Lemma 3.1. *If* $u_0 \in W^{2,p}(R^m)$, $1<p<\infty$, *then the function* $u(x,t)$ *defined by* (2.4) *belongs to* $W^{2,p}(R^m)$ *for all* t *in* R. *Moreover, the time derivative* u_t *of* u *in* $W^{2,p}(R^m)$ *also belongs to* $W^{2,p}(R^m)$ *for all* t *in* R, *while the space derivatives of* u *and* u_t *up to and including the second order can be calculated by differentiating under the integral sign.*

Proof. We first check that $u(x,t)\in W^{2,p}(R^m)$ for all t in R. In fact, the estimate (2.7) for $F(\xi,t)$ insures that $F(\xi,t)$ is a Fourier multiplier for the Fourier multiplier theorem for Fourier integrals [7], [20]. According to this theorem, there is a constant $A_{p,m}$ depending on p and m such that

$$\|u(\cdot,t)\|_{2,p} \leqq M A_{p,m} \|u_0\|_{2,p} \tag{3.1}$$

where

$$M \equiv \max\{M_\alpha, |\alpha| \leqq m\},$$

the M_α's being the constants appearing in (2.7), and where we have adopted the usual norm on Sobolev spaces.

Next, we show that for all t in R,

$$\frac{\partial}{\partial t} u(x,t) = \int_{R^m} \frac{\partial}{\partial t} F(\xi,t)\hat{u}_0(\xi) e^{2\pi i\langle\xi,x\rangle} d\xi \tag{3.2}$$

in $W^{2,p}(R^m)$. From the definition of $u(x,t)$ in (2.4) and from the mean-value theorem applied to $F(\xi,t)$ in the variable t, we see that $u_t(x,t)$ is the limit as $h\to 0$ in $W^{2,p}(R^m)$ of the function

$$\int_{R^m} f(\xi)F(\xi,t)F(\xi,\theta h)\hat{u}_0(\xi)e^{2\pi i\langle\xi,x\rangle} d\xi, \qquad 0<\theta<1.$$

Accordingly, to establish (3.2) it suffices to show that as $h\to 0$

$$\int_{R^m} f(\xi)F(\xi,t)[F(\xi,\theta h)-1]\hat{u}_0(\xi)e^{2\pi i\langle\xi,x\rangle} d\xi \to 0 \quad \text{in } W^{2,p}. \tag{3.3}$$

To see this, we note that $F(\xi,\theta h)-1=O(|h|)$ as $h\to 0$. Also, for vectors β with non-negative integer components satisfying $|\beta|\geqq 1$, we have

$$D^\beta[F(\xi,\theta h)-1] = \frac{P_\beta(\xi,\theta h)}{[1+A(\xi)]^{2|\beta|}} F(\xi,\theta h)\cdot\theta h$$

where $P_\beta(\xi,\theta h)$ is a polynomial in ξ and h of degree at most $3|\beta|$ in ξ. Consequently, Leibniz's rule gives

$$D^\alpha[f(\xi)F(\xi,t)(F(\xi,\theta h)-1)] = \sum_{|\beta|\leqq|\alpha|} \frac{P_{\alpha-\beta}(\xi,t)}{(1+A(\xi))^{2|\alpha-\beta|}} F(\xi,t) \frac{P_\beta(\xi,\theta h)}{[1+A(\xi)]^{2|\beta|}} F(\xi,\theta h)\cdot\theta h$$

where $P_{\alpha-\beta}(\xi,t)$ is a polynomial in ξ and t of degree at most $3|\alpha-\beta|$ in ξ. Hence there is a constant $M_\alpha(t)$ depending only on α and t such that

$$|\xi|^{|\alpha|}\,|D^\alpha[f(\xi)F(\xi,t)(F(\xi,\theta h)-1)]| \leqq M_\alpha(t). \tag{3.4}$$

Again the multiplier theorem for Fourier integrals assures us that

(3.5) $\|(f(\xi)F(\xi,t)(F(\xi,\theta h)-1)\hat{u}_0(\xi))^V\|_{2,p}\leqq M(t)\,|h|\,A_{p,m}\|u_0\|_{2,p}$,

where $(f)^V$ stands for the inverse Fourier transform of f, $M(t)=\max\{M_\alpha(t), |\alpha|\leqq m\}$, $M_\alpha(t)$ are the constants appearing in (3.4), and $A_{p,m}$ is a constant depending only on p and m. The proof of (3.3) is now complete, and hence formula (3.2) is established. Our proof shows moreover that $u_t(x,t)$ also stays in $W^{2,p}(R^m)$ for all time t. Since all time derivatives of $F(\xi,t)$ are of class C^∞ in ξ and are also Fourier multipliers for Fourier integrals, the same reasoning as that used in deriving (3.2) leads to the formula

$$\frac{\partial^n}{\partial t^n}u(x,t)=\int_{R^m}\frac{\partial^n}{\partial t^n}F(\xi,t)\hat{u}_0(\xi)e^{2\pi i\langle\xi,x\rangle}d\xi \tag{3.6}$$

in the space $W^{2,p}(R^m)$, valid for all integers n.

For evaluation of the space derivatives of $u(x,t)$, we first establish the formula,

$$D_j u(x,t)=\int_{R^m}F(\xi,t)(D_j u_0)^\wedge(\xi)e^{2\pi i\langle\xi,x\rangle}d\xi \tag{3.7}$$

in $W^{1,p}(R^m)$ for all t in R. To prove this, we note first that $D_i u_0$ lies in $W^{1,p}(R^m)$ and that $F(\xi,t)$ is a Fourier multiplier. Hence the multiplier theorem for Fourier integrals assures us that the integral in (3.7) defines a function $u^*(x,t)$ in $W^{1,p}(R^m)$. Consequently, to establish the validity of (3.7) it suffices to show that for all t in R

$$\left\|\frac{u(x+h,t)-u(x,t)}{h}-u^*(x,t)\right\|_{1,p}\to 0 \quad\text{as}\quad h\to 0 \tag{3.8}$$

where $x+h=(x_1,\ldots,x_{j-1},x_j+h,x_{j+1},\ldots,x_m)$. Now the expression inside the norm sign in (3.8) is equal to

$$\int_{R^m}F(\xi,t)e^{2\pi i\langle\xi,x\rangle}d\xi\int_{R^m}\left[\frac{u_0(y+h)-u_0(y)}{h}-\frac{\partial u_0(y)}{\partial y_j}\right]e^{-2\pi i\langle\xi,y\rangle}dy; \tag{3.9}$$

thus the function $F(\xi,t)$ has been shown to be a Fourier multiplier for Fourier integrals. It follows immediately that the $\|\cdot\|_{1,p}$-norm of the function in (3.9) is less than or equal to

$$MA_{p,m}\left\|\frac{u_0(y+h)-u_0(y)}{h}-\frac{\partial u_0(y)}{\partial y_j}\right\|_{1,p}$$

where M and $A_{p,m}$ are the constants of (3.2). Formula (3.7) follows from this fact and the assumption that $u_0\in W^{2,p}(R^m)$. Moreover, since $u_0\in W^{2,p}(R^m)$, a repetition of these arguments shows that for $t\in R$

$$D_{ij}u(x,t)=\int_{R^m}F(\xi,t)(D_{ij}u_0)^\wedge(\xi)e^{2\pi i\langle\xi,x\rangle}d\xi \tag{3.10}$$

in the space $L^p(R^m)$. Since $\partial F(\xi,t)/\partial t$ is also a Fourier multiplier, it is now clear from (3.2) that

$$D_{ij}u_t(x,t)=\int_{R^m}F_t(\xi,t)(D_{ij}u_0)^\wedge(\xi)e^{2\pi i\langle\xi,x\rangle}d\xi \tag{3.11}$$

in the space $L^p(R^m)$ for all t in R. The proof of Lemma 3.1 is now complete.

By virtue of (3.10) and (3.11), we have for all t in R

$$L[u(x,t)] = \int_{R^m} L[F(\xi,t)\hat{u}_0(\xi)e^{2\pi i\langle\xi,x\rangle}]\,d\xi \quad \text{in } L^p(R^m),$$

where L is the differential operator in x and t given in (1.1). Thus the function $u(x,t)$ defined in (2.4) obeys

$$\|L[u]\|_{0,p}=0 \qquad \text{for all } t \text{ in } R, \tag{3.12}$$

since $L[F(\xi,t)\exp\{2\pi i\langle\xi,x\rangle\}]=0$ identically in ξ and t.

To show that $u(x,t)=u_0(x)$ in $W^{2,p}(R^m)$ as $t\to 0$, we observe that the Fourier inversion theorem holds in $L^p(R^m)$; see [26]. Hence $u_0(x)=(\hat{u})^\vee(x)$ in $W^{2,p}(R^m)$. Consequently, we may write

$$u(x,t)-u_0(x) = \int_{R^m} [F(\xi,t)-1]\,\hat{u}_0(\xi)e^{2\pi i\langle\xi,x\rangle}\,d\xi \quad \text{in } W^{2,p}(R^m).$$

The reasoning used to derive (3.5) shows that

$$\|u(\cdot,t)-u_0\|_{2,p} \leqq MA_{p,m}t\,\|u_0\|_{2,p},$$

from which our contention follows.

It has been shown that the function $u(x,t)$ defined in (2.4) satisfies (3.12) and that it assumes the initial values $u_0(x)$ in $W^{2,p}(R^m)$. To prove the uniqueness of the solution in $W^{2,p}(R^m)$ for all t in R, let $u_1(x,t)$ and $u_2(x,t)$ be two such solutions. Their difference $v(x,t)$ is a solution in $W^{2,p}(R^m)$ for all t in R which takes on homogeneous initial data. Accordingly, the Fourier transform $\hat{v}(\xi,t)$ of $v(x,t)$ satisfies the equations

$$\hat{v}(\xi,0)=0 \quad \text{in } W^{2,p}(R^m)$$

$$(1+A(\xi))\,\hat{v}_t(\xi,t) = -[B(\xi)+b(\xi)+b]\,\hat{v}(\xi,t)$$

in $W^{2,p}(R^m)$ for all t in R. As a consequence, $v(\xi,t)=0$ in $W^{2,p}(R^m)$ for all t in R. Thus we conclude from the uniqueness theorem for Fourier transforms that $v(x,t)=0$ in $W^{2,p}(R^m)$.

Suppose now that $u_0\in W^{k,p}(R^m)$ with $k\geqq 2$. For $k=3$, the reasoning used to derive (3.7) and (3.8) leads to the conclusion that for all t in R

$$D_{ijk}u(x,t) = \int_{R^m} F(\xi,t)\,(D_{ijk}u_0)^\wedge(\xi)\,e^{2\pi i\langle\xi,x\rangle}\,d\xi$$

in $L^p(R^m)$. Since $F(\xi,t)$ is a Fourier multiplier, this shows that $u(x,t)\in W^{3,p}(R^m)$ for all t in R. Finally, by induction, we see that for all t in R and $|\alpha|=k$

$$D^\alpha u(x,t) = \int_{R^m} F(\xi,t)\,(D^\alpha u_0)^\wedge(\xi)\,e^{2\pi i\langle\xi,x\rangle}\,d\xi$$

in $L^p(R^m)$; hence $u(x,t)\in W^{k,p}(R^m)$ for all t in R provided $u_0\in W^{k,p}(R^m)$. This proves that *the solution $u(x,t)$ of the Cauchy problem* (1.1) *and* (1.2) *is just as regular as the initial data, but no more.* Although we cannot assert that $u(x,t)$ is differentiable in x in the ordinary sense even if k is sufficiently large, nevertheless much can still be said about the differentiability of the solution.

If $E(t)$ denotes the bounded linear operator (mapping $W^{2,p}(R^m)$ into itself) such that $u(x, t) \equiv E(t)\, u_0(x)$ solves the Cauchy problem, then for every t in R the operator $E(t)$ *leaves each of the subspaces* $W^{k,p}(R^m)$, $k \geqq 2$, *invariant*.

For the asymptotic behavior of the solution $u(x, t)$ as $t \to \infty$, we turn to the estimates in (3.1), (2.9), (2.7) and (2.6). It is clear that if $u_0 \in W^{2,p}(R^m)$, then for all t in R

$$\|u(\cdot, t)\|_{2,p} \leqq M' e^{-kt} A_{p,m} \|u_0\|_{2,p} \tag{3.13}$$

where M' is a constant, k is the constant in (2.9), and $A_{p,m}$ is the constant in (3.1). We now observe that if $u_0 \in W^{k,p}(R^m)$ with $k \geqq 2$, then the estimates in (3.13) still hold with $\|\cdot\|_{2,p}$-norm replaced by $\|\cdot\|_{k,p}$-norm. We summarize the preceeding results in

Theorem 3.2. *If* $u_0(x)$ *belongs to* $W^{k,p}(R^m)$, $k \geqq 2$, $1 < p < \infty$, *then the Cauchy problem* (1.1) *and* (1.2) *has a unique solution* $u(x, t)$ *which, together with its time derivatives of all orders, stays in* $W^{k,p}(R^m)$ *for all* t *in* R. *Moreover, both* u *and its derivatives decay to zero expontially in* t *as* $t \to \infty$.

4. Solutions in Banach Spaces

We introduce the Banach spaces $C^{k+\alpha}(R^m)$ consisting of functions whose derivatives of order k are uniformly Hölder continuous in R^m with exponent α, $0 < \alpha < 1$. More precisely, for every $v(x)$ in $C^{k+\alpha}(R^m)$, the quantity

$$H^k_{\alpha}(v) \equiv \sup \left\{ \frac{|D^j v(x) - D^j v(y)|}{|x-y|^{\alpha}};\ x, y \text{ in } R^m, |j| = k \right\}$$

is supposed finite. The Hölder norm on $C^{k+\alpha}(R^m)$ is defined by

$$\|v\|_{k+\alpha} \equiv |v|_k + H^k_{\alpha}(v)$$

$$|v|_k \equiv \sum_{i=0}^{k} \sup\{|D^j v(x)|;\ x \in R^m, |j| = i\}.$$

Evidently, $C^{k+\alpha}(R^m)$ is a Banach space under this norm.

Suppose that the Cauchy data $u_0(x)$ in (1.2) belongs to $C^{k+\alpha}(R^m)$. We wish to construct the unique pointwise solution of (1.1) which (together with its time derivatives of all orders) lies in $C^{k+\alpha}(R^m)$ for all t in R. An essential step in this construction is to derive an estimate for solutions of an elliptic equation with real constant coefficients in the whole space.

Let A be the differential operator in (1.1),

$$Au \equiv \sum_{i,j=1}^{m} a_{ij} D_{ij} u.$$

The *principal fundamental solution* $H(x-y)$, [8, 9, 14, 15, 22], of the equation $Au - u = 0$ in R^m, $m > 1$, is given explicitly by

$$H(x-y) = C^{-1} \rho^{\frac{1-m}{2}} e^{-\rho} \int_0^{\infty} t^{\frac{m-3}{2}} \left(1 + \frac{t}{2\rho}\right)^{\frac{m-3}{2}} e^{-t}\, dt \tag{4.1}$$

where

$$C \equiv 2A^{\frac{1}{2}}(2\pi)^{\frac{m-1}{2}}\,\Gamma\left(\frac{m-1}{2}\right), \qquad A \equiv \det(a_{ij})$$

(4.2)
$$\rho^2 \equiv \sum_{i,j=1}^{m} A_{ij}(x_i - y_i)(x_j - y_j),$$

the matrix (A_{ij}) being the inverse of (a_{ij}). We note that C is only a normalizing factor and that $H(x-y)$ *is a function of the vector* $x-y$ *alone. Thus* $H(x-y)$ *is invariant under translation.*

From the explicit expressions in (4.1) and (4.2) we see that $H(x-y)$ is defined and of class C^∞ on $R^m \times R^m - D$, where D stands for the diagonal of the Cartesian product $R^m \times R^m$. Using the explicit expression for $H(x, y)$, we obtained (by means of elementary estimates) the following useful results:

There exist *positive* constants a, R_0 such that for vectors α with $0 \leqq |\alpha| \leqq 2$,

(4.3)
$$|D_x^\alpha H(x-y)| \leqq \text{const.}(\exp\{-a\,|x-y|\}) \quad \text{for all } |x-y| \geqq R_0;$$

moreover

(4.4)
$$|D_x^\alpha H(x-y)| \leqq \text{const.}\,|x-y|^{2-m-|\alpha|}(1+\delta_{2,m}\,\delta_{0,|\alpha|}\log|x-y|)$$

for $0 \leqq |\alpha| \leqq 2$ in every neighborhood of D, where $\delta_{2,m}$, $\delta_{0,|\alpha|}$ are Kronecker deltas.

The existence of the constants a and R_0 was proved by GIRAUD [14] for principal fundamental solutions of elliptic equations with variable coefficients, as was the singular behavior (4.4) near the diagonal D. For later applications, we emphasize the crucial fact that the constants appearing in (4.3) and (4.4) are independent of x and y. In other words, (4.3) and (4.4) hold when x varies in the whole space R^m.

Let $L(x-y)$ be defined on $R^m \times R^m - D$ by the formula

(4.5)
$$L(x-y) = \begin{cases} \dfrac{\Gamma(m/2)}{2A^{\frac{1}{2}}\pi^{m/2}(m-2)}\,\dfrac{1}{\rho^{m-2}}, & m>2 \\[2ex] \dfrac{1}{2\pi A^{\frac{1}{2}}}\log(1/\rho), & m=2 \end{cases}$$

where $\rho(x-y)$ is given by (4.2) and $A \equiv \det(a_{ij})$. Then near the diagonal D we have for $0 \leqq |\alpha| \leqq 2$, $0 < \beta < 1$,

(4.6)
$$|D_x^\alpha(H-L)(x-y)| \leqq \text{const.}\,|x-y|^{3-m-|\alpha|-\delta_{2,m}\,\delta_{0,|\alpha|}\,\beta}$$

when x varies on the whole space R^m. Such weakly singular behavior was proved by GIRAUD [14] for equations with variable coefficients in every bounded subset of $R^m \times R^m$. The truth of (4.6) is then an obvious consequence of the fact that $H-L$ is a function of $x-y$ alone.

Lemma 4.1. *Let* $H(x-y)$ *be defined by* (4.1). *Then for all* $v(x)$ *in* $C^{0+\alpha}(R^m)$ *and for all* x *in* R^m, *we have*

(4.7)
$$D_{x_i x_j}\int_{R^m} H(x-y)\,v(y)\,dy = \int_{R^m} D_{x_i x_j} H(x-y)\,v(y)\,dy - \frac{1}{m}A_{ij}\,v(x)$$

where the integral on the right-hand side is taken in the principal value sense and $D_{x_i x_j}$ *denotes* $\partial^2/dx_i\,\partial x_j$.

Proof. The properties of $H(x-y)$ in (4.3) and (4.4) assure us that

$$(4.8)\quad D_{x_i}\int_{R^m} H(x-y)\,v(y)\,dy=\int_{R^m} D_{x_i}H(x-y)\,v(y)\,dy,\qquad x\in R^m,\ i=1,\dots,m,$$

To prove (4.7) we first put

$$(4.9)\quad \varphi(x)\equiv\int_{R^m} H(x-y)\,v(y)\,dy,\qquad \varphi_i(x;\delta)\equiv\int_{R^m-I(x,\delta)} D_{x_i}H(x-y)\,v(y)\,dy$$

where $I(x,\delta)$ is the ellipsoid centered at x and defined by $\rho(x-y)\leqq\delta$. From (4.8) we see that $\varphi_i(x;\delta)\to D_i\varphi(x)$ uniformly in x as $\delta\to 0$. By virtue of (4.3) we have, for every $\delta>0$ and for all x in R^m,

$$(4.10)\quad \begin{aligned} D_j\varphi_i(x,\delta)=&\int_{R^m-I(x,\delta)} D_{x_i x_j}H(x-y)\,v(y)\,dy\\ &-\int_{\partial I(x,\delta)} D_{x_i}H(x-y)\,v(y)\,X_j(y)\,d\sigma_y\end{aligned}$$

where $\partial I(x,\delta)$ denotes the boundary of $I(x,\delta)$ and X_j the direction cosines of the exterior normal to $\partial I(x,\delta)$.

The second integral in (4.10) can be written as a sum

$$\begin{aligned}&\int_{\partial I(x,\delta)} D_{x_i}L(x-y)\,v(y)\,X_j(y)\,d\sigma_y\\ &\quad+\int_{\partial I(x,\delta)} D_{x_i}(H-L)(x-y)\,v(y)\,X_j(y)\,d\sigma_y\equiv I_1+I_2.\end{aligned}$$

It is not difficult to check [22] that I_1 may be written as the sum of two integrals, one of which goes to zero as $\delta\to 0$ uniformly with respect to x while the other is equal to $-A_{ij}v(x)/m$ for all x in R^m. In the integral I_2 the function $v(y)$ is bounded while $H-L$ satisfies (4.6). Hence the value of I_2 is bounded by $c\delta$, with the constant c independent of x and δ. Thus $I_2\to 0$ uniformly with respect to x as $\delta\to 0$.

Since $\varphi_i(x,\delta)\to\partial\varphi(x)/\partial x_i$ uniformly, formula (4.7) will be established if we can show that as $\delta\to 0$ the first integral in (4.10) converges uniformly with respect to x. To prove this, we write this integral as the sum

$$I_3+I_4\equiv\Big\{\int_{R^m-B(x,k)}+\int_{B(x,k)-I(x,\delta)}\Big\}D_{x_i x_j}H(x-y)\,v(y)\,dy$$

where $B(x,k)$ is a ball with center at x and radius k. It suffices now to establish the uniform convergence of I_4 as $\delta\to 0$. To this end, we write

$$\begin{aligned}I_4\equiv I_5+I_6\equiv&\int_{B(x,k)-I(x,\delta)} D_{x_i x_j}(H-L)(x-y)\,v(y)\,dy\\ &+\int_{B(x,k)-I(x,\delta)} D_{x_i x_j}L(x-y)\,v(y)\,dy.\end{aligned}$$

From (4.6) we see that I_5 converges uniformly in x as $\delta\to 0$. Since we may keep k fixed, the uniform convergence of I_6 as $\delta\to 0$ follows in a standard way; see, e.g., [20, 22]. The lemma is thus established.

Remark 4.2. The lemma just proved shows that if $v \in C^{0+\alpha}(R^m)$, then $\varphi = H * v$ satisfies the equation $A\varphi - \varphi = v$ in R^m, where "$*$" denotes convolution.

Lemma 4.3. *Let $H(x-y)$ be the principal fundamental solution given in* (4.3). *Then, for every v in $C^{0+\alpha}(R^m)$,*

$$H_\alpha^0\Big(\int_{R^m} D_{x_i x_j} H(x-y)\, v(y)\, dy\Big) \leqq \text{const.}\ \|v\|_{0+\alpha}$$

where the integral is understood to be singular.

Proof. Write the given singular integral as the sum of three integrals

$$\Big\{\int_{R^m - B(x,k)} D_{x_i x_j} H(x-y) + \int_{B(x,k)} D_{x_i x_j}(H-L)(x-y) + \int_{B(x,k)} D_{x_i x_j} L(x-y)\Big\}\, v(y)\, dy$$

and call them $f_1(x)$, $f_2(x)$ and $f_3(x)$, respectively. Here $B(x, k)$ is a ball with center at x and radius k. The constant k will be chosen so large that (4.3) holds. Since $H(x-y)$ is a function of $x-y$ alone, it can be checked that for all vectors h

$$\begin{aligned} |f_1(x+h) - f_1(x)| &= \Big|\int_{R^m - B(x,k)} D_{x_i x_j} H(x-y)\big(v(y+h) - v(y)\big)\, dy\Big| \\ &\leqq C_1 H_\alpha^0(v)\, |h|^\alpha \int_k^\infty r^{m-1} e^{-ar}\, dr, \end{aligned}$$

where $H_\alpha^0(v)$ is the Hölder constant of v with exponent α, and C_1 is a constant independent of v and x. Since the integral is convergent it is clear that

$$\text{(4.11)} \qquad H_\alpha^0(f_1) \leqq \text{const.}\, H_\alpha^0(v) \quad \text{for all } v \text{ in } C^{0+\alpha}(R^m).$$

For a fixed choice of k, we note from (4.6) that

$$|\partial^2 (H-L)(x-y)/\partial x_i \partial x_j| \leqq k_1 / r^{m-1+\delta_{2,m}\beta} \quad \text{in } B(x, k)$$

where the constant k_1 is independent of x. Since $H-L$ is also a function of $x-y$ alone, we have for all vectors h

$$|f_2(x+h) - f_2(x)| = \Big|\int_{B(x,k)} D_{x_i x_j}(H-L)(x-y)\big(v(y+h) - v(y)\big)\, dy\Big| \leqq C_2 H_\alpha^0(v)\, |h|^\alpha$$

where the constant C_2 is independent of v and x. Thus,

$$\text{(4.12)} \qquad H_\alpha^0(f_2) \leqq \text{const}\, H_\alpha^0(v) \quad \text{for all } v \text{ in } C^{0+\alpha}(R^m).$$

To bound the Hölder constant of $f_3(x)$ in R^m, we write the kernel of the integral defining $f_3(x)$ in the form:

$$\frac{\partial^2 L(x-y)}{\partial x_i \partial x_j} = \frac{F(\theta)}{|x-y|^m}$$

where θ varies over the surface of a unit sphere of dimension $m-1$. Here $F(\theta)$ is infinitely differentiable in θ and has mean value zero, $\int F(\theta)\, d\theta = 0$. For a detailed justification of these statements, see for example [20, p. 59]. Now we show that the function

$$\text{(4.13)} \qquad f_3(x) \equiv \int_{B(x,k)} \frac{F(\theta)}{|x-y|^m} v(y)\, dy = \int_{B(x,k)} \frac{F(\theta)}{|x-y|^m}\big(v(y) - v(x)\big)\, dy$$

is uniformly Hölder continuous in R^m and that its Hölder constant is less than or equal to a constant multiple of $\|v\|_{0+\alpha}$. A proof of this non-obvious fact is carried out in detail for Theorem 1.6 in [20]. We make two observations. First, we can choose k so large that Mikhlin's proof gives

$$|f_3(x+h)-f_3(x)| \leqq C_3' H_\alpha^0(v)\,|h|^\alpha \quad \text{for all } |h| \leqq 1 < \frac{k}{2}. \tag{4.14}$$

The constant C_3' is independent of v. It is, moreover, independent of x because the characteristic $F(\theta)$ in (4.13) is independent of the pole x. Secondly, $f_3(x)$ is bounded. In fact, (4.13) shows that

$$\sup |f_3(x)| \leqq \text{const.}\, H_\alpha^0(v).$$

Hence, we conclude that

$$|f_3(x+h)-f_3(x)| \leqq C_3''\,|v|_0\,|h|^\alpha \quad \text{for all } |h|>1.$$

Accordingly, for all vectors h,

$$|f_3(x+h)-f_3(x)| \leqq C_3\,\|v\|_{0+\alpha}\,|h|^\alpha$$

where the constant C_3 is independent of v and x. This means that

$$H_\alpha^0(f_3) \leqq \text{const.}\,\|v\|_{0+\alpha}. \tag{4.15}$$

The lemma now follows directly from (4.11), (4.12) and (4.15).

Lemma 4.4. *The principal fundamental solution given in* (4.1) *generates a bounded convolution integral operator which maps $C^{0+\alpha}(R^m)$ into $C^{2+\alpha}(R^m)$. That is, if*

$$u(x) \equiv \int_{R^m} H(x-y)\,v(y)\,dy, \qquad v \in C^{0+\alpha}(R^m), \tag{4.16}$$

then

$$\|u\|_{2+\alpha} \leqq \text{const.}\,\|v\|_{0+\alpha}. \tag{4.17}$$

The same constant holds for all v in $C^{0+\alpha}(R^m)$.

Proof. From the estimates in (4.3) and (4.4) for $H(x-y)$ we see that

$$|u|_0 \leqq \text{const.}\,|v|_0. \tag{4.18}$$

Similarly, it follows from (4.8), (4.3) and (4.4) that

$$|u|_1 \leqq \text{const.}\,|v|_0. \tag{4.19}$$

The constants in these two inequalities are valid for all v.

From formula (4.7) we have, for $|\alpha|=2$,

$$|u|_2 \leqq \sum_{|\alpha|=2} \sup_x \Big|\int_{R^m} D_x^\alpha H(x-y)\,v(y)\,dy\Big| + \text{const.}\,|v|. \tag{4.20}$$

For each of these integrals (with $|\alpha|=2$) we have

$$\int_{R^m} D_x^\alpha H(x-y)\, v(y)\, dy$$
$$=\Big\{\int_{|x-y|\geqq k} D_x^\alpha H(x-y)+\int_{|x-y|<k} D_x^\alpha (H-L)(x-y)+\int_{|x-y|<k} D_x^\alpha L(x-y)\Big\}\, v(y)\, dy$$
$$\equiv I_1+I_2+I_3.$$

By virtue of (4.3), $|I_1|$ is bounded by a constant multiple of $|v|_0$ if k is large enough. By virtue of (4.4), $|I_2|$ is also bounded by a multiple of $|v|_0$. Since the characteristic of the kernel $D_x^\alpha L(x-y)$, with $|\alpha|=2$, has mean value zero, the absolute value of the singular integral I_3 is less than const. $H_\alpha^0(v)$, as is easily seen from (4.13). Hence

$$\Big|\int_{R^m} D_x^\alpha H(x-y)\, v(y)\, dy\Big| \leqq \text{const.}\, \|v\|_{2+\alpha}, \qquad |\alpha|=2, \tag{4.21}$$

for all v in $C^{0+\alpha}(R^m)$. Thus we conclude from (4.20) and (4.21) that

$$|u|_2 \leqq \text{const.}\, \|v\|_{0+\alpha} \quad \text{for all } v \text{ in } C^{0+\alpha}(R^m). \tag{4.22}$$

By combining the estimates in (4.18)–(4.22) and Lemma 4.3, we obtain the desired estimate in (4.17).

Theorem 4.5. *Let A and B be the second order differential operators in* (1.1′) and (1.1″), *respectively. Suppose that $u_0(x)$ is a given function in $C^{2+\alpha}(R^m)$. Then the Cauchy problem*

$$\begin{aligned} L[u]&=(A-I)u_t+Bu=0 && \text{in } R^m\times R\\ u(x,0)&=u_0(x) && \text{in } R^m \end{aligned} \tag{4.23}$$

has a unique solution $u(x,t)$ which, together with its time derivatives of all orders, belongs to $C^{2+\alpha}(R^m)$ for all t in R. Moreover, it is analytic in t for all t.

Proof. Let $H(x-y)$ be the principal fundamental solution of $Au-u=0$ in R^m. Then $H(x-y)$ is given by the formula in (4.1). Define the linear operator

$$(A-I)^{-1}B\colon\ C^{2+\alpha}(R^m)\to C^{2+\alpha}(R^m)$$

by the formula

$$(A-I)^{-1}Bu \equiv -\int_{R^m} H(x-y)\, Bu(y)\, dy \qquad \text{for all } u \text{ in } C^{2+\alpha}(R^m). \tag{4.24}$$

According to Lemma 4.4, the linear operator $(A-I)^{-1}B$ is bounded. By virtue of the completeness of the space $C^{2+\alpha}(R^m)$, the operator $E(t)$ defined for all t in R by

$$E(t)\equiv \exp\{-t(A-I)^{-1}B\} \equiv \sum_{n=0}^{\infty}\big(-t(A-I)^{-1}B\big)n/n! \tag{4.25}$$

is also a bounded linear operator mapping $C^{2+\alpha}(R^m)$ into itself. It follows in the usual way (see, e.g., [30]) that

(a) $\{E(t),\, t\in R\}$ is an analytic Abelian group such that $E(t_1+t_2)=E(t_1)\,E(t_2)$ for all t_1, t_2 in R, and that $E(0)=I$;

(b) $E(t)$ is continuous in t with respect to the uniform operator topology;

(c) $E(t)$ is differentiable in the uniform operator topology and

$$E'(t) = -(A-I)^{-1} BE(t), \quad t \in R. \tag{4.26}$$

Now for every given $u_0(x)$ in $C^{2+\alpha}(R^m)$ we can define

$$u(x,t) \equiv E(t) u_0(x) \quad \text{for all } t \text{ in } R. \tag{4.27}$$

Then $u(x,t) \in C^{2+\alpha}(R^m)$ for all t in R. If follows from (4.26) that, for all t in R,

$$u_t(x,t) = -(A-I)^{-1} Bu(x,t) \quad \text{in } C^{2+\alpha}(R^m).$$

By applying the differential operator $(A-I)$ to both sides (from the left), we find that, for all t in R,

$$(A-I) u_t(x,t) + Bu(x,t) = 0 \quad \text{in } C^{0+\alpha}(R^m).$$

Property (a) of $E(t)$ insures that $u(x,0) = u_0(x)$ in $C^{2+\alpha}(R^m)$. This shows that the function $u(x,t)$ defined in (4.27) solves the Cauchy problem (4.23).

Since the power series in (4.25) converges uniformly for t in R with respect to the uniform topology on $\mathscr{L}[C^{2+\alpha}(R^m)]$, it is clear that $u(x,t)$ is analytic in t for all x in R^m. Verification of the uniqueness of the solution of (4.23) can be carried out in the same way as in [30]. The proof of the theorem is now complete.

In conclusion, we raise the question whether the solution $u(x,t)$ will have higher order differentiability if its initial data $u_0(x)$ does. The answer is given by

Theorem 4.6. *If the initial data $u_0 \in C^{k+\alpha}(R^m)$, $k \geqq 2$, then the solution $u(x,t)$ belongs to $C^{k+\alpha}(R^m)$ for all t in R.*

Proof. The properties listed in (4.3) and (4.4) for $H(x-y)$ insure that for all v in $C^{1+\alpha}(R^m)$,

$$\frac{\partial}{\partial x_j} \int_{R^m} H(x-y) v(y)\, dy = \int_{R^m} \frac{\partial}{\partial x_j} H(x-y) v(y)\, dy. \tag{4.28}$$

Now for $(x,y) \in R^m \times R - D$ we have

$$\frac{\partial}{\partial y_j} [H(x-y) v(y)] = -\frac{\partial}{\partial x_j} H(x-y) v(y) + H(x-y) \frac{\partial v(y)}{\partial y_j}. \tag{4.29}$$

By virtue of (4.3) and (4.4), we have also

$$\begin{aligned} \lim_{R\to\infty} \lim_{\varepsilon\to 0} \int_{\varepsilon \leqq |x-y| \leqq R} & \frac{\partial}{\partial y_j} [H(x-y) v(y)]\, dy \\ &= \lim_{R\to\infty} \int_{|x-y|=R} H(x-y) v(y) X_j(y)\, d\sigma_y \\ &\quad - \lim_{\varepsilon\to 0} \int_{|x-y|=\varepsilon} H(x-y) v(y) X_j(y)\, d\sigma_y = 0 \end{aligned}$$

where $X_j(y)$ stands for the direction cosines of the exterior normals to $|x-y| = R$ and $|x-y| = \varepsilon$. Thus,

$$\int_{R^m} \frac{\partial}{\partial y_j} [H(x-y) v(y)]\, dy = 0. \tag{4.30}$$

Hence from (4.29) and (4.30) we obtain the useful identity

$$\int_{R^m} \frac{\partial}{\partial x_j} H(x-y)v(y)\,dy = \int_{R^m} H(x-y)\frac{\partial v(y)}{\partial y_j}\,dy \tag{4.31}$$

provided $v \in C^{1+\alpha}(R^m)$.

Suppose now that $u_0(x) \in C^{3+\alpha}(R^m)$. Then $Bu_0 \in C^{1+\alpha}$, where B is the differential operator defined in (1.1″). As in the derivation of (4.28), we have

$$\frac{\partial}{\partial x_j}\int_{R^m} H(x-y)Bu_0(y)\,dy = \int_{R^m}\frac{\partial}{\partial x_j} H(x-y)Bu_0(y)\,dy. \tag{4.30'}$$

By virtue of (4.31) and (4.30) we have

$$\begin{aligned} &\frac{\partial^2}{\partial x_i \partial x_j}\int_{R^m} H(x-y)Bu_0(y)\,dy \\ &= \frac{\partial}{\partial x_i}\int_{R^m}\frac{\partial}{\partial x_j} H(x-y)Bu_0(y)\,dy \\ &= \frac{\partial}{\partial x_i}\int_{R^m} H(x-y)\frac{\partial}{\partial y_j} Bu_0(y)\,dy \\ &= \int_{R^m}\frac{\partial H(x-y)}{\partial x_i}\frac{\partial}{\partial y_j} Bu_0(y)\,dy. \end{aligned} \tag{4.31'}$$

Using formula (4.31) and the proof of Lemma 4.3, we conclude that

$$\begin{aligned} &\frac{\partial^2}{\partial x_i \partial x_j \partial x_k}\int_{R^m} H(x-y)Bu_0(y)\,dy \\ &= \int_{R^m}\frac{\partial^2}{\partial x_i \partial x_j} H(x-y)\frac{\partial}{\partial y_k} Bu_0(y)\,dy \\ &\quad -\frac{1}{m}A_{ij}\frac{\partial}{\partial x_k} Bu_0(x). \end{aligned} \tag{4.32}$$

We conclude now from formulas (4.30)–(4.32) and Lemma 4.4 that the operator $(A-I)^{-1}B$ defined by

$$(A-I)^{-1}Bu_0 \equiv -\int_{R^m} H(x-y)Bu_0(y)\,dy$$

maps $C^{3+\alpha}(R^m)$ into $C^{3+\alpha}(R^m)$ in such a way that

$$\|(A-I)^{-1}Bu_0\|_{3+\alpha} \leqq \text{const.}\, \|u_0\|_{3+\alpha}. \tag{4.33}$$

It now suffices to remark that, if $u_0 \in C^{3+\alpha}(R^m)$, then the solution $u(x, t)$ defined by

$$u(x, t) = \exp\{-t(A-I)^{-1}B\}\cdot u_0(x)$$

belongs to $C^{3+\alpha}(R^m)$ for all t in R.

By induction on k, it follows that if $u_0 \in C^{k+\alpha}$, then

$$\|(A-I)^{-1}Bu_0\|_{k+\alpha} \leqq \|u_0\|_{k+\alpha}$$

for all integers $k \geqq 2$. The proof is now complete.

5. Unification of Methods. Hölder Continuity of Weak Solutions

In §2 and §3, the Cauchy problem was solved by means of Fourier transforms. The solutions so obtained are either infinitely differentiable or belong to certain Sobolev spaces. In either case, they vanish at infinity. The fact that these solutions are real-valued has not yet been confirmed. In §4, we constructed the solutions by using the principal fundamental solution. These solutions are obviously real-valued and solve the problem in the pointwise sense, though they need not vanish at infinity. We wish to show that the results in §2 and §3 can also be obtained by using the principal fundamental solution. In this way, we shall be able to derive more information about the solutions.

Theorem 5.1. *Suppose that, as functions of* x, $u(x,t)$ *and* $u_t(x,t)$ *belong to* $C^{2+\alpha}(R^m)$ *for all* t *in* R *and that* $u(x,t)$ *is a pointwise solution of equation* (1.1) *in* $R^m\times R$. *If the Cauchy data* $u(x,0)=u_0(x)$ *belongs to the space* S *of rapidly decreasing functions, then, as a function of* x, $u(x,t)\in S$ *for all* t *in* R.

Proof. Recall that the principal fundamental solution $H(x-y)$ is of class C^∞ on $R^m\times R^m-D$ and possesses the properties listed in (4.3) and (4.4). Accordingly, it is a *rapidly decreasing distribution* [10, 26], and hence a *temperate distribution*. As a temperate distribution on R^m, it satisfies the equation

$$AH-H=-\delta \tag{5.1}$$

where δ is the Dirac measure. Forming the Fourier transform of (5.1), we get

$$\hat{H}(\xi)=1/[A(\xi)+1]. \tag{5.2}$$

The fact that H is a rapidly decreasing distribution insures that, for every u_0 in the space S, the extended convolution theorem for Fourier transforms is applicable to $H*Bu_0$, where B is the differential operator in (1.1). Thus

$$(H*Bu_0)^\wedge=\hat{H}(Bu_0)^\wedge=-\frac{B(\xi)+b(\xi)+b}{A(\xi)+1}\hat{u}_0(\xi). \tag{5.3}$$

From the detailed analysis in §1, we see that the right-hand side in (5.3) is in S. Hence for all $u_0\in S$,

$$(H*Bu_0)(x)\in S. \tag{5.4}$$

In the notation of (4.24), the relations (5.3) and (5.4) can be written

$$\begin{gathered}((A-I)^{-1}Bu_0)^\wedge(\xi)=\frac{B(\xi)+b(\xi)+b}{A(\xi)+1}\hat{u}_0(\xi),\\ \int_{R^m}H(x-y)(Bu_0)(y)\,dy\in S.\end{gathered} \tag{5.5}$$

By induction on n, we conclude from (5.5) that for all integers $n>0$

$$([(A-I)^{-1}B]^n u_0)^\wedge(\xi)=\left[\frac{B(\xi)+b(\xi)+b}{A(\xi)+1}\right]^n\hat{u}_0(\xi)\in S, \tag{5.6}$$

since the factor of $\hat{u}_0(\xi)$ on the right-hand side in (5.6) is a multiplier in S for all n. It follows that, for all t in R, the series

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} \int_{R^m} (-1)^n \left[\frac{B(\xi)+b(\xi)+b}{A(\xi)+1}\right]^n \hat{u}_0(\xi)\, e^{2\pi i\langle \xi, x\rangle}\, d\xi$$

defines a function in S. Since the order of summation and integration may be interchanged, we have for all t in R

$$\exp\{-t(A-I)^{-1}B\}\cdot u_0(x) = \int_{R^m} \exp\left\{-\frac{B(\xi)+b(\xi)+1}{A(\xi)+1}\, t\right\} \hat{u}_0(\xi)\, e^{2\pi i\langle \xi, x\rangle}\, d\xi,$$

which is what we set out to prove.

Theorem 5.2. *Suppose that, as functions of x, $u(x, t)$ and $u_t(x, t)$ belong to $C^{2+\alpha}(R^m)$ for all t in R and that $u(x, t)$ is a pointwise solution of equation* (1.1) *in $R^m \times R$. If the Cauchy data $u(x, 0) = u_0(x)$ is in $C^{2+\alpha}(R^m) \cap W^{2,p}(R^m)$, then as a function of x, $u(x, t)$ stays there for all t in R.*

Proof. Since $H(x)$ is a temperate distribution, the Fourier transform is given by the expression in (5.2). Since for $u_0 \in W^{2,p}(R^m)$, Bu_0 is certainly a temperate distribution and since $H(x)$ is actually a rapidly decreasing distribution, the convolution $H(x-y) * Bu_0(y)$ is well-defined. Moreover, the convolution theorem for Fourier transforms is applicable [10, 20]. Hence formula (5.3) still holds in the present case. Since the factor $\big(B(\xi)+b(\xi)+b\big)/\big(A(\xi)+1\big)$ is a Fourier multiplier for integrals, we conclude from the multiplier theorem and (5.3) that

$$\|H * Bu_0\|_{2,p} \leqq M\, \|u\|_{2,p} \tag{5.7}$$

where M is a constant depending only on p, m and the multiplier. Using the notation introduced in (4.24), we can write (5.6) as

$$\|(A-I)^{-1} Bu_0\|_{2,p} \leqq M\, \|u_0\|_{2,p}. \tag{5.8}$$

By induction on n, it follows that for all integers $n > 0$

$$[(A-I)^{-1}B]^n u_0(x) = \int_{R^m} \left[\frac{B(\xi)+b(\xi)+b}{A(\xi)+1}\right]^n \hat{u}_0(\xi)\, e^{2\pi i\langle \xi, x\rangle}\, d\xi \tag{5.9}$$

in the space $W^{2,p}(R^m)$. Moreover, by repeatedly applying the estimate in (5.8) to the equation (5.9), we obtain the estimate

$$\|[(A-I)^{-1}B]^n u_0\|_{2,p} \leqq M^n\, \|u_0\|_{2,p} \tag{5.10}$$

where the constant M is the same as that in (5.8). Hence for all t

$$\begin{aligned} E(t)u_0 &\equiv \exp\{-t(A-I)^{-1}B\}\cdot u_0 \\ &= \sum_{n=0}^{\infty} \frac{(-t)^n}{n!} \int_{R^m} \left[\frac{B(\xi)+b(\xi)+b}{A(\xi)+1}\right]^n \hat{u}_0(\xi)\, e^{2\pi i\langle \xi, x\rangle}\, d\xi \\ &= \int_{R^m} \exp\left\{-t\,\frac{B(\xi)+b(\xi)+b}{A(\xi)+1}\right\} \hat{u}_0(\xi)\, e^{2\pi i\langle \xi, x\rangle}\, d\xi \end{aligned} \tag{5.11}$$

in $W^{2,p}(R^m)$. In fact, it follows from (5.10) and (5.11) that

$$\|E(t)u_0\|_{2,p} \leqq e^{M|t|} \|u_0\|_{2,p} \quad \text{for all } t \text{ in } R.$$

The proof is now complete. It may be noted that the last inequality is a good estimate for $t<0$.

For weak solutions of elliptic and parabolic partial differential equations, the regularity theorem [23–25] asserts that weak solutions can be identified with Hölder continuous ones. We might suspect that similar results hold for weak solutions of pseudo-parabolic differential equations. Simple examples show, however, that solutions of pseudo-parabolic equations cannot be smoother than their initial data. On the other hand, there is a big gap between the spaces $W^{2,p}$ and $C^{2+\alpha}$. It is interesting to ask the question whether the solution $u(x, t)$ belongs to $W^{2,p} \cap C^{0+\alpha}(R^m)$ for all t if it is so at any fixed instant.

Theorem 5.3. *Suppose, that, as a function of x, $u(x, t)$ belongs to $W^{2,p}(R^m)$ for all t in R and that $u(x, t)$ is strongly differentiable in t and satisfies equation* (1.1) *in $L^p(R^m)$ for all t in R. If $u_0(x) \equiv u(x, 0)$ is in $W^{2,p}(R^m) \cap C^{0+\alpha}(R^m)$, then, as a function of x, $u(x, t)$ stays there for all t in R.*

This result implies that all weak solutions are Hölder continuous, if it is so at any fixed instant. We expect the same is true for the solutions obtained in [30], although we are unable to prove this at the present time.

Proof of Theorem 5.3. Since $u_0(x) \in W^{2,p}(R^m)$, the existence of a unique solution $u(x, t)$ which belongs to $W^{2,p}(R^m)$ for all t in R has been established in §3. We shall show that under the additional restriction $u_0(x) \in C^{0+\alpha}(R^m)$, the solution $u(x, t)$ also belongs to $C^{0+\alpha}(R^m)$ for all t in R.

It was shown in §3 that the solution in $W^{2,p}(R^m)$ for all t in R is given by the formula

$$u(x, t) = \int_{R^m} \exp\left\{-\frac{B(\xi)+b(\xi)+b}{A(\xi)+1}\, t\right\} \hat{u}_0(\xi)\, e^{2\pi i\langle \xi, x\rangle}\, d\xi, \qquad t \in R$$

(the equality holding in $W^{2,p}(R^m)$). In the proof of Theorem 5.2 we have shown that, for all integers $n \geqq 1$,

$$[(A-I)^{-1}B]^n \cdot u_0 = \int_{R^m} \left[\frac{B(\xi)+b(\xi)+b}{A(\xi)+1}\right]^n \hat{u}_0(\xi)\, e^{2\pi i\langle \xi, x\rangle}\, d\xi$$

in $W^{2,p}(R^m)$. Hence

$$\begin{aligned} u(x, t) &= \sum_{n=0}^{\infty} \frac{(-t(A-I)^{-1}B)^n}{n!} u_0 \\ &= u_0(x) + t \int_{R^m} H(x-y)\, B u_0(y)\, dy \qquad (5.12) \\ &\quad + \frac{t^2}{2!} \int_{R^m} H(x-y_1)\, dy_1 \cdot B \int_{R^m} H(y_1-y)\, B u_0\, dy + \cdots \end{aligned}$$

where B stands for the differential operator in (1.1″), and the equality holds in $W^{2,p}(R^m)$.

To simplify the notation, we write

(5.13) $$u_n(x)=\int_{R^m} H(x-y)Bu_{n-1}(y)\,dy, \qquad n=1,2,\ldots.$$

Then (5.12) can be written

(5.14) $$u(x,t)=u_0(x)+\sum_{n=1}^{\infty}\frac{t^n}{n!}u_n(x) \quad \text{in } W^{2,p}(R^m),\ t\in R.$$

We wish to show that each $u_n(x)$ is Hölder continuous with exponent α, and that its Hölder constant does not exceed a constant multiple of $\|u_0\|_{0+\alpha}$.

By virtue of the properties of $H(x-y)$ listed in (4.3) and (4.4),

(5.15) $$\begin{aligned} u_1(x) &\equiv \int_{R^m} H(x-y)Bu_0(y)\,dy \\ &= \sum_{i,j=1}^{m} b_{ij}\int_{R^m}\frac{\partial}{\partial x_i}H(x-y)\frac{\partial}{\partial y_j}u_0(y)\,dy \\ &\quad+\sum_{i=1}^{m} b_i\int_{R^m}\frac{\partial}{\partial x_i}H(x-y)u_0(y)\,dy-b\int_{R^m}H(x-y)u_0(y)\,dy, \end{aligned}$$

that is, we can shift one differentiation from u_0 to H under the integral sign. For $u_0\in C^{0+\alpha}\cap W^{2,p}$, we have

(5.16) $$\frac{\partial}{\partial y_j}\left[\frac{\partial}{\partial x_i}H(x-y)u_0(y)\right]=-\frac{\partial^2 H(x-y)}{\partial x_i\partial x_j}u_0(y)+\frac{\partial H(x-y)}{\partial x_i}\frac{\partial u_0(y)}{\partial y_j}$$

in the sense of distributions. Moreover, using (4.3) and (4.4), we get

(5.17) $$\begin{aligned} &\lim_{\rho\to 0}\int_{R^m-I(x,\rho)}\frac{\partial}{\partial y_j}\left[\frac{\partial}{\partial x_i}H(x-y)u_0(y)\right]dy \\ &=\lim_{\rho\to 0}\int_{\partial I(x,\rho)}\frac{\partial}{\partial x_i}H(x-y)X_j(y)u_0(y)\,d\sigma_y \\ &=-\frac{1}{m}A_{ij}u_0(x) \end{aligned}$$

where $I(x,\rho)$ is the ellipsoid defined by $\Sigma A_{ij}(x_i-y_i)(x_j-y_j)\leqq\rho^2$, $X_j(y)$ are direction consines of the exterior normals to $\partial I(x,\rho)$, and $(A_{ij})=(a_{ij})^{-1}$. Consequently, we have from (5.16) and (5.17)

(5.18) $$\int_{R^m}\frac{\partial H(x-y)}{\partial x_i}\frac{\partial u_0(y)}{\partial y}\,dy=\int_{R^m}\frac{\partial^2 H(x-y)}{\partial x_i\partial x_j}u_0(y)\,dy-\frac{1}{m}A_{ij}u_0(x);$$

in fact, the existence of the singular integral on the right-hand side has been proved in Lemma 4.1. By combining (5.15) and (5.18), we obtain the useful formula

(5.19) $$\begin{aligned} u_1(x) &\equiv \int_{R^m} H(x-y)Bu_0(y)\,dy \\ &=\int_{R^m}(BH(x-y))u_0(y)\,dy-\sum_{i,j=1}^{m} b_{ij}A_{ij}\frac{u_0(x)}{m}. \end{aligned}$$

With the help of the notations introduced in §4 for Hölder norms, it follows from (5.18) and Lemma 4.3 that

$$
\begin{aligned}
(5.20)\qquad H^0_\alpha(u_1)&=H^0_\alpha(H*Bu_0)\leqq H^0_\alpha(BH*u_0)+\text{const.}\,H^0_\alpha(u_0)\\
&\leqq \text{const.}\,\|u_0\|_{0+\alpha}+\text{const.}\,H^0_\alpha(u_0)\leqq \text{const.}\,\|u_0\|_{0+\alpha}.
\end{aligned}
$$

Moreover, (5.18) and (4.21) imply

$$
\begin{aligned}
(5.21)\qquad |u_1|_0&=|H*Bu_0|_0\leqq |BH*u_0|_0+\text{const.}\,|u_0|_0\\
&\leqq \text{const.}\,\|u_0\|_{0+\alpha}+\text{const.}\,\|u_0\|_{0+\alpha}.
\end{aligned}
$$

Thus, (5.20) and (5.21) imply that

$$
(5.22)\qquad \|u_1\|_{0+\alpha}\leqq \text{const.}\,\|u_0\|_{0+\alpha}.
$$

Clearly, we can choose the constants in (5.20)–(5.22) to be the same. Calling this constant C, and using (5.22) and (5.20), we find by induction on n that

$$
H^0_\alpha(u_n)\leqq C^n\,\|u_0\|_{0+\alpha}.
$$

It follows from this and (5.14) that

$$
(5.23)\qquad H^0_\alpha(u(x,t))\leqq H^0_\alpha(u_0)+\sum_{n=1}^{\infty}\frac{|t|^n}{n!}H^0_\alpha(u_n)\leqq \|u_0\|_{0+\alpha}\,e^{C|t|}.
$$

This proves that the solution $u(x,t)$, given by (5.12), is Hölder continuous with exponent α.

Remark 5.4. It follows from (5.21) and (5.23) that

$$
\|u(\cdot,t)\|_{0+\alpha}\leqq \|u_0\|_{0+\alpha}\,e^{C|t|},\qquad u_0\in C^{0+\alpha}(R^m).
$$

This is a good estimate for $t<0$.

Remark 5.5. Since for real-valued functions $u_0(x)$, the function

$$
\int_{R^m} H(x-y)\,Bu_0(y)\,dy
$$

is also real-valued, Theorems 5.1 and 5.2 insure that the solutions obtained in §2 and §3 are all real-valued.

Remark 5.6. The arguments used in the proof of Theorem 5.3 show also that if $u(x,t)$ is a solution in $W^{2,p}(R^m)$ for all time t, then u belongs to $C^{k+\alpha}(R^m)$, $k\geqq 0$, for all time t if its restriction to any fixed t belongs to $C^{k+\alpha}(R^m)$.

Remark 5.7. By Lemma 4.1, the right-hand side of (5.18) is equal to $D_{ij}(H*u_0)$. Consequently, (5.19) can be written $(A-I)^{-1}B\cdot u_0=B(A-I)^{-1}\cdot u_0$ for u_0 in $W^{2,p}(R^m)$, or in $C^{2+\alpha}(R^m)$, or in $C^{0+\alpha}(R^m)\cap W^{2,p}(R^m)$. In other words, the operators $(A-I)^{-1}$ and B commute in these spaces (of course, this will not be the case for differential operators with variable coefficients).

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