A Non-Linear Eigenvalue Problem: The Shape at Equilibrium of a Confined Plasma

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Communicated by S. ANTMAN

Abstract

A free boundary value problem arising in plasma physics is reduced to a non-linear eigenvalue problem of a non-classical type. We establish the existence of solutions of the non-linear eigenvalue problem; these solutions are critical points of appropriate functionals.

Introduction

This paper studies the equations governing the equilibrium of a plasma in a cavity. The equilibrium is described by a free boundary value problem related to the Maxwell equations; the unknowns are the shape at equilibrium of the plasma and the values of some characteristic function (the flux function) inside the plasma and in the surrounding vacuum. Once the shape of the plasma is known (*i.e.*, once the free boundary is determined), the flux function is defined by the solution of a linear elliptic problem in the vacuum and by the solution of a linear elliptic eigenvalue problem in the plasma (*cf.* Sections 1 and 2 for the precise equations).

Several free boundary value problems have been solved by the methods of variational inequalities (*cf.*, for instance, [2], [3], [8], [10], [13], [22]), but the present problem seems to rely on different methods. Our goals in this paper are to show that the problem is equivalent to finding critical points of some functional k_1 with respect to some other functional k_2 and to show the existence of these critical points.

In Sections 1 and 2 we treat two versions of the problem. In Section 1 we consider a simplified (but classical) form of the problem involving linear equations. The critical points are obtained as the points where the minimum of k_1 on a set $k_2=$constant is attained. Section 2 deals with a more general form of the problem. In applying the same procedure to this problem we encounter the serious difficulty that the functional k_1 is not bounded from below on the whole function space considered. We overcome this difficulty by showing that the functional k_1 is bounded from below on any set $k_2(v)=$const. To do this we employ the functional inequality established in Section 3. This inequality gives a bound for the L^2 (or L^q) norm of the positive part $u_+=\max(u,0)$ of a function u

in terms of the L^2 norm of its gradient grad u and the L^2 (or L^q) norm of its negative part $u_- = \max(-u, 0)$. Other forms of these inequalities may perhaps be proved but we restrict ourselves to the inequalities necessary for Section 2. Finally, in the Appendix we recall the derivation of the Tokomak equilibrium equations, which are based on the equations of magnetohydrodynamics and on the axisymmetric configuration of the machine. (This presentation follows C. MERCIER [19]; see also [4].)

There are several remaining open problems. These include the uniqueness of solutions, the regularity of the free boundary (to be studied in [14]), and the numerical approximation of the problem (to be studied in [7]).

The problem treated here arose in recent work of the fusion of plasma and on the Tokomak machine done at the C.E.A. and the C.I.S.I. (in France). The author is grateful to C. MERCIER and his group for bringing this problem to his attention, and to S. ANTMAN for his remarks.

1. A Model with Linear Equations

1.1. The Problem

Let Ω be an open set in $\mathbb{R}^2$ with a boundary Γ of class $\mathscr{C}^4$ such that

$$0 < x_* \leqq x_1 \leqq x_{**} < \infty, \qquad \forall\, x = (x_1, x_2) \in \Omega. \tag{1.1}$$

We define the self-adjoint operator $\mathscr{L}$ by

$$\mathscr{L}u = \sum_{i=1}^{2} \frac{\partial}{\partial x_i}\left(\frac{1}{x_1}\frac{\partial u}{\partial x_i}\right); \tag{1.2}$$

$\mathscr{L}$ is regular and uniformly elliptic in $\bar{\Omega}$.[1]

The problem of the equilibrium of a plasma in a cavity is governed by the following equations (see C. MERCIER [19] and the Appendix).

Let $I > 0$ be given. We seek an open set Ω_p (occupied by the plasma) with $\bar{\Omega}_p \subset \Omega$ and we seek a function $u: \Omega \to R$ such that

$$\mathscr{L}u = -\lambda b u \quad \text{in } \Omega_p, \tag{1.3}$$

$$\mathscr{L}u = 0 \quad \text{in } \Omega_v = \Omega - \bar{\Omega}_p \text{ (the vacuum)}, \tag{1.4}$$

$$u = 0 \quad \text{on } \Gamma_p = \partial\Omega_p, \tag{1.5}$$

$$\frac{\partial u}{\partial \nu} \text{ is continuous on } \Gamma_p,^{2} \tag{1.6}$$

$$u = \text{constant} = \gamma \quad \text{on } \Gamma \ (\gamma \text{ unknown}), \tag{1.7}$$

$$\int_{\Gamma_p} \frac{1}{x_1}\frac{\partial u}{\partial \nu}\, d\Gamma = I, \tag{1.8}$$

$$u \text{ does not vanish in } \Omega_p; \tag{1.9}$$

[1] We may also consider an abstract problem with a more general second order self-adjoint elliptic operator.

[2] ν is the unit normal on Γp (or Γ) pointing outward from Ω_p (or Ω).

b is a given, continuously differentiable function on $\bar{\Omega}$ that satisfies

(1.10) $$0 < b_0 \leqq b(x) \leqq b_1, \quad \forall x \in \bar{\Omega}.$$

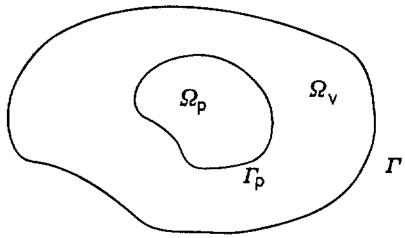

Fig. 1

The unknowns are u, λ, Ω_p; λ plays the role of an eigenvalue. Although the equations (1.3) and (1.4) are linear, the whole boundary value problem (1.3)–(1.9) is *non-linear*.

Remark 1.1. For Ω_p fixed the restriction of u to Ω_p is an eigenfunction for the operator $\mathscr{L}$ with a homogeneous Dirichlet boundary condition. Relation (1.9) implies that u is of constant sign on Ω_p and thus λ is the first eigenvalue of $\mathscr{L}$ in Ω_p:

(1.11) $$\lambda = \operatorname*{Inf}_{v \in H_0^1(\Omega_p)} \frac{a_{\Omega_p}(v, v)}{(bv, v)} = \operatorname*{Inf}_{\substack{v \in H_0^1(\Omega_p) \\ v \geqq 0}} \frac{a_{\Omega_p}(v, v)}{(bv, v)},$$

where, for $\mathcal{O} \subset \Omega$,

$$a_{\mathcal{O}}(u, v) = \int_{\mathcal{O}} \frac{1}{x_1} \operatorname{grad} u \cdot \operatorname{grad} v \, dx,$$

and $a(u, v) = a_\Omega(u, v)$.

Remark 1.2. The problem is positively homogeneous with respect to I: If we replace I by ρI with $\rho > 0$, then u is changed into ρu and Ω_p remains unchanged. □

Assume that u is a smooth solution of the problem (which means that Γ_p is a sufficiently smooth curve and u is smooth in $\bar{\Omega}$). By application of the maximum principle we first deduce some information about the sign of u.

Lemma 1.1.

(1.12) $$\gamma > 0 \quad \textit{and} \quad u > 0 \quad \textit{in } \Omega_v,$$

(1.13) $$u < 0 \quad \textit{in } \Omega_p,$$

and thus

(1.14) $$\begin{aligned} \Omega_p(u) &= \Omega_-(u) = \{x \in \Omega, u(x) < 0\}, \\ \Omega_v(u) &= \Omega_+(u) = \{x \in \Omega, u(x) > 0\}, \\ \Gamma_p(u) &= \Gamma_0(u) = \{x \in \Omega, u(x) = 0\}. \end{aligned}$$

Proof. If $\gamma = 0$, then $u = 0$ in Ω_v in contradiction to (1.9). If $\gamma < 0$, then $u < 0$ in Ω_v, and by the strong maximum principle [9], $\partial u / \partial \nu < 0$ on Γ_p; this contradicts (1.8) ($I > 0$). Thus $\gamma > 0$ and $\partial u / \partial \nu > 0$ on Γ_p. Since the sign of u is constant in Ω_p, this sign must be the negative one.

1.2. Equivalence with a Critical Point Problem

It is convenient to begin by reducing Problem (1.3)–(1.9) to the following problem.

Problem 1.1. *Find* u *and* Ω_p *such that* (1.3)–(1.6) *and* (1.9) *are satisfied and*

$$u=1 \quad \text{on } \Gamma. \tag{1.15}$$

If u' is a solution of (1.3)–(1.9), then $u=u'/\gamma$ is a solution of Problem 1.1 (with the same Ω_p). Conversely, if u is a solution of Problem 1.1, then $u'=\gamma u$ is a solution of (1.3)–(1.9) with

$$\gamma=\frac{I}{\int_{\Gamma_p} \frac{1}{x_1}\frac{\partial u}{\partial v}\,d\Gamma}. \tag{1.16}$$

It is easier to solve Problem 1.1 than to solve (1.3)–(1.9) (see Remark 1.8). Lemma 1.1 holds for a solution u of Problem 1.1.

In order to obtain a variational (or weak) formulation of Problem 1.1 we assume that u is a smooth solution of this problem; assuming also that Γ_p is a sufficently smooth curve, we multiply (1.3)–(1.4) by $v\in\mathscr{C}_0(\Omega)$ and integrate the resulting expression over Ω_p and Ω_v. ($\mathscr{C}_0^\infty(\Omega)$ is the collection of $\mathscr{C}^\infty$ functions with compact support in Ω.) We find

$$-\int_{\Gamma_p} \frac{1}{x_1}\frac{\partial u}{\partial v}\,v\,d\Gamma+a_{\Omega_-}(u,v)=-\lambda(bu_-,v),$$

$$\int_{\Gamma_p} \frac{1}{x_1}\frac{\partial u}{\partial v}\,v\,d\Gamma+a_{\Omega_+}(u,v)=0.$$

We add these relations and use (1.6) to get

$$a(u,v)=-\lambda(bu_-,v), \tag{1.17}$$

which holds for each $v\in\mathscr{C}_0^\infty(\Omega)$ and, by continuity, for each $v\in H_0^1(\Omega)$.[3]

Conversely, let us assume that u is sufficently smooth, that

$$\Gamma_0=\Gamma_0(u)=\{x, u(x)=0\}$$

is a smooth curve, and that $u-1\in H_0^1(\Omega)$ satisfies (1.17) for each v in $H_0^1(\Omega)$. Then, again, it is a standard exercise in variational formulations to check that u is solution of Problem 1.1 (see also the proof of Theorem 1.1).

Setting $u=u_0+1$, $u_0\in H_0^1(\Omega)$, we consider the following variational problem.

Problem 1.2. *Find* u_0 *in* $H_0^1(\Omega)$ *and* $\lambda\in R$ *such that*

$$a(u_0,v)=-\lambda\big(b(u_0+1)_-,v\big), \qquad \forall\ v\in H_0^1(\Omega). \tag{1.18}$$

In the sequel we shall solve Problem 1.1 in its weak form, Problem 1.2: The fact that $u=u_0+1$with a u_0 solution of (1.18) is a classical (or strong) solution of

[3] $H^m(\Omega)$, m an integer, is the Sobolev space

$$\{u\in L^2(\Omega),\ D^\alpha u\in L^2(\Omega),\ \forall\,\alpha=(\alpha_1,\dots,\alpha_n),\ |\alpha|\leq m\}.$$

$H_0^m(\Omega)$ is the closure in $H^m(\Omega)$ of the space $\mathscr{C}_0^\infty(\Omega)$. For all the properties of Sobolev spaces (density, compactness, imbedding theorems, ...) see LIONS & MAGENES [18], NEČAS [21].

Problem 1.1 is then a problem of regularity of u_0 and of $\Gamma_0(u)=\{x\in\Omega, u_0(x)+1=0\}$ (see below).

The next step is to transform Problem 1.2 into a critical point problem. For this purpose we introduce the functionals

$$k_1(v)=\tfrac{1}{2}a(v,v), \tag{1.19}$$

$$k_2(v)=\tfrac{1}{2}(b(v+1)_-,(v+1)_-)=\int_\Omega \frac{b}{2}[(v+1)_-]^2\,dx. \tag{1.20}$$

The functional k_1 is differentiable on $H_0^1(\Omega)$ with differential[4]

$$\langle k_1'(v),w\rangle=(v,w), \qquad \forall\, v,w\in H_0^1(\Omega), \tag{1.21}$$

and we infer from Lemma 1.2 below that k_2 is differentiable on $H_0^1(\Omega)$ with differential

$$\langle k_2'(v),w\rangle=-(b(v+1)_-,w)=-\int_\Omega b[(v+1)_-]\,w\,dx, \qquad \forall\, w\in H_0^1(\Omega). \tag{1.22}$$

The relation (1.18) is equivalent to

$$\langle k_1'(u_0),v\rangle=\lambda\langle k_2'(u_0),v\rangle \qquad \forall\, v\in H_0^1(\Omega) \tag{1.23}$$

or

$$k_1'(u_0)=\lambda k_2'(u_0), \tag{1.24}$$

and Problem 1.2 is now equivalent (*cf.* [12]) to

Problem 1.3. *Find the critical points of k_1 on the subsets $k_2(v)=const.$ of $H_0^1(\Omega)$; λ is the corresponding critical value.*

In the next subsection we establish the existence of critical points. We conclude this paragraph with

Lemma 1.2. *Assume that $g\in L^\infty(\Omega)$, $\Phi\in L^2(\Omega)$. Then the functional*

$$k(u)=\tfrac{1}{2}\int_\Omega [(u+\Phi)_-]^2\,g\,dx$$

is Gâteaux differentiable on $L^2(\Omega)$ with differential

$$\langle k'(u),v\rangle=-\int_\Omega g[(u+\Phi)_-]\,v\,dx.$$

Proof. We can restrict ourselves to the case $\Phi=0$. The function $s\to(s_-)^2$ is differentiable with differential $-2s_-$. Hence, for almost every $x\in\Omega$,

$$e_\lambda(x)=\frac{[(u(x)+\lambda v(x))_-]^2-[u(x)_-]^2}{2\lambda}\to -u_-(x)\,v(x)$$

as $\lambda\searrow 0$.

[4] Gâteaux differentiability (*cf.* [11], [17]) is sufficient for our purposes; $\langle,\rangle$ denotes the pairing between the space ($H_0^1(\Omega)$ here) and its dual.

On the other hand, since s_- is Lipschitzian, the modulus of $e_\lambda(x)$ is bounded by $|u(x)|\,|v(x)|$ (for $\lambda \leqq 1$). The dominated convergence theorem is thus applicable and we obtain the result.

1.3. Existence of Critical Points

We recall the following well known property.

Lemma 1.3. *Let h_1 and h_2 be two differentiable real-valued functionals on a Banach space X. Assume that u minimizes (or maximizes) h_1 on the set $h_2(u)=c$ and that $h_2'(u) \neq 0$. Then there exists $\lambda \in R$ such that*

$$h_1'(u) = \lambda h_2'(u)$$

(u is a critical point).

Lemma 1.4. (i) *For every $c>0$, the minimum of $k_1(v)$ on the set $k_2(v)=c$ is attained at a point u_0 which is a critical point of k_1 on the set $\{k_1(v)=c\}$.*

(ii) *For every $c>0$, the maximum of $k_2(v)$ on the set $k_1(v)=c$ is attained at a point u_0 which is a critical point of k_2 on the set $\{k_1(v)=c\}$.*

Proof. (i) Let

$$\alpha = \operatorname{Inf}\{k_1(v), k_2(v)=c\}. \tag{1.25}$$

Obviously $\alpha \geqq 0$ is finite. Let u_{0m} be a minimizing sequence for (1.25). The sequence is bounded in $H_0^1(\Omega)$ (due to the Poincaré inequality) and is thus relatively compact in $L^2(\Omega)$. Therefore there exists a subsequence (still denoted u_{0m}) such that

$$u_{0m} \to u_0 \ \text{ weakly in } H_0^1(\Omega) \ \text{ and strongly in } \ L^2(\Omega). \tag{1.26}$$

Then $(u_{0m}+1)_- \to (u_0+1)_-$ strongly in $L^2(\Omega)$ and $k_1(u_0) = \lim k_2(u_{0m}) = c$. On the other hand, by the weak lower semi-continuity of k_1 (see, for instance, [17], [20])

$$k_1(u_0) \leq \varliminf_{m\to\infty} k_1(u_{0m}) = \alpha$$

so that $\alpha = k_1(u_0)$.

We must still show that $k_2'(u_0) \neq 0$ (*cf.* Lemma 1.3). We infer from (1.22) that $k_2'(u_0)=0$ implies that $(u_0+1)_-=0$, and this is impossible since $k_2(u_0)=c\neq 0$.

(ii) The proof is similar.

1.4. The Main Result

Theorem 1.1. *For every fixed $c>0$, Problem 1.1 possesses at least one solution $u=u_0+1$, with*

$$\int_\Omega b(x)[u_-(x)]^2\,dx = c. \tag{1.27}$$

The function u belongs to $W^{3,\alpha}(\Omega)$ for all $\alpha \geqq 1$ and to $\mathscr{C}^{2,\eta}(\bar\Omega)$ for all η satisfying $0 \leqq \eta < 1$.[5]

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[5] $W^{m,r}(\Omega)$, m integer, $1 \leqq r \leqq \infty$, is the Sobolev space

$$\{u \in L^r(\Omega),\ D^\alpha u \in L^r(\Omega),\ \forall \alpha = (\alpha_1, \dots, \alpha_n, |\alpha| \leqq m\}.$$

$\mathscr{C}^{m,\eta}(\bar\Omega)$, m integer, $0 \leqq \eta \leqq 1$ is the space of m times continuously differentiable functions in $\bar\Omega$, such that the m^{th} derivatives are Hölder continuous with exponent η.

Let the sets $\Omega_p, \Omega_v, \Gamma_p$ be defined by (1.14). *Then u satisfies* (1.3), (1.4), (1.5) *and u is analytic in Ω_p and in Ω_v. In the neighborhood of each point $x \in \Gamma_p$ such that* grad $u(x) \neq 0$, *Γ_p is a $\mathscr{C}^2$ curve, and* (1.6) *is satisfied.*

Finally if Γ_p is a $\mathscr{C}^1$ curve and if

(1.28) $$\gamma = \frac{I}{\int_{\Gamma_p} \frac{1}{x_1} \frac{\partial u_0}{\partial v} d\Gamma},$$

then $u' = \gamma u = \gamma(u_0 + 1)$ is a solution of (1.3)–(1.9).

Proof. We choose a u_0 that minimizes k_1 on $\{k_2(v) = c\}$. It is a critical point of k_1 on this set and is thus a solution of Problems 1.2 and 1.3; $u = u_0 + 1$ is a solution of Problem 1.1.

Writing (1.18) with $v \in \mathscr{C}_0^\infty(\Omega)$, we see that $u = u_0 + 1$ is such that

(1.29) $$\mathscr{L} u = \lambda b u_-$$

in the distributional sense in Ω. Since $u \in H^1(\Omega)$, it follows from Stampacchia's results [23] that u_- also belongs to $H^1(\Omega)$. The standard regularity results for elliptic problems ($u = 1$ on Γ) then imply that u also belongs to $H^3(\Omega)$. The Sobolev imbedding theorems imply that $H^1(\Omega) \subset L^\alpha(\Omega)$ for all α satisfying $1 \leqq \alpha < \infty$ and that $H^3(\Omega) \subset W^{1,\alpha}(\Omega)$ for all α satisfying $1 \leqq \alpha < \infty$. Again using Stampacchia's results [23] and the results on the regularity of solutions of elliptic problems [1], we find that $u_- \in W^{1,\alpha}(\Omega)$, $\lambda b u_- \in W^{1,\alpha}(\Omega)$, and $u \in W^{3,\alpha}(\Omega)$ for all α satisfying $1 \leqq \alpha < \infty$. Because of the Sobolev imbedding theorems, $u \in \mathscr{C}^{2,\eta}(\bar{\Omega})$ for all η satisfying $0 \leqq \eta < 1$. In general u_- belongs neither to $\mathscr{C}^1(\bar{\Omega})$ nor to $W^{2,\alpha}(\Omega)$ for any $\alpha \geqq 1$; we cannot expect to improve these results on the global regularity of u in Ω.

Since u is continuous in $\bar{\Omega}$, the sets $\Omega_p, \Omega_v, \Gamma_p$ as defined in (1.14) make sense; Ω_p and Ω_v are open and Γ_p is closed. Setting $v = \Phi \in \mathscr{C}_0^\infty(\Omega_p)$ (or $\mathscr{C}_0^\infty(\Omega_v)$) in (1.18), we see that $u = u_0 + 1$ satisfies (1.3) (or (1.4)). Thus u is an analytic function in Ω_p and in Ω_v.

We observe that Ω_p is not empty because $k_2(u) = c > 0$ and that Ω_v is not empty because $u = 1$ on Γ. We note that Γ_p has an empty interior since otherwise, by writing (1.18) with $v \in \mathscr{C}_0^\infty(\Omega - \bar{\Omega}_p)$, we should find

$$\mathscr{L} u = 0 \quad \text{in } \Omega - \bar{\Omega}_p.$$

Then u would be analytic in $\Omega - \bar{\Omega}_p$ and equal to zero on an open subset, the interior of Γ_p; then u would vanish in Ω_v, and this is impossible since $u = 1$ on Γ.

Finally, at each point $x \in \Gamma_p$ such that grad $u(x) \neq 0$, the implicit function theorem implies that Γ_p is a smooth curve in some neighborhood of x, say $\mathcal{O}$. In this case (1.8) is satisfied in $\mathcal{O}$. Indeed, multiplying (1.19) by $v \in \mathscr{C}_0^\infty(\mathcal{O})$ and using Green's formula in $\mathcal{O} \cap \Omega_p$ and in $\mathcal{O} \cap \Omega_v$, we find

$$-(\mathscr{L} u, v) = a(u, v) + \int_{\Gamma_p} \left[\frac{1}{x_1} \frac{\partial u}{\partial v}\right] v \, d\Gamma = \lambda (b u_-, v),$$

where

(1.30) $$\left[\frac{1}{x_1} \frac{\partial u}{\partial v}\right] = \text{the jump of } \frac{1}{x_1} \frac{\partial u}{\partial v} \text{ on } \Gamma_p.$$

A comparison of this expression with (1.18) shows that

$$\int_{\Gamma_p} \left[\frac{1}{x_1}\frac{\partial u}{\partial v}\right] v \, d\Gamma = 0.$$

Since v is an arbitrary function in $\mathscr{C}_0^\infty(\mathcal{O})$, the jump (1.30) vanishes.

1.5. Miscellaneous Remarks

Remark 1.3. It is clear that λ is the first eigenvalue of $\mathscr{L}$ in Ω_p for the Dirichlet boundary condition and that the restriction of u to Ω_p is the corresponding eigenfunction (u has a constant sign in Ω_p). Hence

$$(1.31)\qquad \lambda = \frac{a(u_-, u_-)}{(bu_-, u_-)} = \frac{a_{\Omega_p}(u, u)}{(bu_-, u_-)} \leqq \frac{a_{\Omega_p}(v, v)}{(bv, v)}, \qquad \forall\, v \in H_0^1(\Omega_p),\ v \geqq 0.$$

This can also be deduced from (1.18). Since $u=1$ on Γ, u_- belongs to $H_0^1(\Omega)$ and we can set $v_1 = u_-$ in (1.18). We find the first equality in (1.31). For $v \in H_0^1(\Omega_p)$, let $\tilde v(x) = v(x)$ for $x \in \Omega_p$ and let $\tilde v(x) = 0$ for $x \in \Omega - \Omega_p$. Then $\tilde v \in H_0^1(\Omega)$ and $w = u_+ + \rho \tilde v - 1 \in H_0^1(\Omega)$. We have $k_2(w) = c$ provided

$$\rho^2 = \frac{2c}{(b\tilde v, \tilde v)}.$$

Thus

$$k_1(u_0) = k_1(u-1) \leqq k_1(u_+ + \rho \tilde v - 1),$$

which yields the inequality contained in (1.31).

By homogeneity (1.31) also holds for γu.

Remark 1.4. Due to (1.29) and Remark 1.4, $u = u_0 + 1$ is solution of the non-linear Dirichlet problem

$$(1.32)\qquad \begin{aligned} &\mathscr{L} u = b \lambda(u) u_- \quad \text{in } \Omega,\\ &u = 1 \quad \text{on } \Gamma,\\ &\lambda(u) = \frac{a(u_-, u_-)}{(bu, u_-)}. \end{aligned}$$

Conversely one can show that any solution of (1.32) defines a solution $u_0 = u - 1$ of Problems 1.2, 1.3. Hence for each fixed $c > 0$, the Dirichlet-type Problem (1.32) possesses at least one solution u such that $(bu_-, u_-) = c$.

We see that $u' = \gamma u$ (see (1.28)) is solution of

$$(1.33)\qquad \begin{aligned} &\mathscr{L} u' = b \lambda(u') u'_- \quad \text{in } \Omega,\\ &u' = \gamma \quad \text{(unknown constant) on } \Gamma,\\ &\int_\Gamma \frac{1}{x_1}\frac{\partial u'}{\partial v}\, d\Gamma = I. \end{aligned}$$

Remark 1.5. The set Ω_v is *connected*.

It is clear that one of the connected components of Ω_v contains Γ in its boundary ($u=1$ on Γ). If $\mathcal{O}$ is any other connected component of Ω_v, then $\mathcal{O}$ does not intersect Γ, and we should have $\bar{\mathcal{O}}\subset\Omega$, $\mathscr{L}u=0$ in $\mathcal{O}$, and $u=0$ on $\partial\mathcal{O}$ which implies $u=0$ in $\mathcal{O}$, leading to a contradiction $(\mathcal{O}\not\subset\Omega_v)$.

Remark 1.6. We can pose (1.3)–(1.9) directly as a critical value problem without using the intermediate unknown $u\gamma$. This approach will be used in the next section for a more general problem. Here we confront the difficulty that the functionnal h_1 is not bounded in the whole space.

Remark 1.7. For the regularity of Γ_p, *cf.* [14].

2. The General Case

In this section we study a more elaborate model describing the equilibrium of a plasma in a cavity.

2.1. The Problem

Let there be given a twice continuously differentiable real-valued function g on $\bar{\Omega}\times R^6$ that satisfies the following conditions:

(2.1) There exist numbers $\beta>1, b_1, b_2>0$ such that

$$b_1(|u|^\beta-1)\leqq\frac{\partial g}{\partial u}(x,u)\leqq b_2(|u|^\beta+1),\qquad \forall x, \forall u,$$

(2.2)
$$\frac{\partial g}{\partial u}(x,0)=0,^7$$

(2.3)
$$\frac{\partial g}{\partial u}(x,u)>0 \quad \text{for } u<0.^7$$

We retain the assumptions on Ω and $\mathscr{L}$ that are stated in Section 1. We consider the following modification of the problem (1.3)–(1.9), obtained by replacing (1.3) with a non-linear equation.

Problem 2.1. *Find an open set* Ω_p, $\bar{\Omega}_p\subset\Omega$, *and a function* u: $\Omega\to\mathbb{R}$ *such that*

(2.4)
$$\mathscr{L}u=+\lambda\frac{\partial g}{\partial u}(x,u)\quad \textit{in } \Omega_p,$$

(2.5)
$$\mathscr{L}u=0\quad \textit{in } \Omega_V=\Omega-\bar{\Omega}_p,$$

(2.6)
$$u=0\quad \textit{on } \Gamma_p=\partial\Omega_p,$$

(2.7)
$$\frac{\partial u}{\partial v}\quad \textit{is continuous on } \Gamma_p,$$

(2.8)
$$u=\textit{constant}=\gamma \textit{ on } \Gamma\ (\gamma \textit{ unknown}).$$

(2.9)
$$\int_{\Gamma_p}\frac{1}{x_1}\frac{\partial u}{\partial v}d\Gamma=I,$$

(2.10)
$$u \textit{ does not vanish in } \Omega_p.$$

[6] Lemma 2.1 will show that it suffices to have g defined only on $\bar{\Omega}\times]-\infty,0]$.

[7] *Cf.* Remark 2.1.

As before, the unknowns are u, λ, Ω_p; λ plays the role of an eigenvalue. For Ω_p fixed, λ is a critical value of the functional $a_{\Omega_p}(u,u)$ on the subset of $H_0^1(\Omega_p)$ defined by

$$\int_{\Omega_p} g(x, u(x))\, dx = \text{constant}.$$

The formulation of the whole Problem 2.1 as a critical point problem will be obtained below. We observe that this problem is not homogeneous with respect to I so that its reduction to a problem like 1.1 is not possible.

A priori information about the sign of u can be obtained exactly as in Lemma 1.1.

Lemma 2.1. *If u is a sufficiently regular solution of Problem 2.1, then $\gamma>0$, $u>0$ in Ω_v, and $u<0$ in Ω_p. Hence*

$$\begin{aligned} \Omega_p(u) &= \Omega_-(u) = \{x\in\Omega, u(x)<0\},\\ \Omega_v(u) &= \Omega_+(u) = \{x\in\Omega, u(x)>0\},\\ \Gamma_p(u) &= \Gamma_0(u) = \{x\in\Omega, u(x)=0\}. \end{aligned} \tag{2.11}$$

2.2. Equivalence with a Critical Point Problem

Let v denote a function in $\mathscr{C}^\infty(\bar\Omega)$ that is constant on Γ (the value on Γ is denoted $v(\Gamma)$). Assuming that u is a sufficiently regular solution of Problem 2.1 and that Γ_p is sufficiently smooth, we can multiply (2.4) and (2.5) by v, integrate the resulting expression over Ω_p and Ω_v, and apply the Green formula. We obtain

$$-\int_{\Gamma_p} \frac{1}{x_1}\frac{\partial u}{\partial \nu} v\, d\Gamma + a_{\Omega_p}(u,v) = -\lambda \int_{\Omega_p} \frac{\partial g}{\partial u}(x,u) v\, dx,$$

$$-\int_{\Gamma_p} \frac{1}{x_1}\frac{\partial u}{\partial \nu} v\, d\Gamma + \int_{\Gamma_p} \frac{1}{x_1}\frac{\partial u}{\partial \nu} v\, d\Gamma + a_{\Omega_v}(u,v) = 0.$$

Adding these relations and using (2.2), (2.9), and (2.11), we find

$$a(u,v) = -\lambda \int_{\Omega} \frac{\partial g}{\partial u}(x, -u_-(x))\, v(x)\, dx + I\, v(\Gamma). \tag{2.12}$$

By continuity this relation holds for each v in

$$W = \{v\in H^1(\Omega),\ v=\text{constant on } \Gamma\}.^8 \tag{2.13}$$

Conversely, if $u\in W$ satisfies (2.12) for each $v\in W$ and if u is sufficiently smooth, then we can prove that u satisfies (2.4)–(2.9) (see the proof of Theorem 2.1). We therefore arrive at a weaker form of Problem 2.1:

Problem 2.2. *Find u in W and $\lambda\in R$ such that*

$$a(u,v) = \lambda \int_{\Omega} \frac{\partial g}{\partial u}(x, -u_-)\, v\, dx + I\, v(\Gamma), \qquad \forall\, v\in W. \tag{2.14}$$

[8] It is clear that W is a closed subspace of $H^1(\Omega)$ and that $\mathscr{C}^\infty(\Omega)\cap W$ is dense in W (since $\mathscr{C}_0^\infty(\bar\Omega)$ is dense in $H_0^1(\Omega)$).

We now introduce the following functionals:

$$k_1(v)=\tfrac{1}{2}a(v,v)-I\,v(\Gamma). \tag{2.15}$$

$$k_2(v)=\int_\Omega [g(x,-v_-(x))-g(x,0)]\,dx. \tag{2.16}$$

The functional k_1 is defined, continuous, and differentiable on W with differential

$$\langle k_1'(v),w\rangle=a(v,w)-I\,w(\Gamma). \tag{2.17}$$

We prove in Lemma 2.2 below that k_2 is also defined and differentiable on W, with a differential

$$\langle k_2'(v),w\rangle=\int_\Omega \frac{\partial g}{\partial u}(x,-v_-(x))\,w(x)\,dx. \tag{2.18}$$

Admitting for the moment the conclusions of Lemma 2.2, we observe that the relation (2.14) is equivalent to

$$k_1'(u)=\lambda k_2'(u), \tag{2.19}$$

and Problem 2.2 is equivalent to

Problem 2.3. *Find the critical points of k_1 on the subsets $k_2(v)=$ constant of W; λ is the corresponding critical value.*

Lemma 2.2. *The function k_2 in* (2.16) *is defined, continuous, and differentiable on $L^{\beta+1}(\Omega)$ and thus on W. Its differential is given by* (2.18).

Proof. By integrating (2.1) we find that,

$$b_1\left(\frac{|u|^{\beta+1}}{\beta+1}-|u|\right)\leqq-[g(x,u)-g(x,0)]\leqq b_2\left(\frac{|u|^{\beta+1}}{\beta+1}+|u|\right)\quad \text{for } x\in\bar\Omega,\ u<0, \tag{2.20}$$

so that there exists $b_3>0$ such that [9]

$$|g(x,u)-g(x,0)|\leqq b_3(|u|^{\beta+1}+1). \tag{2.21}$$

Thus, for each $v\in L^{\beta+1}(\Omega)$, the function $x\mapsto g(x,-v_-(x))$ is integrable, and by a theorem of KRASNOSEL'SKIĬ [15], the functional

$$v\mapsto\int_\Omega g(x,-v_-(x))\,dx$$

is continuous on $L^{\beta+1}(\Omega)$, as is k_2. Using (2.1) and similar arguments, we see that

$$\{v,w\}\mapsto\int_\Omega \frac{\partial g}{\partial u}(x,-v_-(x))\,w(x)\,dx$$

is a continuous mapping from $L^{\beta+1}(\Omega)\times L^{\beta+1}(\Omega)$ into $\mathbb{R}$. That k_2 is Gâteaux-differentiable on $L^{\beta+1}(\Omega)$ with (2.18) as its differential then follows from the Lebesgue dominated convergence theorem, as in Lemma 1.2.

[9] By (2.3), $g(x,u)-g(x,0)<0$ for $x\in\bar\Omega$, $u<0$.

2.3. Existence of Critical Points

In marked contrast to the situation of Section 1, the functional k_1 is not bounded from below on the whole space W. However, due to a special inequality proved in Section 3, k_1 is bounded from below on the sets $k_2(v)=c$ $(c \neq 0)$ of W. This permits us to prove

Lemma 2.3. *For every $c>0$, k_1 is bounded from below on the set $\{v, k_2(v)=c\}$ of W and the minimum of k_1 is attained at at least one point u, which is a critical point of k_1 on this set.*

Proof. We first show that there exist two constants c', c'' depending on g, c, and Ω such that

$$0<c' \leqq |v_-|_{L^{\beta+1}(\Omega)} \leqq c'' < +\infty \tag{2.22}$$

for each v in W satisfying $k_2(v)=c$.

The existence of c'' follows easily from the left inequality in (2.20). For the existence of c', we proceed by contradiction, and we assume the existence of a sequence $v_m \in W$ such that $k_2(v_m)=c$ and $|(v_m)_-|_{L^{\beta+1}(\Omega)} \to 0$. Replacing v_m by $-(v_m)_-$, we may assume that v_m converges to 0 in $L^{\beta+1}(\Omega)$, and since k_2 is continuous on $L^{\beta+1}(\Omega)$, we should have $k_2(0)=c$, in contradiction to the assumption $c \neq 0$.

The next step consists in the proof that

(2.23) $k_1(v)$ is bounded from below on the set $k_2(v)=c$ in $W (c>0)$.

We infer from (2.22) and the inequality (3.29) of Section 3 that

$$|v_+| \leqq \alpha |\text{grad } v|^2 + c'' \delta(\alpha, \beta+1, c'),$$

for each $\alpha>0$ and for each v in the set $k_2(v)=c$. ($|.|$ is the L^2 norm and $\delta = \delta(\alpha, \beta+1, c')>0$ is a constant depending on α, β, c'). Thus

$$|v| \leqq |v_+| + |v_-| \leqq \alpha |\text{grad } v|^2 + c''(1+\delta). \tag{2.24}$$

Now, by the trace theorem, there exists a constant c_1 depending only on Ω such that

$$|v(\Gamma)| \leqq c_1 \{|\text{grad } v|^2 + |v|^2\}^{\frac{1}{2}}, \qquad \forall v \in W.$$

Relation (1.1) then implies

$$k_1(v) \geqq \frac{1}{2x_{**}} |\text{grad } v|^2 - I c_1 |\text{grad } v| - I c_1 |v|.$$

From (2.24) we obtain

$$k_1(v) \geqq \left(\frac{1}{4x_{**}} - I c_1 \alpha\right) |\text{grad } v|^2 - I^2 c_1^2 x_{**} - I c_1 c''(1+\delta). \tag{2.25}$$

Choosing $\alpha = 1/8 I c_1 x_{**}$, we obtain

$$k_1(v) \geqq \frac{1}{8x_{**}} |\text{grad } v|^2 - c''', \tag{2.26}$$

where

$$c''' = -I^2 c_1^2 x_{**} - I c_1 c''\big(1+\delta(\alpha, \beta+1, c')\big).$$

Thus k_1 is bounded from below on the set under consideration. Now set

$$\xi = \operatorname{Inf}\{k_1(v), v \in W, k_2(v) = c\} \tag{2.27}$$

and consider a minimizing sequence u_m of (2.27). From (2.26) and the preceding remarks, this sequence is bounded in $H^1(\Omega)$ and by the weak compactness of $H^1(\Omega)$ we can extract a subsequence (still denoted by u_m), such that u_m converges to some limit u, weakly in W and strongly in $L^q(\Omega)$ for $1 \leqq q < \infty$. Clearly $k_2(u_m) \to k_2(u) = c$. The weak lower semi-continuity of k_1 implies that $k_1(u) = \xi$. Thus u minimizes k_1 on $\{v, k_2(v) = c\}$.

To finish our proof we must show that $k_2'(u) \neq 0$. This follows from (2.3): If $k_2'(u) = 0$, then $u_- = 0$ and $k_2(u) = 0$ in contradiction to $c \neq 0$.

2.4. The Main Result

Theorem 2.1. *For every fixed $c > 0$, Problem 2.2 possesses at least one solution u with*

$$k_2(u) = c.$$

The function u belongs to $W^{3,\alpha}(\Omega)$ for all $\alpha \geqq 1$, and to $\mathscr{C}^{2,\eta}(\bar{\Omega})$ for all η satisfying $0 \leqq \eta < 1$.

Let the sets $\Omega_p, \Omega_v, \Gamma_p$ be defined by (2.11). *Then u satisfies* (2.4), (2.5), (2.6), (2.8) *and u is analytic in Ω_v. The set Γ_p has an empty interior in $\mathbb{R}^2$. In the neighborhood of each $x \in \Gamma_p$ such that* $\operatorname{grad} u(x) \neq 0$, *$\Gamma_p$ is a $\mathscr{C}^2$ curve and* (2.7) *is satisfied. Finally if Γ_p is a piecewise $\mathscr{C}^2$ curve, then* (2.9) *is also satisfied.*

Proof. The proof is essentially the same as that of Theorem 1.1. By restricting (2.14) to functions v in $\mathscr{C}_0^\infty(\Omega)$, we find that

$$\mathscr{L} u = +\lambda \frac{\partial g}{\partial u}(x, -u_-) \tag{2.28}$$

in the distributional sense in Ω.

Since $u \in H^1(\Omega)$, the function $-u_-$ belongs to $L^q(\Omega)$ for any $q \geqq 1$, and by (2.1) $\frac{\partial g}{\partial u}(x, -u_-)$ also belongs to $L^q(\Omega)$ for any $q \geqq 1$. Using the results of [1], we obtain $u \in W^{2,q}(\Omega)$ for any q. Hence $u \in \mathscr{C}^1(\bar{\Omega})$ and $-u_-$ is a Lipschitzian function belonging to $W^{1,q}(\Omega)$ for any $q \geqq 1$ (*cf.* [23]). It is then easy to show that $\frac{\partial g}{\partial u}(x, -u_-)$ is in $W^{1,q}(\Omega)$ for any $q \geqq 1$. It follows from [1] that u is in $W^{3,q}(\Omega)$. The other points are proved exactly as in Theorem 1.1.

Remark 2.1. (i) With our assumption on g, we cannot obtain the analyticity of u in Ω_p. As usual, if we require that g have more regularity, then we can prove that u has correspondingly more regularity. In particular, if g is analytic, then we can prove that u is also analytic.

3. A Non-Classical Inequality

In this section, we prove a non-classical inequality that relates the positive part of a function to its gradient and to its negative part (assumed to be different from 0).

Theorem 3.1. *We assume that* $\Omega \subset \mathbb{R}^n$, $n=2, 3$, *is a bounded open set of class* $\mathscr{C}^2$. *For each* $\alpha>0$ *there exists a function* δ_α: $]0, +\infty[\to \mathbb{R}$, *such that*

$$(3.1) \qquad |u_+|_{L^2(\Omega)} \leqq \alpha\,|\operatorname{grad} u|^2_{L^2(\Omega)^n} + \delta_\alpha(|u_-|_{L^2(\Omega)})$$

for every u *in* $H^1(\Omega)$ *such that* $|u_-|_{L^2(\Omega)} \neq 0$.

Remark 3.1. (i) An inequality like (3.1) is not possible for the functions u such that $u_-=0$.

(ii) For each $\alpha>0$, $a>0$ there exists a constant $\delta(\alpha, a)$ such that

$$(3.2) \qquad |u_+|_{L^2(\Omega)} \leqq \alpha\,|\operatorname{grad} u|^2_{L^2(\Omega)^n} + \delta(\alpha, a)\,|u_-|_{L^2(\Omega)}$$

for every u in $H^1(\Omega)$ satisfying

$$(3.3) \qquad 0<a\leqq |u_-|_{L^2(\Omega)}.$$

We obtain (3.2) by writing (3.1) with α replaced by αa and with u replaced by $v=\dfrac{u}{|u_-|_{L^2(\Omega)}}$. We find (3.2) with $\delta(\alpha, a)=\delta_{\alpha a}(1)$.

(iii) It follows from the inequality (3.1) that Dirichlet integrals like

$$(3.4) \qquad \int_\Omega |\operatorname{grad} u|^2\,dx - 2\int_\Omega f u\,dx$$

($f\in L^2(\Omega)$) are bounded from below on the subsets (3.3) of $H^1(\Omega)$ and on manifolds of the type $|u_-|_{L^2(\Omega)}=a>0$.

Proof.* (i) The proof of (3.1) amounts showing that the functional

$$(3.5) \qquad e_1(u)=\alpha\,|\operatorname{grad} u|^2 - |u_+|^{10}$$

is bounded from below on the set

$$(3.6) \qquad \mathscr{C}_a=\{u\in H^1(\Omega)\,\|u_-|=a\} \qquad (a>0).$$

Since this result is not obvious, we introduce a regularized functional

$$e_{1\varepsilon}(u)=\alpha\,|\operatorname{grad} u|^2+\varepsilon\,|u|^2-|u_+|, \qquad \varepsilon>0.$$

We shall show that $e_{1\varepsilon}$ is bounded from below on the whole space $H^1(\Omega)$ and attains its minimum on $\mathscr{E}_a$. After establishing the properties of the minimizer, we shall then allow ε to approach 0.

(ii) We observe that

$$(3.7) \qquad |u_+|\leqq|u|\leqq\frac{\varepsilon}{2}\,|u|^2+\frac{1}{2\varepsilon},$$

[10] We omit the subscript $L^2(\Omega)$, and as before $(\cdot,\cdot)$ and $|\cdot|$ denote the scalar product and the norm either in $L^2(\Omega)$ or in $L^2(\Omega)^n$.

* *Note Added in Proof.* See in [27] an alternate proof of this inequality, due to H. BREZIS.

so that the functional $e_{1\varepsilon}$ is indeed bounded from below on $H^1(\Omega)$ and thus on $\mathscr{E}_a$. Let $u_{\varepsilon m}$ denote a minimizing sequence for $e_{1\varepsilon}$ on $\mathscr{E}_a$. Because of (3.7), $u_{\varepsilon m}$ is bounded in $H^1(\Omega)$ so that there exists a subsequence (still denoted by $u_{\varepsilon m}$) that converges weakly in $H^1(\Omega)$ to some limit u_ε. The injection of $H^1(\Omega)$ into $L^2(\Omega)$ is compact and the mapping $\Phi \to \Phi_-$ is continuous on $L^2(\Omega)$. Hence $u_{\varepsilon m} \to u_\varepsilon$ strongly in $L^2(\Omega)$ and $(u_{\varepsilon m})_- \to (u_\varepsilon)_-$ strongly in $L^2(\Omega)$. It is then clear that $|(u_\varepsilon)_-| = a$. By the weak lower semi-continuity of $e_{1\varepsilon}$, we have

$$e_{1\varepsilon}(u_\varepsilon) \leqq \varliminf_{m\to\infty} e_{1\varepsilon}(u_{\varepsilon m}),$$

so that u_ε minimizes $e_{1\varepsilon}$ on $\mathscr{E}_a$.

(iii) We note that $su_{\varepsilon+} - u_{\varepsilon-}$ belongs to the set $\mathscr{E}_a$ for every $s>0$; hence $e_{1\varepsilon}(su_{\varepsilon+} - u_{\varepsilon-})$ attains its minimum at $s=1$ and

$$\frac{d}{ds}\, e_{1\varepsilon}(su_{\varepsilon+} - u_{\varepsilon-})|_{s=1} = 0. \tag{3.8}$$

By a result of G. STAMPACCHIA [23]

$$|\operatorname{grad}(su_{\varepsilon+} - u_{\varepsilon-})|^2 = s^2 |\operatorname{grad} u_{\varepsilon+}|^2 + |\operatorname{grad} u_{\varepsilon-}|^2.$$

It is clear that

$$|su_{\varepsilon+} - u_{\varepsilon-}|^2 = s^2 |u_{\varepsilon+}|^2 + |u_{\varepsilon-}|^2.$$

Thus (3.8) becomes

$$|u_{\varepsilon+}| = 2\alpha |\operatorname{grad} u_{\varepsilon+}|^2 + 2\varepsilon |u_{\varepsilon+}|^2 \tag{3.9}$$

and we have

$$e_{1\varepsilon}(u_\varepsilon) = \alpha |\operatorname{grad} u_{\varepsilon+}|^2 + \varepsilon |u_{\varepsilon-}|^2 - \tfrac{1}{2} |u_{\varepsilon+}|. \tag{3.10}$$

(iv) If $u_{\varepsilon+}$ is different from zero, the functional $e_{1\varepsilon}$ is differentiable at u_ε and its differential is defined by

$$\langle e'_{1\varepsilon}(u_\varepsilon), v\rangle = 2\alpha(\operatorname{grad} u_\varepsilon, \operatorname{grad} v) + 2\varepsilon(u_\varepsilon, v) - \frac{1}{|u_{\varepsilon+}|}(u_{\varepsilon+}, v) \qquad \forall\, v \in H^1(\Omega). \tag{3.11}$$

The set $\mathscr{E}_a$ is also the set of u in $H^1(\Omega)$ such that

$$e_2(u) = |u_-|^2 = a^2. \tag{3.12}$$

Because of Lemma 1.2 and Lemma 1.3, there exists a $\lambda_\varepsilon \in \mathbb{R}$ such that

$$e'_{1\varepsilon}(u_\varepsilon) = \lambda_\varepsilon\, e'_2(u_\varepsilon)$$

($u_{\varepsilon+} \neq 0$), and this amounts to

$$\langle e'_{1\varepsilon}(u_\varepsilon), v\rangle = \lambda_\varepsilon \langle e'_2(u_\varepsilon), v\rangle, \qquad \forall\, v \in H^1(\Omega)$$

or

$$\alpha(\operatorname{grad} u_\varepsilon, \operatorname{grad} v) + \varepsilon(u_\varepsilon, v) = \frac{1}{2|u_{\varepsilon+}|}(u_{\varepsilon+}, v) - \lambda_\varepsilon(u_{\varepsilon-}, v). \tag{3.13}$$

It is easy to deduce from (3.13) that u_ε is solution of the non-linear Neumann problem

$$-\alpha \Delta u_\varepsilon + \varepsilon u_\varepsilon = \frac{u_{\varepsilon+}}{2|u_{\varepsilon+}|} - \lambda_\varepsilon (u_\varepsilon)_- \quad \text{in } \Omega, \tag{3.14}$$

$$\frac{\partial u_\varepsilon}{\partial \nu} = 0 \quad \text{on } \partial\Omega. \tag{3.15}$$

(v) As $\varepsilon \to 0$, $\varepsilon u_\varepsilon$ remains bounded in $L^2(\Omega)$. Indeed, if u is some fixed element of $\mathscr{E}_a$

$$e_{1\varepsilon}(u_\varepsilon) \leqq e_{1\varepsilon}(u) \leqq c_0,$$

whence

$$\alpha\varepsilon\,|\text{grad}\, u_\varepsilon|^2 + \varepsilon^2\,|u_\varepsilon|^2 \leqq \varepsilon\,|u_{\varepsilon+}| + \varepsilon c_0 \leqq \frac{\varepsilon^2}{2}\,|u_\varepsilon|^2 + c_1.$$

The integration of (3.14) over Ω gives

$$\int_\Omega \lambda_\varepsilon u_{\varepsilon-}\, dx = \int_\Omega \left(\frac{u_{\varepsilon+}}{2|u_{\varepsilon+}|} - \varepsilon u_\varepsilon\right) dx,$$

and since the right-hand side of this relation is bounded, $\lambda_\varepsilon u_{\varepsilon-}$ is bounded in $L^1(\Omega)$:

$$|\lambda_\varepsilon (u_\varepsilon)_-|_{L^1(\Omega)} \leqq c_2. \tag{3.16}$$

On the other hand, since $|(u_\varepsilon)_-| = |(u_\varepsilon)_-|_{L^2(\Omega)} = a$, we have

$$|\lambda_\varepsilon (u_\varepsilon)_-|_{L^2(\Omega)} = \lambda_\varepsilon a. \tag{3.17}$$

By the interpolation theorem of M. RIESZ (*cf.* [16], for instance), we infer from (3.16) and (3.17) that

$$|\lambda_\varepsilon (u_\varepsilon)_-|_{L^{\frac{5}{4}}(\Omega)} \leqq c_3\,|\lambda_\varepsilon|^{\frac{2}{5}}. \tag{3.18}$$

We deduce from (3.14) and the preceding results, that the norm of Δu_ε in $L^{\frac{5}{4}}(\Omega)$ is majorized by $c_4 + c_3 \lambda_\varepsilon^{\frac{2}{5}}$. From (3.15) and the classical regularity results for elliptic problems (*cf.* AGMON, DOUGLIS & NIRENBERG [1]) we conclude that

$$\|u_\varepsilon\|_{W^{2,\frac{5}{4}}(\Omega)/\mathbb{R}} \leqq c_5 (1 + \lambda_\varepsilon^{\frac{2}{5}}).$$

By the Sobolev imbedding theorems, $W^{2,\frac{5}{4}}(\Omega) \subset H^1(\Omega)$ $\left(\frac{4}{5} - \frac{1}{n} < \frac{1}{2}\right)$ and therefore

$$\|u_\varepsilon\|_{H^1(\Omega)/\mathbb{R}} \leqq c_6 (1 + \lambda_\varepsilon^{\frac{2}{5}}). \tag{3.19}$$

We now set $v = u_\varepsilon$ in (3.13) and then use (3.10) to obtain

$$\lambda_\varepsilon |u_{\varepsilon-}|^2 = \alpha\,|\text{grad}\, u_\varepsilon|^2 + \varepsilon\,|u_\varepsilon|^2 - \tfrac{1}{2}|u_{\varepsilon+}| = \alpha\,|\text{grad}\, u_{\varepsilon-}|^2 + \varepsilon\,|u_{\varepsilon-}|^2.$$

Thus $\lambda_\varepsilon \geqq 0$. Inequality (3.19) then implies that

$$\lambda_\varepsilon a^2 < \varepsilon a^2 + c_7 + c_8 \lambda_\varepsilon^{\frac{4}{5}}. \tag{3.20}$$

This inequality shows that λ_ε remains bounded as $\varepsilon \to 0$ and (3.19) then implies that

$$\|u_\varepsilon\|_{H^1(\Omega)/\mathbb{R}} \leqq \text{const}. \tag{3.21}$$

(vi) Before passing to the limit as $\varepsilon \to 0$, we establish that the family u_ε itself is bounded in $H^1(\Omega)$. Because of (3.21), there exists numbers k_ε such that $u_\varepsilon + k_\varepsilon$ remains bounded in $H^1(\Omega)$ as $\varepsilon \to 0$; hence a sequence $\varepsilon_m \to 0$ can be extracted for which

(3.22) $$u_{\varepsilon_m} + k_{\varepsilon_m} \to v, \qquad \text{weakly in } H^1(\Omega) \text{ and strongly in } L^2(\Omega).$$

We must show that the k_ε remain bounded. If not, the sequence ε_m can be chosen so that $|k_\varepsilon| \to \infty$. Let $\theta_\varepsilon(x)=1$ if $u_\varepsilon(x)<0$ and let $\theta_\varepsilon(x)=0$ otherwise. By (3.22), $\theta_\varepsilon(u_\varepsilon + k_\varepsilon - v) \to 0$ in $L^2(\Omega)$ (we write ε instead of ε_m):

$$\int_\Omega \theta_\varepsilon (u_\varepsilon + k_\varepsilon - v)^2\, dx \to 0.$$

We expand the integrand of this expression, observe that $\theta_\varepsilon u_\varepsilon = -(u_\varepsilon)_-$ and $|(u_\varepsilon)_-| = a$, divide by k_ε^2, and let $\varepsilon \to 0$ to obtain

$$\int_\Omega \theta_\varepsilon\, dx \to 0.$$

The Schwarz inequality then implies that

$$|(u_\varepsilon)_-|_{L^1(\Omega)} \leqq a\Big(\int_\Omega \theta_\varepsilon\, dx\Big)^{\frac{1}{2}} \to 0,$$

which is impossible because the sequence $(u_\varepsilon)_-$ is bounded in $H^1(\Omega)$ (by (3.6) and (3.21)) and is relatively compact in $L^2(\Omega)$ and because $|(u_\varepsilon)_-| = a$. Thus by contradiction, the k_ε's are bounded and

$$\|u_\varepsilon\|_{H^1(\Omega)} \leqq \text{const.}^{11}$$

(vii) The passage to the limit $\varepsilon \to 0$ is now elementary. There exists a sequence (still denoted by ε) which converges to 0 and such that

$$u_\varepsilon \to \bar{u} \qquad \text{weakly in } H^1(\Omega).$$

The convergence is also strong in $L^2(\Omega)$ and $(u_\varepsilon)_- \to \bar{u}_-$ strongly in $L^2(\Omega)$ so that $|\bar{u}_-| = a$. Now for any fixed v in the set (3.6), we have

$$e_1(\bar{u}) = \alpha |\text{grad}\, \bar{u}|^2 - |\bar{u}_+| \leqq \varliminf_{\varepsilon \to 0} e_1(u_\varepsilon) \leqq \varliminf_{\varepsilon \to 0} e_{1\varepsilon}(u_\varepsilon) \leqq \varliminf_{\varepsilon \to 0} e_{1\varepsilon}(v) = e_1(v).$$

Hence the infinum of e_1 on the set $\mathscr{E}_a$ is finite and is attained at the point $\bar{u}$. The passage to the limit in (3.13) ($\lambda_\varepsilon \to \lambda$) also gives

(3.23) $$\alpha(\text{grad}\, \bar{u}, \text{grad}\, v) = \frac{1}{2|\bar{u}_+|}(\bar{u}_+, v) - \lambda(\bar{u}_-, v), \qquad \forall v \in H^1(\Omega).$$

The relation (3.1) is finally proved with

(3.24) $$-\delta(\alpha, a) = \inf_{v \in \mathscr{E}_a} \{\alpha |\text{grad}\, v|^2 - |v_+|\} = a |\text{grad}\, \bar{u}|^2 - |\bar{u}_+|.$$

Remark 3.2 *(L^q norm inequalities).* (i) An inequality similar to (3.1) or (3.3), but involving L^q norms can be obtained. For example (see also point (ii)), assume that

[11] We only consider the case $u_{\varepsilon_+} \neq 0$. If $u_{\varepsilon_+} = 0$, this conclusion is very easy.

$\Omega \subset \mathbb{R}^2$ is a bounded open set of class $\mathscr{C}^2$. Then for each $\alpha > 0$, for each $q \geqq 1$, there exists a function $\delta_{\alpha,q}$: $]0, +\infty[\to \mathbb{R}$, such that

$$|u_+|_{L^2(\Omega)} \leqq \alpha |\text{grad } u|^2_{L^2(\Omega)^2} + \delta_{\alpha,q}(|u_-|_{L^q(\Omega)}), \tag{3.25}$$

for every u in $H^1(\Omega)$ such that $u_- \neq 0$.

The proof of (3.25) is exactly the same as the proof of (3.1). We consider the functionals e_1 and $e_{1\varepsilon}$ as before, but the set $\mathscr{E}_a$ is replaced by

$$\mathscr{F}_a = \{u \in H^1(\Omega) \,|\, |u_-|_{L^q(\Omega)} = a\} \qquad (a > 0).$$

The minimization of $e_{1\varepsilon}$ on $\mathscr{F}_a$ leads to the existence of $u_\varepsilon \in H^1(\Omega)$ satisfying

$$-\alpha \Delta u_\varepsilon + \varepsilon u_\varepsilon = \frac{u_{\varepsilon+}}{2|u_{\varepsilon+}|} - \lambda_\varepsilon [(u_\varepsilon)_-]^{q-1}, \tag{3.26}$$

$$\frac{\partial u_\varepsilon}{\partial \nu} = 0 \tag{3.27}$$

instead of (3.14), (3.15).

Only minor modifications of the arguments are necessary in the subsequent steps of the proof: λ_ε is bounded, u_ε is bounded in $H^1(\Omega)/\mathbb{R}$, u_ε is bounded in $H^1(\Omega)$, and u_ε converges weakly to $\bar{u}$ which minimizes e_1 on $\mathscr{F}_a$.

(ii) From (3.25) and the imbedding of $H_1(\Omega)$ into $L^r(\Omega)$, $r \geqq 1$, we deduce that

$$|u_+|_{L^q(\Omega)} \leqq \alpha |\text{grad } u|^2_{L^2(\Omega)} + \delta'_{\alpha,q}(|u_-|_{L^q(\Omega)}). \tag{3.28}$$

Indeed, if $q > 2$, we have

$$|u_+|_{L^q(\Omega)} \leqq c(q, \Omega) \, \|u\|_{H^1(\Omega)},$$

and the passage to (3.28) is easy.

Using the same homogeneity arguments as in Remark 3.1 (ii), one can infer from (3.25) or (3.28) that

$$|u_+|_{L^2(\Omega)} \leqq \alpha |\text{grad } u|^2_{L^2(\Omega)^2} + \delta(\alpha, q, a) |u_-|_{L^q(\Omega)}, \tag{3.29}$$

$$|u_+|_{L^q(\Omega)} \leqq \alpha |\text{grad } u|^2_{L^2(\Omega)^2} + \delta'(\alpha, q, a) |u_-|_{L^q(\Omega)} \tag{3.30}$$

for every u in $H^1(\Omega)$ such that

$$\alpha \leqq |u_-|_{L^q(\Omega)} \qquad (a > 0). \tag{3.31}$$

Appendix. Equilibrium Equations in the Tokomak*

The Tokomak machine is represented as an axisymmetric torus with axis $0z$. In a plane $0xz$ the cross-section of the Tokomak is an open set Ω with boundary Γ representing the cross-section of the shell. The plasma fills a subdomain Ω_p of Ω, with boundary Γ_p, and the complementary region $\Omega - (\Omega_p \cup \Gamma_p) = \Omega_v$ is empty.[12]

* According to [19]; see also [4].

[12] We do not consider here equilibrium in the presence of electric currents circulating in Ω_v. See [5], [6].

The Equations

The space is referred to the cylindrical coordinate system (r, θ, z); $\boldsymbol{e}_r$, $\boldsymbol{e}_\theta$, $\boldsymbol{e}_z$ denote the usual orthonormal vectors.

In the vacuum, Maxwell's equations are

$$\text{div}\,\boldsymbol{B}=0 \quad \text{in } \Omega_v \tag{1}$$

$$\text{rot}\,\boldsymbol{B}=0 \quad \text{in } \Omega_v, \tag{2}$$

In the plasma, the magnetohydrodynamic (MHD) description of the macroscopic equilibrium gives the equations

$$\text{div}\,\boldsymbol{B}=0 \quad \text{in } \Omega_p, \tag{1'}$$

$$\text{rot}\,\boldsymbol{B}=\mu_0 \boldsymbol{J} \quad \text{in } \Omega_p, \tag{2'}$$

$$\text{grad}\, p=\boldsymbol{J}\times\boldsymbol{B} \quad \text{in } \Omega_p, \tag{3}$$

where

$$\boldsymbol{B}=B_r\boldsymbol{e}_r+B_\theta\boldsymbol{e}_\theta+B_z\boldsymbol{e}_z, \qquad \boldsymbol{J}=J_r\boldsymbol{e}_r+J_\theta\boldsymbol{e}_\theta+J_z\boldsymbol{e}_z, \quad \text{and} \quad p$$

are the magnetic flux, the current, and the pressure. Due to the axisymmetry, all the quantities B_r, B_θ, B_z, J_r, J_θ, J_z and p are independant of θ. Therefore equation (1) is equivalent to

$$\text{div}\,\boldsymbol{B}=\frac{1}{r}\frac{\partial}{\partial r}(r B_r)+\frac{\partial B_z}{\partial z}=0, \tag{4}$$

and we conclude that there exists a locally defined function ϕ (the flux function) such that

$$B_r=+\frac{1}{r}\frac{\partial\phi}{\partial z}, \qquad B_z=-\frac{1}{r}\frac{\partial\phi}{\partial r}. \tag{5}$$

Later, using the boundary conditions, we shall see that ϕ is a single-valued function in the whole domain Ω. For convenience we set $f=f(r,z)=rB_\theta$. Then

$$\boldsymbol{B}=\nabla\phi\times\boldsymbol{e}_{\theta/r}+(f/r), \qquad \nabla=\left\{\frac{\partial}{\partial r}, 0, \frac{\partial}{\partial z}\right\}. \tag{6}$$

Now, with this representation of $\boldsymbol{B}$, the equation (2) (or (2′)) becomes

$$\mathscr{L}\phi=0 \quad \text{in } \Omega_V, \tag{7}$$

$$\mathscr{L}\phi\,\boldsymbol{e}_\theta+\nabla f\times\boldsymbol{e}_\theta/r=\mu_0\boldsymbol{J} \quad \text{in } \Omega_p, \tag{7'}$$

where

$$\mathscr{L}=\nabla\left(\frac{1}{r}\nabla\right)=\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}\right)+\frac{1}{r}\frac{\partial^2}{\partial z^2}. \tag{8}$$

The equation (3) can now be written as

$$\mu_0 \frac{\partial p}{\partial r} = -\frac{1}{r} \mathscr{L}\psi \cdot \frac{\partial \psi}{\partial r} - \frac{1}{2r^2} \frac{\partial f^2}{\partial r},$$

(9) $$0 = \frac{1}{r^2} \left(+ \frac{\partial \psi}{\partial z} \frac{\partial f}{\partial r} - \frac{\partial f}{\partial z} \frac{\partial \psi}{\partial r} \right),$$

$$\mu_0 \frac{\partial p}{\partial z} = -\frac{1}{r} \mathscr{L}\psi \cdot \frac{\partial \psi}{\partial z} - \frac{1}{2r^2} \frac{\partial f^2}{\partial z}.$$

The second equation (9) implies that ∇f is parallel to $\nabla \psi$, in Ω_p so that f depends only on ψ:

(10) $$f^2 = g_0(\psi), \quad \nabla f^2 = g_0' \cdot \nabla \psi, \quad \text{where } g_0' = \frac{dg_0}{d\psi}.$$

From the first and third equations of (9) we now deduce that

$$\nabla p = \left(-\frac{1}{r} \mathscr{L}\psi - \frac{1}{2r^2} g_0' \right) \nabla \psi.$$

Again ∇p is parallel to $\nabla \psi$, and p depends only on ψ:

(11) $$p = g_1(\psi), \quad \nabla p = g_1' \cdot \nabla \psi.$$

Relations (10) and (11) imply that the equations (9) are now equivalent to the single equation

(12) $$\mathscr{L}\psi = -\mu_0 r\, g_1'(\psi) - \frac{1}{2r} g_0'(\psi), \quad \text{in } \Omega_p.$$

Due to (5), (7), (10), (11), all the unknowns are functions of ψ, and ψ is solution of (7) and (12). Of course g_0 and g_1 are unknown functions of ψ, but they cannot be determined by using only Maxwell equations. *They are constitutive functions for the plasma.* Below and in the text, we impose very mild restrictions on the form of these functions.

The Boundary Conditions

Let $\boldsymbol{v}$ be the unit outer normal vector and $\boldsymbol{\tau}$ be the unit tangent vector to Γ_p or Γ. Since Γ is a conducting shell, we have

(13) $$\boldsymbol{B} \cdot \boldsymbol{v} = 0 \quad \text{on } \Gamma.$$

We similarly require that

(14) $$\boldsymbol{B} \cdot \boldsymbol{v} = 0 \quad \text{on } \Gamma_p,$$

(15) $$\boldsymbol{B} \cdot \boldsymbol{\tau} \quad \text{is continuous on } \Gamma_p.$$

By (5), $\boldsymbol{B} \cdot \boldsymbol{v} = -\frac{1}{r} \frac{\partial \psi}{\partial \tau}$ and $\boldsymbol{B} \cdot \boldsymbol{\tau} = \frac{1}{r} \frac{\partial \psi}{\partial v}$. Hence $\frac{\partial \psi}{\partial \tau} = 0$ locally on Γ_p and Γ; this shows that ψ is a single-valued function in Ω and ψ is constant on Γ_p and on Γ.

Since ψ is defined up to an additive constant, we choose[13]

(16) $$\psi=0 \quad \text{on } \Gamma_p,$$

(17) $$\psi=\text{constant}=\gamma \quad \text{on } \Gamma.$$

Finally (15) reduces to

(18) $$\frac{\partial\psi}{\partial\nu} \quad \text{is continuous on } \Gamma_p.$$

Other Assumptions

The preceding equations do not assume any physical hypothesis on the plasma, except its description by the M.H.D. model. Some simplifying assumptions are now necessary in order to obtain the models treated in the text.

Following MERCIER [19], we make the following assumptions, considered as very realistic:

(19) $$p=0 \quad \text{and} \quad J_\theta=0 \quad \text{on } \Gamma_p,$$

(20) $$J_\theta \neq 0 \quad \text{in } \Omega_p.$$ [14]

Thus by (11) and (16), we have

(21) $$g_1(0)=0$$

and by (7′) and (12), we have

$$\mu_0 J_\theta=\mathscr{L}\psi=-\mu_0 r g_1'(\psi)-\frac{1}{2r}g_0'(\psi)=0 \quad \text{on } \Gamma_p,$$

i.e.,

(22) $$g_0'(0)=g_1'(0)=0.$$

The assumptions (20) and (21) imply that necessarily

(23) $$\psi \neq 0 \quad \text{in } \Omega_p.$$

Two Models

In Section 1 we consider the simplest model: $p(\psi)$ and $f^2(\psi)$ (*i.e.*, g_0 and g_1) are *quadratic functions* of ψ. Then (21) and (22) imply

(24) $$\begin{aligned} f^2(\psi)&=g_0(\psi)=b_0+b_2\psi^2,\\ p(\psi)&=g_1(\psi)=a_2\psi^2. \end{aligned}$$

Since f^2 and p (the pressure) are $\geqq 0$, we must have

(25) $$b_0, b_2, a_2 \geqq 0.$$

[13] In doing so, we restrict ourselves to the case in which Γ_p is connected, *i.e.*, in which Ω_p is simply-connected.

[14] *I.e.*, in particular, J_θ does not change its sign in Ω_p.

In this case (12) reduces to

$$\mathscr{L}\psi = -\left(2\mu_0 r a_2 + \frac{b_2}{r}\right)\psi \quad \text{in } \Omega_p. \tag{26}$$

Since $\mathscr{L}\psi = 0$ in Ω_v (*cf.* (7)), *the equations are linear (although the whole boundary value problem for ψ is nonlinear).*

In Section 2 we consider a much more general model with only mild mathematical restrictions on g_0 and g_1. It is also assumed in both Sections that the total current in the plasma is given:

$$I = \mu_0 \int_{\Omega_p} J_\theta \, dr\, dz = \int_{\Omega_p} \mathscr{L}\psi \, dr\, dz = \int_{\Gamma_p} \frac{1}{r}\frac{\partial\psi}{\partial\nu}\, d\Gamma.$$

After this work was completed, appeared the paper of GRAD, KADISH & STEVENS [25], *which treats the free boundary Tokomak equilibrium problem under different assumptions (in particular, cylindrical geometry, no currents, and constant pressure inside the plasma). A similar problem arises in the theory of vortex rings in an ideal fluid* (cf. [26]).

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(Received July 1, 1975)