A Minimax Theory for Overdamped Systems

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1. Introduction

The characterization of the least eigenvalue of a symmetric linear operator given by Rayleigh [1] in 1873 has been the basis of a considerable amount of investigation in the years since. This is due to the extremely convenient technique implied by it for the estimation of the least eigenvalue and, under modification by Poincaré [2], Fischer [3], Weyl [4], Courant [5] and others, for the higher eigenvalues of linear operators.

Rayleigh made his discovery while considering conservative mechanical systems governed by Lagrange's equations of small free motion:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0, \qquad i = 1, 2, \ldots, N,$$

where L is the sum of a quadratic form in $q_1, \ldots, q_N$ and of one in $\dot{q}_1, \ldots, \dot{q}_N$, the time ($t$) derivatives of the former. This reduces to a system of equations of the form

$$A\ddot{q} + Cq = 0,$$

where q is the vector $(q_1, \ldots, q_N)$ and A and C are constant symmetric matrices with positive definite quadratic forms. The solution to such a system is expandable in functions of the form $q = v\, e^{ikt}$, where v is a fixed N-vector and k is a real constant. These are referred to as the normal modes of the system. Rayleigh consequently sought pairs k, v satisfying

$$(1) \qquad -k^2 A v + C v = 0.$$

He discovered that the minimum of the "Rayleigh quotient" $(Cu, u)^{\frac{1}{2}}/(Au, u)^{\frac{1}{2}}$, computed over all non-zero N-vectors u is a value of k for which there is a non-zero v satisfying (1). The notation (,) represents a positive definite inner product on N-space R^N.

Rayleigh did not have a similar success with non-conservative (dissipative) systems. However, recently Duffin [6, 7] has found a class of dissipative systems to which Rayleigh's technique may be applied. A dissipative Lagrangian system of small free motion may be given the form

$$A\ddot{q} + B\dot{q} + Cq = 0,$$

where A, B, and C are semi-definite, symmetric matrices. The "eigenvalue" problem is that of finding pairs k, v, where k is a complex scalar and v is a non-zero complex N-vector, such that

$$k^2 A v + k B v + C v = 0. \tag{2}$$

Assuming an "overdamping" condition

$$d(u) = (Bu, u)^2 - 4(Au, u)(Cu, u) > 0 \tag{3}$$

for all non-zero N-vectors u, DUFFIN uses the roots of the quadratic

$$x^2(Au, u) + x(Bu, u) + (Cu, u) = 0 \tag{4}$$

viewed as functionals of u, as generalized "Rayleigh quotients". These "Rayleigh functionals" are necessarily real. Consequently, the eigenvalue problem (2) will possess real eigenvectors. Thus the vector space dealt with is taken to be real N-space.

The overdamping condition (3) guarantees that one of the functionals p, defined implicitly with (4), is defined for all non-zero vectors in N-space. DUFFIN [6] shows minimax values defined in terms of p to be eigenvalues.

As in DUFFIN'S paper, no consideration of complex eigenvalues is undertaken here. Instead a more general eigenvalue problem is analyzed and the equivalence between minimax values and certain eigenvalues demonstrated. The techniques employed are not essentially different from those used by DUFFIN on the operator $x^2 A + x B + C$.

Attention is focused on the symmetric linear operator T_x, dependent on the real parameter x. The pair k, v is referred to as an *eigenpair* (k is the *eigenvalue* and v, the *eigenvector*), if $T_k v = 0$. An assumption of the same force as the overdamping condition (3) is required. The direct assumption is made that a continuous, real functional p is definable on all non-zero N-vectors such that

$$p(\alpha u) = p(u) \quad \text{for} \quad \alpha \neq 0, \tag{5}$$

$$(T_{p(u)} u, u) = 0, \tag{6}$$

and

$$(T'_{p(u)} u, u) = \frac{\partial}{\partial x} (T_x u, u)\big|_{x = p(u)} > 0, \tag{7}$$

for all non-zero $u \in R^N$. We call such systems "overdamped", though they may arise in connection with non-Lagrangian problems.

The assumption of positiveness in (7) is not critical. If $(T'_{p(u)} u, u) < 0$ for all non-zero u, then the operator $-T_x$ would satisfy (6) and (7) using the same functional p.

The critical observation in the abstract development is a strengthened analogue (Lemma 1) of Lemma 2 in DUFFIN [6]. It characterizes the sets $\{u \mid p(u) < h\}$, $\{u \mid p(u) = h\}$, and $\{u \mid p(u) > h\}$ in terms of the quadratic form of T_x. Because of the fact that the functional p is independent of the length of its argument, Lemma 1 may be used as the basis of a geometric development on the unit sphere in N-space. However, a more direct approach is undertaken.

A degree of completeness is demonstrated in that all eigenvalues in the range of p are obtained as minimax values of p and their eigenvectors span N-space.

Rayleigh's Principle on constraints asserts that a single linear constraint on a linear system vibrating about a static equilibrium causes the eigenfrequencies of normal vibration of the constrained system to be spaced between the eigenfrequencies of the original system. A consequence of the minimax theory given here is a Rayleigh's Principle for more general linear systems. Specifically, when T_x is restricted to operate on an $(N-1)$-dimensional subspace of R^N, the $(N-1)$ eigenvalues in the range of p on the subspace separate the N eigenvalues corresponding to no constraint.

As an example of a class of operators T_x fulfilling the hypotheses of this paper, the forementioned $x^2 A + xB + C$ with the overdamping condition (3) may be cited. The functional p may be defined by $p(u) = -2(Cu, u)/[(Bu, u) + \sqrt{d(u)}]$, where $d(u)$ is defined by (3). Obviously, $p(\alpha u) = p(u)$ for $\alpha \neq 0$. Moreover, because B is positive semi-definite, p is defined for all non-zero u. We see that condition (7) is satisfied by computing $(T'_{p(u)} u, u) = 2p(u)(Au, u) + (Bu, u)$. This is $+\sqrt{d(u)}$.

A second example is $T_x = -(xA + \sqrt{1-x^2}B + C)$, where A, B, and C are symmetric operators satisfying $(Au, u) < 0$, $(Cu, u) < 0$, and $((A + B + C)u, u) > 0$ for all non-zero u. Then $p(u)$ may be defined as

$$[-(Au, u)(Cu, u) + (Bu, u)\sqrt{(Au, u)^2 + (Bu, u)^2 - (Cu, u)^2}]/[(Au, u)^2 + (Bu, u)^2].$$

It is a consequence of the conditions on A, B, and C that $(Au, u)^2 + (Bu, u)^2 - (Cu, u)^2 > 0$ for all non-zero u. The expression for $(T'_{p(u)} u, u)$ is complicated but can nonetheless be shown to be positive for all non-zero u.

The operator $x^3 A + xB + C$ can be treated, also. To guarantee that a continuous functional p exists, it is sufficient to know that $4(Bu, u)^3 + 27(Au, u) \times (Cu, u)^2 < 0$ and $(Au, u) > 0$ for all non-zero u.

2. The Rayleigh Functional p

To be precise, the symmetric linear transformation T_x of R^N into R^N is taken to be dependent on the real parameter x so that the derivative T'_x exists and is continuous in x. It is assumed that a continuous functional p exists satisfying (5), (6), and (7). From relation (6) one observes that allowing a variation δu of u,

$$0 = \delta p(u)(T'_{p(u)} u, u) + 2(T_{p(u)} u, \delta u).$$

Consequently, $p(v)$, v is an eigenpair if, and only if, $\delta p(v) = 0$ for all δv; that is, if, and only if, p is stationary at v.

For each h in the range of p it is clear that $\{u \mid p(u) = h\} \cup \{0\} \subset \{u \mid (T_h u, u) = 0\}$. That the inclusion is actually equality is one consequence of the following fundamental lemma.

Lemma 1. *For* $\min p(u) \leqq h \leqq \max p(u)$

$$\{u \mid p(u) < h\} = \{u \mid (T_h u, u) > 0\},$$
$$\{0\} \cup \{u \mid p(u) = h\} = \{u \mid (T_h u, u) = 0\},$$
$$\{u \mid p(u) > h\} = \{u \mid (T_h u, u) < 0\}.$$

Proof. Let us suppose there to be a $u_1 \in \{u \mid (T_h u, u) \geqq 0\}$ such that $p(u_1) > h$. Let u_0 represent a vector satisfying $p(u_0) = h$. The sign of u_0 is chosen so that $(T_h u_0, u_1) \geqq 0$. Then

$$u_\alpha = (1-\alpha) u_0 + \alpha u_1 \in \{u \mid (T_h u, u) \geqq 0\}$$

for all α in $0 \leqq \alpha \leqq 1$. Since $p(u_1) > h$, there is a largest scalar β in $0 \leqq \beta < 1$ satisfying $p(u_\beta) = h$. Then $p(u_\alpha) > h$ for $\beta < \alpha \leqq 1$. The following relations are assembled in the diagram: $(T_h u_\alpha, u_\alpha) \geqq 0$, $(T_{p(u_\alpha)} u_\alpha, u_\alpha) = 0$, $(T'_{p(u_\alpha)} u_\alpha, u_\alpha) > 0$ for $\beta < \alpha \leqq 1$. Clearly there is an x_α in $h \leqq x_\alpha < p(u_\alpha)$ at which $(T'_{x_\alpha} u_\alpha, u_\alpha) = 0$. Now letting $\alpha \to \beta +$, we see

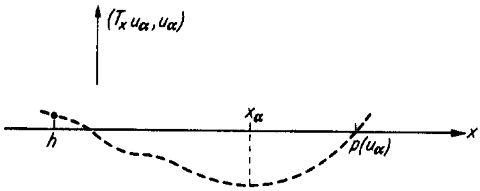

Fig. 1

$$(T'_h u_\beta, u_\beta) = \lim_{\alpha \to \beta +} (T'_{x_\alpha} u_\alpha, u_\alpha) = 0.$$

Since $p(u_\beta) = h$, this is a contradiction of (7). We are forced to conclude that $p(u) \leqq h$ whenever $(T_h u, u) \geqq 0$. A similar argument shows $p(u) \geqq h$ whenever $(T_h u, u) \leqq 0$. Consequently,

$$\{u \mid p(u) = h\} \cup \{0\} = \{u \mid (T_h u, u) = 0\},$$

and the other conclusions of the lemma follow.

Lemma 2. *If $u_1, u_2, \ldots, u_n$ are eigenvectors corresponding to eigenvalues $p(u_1) < p(u_2) < \cdots < p(u_n)$, then the vectors $u_1, u_2, \ldots, u_n$ are linearly independent and $p(u_1) < p(u_1 + \cdots + u_n) < p(u_n)$.*

Proof. Since $p(\alpha u) = p(u)$ for $\alpha \neq 0$ and $p(u_1) < p(u_2)$, u_1 and u_2 are linearly independent. Because by Lemma 1

$$\big(T_{p(u_1)}(u_1 + u_2), (u_1 + u_2)\big) = (T_{p(u_1)} u_2, u_2) < 0,$$

$p(u_1) < p(u_1 + u_2)$, also by Lemma 1. Similarly $p(u_1 + u_2) < p(u_2)$. Induction on the length n of the sum is used.

Suppose $v = u_1 + \cdots + u_{n-1}$ is non-zero and $p(u_1) < p(v) < p(u_{n-1})$. Because $p(u_{n-1}) < p(u_n)$, v and u_n are linearly independent. As before, $\big(T_{p(u_n)}(v + u_n), (v + u_n)\big) = (T_{p(u_n)} v, v) > 0$ by Lemma 1. Thus $p(v + u_n) < p(u_n)$. In like fashion, we set $w = u_2 + \cdots + u_n$ and assume $w \neq 0$ and $p(u_2) < p(w) < p(u_n)$. Independence of u_1 and w and $p(u_1) < p(u_1 + w)$ follow by the twice-used arguments.

Since $v + u_n = u_1 + \cdots + u_n = u_1 + w$, the assertion of the lemma is verified.

3. The Minimax Theory

The possibly non-linear occurrence of the parameter x in T_x does not allow us to expect orthogonality for the eigenvectors of distinct eigenvalues. The minimax values therefore are not defined using orthogonality, as is commonly done. The distinction between minimax values is caused by dimensional differences in subspaces used to define them. Given a subspace U of R^N, we define:

$$P(U) = \max p(u), \quad \text{over the non-zero } u \in U,$$

$$Q(U) = \min p(u), \quad \text{over the non-zero } u \in U,$$

and

$$k_j = \min P(U), \quad \text{over the } j\text{-dimensional subspaces } U,$$

$$h_j = \max Q(U), \quad \text{over the } j\text{-dimensional subspaces } U.$$

Compactness properties of finite-dimensional spaces guarantee that $P(U)$, $Q(U)$, k_j, and h_j are all achieved for some vector or subspace. Hence the use of the max and min terminology is justified. The k's are referred to as min-max values and the h's, as max-min values. Separate investigations may be carried out for each, but the usual duality is valid.

As a first step toward proving an equivalence between the min-max and max-min values and a class of the eigenvalues, the following lemma is demonstrated:

Lemma 3. *The min-max values $k_1 \leqq k_2 \leqq \cdots \leqq k_N$ are precisely all of the eigenvalues in the range of p.*

Proof. Let $k = k_j$, for an arbitrary j in $1 \leqq j \leqq N$. Let $k = P(V)$, where $\dim V = j$. The linear operator $J_m = I + m^{-1} T_k$ is formed for an arbitrary positive integer m. I represents the identity transformation. For m sufficiently large ($\geqq M$) J_m is non-singular. Hence the sequence $\{V_m\}_{m=M}^{\infty}$, defined by $V_m = J_m V$, is composed of subspaces of dimension j converging to V. For each m a unit vector v_m may be chosen in V_m such that $p(v_m) = P(V_m)$. A second sequence of vectors $\{u_m\}_{m=M}^{\infty}$ is composed of the predecessors of the v_m under J_m; $v_m = J_m u_m$. Since the unit sphere of R^N is topologically compact, a convergent subsequence $\{v_m'\}^{\infty}$ of $\{v_m\}_{m=M}^{\infty}$ may be chosen. Then $\{v_m'\}^{\infty}$ and $\{u_m'\}^{\infty}$ have a common limit u in V, because $\|v_m' - u_m'\| = \|u_m' + m^{-1} T_k u_m' - u_m'\| = m^{-1} \|T_k u_m'\|$ tends to zero.

For each m (the prime is dropped; the use of the subsequence being understood)

$$(8) \qquad (T_x v_m, v_m) = (T_x u_m, u_m) + 2m^{-1} (T_x u_m, T_k u_m) + m^{-2} (T_x T_k u_m, T_k u_m)$$

and

$$(9) \qquad (T_x v_m, v_m) = (T_x v_m, v_m) - (T_{p(v_m)} v_m, v_m) = \big(x - p(v_m)\big) (T_{x_0}' v_m \; v_m)$$

for some x_0 between x and $p(v_m)$. Letting $x = p(u_m)$, we note $p(v_m) = P(V_m) \geqq k = P(V) \geqq p(u_m)$, so that $p(u_m)$, x_0, and $p(v_m)$ all tend to k as m increases and $k = p(u)$. Also, from (8) and (9)

$$(10) \quad \big(p(u_m) - p(v_m)\big) (T_{x_0}' v_m, v_m) = 2m^{-1} (T_{p(u_m)} u_m, T_k u_m) + m^{-2} (T_{p(u_m)} T_k u_m, T_k u_m).$$

As m increases, $(T_{x_0}' v_m, v_m) \to (T_{p(u)}' u, u) > 0$, so that the left side of (10) is eventually non-positive. We obtain

$$0 \geqq 2 (T_{p(u_m)} u_m, T_k u_m) + m^{-1} (T_{p(u_m)} T_k u_m, T_k u_m),$$

which tends to $0 \geqq 2 \|T_k u\|^2$. Thus $T_k u = 0$ and $k = k_j$ is an eigenvalue for each j in $1 \leqq j \leqq N$.

Finally, we let k, v be an eigenpair with $\min p(u) \leqq k \leqq \max p(u)$. Then by Lemma 1, $(T_k v, v) = 0$ implies $k = p(v)$. The letter δ represents the largest of the dimensions of the subspaces contained in $\{u \mid (T_k u, u) > 0\} \cup \{0\}$ and V is taken to be such a δ-dimensional subspace. The subspace W generated by V

and the eigenvector v clearly belongs to $\{u \mid (T_k u, u) \geqq 0\}$. Consequently, $k_{\delta+1} \leqq P(W) = k$. However, because of the maximality of δ, $k_{\delta+1} \geqq k$. Therefore, $k = k_{\delta+1}$, and the lemma is proved.

Now the duality between the min-max and max-min values may be shown.

Lemma 4. *For each j in $1 \leqq j \leqq N$,*

$$k_j = h_{N-j+1}.$$

Proof. The transformation $S_x = -T_{(-x)}$ with Rayleigh functional q defined by $q(u) = -p(u)$ satisfies the hypotheses laid down for T_x. Let $k_1' \leqq k_2' \leqq \cdots \leqq k_N'$ represent the min-max values of q. By their definition,

$$k_j' = \min_{\dim U = j} \max_{u \in U} \left(-p(u)\right) = \min\left(-\min p(u)\right)$$
$$= -\max \min p(u) = -h_j.$$

By Lemma 3, $k_1' \leqq k_2' \leqq \cdots \leqq k_N'$ is precisely the ordered family of eigenvalues of S_x in the range of q. Because $S_x = -T_{(-x)}$, the eigenvalues of T_x in the range of $p = -q$ are $-k_N' \leqq \cdots \leqq -k_1'$. However, Lemma 3 asserts that these are $k_1 \leqq \cdots \leqq k_N$. Equating these gives $k_j = -k_{N-j+1}' = h_{N-j+1}$.

Lemmas 3 and 4 together yield:

Theorem 1. *The value k is a minimax value if, and only if, k is in the range of p and k, v is an eigenpair for some non-zero v.*

Although the eigenvectors of distinct eigenvalues are not orthogonal, Lemma 2 shows their independence. It is a simple step to

Theorem 2. *The eigenvectors of T_x with eigenvalues in the range of p span R^N.*

Proof. Suppose $k = k_{j+1} = \cdots = k_{j+i}$ and let V_1 be a subspace of dimension $j+i$ such that $P(V_1) = k$. The proof of Lemma 3 shows that an eigenvector $w_1 \in V_1$ may be found. Denoting the orthogonal complement of w_1 in V_1 by V_2, we note that

$$k_{j+i-1} \leqq P(V_2) \leqq P(V_1) = k_{j+i} = k_{j+i-1}.$$

Thus an eigenvector $w_2 \in V_2$ may be found. In this way i mutually orthogonal, and thus independent, eigenvectors are generated.

Now for each distinct minimax value a maximal set of independent eigenvectors is chosen. The set of all vectors so chosen is denoted by $v_1, v_2, \ldots, v_N$. If the vectors of any subset all correspond to the same eigenvalue, then all linear combinations of them are eigenvectors of the common eigenvalue, also. Thus for an arbitrary choice of the ϱ_j's, $w = \sum_{j=1}^{N} \varrho_j v_j$ may be regarded as a sum of eigenvectors whose eigenvalues are distinct. By Lemma 2, $w \neq 0$ and consequently, $v_1, v_2, \ldots, v_N$ constitute a basis for R^N.

4. Constraints

Mathematically, a linear constraint is taken to be the restriction of T_x to a subspace of R^N and the consideration of k, v as an eigenpair in the constrained system if, and only if, $T_x v$ is orthogonal to the restricted domain of T_x. The result to be given concerns a one-dimensional constraint; that is, T_x is restricted to the subspace $[w]^{\perp}$, $w \neq 0$, of all vectors orthogonal to w. The consequences of multi-dimensional linear constraints may be deduced inductively from the case of a one-dimensional constraint.

The symbols k_j^* and h_j^* will be used to represent the minimax values of the functional p restricted to $[w]^{\perp}$. Since the constrained system fulfills all of the requirements, the lemmas and theorems already derived apply to it. Thus the k^*'s and h^*'s are all eigenvalues and $k_j^* = h^*_{(N-1)-j+1} = h^*_{N-j}$.

Because of the restriction in the available subspaces $k_j \leqq k_j^*$ and $h_j^* \leqq h_j$ for $1 \leqq j \leqq N-1$. Consequently,

$$k_j \leqq k_j^* = h^*_{N-j} \leqq h_{N-j} = k_{j+1}$$

or $k_j \leqq k_j^* \leqq k_{j+1}$ for $1 \leqq j \leqq N-1$. This proves our analogue of Rayleigh's Principle:

Theorem 3. *The eigenvalues of T_x in the range of p possess the separation property that under a one-dimensional constraint an eigenvalue is found in each closed interval between two adjacent eigenvalues of the unconstrained system.*

5. Remarks

The results obtained heretofore all have analogues in infinite systems; that is, when T_x is a symmetric linear operator on an infinite dimensional space. The transition requires the replacement of certain vectors by sequences of vectors. The "inf-sup" values k_j and the "sup-inf" values h_j have the "spectral" property that sequences of vectors $\{v_i\}_{i=1}^{\infty}$ and $\{w_i\}_{i=1}^{\infty}$ exist which do not tend to zero and which satisfy

$$\lim_{i\to\infty} T_{k_j} v_i = 0 \quad \text{and} \quad \lim_{i\to\infty} T_{h_j} w_i = 0.$$

A new approach to the analogue of Lemma 3 is necessary, and the duality expressed by Lemma 4 is lost.

In both finite and infinite dimensional cases a slight relaxation of the condition expressed by (7) is possible. That $(T'_{p(u)} u, u)$ is non-zero is used only in proving Lemmas 1 and 3. It suffices for Lemma 1 to require only that $\{u \,|\, (T'_{p(u)} u, u) > 0\}$ be connected and dense in R^N. Only a few alterations in the proof given are necessary. With this weakened hypothesis the alternative approach to Lemma 3 mentioned in the previous paragraph is adequate. It does not make use of non-zeroness of $(T'_{p(u)} u, u)$ for any given u.

As one might expect, the development given here is applicable to the case in which T_x is a Hermitian map of complex N-space to itself.

The results of this paper may be regarded from the point of view represented by abstract variational theory. MORSE [*8*, for instance] developed an "inf-sup" minimax theory for the stationary values of a real-valued function on a manifold. His theory deals with "inf-sup" values defined with classes of "cycles of a given dimension". It applies to the problem of this paper when p is regarded as a function on the unit sphere of R^N. Here the "cycles" have been restricted to the intersections of subspaces of a given dimension with the unit sphere. Thus in the problem of this paper a much smaller and more easily described class of "cycles on the unit sphere" suffices.

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