Wave Operators and Asymptotic Solutions of Wave Propagation Problems of Classical Physics

CALVIN H. WILCOX

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Contents

§ 1. Introduction

In the quantum theory of the scattering of elementary particles by a potential, the wave packets describing scattered particles are asymptotically equal, for large time, to corresponding wave packets describing free particles. The correspondence is given by the wave operator of MØLLER [*19, 20*]. In this paper analogous concepts are developed for wave propagation problems of classical physics. It is shown that waves propagating in an inhomogeneous medium are asymptotically equal, for large time, to corresponding waves propagating in a homogeneous medium. The correspondence is given by an analogue of the Møller wave operator. Since wave propagation problems for homogeneous media can be solved explicitly, the results provide asymptotic solutions of wave propagation problems for inhomogeneous media.

The wave propagation phenomena dealt with in this paper include electromagnetic waves, acoustic waves, seismic waves, electric waves on transmission lines, and other wave propagation phenomena of classical physics. A unified discussion of these phenomena is possible because they are all governed by systems of partial differential equations which can be written in the matrix form

$$E(x)\frac{\partial u}{\partial t}=\sum_{j=1}^{n} A^j \frac{\partial u}{\partial x_j}. \tag{1.1}$$

Here $x=(x_1, x_2, \dots, x_n)\in R^n$ (space), $t\in R^1$ (time), $u=u(x,t)$ is a real $m\times 1$ (column) matrix which describes the state of the medium at position x and time t, and $E(x)$, $A^1, A^2, \dots, A^n$ are $m\times m$ matrices with the following properties:

(1.2) $E(x)$ is real, symmetric, and positive definite, and

(1.3) $A^1, A^2, \ldots, A^n$ are real, symmetric, and constant.

Some of the wave equations of classical physics are exhibited in the matrix form (1.1) in the Appendix.

The matrix $E(x)$ defines the quadratic form*

$$\eta = u^* E(x) u \tag{1.4}$$

which is interpreted as an energy density (energy per unit volume) in the applications. The matrices A^j define the quadratic forms

$$\Sigma_j = -u^* A^j u \tag{1.5}$$

which are interpreted as the components of a Poynting vector describing the flow of energy (energy per unit area per unit time). Solutions of (1.1) satisfy a conservation of energy law which in differential form is

$$\frac{\partial \eta}{\partial t} + \sum_{j=1}^{n} \frac{\partial \Sigma_j}{\partial x_j} = 0. \tag{1.6}$$

Integration of (1.6) over $x \in R^n$, $0 \leqq t \leqq T$ gives the conservation law in integral form:

$$\int_{R^n} u^*(x, T) E(x) u(x, T) \, dx = \int_{R^n} u^*(x, 0) E(x) u(x, 0) \, dx. \tag{1.7}$$

It is well known that solutions of (1.1) are uniquely determined by their initial values

$$u(x, 0) = \varphi(x), \qquad x \in R^n. \tag{1.8}$$

The solution $u(x, t)$ of the initial value problem (1.1), (1.8) describes the propagation of waves in a medium whose states are governed by (1.1) and whose initial state is described by $\varphi(x)$. Hence, in what follows the initial value problem (1.1), (1.8) is called the propagation problem for (1.1).

The matrices A^j are assumed to be constant. If $E(x) = E^0$ is also constant, the medium governed by (1.1) is homogeneous. In this case the solution of the propagation problem can be constructed by the Fourier transform method, the method of plane waves and other methods. Such solutions have been studied by many authors; see [*6*] for a bibliography.

If $E(x)$ is not constant, the medium governed by (1.1) is inhomogeneous. There is a large literature concerning the existence, uniqueness, and regularity of solutions of initial value problems which is applicable to this case; see [*6*] for a bibliography. However, explicit methods for constructing the solution, comparable to the Fourier transform method for homogeneous media, are not available for inhomogeneous media.

This paper is concerned with the propagation problem for an inhomogeneous medium which is "homogeneous at ∞" in the sense that

$$\lim_{|x| \to \infty} E(x) = E^0 \tag{1.9}$$

* If M is a matrix, M^* denotes the transpose of M.

exists. If the initial state $\varphi(x)$ has finite energy, then physical intuition suggests that the energy will propagate out to ∞ as $t \to \infty$ (or $t \to -\infty$). Hence, if (1.9) holds, then for large t most of the wave will be in a region $|x| > R$ where $E(x)$ is nearly constant. This suggests that $u(x, t)$ may tend asymptotically, as $t \to \infty$, to a wave propagating in the homogeneous medium characterized by E^0; *i.e.*,

$$u(x, t) \sim u^0(x, t), \qquad t \to \infty, \tag{1.10}$$

where $u^0(x, t)$ solves

$$E^0 \frac{\partial u^0}{\partial t} = \sum_{j=1}^{n} A^j \frac{\partial u^0}{\partial x_j}. \tag{1.11}$$

A wave $u^0(x, t)$ with these properties is called an asymptotic solution of the propagation problem for (1.1). The reasoning leading to (1.10) is, of course, heuristic. It is the purpose of this paper to make precise the meaning of the asymptotic equality (1.10) and to give conditions on $E(x)$ and A^j ($j = 1, 2, \ldots, n$) which guarantee this behavior.

Of course, solutions of (1.11) also are uniquely determined by their initial values

$$u^0(x, 0) = \varphi^0(x), \qquad x \in R^n. \tag{1.12}$$

Thus if (1.10) holds, it follows that φ^0 uniquely determines $u(x, t)$, and hence $u(x, 0) = \varphi(x)$;

$$\varphi = \Omega \varphi^0. \tag{1.13}$$

Moreover, Ω is a linear operator since (1.10) is linear and $u(x, t)$ and $u^0(x, t)$ are linear functions of φ and φ^0 respectively. Ω is an analog of the Møller wave operators of quantum mechanics. It is called the wave operator for (1.1) in what follows. If Ω is known, then the asymptotic solution $u^0(x, t)$ of the propagation problem (1.1), (1.8) can be constructed from $\varphi^0 = \Omega^{-1} \varphi$ by the methods available for homogeneous media.

The energy form (1.4) defines a norm for the initial values $\varphi(x)$,

$$\|\varphi\|^2 = \int_{R^n} \varphi^*(x)\, E(x)\, \varphi(x)\, dx, \tag{1.14}$$

and the linear space $\mathscr{L}$ of initial values with finite energy is a Hilbert space $\mathscr{H}$ with respect to this norm. The correspondence $\varphi \to u(\cdot, t)$ defines a linear transformation $U(t)$ on $\mathscr{H}$ which is an isometry (with respect to the norm (1.14)) by the conservation of energy law (1.7). In fact, as is shown below, $U(t)$ defines a one-parameter group of unitary transformation on $\mathscr{H}$. The propagation problem has been studied from this point of view by Phillips [*15*].

The constant energy form based on E^0 also defines a norm

$$\|\varphi^0\|_0^2 = \int_{R^n} \varphi^{0*}(x)\, E^0\, \varphi^0(x)\, dx \tag{1.15}$$

which, in general, is different from (1.14). The space $\mathscr{L}_0$ of initial values φ^0 with finite energy is a Hilbert space $\mathscr{H}_0$ with respect to the norm (1.15).

In this paper the energy forms based on $E(x)$ and E^0 are assumed to be equivalent; *i.e.*,

$$c^2\, \xi^* E^0 \xi \leqq \xi^* E(x)\, \xi \leqq c'^2\, \xi^* E^0 \xi, \qquad \text{for all } x \in R^n \text{ and } \xi \in R^m, \tag{1.16}$$

where c and c' are positive constants. Under this hypothesis the norms (1.14) and (1.15) are equivalent; *i.e.*,

$$c\,\|\varphi\|_0 \leqq \|\varphi\| \leqq c'\,\|\varphi\|_0 . \tag{1.17}$$

Hence $\mathscr{L}$ and $\mathscr{L}_0$ are the same linear space of initial values and are Hilbert spaces with respect to two different, but equivalent, norms.

If $u(x,t)$ and $u^0(x,t)$ are solutions with finite energy of (1.1) and (1.11) respectively, then $u(x,t)-u^0(x,t)$ is in $\mathscr{H}$ (and also $\mathscr{H}_0$) by (1.17). It is shown below that this difference tends to zero in $\mathscr{H}$ (and, equivalently, in $\mathscr{H}_0$) when $t\to\infty$,

$$\lim_{t\to\infty} \|u(\cdot,t)-u^0(\cdot,t)\| = 0, \tag{1.18}$$

provided the initial values φ and φ^0 are related by an appropriate wave operator. The asymptotic equality (1.10) is interpreted in this sense in what follows.

The purpose of the preceding heuristic discussion is to motivate the work presented below. The remainder of the paper is organized as follows. In § 2 a precise formulation of the propagation problem is given in terms of self-adjoint operators on a Hilbert space, the existence of a solution is demonstrated, and some properties of the solutions that are needed later are derived. In § 3 asymptotic solutions and wave operators for the propagation problems are defined and a criterion for their existence is derived. § 4 describes a class of media, the uniformly propagative media, for which the asymptotic behavior for large time of waves can be estimated. A number of properties of such media which are needed later are derived in this section. In § 5 the Riemann matrix for a homogeneous medium is studied and used to obtain estimates for waves propagating in uniformly propagative media. In § 6 these estimates are used, together with the criterion of § 3, to obtain sufficient conditions for the existence of wave operators, and asymptotic solutions, for inhomogeneous media which are perturbations of uniformly propagative homogeneous media. Two such conditions are given. The first is applicable to perturbations of any uniformly propagative system. The second is applicable only to perturbations of systems of Maxwell type, a class of systems which includes MAXWELL's equations and the equations of acoustics.

This paper opens a new area of research on the wave propagation problems of classical physics, and the results presented here raise many more questions than they answer. Some unsolved problems and directions for future research are discussed briefly in § 7.

§ 2. A Hilbert Space Formulation of the Wave Propagation Problem

In this section a precise formulation of the propagation problem

$$E(x)\frac{\partial u}{\partial t} = \sum_{j=1}^{n} A^j \frac{\partial u}{\partial x_j}, \qquad x\in R^n,\ t>0, \tag{2.1}$$

$$u(x,0)=\varphi(x), \qquad x\in R^n \tag{2.2}$$

is given, and the existence and functional properties of the solution are discussed.

From the physical point of view all initial values $\varphi(x)$ with finite energy are admissible as initial states of a medium. Hence it is desirable to formulate the propagation problem in a way that admits all these initial states. Such a formulation, based directly on the conservation of energy law (1.7), was given by PHILLIPS in 1959 [*15*]. In this work PHILLIPS regards a solution of the propagation problem as a one-parameter group of unitary operators $U(t)$ on the Hilbert space $\mathscr{H}$ of initial values,

$$\varphi \to u(\cdot, t) = U(t)\, \varphi,$$

and proves the existence of such a solution for a large class of coefficient matrices E and A^j. A second formulation, more directly connected with the partial differential equation (2.1), was given by the author in 1962 [*17, 18*]. In this work a class of solutions wFE (=with finite energy) of (2.1), (2.2) was introduced, and the existence and uniqueness of such solutions was proved for all initial states $\varphi(x)$ with finite energy. It is not difficult to show the identity of these two formulations of the problem, but the proof will not be given here. Instead, PHILLIPS' viewpoint is adopted and a simple direct construction of the solution operator $U(t)$ is given. The construction makes use of a number of well-known definitions and theorems concerning self-adjoint and unitary operators on a Hilbert space. These may be found in RIESZ-NAGY [*16*].

If $U(t)$ defines a continuous one-parameter group of unitary operators on $\mathscr{H}$, then by STONE's theorem [*16*]

$$U(t) = e^{-itH} \tag{2.3}$$

where H is a self-adjoint operator on $\mathscr{H}$. Conversely, every self-adjoint operator H on $\mathscr{H}$ generates a continuous one-parameter group of unitary operators on $\mathscr{H}$ defined by (2.3). If (2.3) defines a solution operator for (2.1), (2.2), then formally H is the operator

$$i\, E(x)^{-1} \sum_{j=1}^{n} A^j \frac{\partial}{\partial x_j}. \tag{2.4}$$

In this paper the solution of the propagation problem is defined by constructing a self-adjoint extension H of the differential operator (2.4) and then defining the solution operator by (2.3). The construction is given first for the case of a homogeneous medium ($E(x) = E^0$ constant).

The system of partial differential equations (1.11) describing a homogeneous medium may be written

$$\frac{\partial u^0}{\partial t} = (E^0)^{-1} \sum_{j=1}^{n} A^j \frac{\partial u^0}{\partial x_j} = -i\, H_0\, u \tag{2.5}$$

where

$$H_0 = i (E^0)^{-1} \sum_{j=1}^{n} A^j \frac{\partial}{\partial x_j}. \tag{2.6}$$

The factor $i = \sqrt{-1}$ has been introduced to make H_0 formally self-adjoint on $\mathscr{H}_0$. Because of its presence it is convenient to work with complex-valued initial values $u^0(x, 0) = \varphi^0(x)$ and solutions $u^0(x, t)$. Of course, these solutions include all the real solutions which correspond to real initial values, because E^0 and the A^j are real.

The matrix E^0 was assumed to be real and symmetric (hence Hermitian) and positive definite. It follows that

$$\lambda^2 \sum_{\alpha=1}^{m} |\zeta_\alpha|^2 \leqq \zeta^* E^0 \bar{\zeta} \leqq \lambda'^2 \sum_{\alpha=1}^{m} |\zeta_\alpha| \qquad \text{for all } \zeta \in C^m, \tag{2.7}$$

where

$$C^m = \{\zeta = (\zeta_1, \zeta_2, \dots, \zeta_m) \colon \zeta_\alpha \text{ complex}\},$$

$\bar{\zeta} = (\bar{\zeta}_1, \bar{\zeta}_2, \dots, \bar{\zeta}_m)$ denotes the complex conjugate vector and λ and λ' are positive constants, the smallest and largest eigenvalues of E^0 respectively. (2.7) implies that

$$\lambda^2 \int_{R^n} \sum_{\alpha=1}^{m} |\varphi_\alpha(x)|^2 \, dx \leqq \int_{R^n} \varphi^*(x) E^0 \bar{\varphi}(x) \, dx \leqq \lambda'^2 \int_{R^n} \sum_{\alpha=1}^{m} |\varphi_\alpha(x)|^2 \, dx \tag{2.8}$$

for each Lebesgue-measurable complex vector $\varphi(x) = \varphi^1(x) + i\varphi^2(x)$. If the energy, for homogeneous media, of such vectors is defined by

$$\int_{R^n} \varphi^*(x) E^0 \bar{\varphi}(x) \, dx = \int_{R^n} \varphi^{1*}(x) E^0 \varphi^1(x) \, dx + \int_{R^n} \varphi^{2*}(x) E^0 \varphi^2(x) \, dx,$$

then (2.8) implies that $\varphi(x)$ has finite energy if and only if, for each $\alpha = 1, 2, \dots, m$,

$\varphi_\alpha \in L_2(R^n) =$ the Lebesgue space of complex-valued measurable, square-integrable functions on R^n.

Hence the direct sum

$$\mathscr{L}_0 = L_2(R^n) \oplus L_2(R^n) \oplus \cdots \oplus L_2(R^n), \qquad m \text{ summands}$$

is the appropriate linear space of initial values with finite energy for (2.5). It is easy to verify that $\mathscr{L}_0$ is a Hilbert space $\mathscr{H}_0$ with respect to the energy inner product

$$(\varphi, \psi)_0 = \int_{R^n} \varphi^*(x) E^0 \bar{\psi}(x) \, dx. \tag{2.9}$$

It is shown next that the differential operator (2.6) has an extension which is a self-adjoint operator with respect to the inner product (2.9) on $\mathscr{H}_0$. The construction makes use of the Plancherel theory of the Fourier transform. The basic theorems may be stated as follows [*2*]. If $f(x) \in L_2(R^n)$, then

$$\hat{f}(p) = \underset{R \to \infty}{\text{l.i.m.}} \frac{1}{(2\pi)^{n/2}} \int_{|x| \leqq R} e^{i p \cdot x} f(x) \, dx$$

exists in $L_2(R^n)$ and

$$f(x) = \underset{R \to \infty}{\text{l.i.m.}} \frac{1}{(2\pi)^{n/2}} \int_{|p| \leqq R} e^{-i x \cdot p} \hat{f}(p) \, dp.$$

Here $p = (p_1, p_2, \dots, p_n) \in R^n$, $p \cdot x = p_1 x_1 + p_2 x_2 + \cdots + p_n x_n$, and l.i.m. signifies convergence in $L_2(R^n)$. Moreover, functions f and g in $L_2(R^n)$ satisfy PARSEVAL'S formula

$$\int_{R^n} f(x) \overline{g(x)} \, dx = \int_{R^n} \hat{f}(p) \overline{\hat{g}(p)} \, dp. \tag{2.10}$$

It follows that the Fourier transform defines a unitary transformation of $L_2(R^n)$ onto itself. The Fourier transform of a function $\varphi \in \mathscr{H}_0$ is defined by

$$\hat{\varphi}(p) = (\hat{\varphi}_1(p), \hat{\varphi}_2(p), \dots, \hat{\varphi}_m(p))^*.$$

It follows easily from this definition and (2.10) that

$$(2.11) \qquad (\varphi, \psi)_0 = \int_{R^n} \varphi^*(x)\, E^0\, \bar{\psi}(x)\, dx = \int_{R^n} \hat{\varphi}^*(p)\, E^0\, \bar{\hat{\psi}}(p)\, dp = (\hat{\varphi}, \hat{\psi})_0$$

for all $\varphi, \psi \in \mathscr{L}_0$. Hence the Fourier transform also defines a unitary transformation of $\mathscr{H}_0$ onto itself.

If $f(x)$ and $\dfrac{\partial f(x)}{\partial x_j}$ are in $L_2(R^n)$, it follows that

$$\widehat{\left(\frac{\partial f}{\partial x_j}\right)}(p) = -i\, p_j\, \hat{f}(p).$$

Hence, if $\varphi(x)$ and $\dfrac{\partial \varphi(x)}{\partial x_j}$ are in $\mathscr{H}_0$ for $j = 1, 2, \dots, n$, then

$$H_0\, \varphi = i(E^0)^{-1} \sum_{j=1}^{n} A^j \frac{\partial \varphi}{\partial x_j}$$

is in $\mathscr{H}_0$ and has the Fourier transform

$$(\widehat{H_0\, \varphi})(p) = (E^0)^{-1} \left(\sum_{j=1}^{n} A^j\, p_j\right) \hat{\varphi}(p).$$

This suggests the following theorem.

Theorem 2.1. *The operator H_0 on $\mathscr{H}_0$ with domain*

$$D(H_0) = \left\{\varphi\colon \hat{\varphi}(p) \text{ and } \sum_{j=1}^{n} A^j\, p_j\, \hat{\varphi}(p) \text{ are in } \mathscr{H}_0\right\}$$

and range defined by

$$(2.12) \qquad (H_0\, \varphi)(x) = \underset{R \to \infty}{\text{l.i.m.}} \frac{1}{(2\pi)^{n/2}} \int_{|p| \leq R} e^{-i x \cdot p} (E^0)^{-1} \sum_{j=1}^{n} A^j\, p_j\, \hat{\varphi}(p)\, dp$$

is a self-adjoint operator with respect to the energy inner product (2.9).

Proof. The Plancherel theory implies that (2.12) defines a vector $H_0\, \varphi \in \mathscr{H}_0$ for each $\varphi \in D(H_0)$. Moreover, it is easy to show that $D(H_0)$ is dense in $\mathscr{H}_0$. Hence the adjoint operator H^* is well defined. The proof of the theorem is completed by showing that $H_0 \subset H_0^*$ (H_0^* is an extension of H_0) and $H_0^* \subset H_0$, whence $H_0 = H_0^*$.

To prove $H_0 \subset H_0^*$, let $\varphi, \psi \in D(H_0)$. Then by (2.11)

$$(H_0\, \varphi, \psi) = (\widehat{H_0\, \varphi}, \hat{\psi})_0 = \int_{R^n} (\widehat{H_0\, \varphi})^*(p)\, E^0\, \bar{\hat{\psi}}(p)\, dp.$$

Moreover

$$(\widehat{H_0\, \varphi})^*(p) = \left((E^0)^{-1} \left(\sum_{j=1}^{n} A^j\, p_j\right) \hat{\varphi}(p)\right)^* = \hat{\varphi}^*(p) \left(\sum_{j=1}^{n} A^j\, p_j\right) (E^0)^{-1}$$

because $(E^0)^* = E^0$, $(A^j)^* = A^j$ and $((E^0)^{-1})^* = ((E^0)^*)^{-1}$. Thus

$$(H_0\,\varphi, \psi) = \int_{R^n} \hat{\varphi}^*(p)\, E^0 \overline{(E^0)^{-1}\left(\sum_{j=1}^n A^j p_j\right) \hat{\psi}(p)}\, dp = (\hat{\varphi}, \widehat{H_0\,\psi_0}) = (\varphi, H_0\,\psi)_0 .$$

This proves that if $\psi \in D(H_0)$, then $\psi \in D(H_0^*)$ and $H_0^*\,\psi = H_0\,\psi$; *i.e.*, $H_0 \subset H^*$.

To prove $H_0^* \subset H_0$, let $\psi \in D(H_0^*)$; *i.e.*, $\psi \in \mathscr{H}_0$ and

(2.13) $(H_0\,\varphi, \psi) = (\varphi, \vartheta)$ for some vector $\vartheta \in \mathscr{H}_0$ and all $\varphi \in D(H_0)$.

The vector $\vartheta = H_0^*\,\psi$, by definition. Equations (2.11) and (2.13) imply

$$(H_0\,\varphi, \psi)_0 = \int_{R^n} \hat{\varphi}^*(p) \overline{\left(\sum_{j=1}^n A^j p_j\right) \hat{\psi}(p)}\, dp = \int_{R^n} \hat{\varphi}^*(p)\, \overline{E^0\, \hat{\vartheta}(p)}\, dp = (\varphi, \vartheta)_0$$

for all $\varphi \in D(H_0)$. But $D(H_0)$ is dense in $\mathscr{H}_0$, whence

$$(E^0)^{-1}\left(\sum_{j=1}^n A^j p_j\right) \hat{\psi}(p) = \hat{\vartheta}(p) \in \mathscr{H}_0 .$$

Thus $\psi \in D(H_0)$ and

$$(H_0\,\psi)(x) = \underset{R\to\infty}{\text{l.i.m.}} \frac{1}{(2\pi)^{n/2}} \int_{|p|<R} e^{-i x\cdot p} (E^0)^{-1} \sum_{j=1}^n A^j p_j\, \hat{\psi}(p)\, dp = \vartheta(x) \qquad \text{in } \mathscr{H}_0 .$$

This proves that if $\psi \in D(H_0^*)$, then $\psi \in D(H_0)$ and $H_0\,\psi = \vartheta = H_0^*\,\psi$; *i.e.*, $H_0^* \subset H$.

The propagation problem for an inhomogeneous medium governed by (2.1) is discussed next. System (2.1) may be written

$$\frac{\partial u}{\partial t} = E(x)^{-1} \sum_{j=1}^n A^j \frac{\partial u}{\partial x_j} = -i\,H\,u \tag{2.14}$$

where

$$H = i\,E(x)^{-1} \sum_{j=1}^n A^j \frac{\partial}{\partial x_j} . \tag{2.15}$$

It is easy to verify that H is a formally self-adjoint operator with respect to the energy inner product

$$(\varphi, \psi) = \int_{R^n} \varphi^*(x)\, E(x)\, \bar{\psi}(x)\, dx . \tag{2.16}$$

Now, the energy forms based on $E(x)$ and E^0 were assumed to be equivalent; see (1.16) which implies

$$c^2\, \zeta^*\, E^0\, \bar{\zeta} \leqq \zeta^*\, E(x)\, \bar{\zeta} \leqq c'^2\, \zeta^*\, E^0\, \bar{\zeta}, \qquad \text{for all } x \in R^n \text{ and } \zeta \in C^m.$$

Combining this with (2.7) gives

$$\mu^2 \sum_{\alpha=1}^m |\zeta_\alpha|^2 \leqq \zeta^*\, E(x)\, \bar{\zeta} \leqq \mu'^2 \sum_{\alpha=1}^m |\zeta_\alpha|^2, \qquad \text{for all } x \in R^n \text{ and } \zeta \in C^m, \tag{2.17}$$

where $\mu = \lambda\, c$ and $\mu' = \lambda' c'$. It is assumed that the components $E_{\alpha\beta}(x)$ of $E(x)$ are Lebesgue-measurable functions in R^n. It follows from (2.17) that

$$\mu^2 \int_{R^n} \sum_{\alpha=1}^m |\varphi_\alpha(x)|^2\, dx \leqq \int_{R^n} \varphi^*(x)\, E(x)\, \bar{\varphi}(x)\, dx \leqq \mu'^2 \int_{R^n} \sum_{\alpha=1}^m |\varphi_\alpha(x)|^2\, dx \tag{2.18}$$

for each measurable complex vector $\varphi(x)$. Thus if the energy, for inhomogeneous media, of such vectors is defined by

$$\|\varphi\|^2 = \int_{R^n} \varphi^*(x)\, E(x)\, \overline{\Phi(x)}\, dx,$$

then (2.18) implies that $\varphi(x)$ has finite energy if and only if, as before, each $\varphi_\alpha \in L(R^n)$, $\alpha = 1, 2, \ldots, m$. Hence $\mathscr{L}_0$ is also the appropriate linear space of initial values with finite energy for (2.14). It is easy to verify that $\mathscr{L}_0$ is a Hilbert space $\mathscr{H}$ with respect to the energy inner product (2.16).

It is shown next that the differential operator (2.15) has an extension which is a self-adjoint operator with respect to the inner product (2.16) on $\mathscr{H}$. Notice that formally

$$H = i\, E(x)^{-1} E^0 (E^0)^{-1} \sum_{j=1} A^j \frac{\partial}{\partial x_j} = E(x)^{-1} E^0 H_0 .$$

This suggests the following generalization of Theorem 2.1.

Theorem 2.2. *The operator H on $\mathscr{H}$ with domain $D(H) = D(H_0)$ and range defined by*

$$(H\varphi)(x) = E^{-1}(x)\, E^0 (H_0 \varphi)(x) \tag{2.19}$$

is a self-adjoint operator with respect to the energy inner product (2.16).

The proof makes use of the following lemma which is also needed in § 6 below.

Lemma 2.1. *If $E(x)$ is a real symmetric positive definite matrix which satisfies* (2.17) *with positive constants μ, μ', then*

$$\frac{1}{\mu'^2} \sum_{\alpha=1}^{m} |\zeta_\alpha|^2 \leqq \zeta^* E^{-1}(x)\, \bar{\zeta} \leqq \frac{1}{\mu^2} \sum_{\alpha=1}^{m} |\zeta_\alpha|^2, \qquad \text{for all } x \in R^n \text{ and } \zeta \in C^m. \tag{2.20}$$

Proof. (2.17) is equivalent to the statement that the (real, positive) eigenvalues of $E(x)$ lie between μ and μ'. (2.20) follows immediately since the eigenvalues of $E^{-1}(x)$ are the reciprocals of the eigenvalues of $E(x)$.

Proof of Theorem 2.2. To see that (2.19) defines a mapping from $D(H_0)$ into $\mathscr{H}$, note that for each $\varphi \in D(H_0)$, $H_0 \varphi \in \mathscr{H}$ (by Theorem 2.1) and hence $\psi = E^0 (H_0 \varphi) \in \mathscr{H}$. Now Lemma 2.1 with

$$\zeta = \psi(x), \qquad E^{-1}(x)\,\zeta = E^{-1}(x)\,\psi(x) = \vartheta(x)$$

implies

$$\frac{1}{\mu'^2} \sum_{\alpha=1}^{m} |\psi_\alpha(x)|^2 \leqq \vartheta^*(x)\, E(x)\, \vartheta(x) \leqq \frac{1}{\mu^2} \sum_{\alpha=1}^{m} |\psi_\alpha(x)|^2$$

whence

$$\vartheta^*(x)\, E(x)\, \vartheta(x) \in L_1(R^n), \tag{2.21}$$

since $\varphi \in \mathscr{H}$. (2.21) and (2.18) with φ replaced by ϑ implies that

$$\vartheta(x) = E^{-1}(x)\, \psi(x) = E^{-1}(x)\, E^0 (H_0 \varphi)(x) \in \mathscr{H};$$

i.e., H maps $D(H) = D(H_0)$ into $\mathscr{H}$.

The proof of Theorem 2.2 is completed by showing that $H \subset H^*$ and $H^* \subset H$, whence $H = H^*$. First note that if φ and ψ are in $\mathscr{H}$,

$$(2.22)\qquad \begin{aligned}(\varphi, \psi) &= \int_{R^n} \varphi^*(x) E(x) \overline{\psi(x)}\, dx = \int_{R^n} \varphi^*(x) E^0 \overline{(E^0)^{-1} E(x) \psi(x)}\, dx \\ &= (\varphi, (E^0)^{-1} E \psi)_0 .\end{aligned}$$

In particular, if $\varphi \in D(H)$, $\psi \in \mathscr{H}$, then (2.22) implies

$$(2.23)\qquad (H\varphi, \psi) = \overline{(\psi, H\varphi)} = \overline{(\psi, (E^0)^{-1} E H \varphi)_0} = \overline{(\psi, H_0 \varphi)_0} = (H_0 \varphi, \psi)_0 .$$

To prove $H \subset H^*$, let $\varphi, \psi \in D(H) = D(H^0)$. Then, by (2.23) and the self-adjointness of H_0,

$$\begin{aligned}(H\varphi, \psi) &= (H_0 \varphi, \psi)_0 = (\varphi, H_0 \psi)_0 = \overline{(H_0 \psi, \varphi)_0} \\ &= \overline{(H\psi, \varphi)} = (\varphi, H\psi) .\end{aligned}$$

Thus $\psi \in D(H^*)$ and $H^* \psi = H \psi$; *i.e.*, $H \subset H^*$.

To prove $H^* \subset H$, let $\psi \in D(H^*)$; *i.e.*, $\psi \in \mathscr{H}$ and

(2.24) $(H\varphi, \psi) = (\varphi, \vartheta)$ for some $\vartheta \in \mathscr{H}$ and all $\varphi \in D(H)$. The vector $\vartheta = H^* \psi$, by definition. Equations (2.22) and (2.23), applied to (2.24), give

$$(H_0 \varphi, \psi)_0 = (\varphi, (E^0)^{-1} E \vartheta)_0 \qquad \text{for all } \varphi \in D(H) = D(H^0) .$$

Since H_0 is self-adjoint, this implies that $\psi \in D(H_0^*) = D(H_0) = D(H)$ and

$$H_0^* \psi = H_0 \psi = (E^0)^{-1} E \vartheta = (E^0)^{-1} E H^* \psi .$$

Thus

$$H^* \psi = E^{-1} E^0 (H_0 \psi) = H \psi ;$$

i.e., $H^* \subset H$.

The Spectral Theorem [*16*] implies that the self-adjoint operators H_0 and H have spectral resolutions

$$H_0 = \int_{-\infty}^{\infty} \lambda\, dE_0(\lambda), \qquad H = \int_{-\infty}^{\infty} \lambda\, dE(\lambda)$$

where $E_0(\lambda)$ and $E(\lambda)$ are resolutions of the identity for $\mathscr{H}_0$ and $\mathscr{H}$, respectively.

The solution operators for the propagation problems are defined by

$$(2.25)\qquad U_0(t) = e^{-itH_0} = \int_{-\infty}^{\infty} e^{-it\lambda}\, dE_0(\lambda)$$

for homogeneous media, and by

$$(2.26)\qquad U(t) = e^{-itH} = \int_{-\infty}^{\infty} e^{-it\lambda}\, dE(\lambda)$$

for inhomogeneous media. Thus

$$u^0(x, t) = (U_0(t)\varphi^0)(x)$$

and

$$u(x, t) = (U(t)\varphi)(x)$$

are interpreted as the solutions of the propagation problems for homogeneous and inhomogeneous media, respectively. The following properties of the solutions follow directly from (2.25), (2.26) and the Spectral Theorem: see [*19*, p. 614] for the proofs.

Corollary 2.1.

(a) *$U_0(t)$ and $U(t)$ define one-parameter groups of unitary operators on $\mathscr{H}_0$ and $\mathscr{H}$, respectively. In particular, the following conservation of energy laws hold:*

$$\| u^0(\cdot, t) \|_0 = \| \varphi^0 \|_0 \quad \textit{and} \quad \| u(\cdot . t) \| = \| \varphi \| \qquad \textit{for all } t \in R^1.$$

(b) *$u^0(\cdot, t) \in C(-\infty, \infty; \mathscr{H}_0)$ and $u(\cdot, t) \in C(-\infty, \infty; \mathscr{H})$*; i.e., *for each $t \in R^1$*

$$\lim_{\tau \to 0} \| u^0(\cdot, t+\tau) - u^0(\cdot, t) \|_0 = 0 \quad \textit{and} \quad \lim_{\tau \to 0} \| u(\cdot, t+\tau) - u(\cdot, t) \| = 0.$$

(c) *If $\varphi^0 \in D(H^0)$, then $u^0(\cdot, t) \in D(H^0)$ for every $t \in R^1$. Moreover, $u^0(\cdot, t) \in C^1(-\infty, \infty; \mathscr{H}_0)$, and*

$$\frac{\partial u^0(\cdot, t)}{\partial t} = -i H_0 u^0(\cdot, t) \qquad \textit{for each } t \in R^1.$$

The corresponding statements hold for $u(\cdot, t)$.

§ 3. Wave Operators and the Existence of Asymptotic Solutions

This section deals with the abstract propagation problems formulated in § 2. The asymptotic equality as $t \to \infty$ of the solutions of two such problems is shown to depend on the existence of a wave operator, and a criterion for the existence of the wave operator is derived. The results presented in this section closely parallel analogous results for the quantum mechanical scattering problem due to Cook [*5*] and Kuroda [*12*].

The following notations and hypotheses, suggested by the discussion in § 2, are adopted in this section.

(3.1) $\mathscr{L}_0$ is a linear space over the complex number field.

(3.2) $(\varphi, \psi)_0$ and (φ, ψ) are two inner products on $\mathscr{L}_0$ such that $\mathscr{L}_0$ becomes a Hilbert space $\mathscr{H}_0$ with respect to $(\varphi, \psi)_0$ and a Hilbert space $\mathscr{H}$ with respect to (φ, ψ). ($\mathscr{H}_0$ and $\mathscr{H}$ are assumed to be separable.)

(3.3) The norms associated with the two inner products are equivalent; *i.e.*, there exist positive constants c and c' such that

$$c \| \varphi \|_0 \leqq \| \varphi \| \leqq c' \| \varphi \|_0$$

for all $\varphi \in \mathscr{L}_0$.

(3.4) H and H_0 are linear operators on $\mathscr{L}_0$, self-adjoint with respect to (φ, ψ) and $(\varphi, \psi)_0$ respectively, and

$$D(H) = D(H_0).$$

In § 2

$$u(t) = e^{-itH} \varphi \quad \text{and} \quad u^0(t) = e^{-itH_0} \varphi^0$$

represent solutions of the propagation problems for inhomogeneous and homogeneous media respectively. Conditions on the initial states φ and φ^0 are sought which guarantee the asymptotic equality of $u(t)$ and $u^0(t)$ as $t\to\infty$ in the sense that

$$\lim_{t\to\infty}\|u(t)-u^0(t)\|=\lim_{t\to\infty}\|e^{-itH}\varphi-e^{-itH_0}\varphi^0\|=0. \tag{3.5}$$

By (3.3) this is equivalent to the condition

$$\lim_{t\to\infty}\|u(t)-u^0(t)\|_0=0.$$

Now

$$\|e^{-itH}\varphi-e^{-itH_0}\varphi^0\|=\|\varphi-e^{itH}e^{-itH_0}\varphi^0\|$$

because e^{itH} is a unitary operator on $\mathscr{H}$. Thus (3.5) is equivalent to the condition

$$\lim_{t\to\infty}\|e^{itH}e^{-itH_0}\varphi^0-\varphi\|=0. \tag{3.6}$$

Since $\mathscr{H}$ is complete with respect to the norm $\|\psi\|$, (3.6) holds if and only if

$$\Omega\varphi^0=\lim_{t\to\infty}e^{itH}e^{-itH_0}\varphi^0 \text{ exists} \tag{3.7}$$

in $\mathscr{H}$ for each $\varphi^0\in\mathscr{H}_0$. Thus each solution $u^0(t)$ is asymptotically equal to a solution $u(t)$ if and only if (3.7) holds for each $\varphi^0\in\mathscr{H}_0$; *i.e.*, if and only if the wave operator $\Omega\colon\mathscr{H}_0\to\mathscr{H}$ defined by

$$\Omega=\Omega(H,H_0)=\operatorname*{s-lim}_{t\to\infty}e^{itH}e^{-itH_0} \tag{3.8}$$

exists (s-lim signifies strong limit). If Ω exists, then (3.5) holds with

$$\varphi=\Omega\varphi^0.$$

A number of properties of the wave operator Ω follow directly from its existence. These are summarized in the following theorem (*cf.* KURODA [*12*], pp. 438 – 442).

Theorem 3.1. *If the wave operator $\Omega=\Omega(H,H_0)$ exists, then*

(a) *Ω is one-to-one and bounded. In fact*

$$c\|\varphi\|_0\leqq\|\Omega\varphi\|\leqq c'\|\varphi\|_0 \qquad \textit{for all } \varphi\in\mathscr{H}_0, \tag{3.9}$$

where c and c' are the positive constants defined in (3.3);

(b) *Ω satisfies the following operator identities:*

$$e^{-itH}\Omega=\Omega e^{-itH_0}, \qquad -\infty<t<\infty, \tag{3.10}$$

$$E(\lambda)\Omega=\Omega E_0(\lambda), \qquad -\infty<\lambda<\infty, \tag{3.11}$$

$$H\Omega=\Omega H_0. \tag{3.12}$$

(c) *$\Omega\mathscr{H}_0=\mathscr{H}$*; i.e., *the range of Ω is the whole space $\mathscr{H}$, if and only if $\Omega(H_0,H)$ also exists. In this case $\Omega(H,H_0)$ has an inverse and*

$$\Omega(H,H_0)^{-1}=\Omega(H_0,H). \tag{3.13}$$

Proof. Write

$$\Omega(t)=e^{itH}e^{-itH_0}, \quad \text{so} \quad \Omega=\operatorname*{s-lim}_{t\to\infty}\Omega(t).$$

Then since e^{itH} and e^{-itH_0} are unitary with respect to the norms $\|\varphi\|$ and $\|\varphi\|_0$ respectively, (3.3) implies

$$c\,\|\varphi\|_0=c\,\|e^{-itH_0}\varphi\|_0\leqq\|e^{-itH_0}\varphi\|=\|\Omega(t)\,\varphi\|\leqq c'\,\|e^{-itH_0}\varphi\|_0=c'\,\|\varphi\|_0\,.$$

Making $t\to\infty$ and using the continuity of the norm $\|\varphi\|$ gives (3.9).

Notice that

$$e^{-itH}\,\Omega(s)=e^{-itH}\,e^{isH}\,e^{-isH_0}\,e^{itH_0}\,e^{-itH_0}=\Omega(s-t)\,e^{-itH_0}.$$

Making $s\to\infty$ in this identity gives (3.10). It is known that (3.10) implies (3.11); see, *e.g.*, Riesz & Sz.-Nagy [*16*], pp. 383–384. Moreover, (3.11) implies

$$\int_{-N}^{M}\lambda\,d(\Omega\,E_0(\lambda)\,\varphi)=\int_{-N}^{M}\lambda\,d(E(\lambda)\,\Omega\,\varphi), \qquad \varphi\in\mathscr{H}_0\,. \tag{3.14}$$

The left-hand side of this equation tends to a limit $(=\Omega\,H_0\,\varphi)$ in $\mathscr{H}$ when $M, N\to\infty$ if and only if $\varphi\in D(H_0)=D(\Omega\,H_0)$. Similarly, the right-hand side tends to a limit $(=H\,\Omega\,\varphi)$ if and only if $\varphi\in D(H\,\Omega)$. Thus (3.14) implies (3.12).

If both $\Omega(H, H_0)$ and

$$\Omega(H_0, H)=\operatorname*{s-lim}_{t\to\infty}e^{itH_0}\,e^{-itH}$$

exist, then the identity

$$(e^{itH}\,e^{-itH_0})(e^{itH_0}\,e^{-itH})\,\varphi=\varphi, \qquad \varphi\in\mathscr{H},$$

implies the identity

$$\Omega(H, H_0)\,\Omega(H_0, H)=I \tag{3.15}$$

where I is the identity operator on $\mathscr{H}$. (3.15) implies that $\Omega\,\mathscr{H}_0=\Omega(H, H_0)\,\mathscr{H}_0=\mathscr{H}$. Conversely, $\Omega\,\mathscr{H}_0=\mathscr{H}$ implies that for each $\varphi\in\mathscr{H}$ there is a $\psi\in\mathscr{H}_0$ such that

$$\varphi=\Omega\,\psi=\lim_{t\to\infty}e^{itH}\,e^{-itH_0}\,\psi\,. \tag{3.16}$$

Now (3.3) implies that

$$\begin{aligned}c\,\|e^{itH_0}\,e^{-itH}\,\varphi-\psi\|_0&=c\,\|e^{-itH}\,\varphi-e^{-itH_0}\,\psi\|_0\\&\leqq\|e^{-itH}\,\varphi-e^{-itH_0}\,\psi\|=\|\varphi-e^{itH}\,e^{-itH_0}\,\psi\|\,.\end{aligned}$$

Thus (3.16) implies that $\Omega(H_0, H)\,\varphi=\psi$ exists for each $\varphi\in\mathscr{H}$; *i.e.* $\Omega(H_0, H)$ exists.

The following theorem gives a simple sufficient condition for the existence of the wave operator $\Omega=\Omega(H, H_0)$. It is an analogue of a condition for the existence of quantum mechanical wave operators due to Cook [*5*] (see also Kuroda [*12*, p. 443]).

Theorem 3.2. *Let* $\mathscr{L}_0$, $\|\varphi\|_0$, $\|\varphi\|$, H_0 *and* H *satisfy hypotheses* (3.1), (3.2), (3.3), *and* (3.4). *Let* $\mathscr{D}$ *be a subset of* $\mathscr{L}_0$ *which satisfies*

(3.17) *the linear manifold determined by* $\mathscr{D}$ *is dense in* $\mathscr{H}_0$, *and*

$$\mathscr{D} \subset D(H_0) = D(H). \tag{3.18}$$

Then a sufficient condition for the existence of the wave operator $\Omega = \Omega(H, H_0)$ is

$$\int_T^\infty \| (H - H_0) e^{-itH_0} \varphi \| \, dt < \infty \tag{3.19}$$

for each $\varphi \in \mathscr{D}$ and some finite T (T may depend on φ).

Proof. If $\varphi(t) = e^{itH} e^{-itH_0} \varphi$ defines a Cauchy sequence in $\mathscr{H}$ for $t \to \infty$, then

$$\Omega \varphi = \lim_{t \to \infty} \varphi(t)$$

exists. It will be shown that (3.19) implies that $\varphi(t)$ defines a Cauchy sequence for each $\varphi \in \mathscr{H}_0$.

First, let $\varphi \in \mathscr{D}$. Then, by Corollary 2.1, $\varphi(t) \in D(H_0) = D(H)$ for every t. Moreover, $\varphi(t) \in C^1(-\infty, \infty; \mathscr{H})$ and

$$\frac{d\varphi(t)}{dt} = i H e^{itH} e^{-itH_0} \varphi - e^{itH}(-i H_0 e^{-itH_0} \varphi) = i e^{itH} (H - H_0) e^{-itH_0} \varphi \tag{3.20}$$

since H commutes with e^{itH}. Thus

$$\varphi(t) - \varphi(s) = i \int_s^t e^{i\tau H} (H - H_0) e^{-i\tau H_0} \, d\tau,$$

and it follows that

$$\| \varphi(t) - \varphi(s) \| = \left\| \int_s^t e^{i\tau H} (H - H_0) e^{-i\tau H_0} \, d\tau \right\| \leqq \int_s^t \| (H - H_0) e^{-i\tau H_0} \varphi \| \, d\tau$$

because $e^{i\tau H}$ is unitary with respect to $\| \varphi \|$. Thus (3.19) implies that $\varphi(t)$ defines a Cauchy sequence for $t \to \infty$ if $\varphi \in \mathscr{D}$. If $\varphi' \in \mathscr{H}_0$ is an arbitrary vector, then

$$\begin{aligned} &\| e^{itH} e^{-itH_0} \varphi' - e^{isH} e^{-isH_0} \varphi' \| \\ &\quad \leqq \| \varphi(t) - \varphi(s) \| + \| e^{itH} e^{-itH_0} (\varphi' - \varphi) \| + \| e^{isH} e^{-isH_0} (\varphi' - \varphi) \| \\ &\quad \leqq \| \varphi(t) - \varphi(s) \| + 2c' \, \| \varphi' - \varphi \| \end{aligned}$$

by (3.3) and the unitary properties of e^{itH} and e^{-itH_0}. This inequality and (3.17) imply that $e^{itH} e^{-itH_0} \varphi'$ defines a Cauchy sequence when $t \to \infty$ for every $\varphi' \in \mathscr{H}_0$.

§ 4. Uniformly Propagative Homogeneous Media

This section describes a class of homogeneous media, the uniformly propagative media, for which the asymptotic behavior for large time of waves can be estimated. The definition of this class depends on the concepts of normal speed surface, slowness surface and wave cone for a system

$$E^0 \frac{\partial u^0}{\partial t} = \sum_{j=1} A^j \frac{\partial u^0}{\partial x_j}. \tag{4.1}$$

These concepts are defined, the uniformly propagative media are introduced, and a number of their properties are derived in this section.

The system (4.1) has plane wave solutions; *i.e.*, solutions of the form

$$u^0(x,t)=f(s\,t-\eta\cdot x)\,c \tag{4.2}$$

where $f(\tau)$ is a real-valued function of $\tau\in R^1$, while $s\in R^1$, $\eta=(\eta_1,\eta_2,\dots,\eta_n)\in R^n$ and $c=(c_1,c_2,\dots,c_m)\in R^m$ are constants. If $f'(\tau)\not\equiv 0$, then (4.2) solves (4.1) if and only if

$$\left(E^0 s+\sum_{j=1}^{n} A^j\eta_j\right)c=0. \tag{4.3}$$

If $c\neq 0$, then (4.3) implies

$$\det\left(E^0 s+\sum_{j=1}^{n} A^j\eta_j\right)=0, \tag{4.4}$$

i.e., the hyperplanes $s\,t-\eta\cdot x=$constant are characteristic hyperplanes for the system (4.1). The plane wave (4.2) propagates in the direction of the vector η with the speed $s/|\eta|$, where $|\eta|^2=\eta_1^2+\eta_2^2+\dots+\eta_n^2$. Hence, the possible "normal speeds" of plane waves (4.2) for system (4.1) are given by the roots s of (4.4) corresponding to unit vectors η.

It is well known that there is a non-singular $m\times m$ matrix T such that

$$T^* E^0 T=I. \tag{4.5}$$

Thus

$$\det T^*\det\left(E^0 s+\sum_{j=1}^{n} A^j\eta_j\right)\det T=\det\left(I s-\sum_{j=1}^{n} B^j\eta_j\right)$$

where

$$B^j=-T^* A^j T \tag{4.6}$$

is a real symmetric matrix. It follows that the roots s of (4.4) are all real if η is a real vector. Thus system (4.1) has m normal speeds (not necessarily distinct) for each direction η.

The function

$$P(\eta,s)=\det\left(I s-\sum_{j=1}^{n} B^j\eta_j\right) \tag{4.7}$$

is a homogeneous polynomial of degree m in the variable $(\eta_1,\eta_2,\dots,\eta_n,s)$. Hence the roots s of (4.7) are algebraic functions of $\eta=(\eta_1,\dots,\eta_n)$. If the roots s are functions of $|\eta|$ only, the medium governed by (4.1) is called isotropic (the normal speeds are independent of the direction of propagation). If the roots vary with the direction of propagation, the medium is called anisotropic. The anisotropy of a medium can be visualized by means of the normal speed surface whose points are the terminal points of the normal velocity vectors, defined by

$$v=(v_1,\dots,v_n)=s(\eta_1,\dots,\eta_n)$$

where s is a root of (4.7) and $|\eta|=1$. Since $s^2=|v|^2=v_1^2+\dots+v_n^2$, the normal speed surface has the equation

$$P(v,|v|^2)=\det\left(I\,|v|^2-\sum_{j=1}^{n} B^j v_j\right)=0 \tag{4.8}$$

(multiply (4.7) by s^m where s is a normal speed). Hence the normal speed surface is an algebraic surface of degree $2m$.

The anisotropy of a medium can also be visualized by means of its slowness surface S, which may be defined as the image of the normal speed surface (4.8) under the transformation

$$p=\frac{v}{|v|^2}, \qquad v=\frac{p}{|p|^2}; \tag{4.9}$$

i.e., inversion in the unit sphere. Since points p on S satisfy $|p|\,|v|=1$, the distances from the origin of the point of S are the reciprocals of the normal speeds, *i.e.*, the "slownesses" for the system (4.1). An equation for S is

$$P(p,1)=\det\left(I-\sum_{j=1}^{n} B^j p_j\right)=0, \tag{4.10}$$

by (4.8) and (4.9). Hence S is an algebraic surface of degree not exceeding m.

The polynomial $P(\eta, s)$ has a factorization

$$P(\eta,s)=Q_1^{m_1}(\eta,s)\,Q_2^{m_2}(\eta,s)\dots Q_l^{m_l}(\eta,s) \tag{4.11}$$

where the factors $Q_j(\eta,s)$ are distinct homogeneous polynomials in (η,s), irreducible over the real number field. The factors $Q_j(\eta,s)$ are unique, apart from their order and constant factors. $P(\eta,s)$ is of order m in s and the coefficient of s^m in $P(\eta,s)$ is 1. Hence the factors $Q_j(\eta,s)$ may be defined uniquely by requiring that the coefficient of the highest power of s in each $Q_j(\eta,s)$ be 1. Let

$$Q(\eta,s)=Q_1(\eta,s)\,Q_2(\eta,s)\dots Q_l(\eta,s) \tag{4.12}$$

denote the corresponding polynomial with simple irreducible factors. Then it is clear from (4.10), (4.11) that S may be described as the locus

$$Q(p,1)=Q_1(p,1)\,Q_2(p,1)\dots Q_l(p,1)=0. \tag{4.13}$$

The geometrical properties of the slowness surface S play a decisive part in determining the structure and properties of waves governed by the system (4.1). A class of systems (4.1) for which the asymptotic behavior for large time can be estimated is described by the following definition.

Definition. A system (4.1) (and the medium governed by it) is said to be *uniformly propagative* if

(4.14) the slowness surface S is bounded,

and

$$p\cdot\nabla Q(p,1)=p_1\frac{\partial Q(p,1)}{\partial p_1}+\cdots+p_n\frac{\partial Q(p,1)}{\partial p_n}\neq 0 \qquad \text{when } p\in S. \tag{4.15}$$

Many of the wave propagation phenomena of classical physics are governed by uniformly propagative systems. A number of examples from physics are discussed in the Appendix. The name "uniformly propagative" is motivated by the observation that the normal speeds of such systems have constant multiplicity and constant algebraic sign, independent of the direction of propagation η (Corollary 4.7, below).

Let η^0 be a fixed unit vector, and let $s(\eta^0)$ be a corresponding normal speed; *i.e.*, a root of

$$Q(\eta^0, s) = s^r + Q_1(\eta^0)\, s^{r-1} + \cdots + Q_{r-1}(\eta^0)\, s + Q_r(\eta^0) = 0. \tag{4.16}$$

Lemma 4.1. *For uniformly propagative media, if* $s(\eta^0) \neq 0$, *then* $\partial Q(\eta^0, s(\eta^0))/\partial s \neq 0$. *Hence, the non-zero roots of* (4.16) *are simple.*

Proof. $Q(\eta, s)$ is homogeneous of degree r in (η, s). Hence by EULER's theorem

$$\eta \cdot \nabla Q(\eta, s) + s \frac{\partial Q(\eta, s)}{\partial s} = r\, Q(\eta, s).$$

Putting $\eta = \eta^0$, $s = s(\eta^0)$ in this equation gives

$$\eta^0 \cdot \nabla Q(\eta^0, s(\eta^0)) + s(\eta^0) \frac{\partial Q(\eta^0, s(\eta^0))}{\partial s} = r\, Q(\eta^0, s(\eta^0)) = 0, \tag{4.17}$$

by the definition of $s(\eta^0)$. Now if $p^0 = s(\eta^0)^{-1} \eta^0$, then

$$Q(p^0, 1) = s(\eta^0)^{-r}\, Q(\eta^0, s(\eta^0)) = 0;$$

i.e., $p^0 \in S$. Multiplying (4.17) by $s(\eta^0)^{-r}$ and using the homogeneity of the derivatives of Q gives

$$p^0 \cdot \nabla Q(p^0, 1) + \frac{\partial Q(p^0, 1)}{\partial s} = 0.$$

Thus

$$\frac{\partial Q(\eta^0, s(\eta^0))}{\partial s} = s(\eta^0)^{r-1} \frac{\partial Q(p^0, 1)}{\partial s} = -s(\eta^0)^{r-1}\, p^0 \cdot \nabla Q(p^0, 1) \neq 0$$

by (4.15).

Lemma 4.2. *For uniformly propagative media, one of the following two alternatives holds. Either*

Case 1. *There is a unit vector* η^0 *such that the roots* $s_1^0, s_2^0, \ldots, s_r^0$ *of* $Q(\eta^0, s)$ *are all different from zero (and therefore simple, by Lemma 4.1), or*

Case 2. $s(\eta^0) = 0$ *is a root for every* η^0, *so that* $Q_r(\eta^0) \equiv 0$, *and there is a unit vector* η^0 *such that the* $r-1$ *roots of* $s^{-1} Q(\eta^0, s)$ *are all different from zero (and therefore simple).*

In particular, there is a unit vector η^0 *such that* $Q(\eta^0, s)$ *has r simple roots in both cases.*

Proof. If Case 1 does not hold, then $Q_r(\eta) \equiv 0$. If $Q_{r-1}(\eta) \equiv 0$ also, then $Q(\eta, s) = s^2\, Q'(\eta, s)$ has a repeated irreducible factor contrary to the definition of $Q(\eta, s)$. Thus $Q_{r-1}(\eta^0) \neq 0$ for some unit vector η^0, and the $r-1$ roots of $s^{-1} Q(\eta, s)$ are all different from zero for $\eta = \eta^0$. This proves Lemma 4.2.

Now fix a unit vector η^0 as in Lemma 4.2, and let $s_1^0, s_2^0, \ldots, s_r^0$ be the corresponding set of roots of $Q(\eta^0, s)$. They are distinct simple roots by Lemma 4.2. If $s_k^0 \neq 0$, then since

$$Q(\eta^0, s_k) = 0, \qquad \frac{\partial Q(\eta^0, s_k^0)}{\partial s} \neq 0,$$

the equation $Q(\eta, s)=0$ has a unique analytic solution $s=s_k(\eta)$ defined in a neighborhood of $\eta=\eta^0$ and satisfying $s_k(\eta^0)=s_k^0$ (Implicit Function Theorem for Analytic Functions [*1*, p. 39]). If $s_k^0=0$, then zero is a root of $Q(\eta, s)$ for every η and $s_k(\eta)$ is defined to be identically zero. Note that in both cases

$$s_k(\mu\,\eta)=\mu\, s_k(\eta)\,, \qquad \mu>0\,, \tag{4.18}$$

by the homogeneity of $Q(\eta, s)$ and the uniqueness of the function $s_k(\eta)$.

If $s_k(\eta)\neq 0$, then

$$Q(s_k(\eta)^{-1}\,\eta, 1)=s_k(\eta)^{-r}\,Q(\eta, s_k(\eta))=0;$$

whence

$$p=s_k(\eta)^{-1}\,\eta\in S\,.$$

It follows that $|p|=|s_k(\eta)|^{-1}$ and hence, by (4.18),

$$|s_k(p)|=|p|\,|s_k(\eta)|=1. \tag{4.19}$$

Thus (4.19) is an equation for a portion of S. These facts lead to the following theorem.

Theorem 4.1. *For uniformly propagative media, the r roots $s_1(\eta), s_2(\eta), \ldots, s_r(\eta)$ of $Q(\eta, s)=0$ defined above are analytic functions of η for all real $\eta\neq 0$.*

Proof. Each function $s_k(\eta)$, $k=1, 2, \ldots, r$, is defined by the implicit function theorem near $\eta=\eta^0$ and then extended by analytic continuation. The only obstacle to the analytic continuation would be the occurrence of a branch point. Branch points $\eta^1\neq 0$ with a non-zero root $s_k(\eta^1)$ do not occur, by Lemma 4.1. If $s_k(\eta^1)=0$, then $s_k(\eta)\equiv 0$. For if $s_k(\eta^0)\neq 0$ and $s_k(\eta)\to 0$ when $\eta\to\eta^1$, then, by (4.19), $|p|\to\infty$ when $\eta\to\eta^1$ and S is unbounded, contrary to hypothesis. This same argument also proves

Corollary 4.1. *Each root $s_k(\eta)$, $k=1, 2, \ldots, r$, is of constant algebraic sign. In particular, a root can vanish only if it is identically zero.*

The occurrence of a root $s_k(\eta)\equiv 0$ is associated with the existence of static solutions with finite energy of the system (4.1); *i.e.*, solutions $u^0=u^0(x)$ such that

$$\sum_{j=1}^{n} A^j \frac{\partial u^0}{\partial x_j}=0 \quad\text{and}\quad \int_{R^n} u^{0*}(x)\,E^0\,u^0(x)\,dx<\infty\,.$$

Indeed, it can be shown by the Fourier transform method (see § 2) that such solutions exist for uniformly propagative systems if and only if a root $s_k(\eta)\equiv 0$ exists. Thus Case 2 (Lemma 4.2) is applicable to uniformly propagative systems which have static solutions. It is important to include this case in the discussion because many of the systems of wave equations from classical physics have static solutions. Examples include MAXWELL's equations and the equations for acoustic and seismic waves (see the Appendix).

Corollary 4.2. *If the roots s_k^0 $(k=1, 2, \ldots, r)$ are enumerated so that*

$$s_1^0>s_2^0>\cdots>s_r^0,$$

then

$$s_1(\eta)>s_2(\eta)>\cdots>s_r(\eta) \qquad \textit{for all real } \eta\neq 0\,. \tag{4.2}$$

Proof. (4.20) holds for $\eta=\eta^0$ by hypothesis. If any of these inequalities fail for some $\eta \neq 0$, then, by continuity, there will be an $\eta^1 \neq 0$ such that $s_j(\eta^1)=s_k(\eta^1)$ with $1 \leqq j, k \leqq r$ and $j \neq k$. If $s_j(\eta^1) \neq 0$, then it is a double root, contradicting Lemma 4.1. Suppose that $s_j(\eta^1)=0$. Since the functions $s_j(\eta)$ and $s_k(\eta)$ are distinct, one of them is not identically zero and vanishes for $\eta=\eta^1$, contradicting Corollary 4.1. Thus (4.20) holds.

Henceforth it is assumed that the roots $s_\alpha(\eta)$ are numbered as in (4.20). A polynomial $Q(\eta, s)$ is said to be *strictly hyperbolic* (with respect to the vector $(\eta, s)=(0, \ldots, 0, 1)$ if for each fixed real vector $\eta \neq 0$, the roots s of $Q(\eta, s)=0$ are real, distinct and different from zero [*10*, p. 137]). Thus Theorem 4.1 implies

Corollary 4.3. *For uniformly propagative media, either $Q(\eta, s)$ is strictly hyperbolic (Case 1) or $s^{-1}Q(\eta, s)$ is strictly hyperbolic (Case 2).*

Corollary 4.4. *For uniformly propagative media, the r distinct roots $s_1(\eta), \ldots, s_r(\eta)$ satisfy*

$$s_k(-\eta)=-s_{r-k+1}(\eta) \qquad \text{for } k=1,2,\ldots,r \text{ and all } \eta. \tag{4.21}$$

Proof. Since $Q(-\eta, -s)=(-1)^r Q(\eta, s)$, the numbers $-s_1(\eta), \ldots, -s_r(\eta)$ are the roots corresponding to the vector $-\eta$. Also, by assumption

$$-s_1(\eta)<-s_2(\eta)<\cdots<-s_r(\eta).$$

Thus it follows that $-s_1(\eta)=s_r(-\eta)$, $-s_2(\eta)=s_{r-1}(-\eta), \ldots$, *i.e.*, (4.21) holds.

Corollary 4.5. *For uniformly propagative media, one of the following two alternatives holds. Either*

Case 1. $r=2\rho$ *is even and the roots* $s_k(\eta)$ *satisfy*

$$s_1(\eta)>\cdots>s_\rho(\eta)>0>s_{\rho+1}(\eta)=-s_\rho(-\eta)>\cdots>s_{2\rho}(\eta)=-s_1(-\eta), \tag{4.22}$$

or

Case 2. $r=2\rho+1$ *is odd and the roots* $s_k(\eta)$ *satisfy*

$$\begin{aligned} &s_1(\eta)>\cdots>s_\rho(\eta)>s_{\rho+1}(\eta)\equiv 0>s_{\rho+2}(\eta) \\ &\qquad =-s_\rho(-\eta)>\cdots>s_{2\rho+1}(\eta)=-s_1(-\eta). \end{aligned} \tag{4.23}$$

Proof. (4.21) implies that for every positive root $s_k(\eta)$ there is a negative root $s_{r-k+1}(\eta)$, because the roots have constant sign (Corollary 4.1). Because of convention (4.20), $s_1(\eta)=-s_r(-\eta)>0$. Similarly $s_2(\eta)=-s_{r-1}(-\eta)>0$, *etc.* If $r=2\rho$ (Case 1), then $k=\rho$ implies $r-k+1=\rho+1$. Thus $s_\rho(\eta)=-s_{\rho+1}(-\eta)>0$. Indeed, $s_\rho(\eta)<0$ would imply $s_{\rho+1}(\eta)>0>s_\rho(\eta)$, by (4.21), contrary to (4.20), while $s_\rho(\eta)=0$ would imply $s_{\rho+1}(\eta)=0$, contrary to the fact that the roots are simple. If $r=2\rho+1$ (Case 2), then $k=\rho$ implies $r-k+1=\rho+2$ and $k=\rho+1$ implies $r-k+1=\rho+1$. Thus $s_\rho(\eta)=-s_{\rho+2}(-\eta)>0$, by the argument given in Case 1, and $s_{\rho+1}(\eta)=-s_{\rho+1}(-\eta)=0$, because $s_{\rho+1}(\eta)$ does not change sign.

The properties of the roots proved above imply the following theorem.

Theorem 4.2. *For uniformly propagative media, the slowness surface S consists of $\rho=[r/2]$ disjoint bounded sheets which are analytic surfaces. Equations for them are*

$$s_k(p)=1, \qquad k=1,2,\ldots,\rho. \tag{4.24}$$

Proof. (4.24) follows from (4.19) and the fact that $s_k(\eta)>0$ for $k=1,2,\ldots,\rho$. The sheets of S are disjoint because the roots $s_k(\eta)$ are distinct. The analyticity of the sheets follows from Theorem 4.1.

Another equation for the sheet corresponding to $s_k(\eta)$ is, by (4.18),

$$|p|\, s_k(\eta)=1, \qquad |\eta|=1.$$

This implies

Corollary 4.6. *The ρ sheets of S are non-intersecting closed surfaces, enclosing the origin. $|p|\, s_1(\eta)=1$ defines the innermost sheet, $|p|\, s_2(\eta)=1$ defines the next,* etc., *and $|p|\, s_\rho(\eta)=1$ defines the outermost sheet.*

The discussion above shows that the roots $s_1(\eta), \ldots, s_r(\eta)$ are just the distinct roots of

$$\det\left(E^0 s+\sum_{j=1}^{n} A^j \eta_j\right)=0. \tag{4.25}$$

The latter can be interpreted as the possible normal speeds for plane waves propagating in the direction of the unit vector η. Thus Corollaries 4.1 and 4.2 above imply

Corollary 4.7. *For uniformly propagative systems the m normal speeds $s_\alpha(\eta)$, $\alpha=1,2,\ldots,m$, defined by* (4.25) *have constant multiplicity and constant algebraic sign, independent of η.*

It is these properties that motivate the term "uniformly propagative" system. Another important property of the system is described by

Theorem 4.3. *The matrix*

$$B(\eta)=\sum_{j=1}^{n} B^j \eta_j$$

(*see* (4.6)) *satisfies the identity*

$$Q(\eta, B(\eta))=B(\eta)^r+Q_1(\eta)\, B(\eta)^{r-1}+\cdots+Q_{r-1}(\eta)\, B(\eta)+Q_r(\eta)\, I=0 \tag{4.26}$$

for every η. Hence $Q(\eta, s)$ is the minimal polynomial for $B(\eta)$.

Proof. The polynomial

$$P(\eta, s)=\det\big(I s-B(\eta)\big)=Q_1^{m_1}(\eta, s)\ldots Q_l^{m_l}(\eta, s)$$

is the characteristic polynomial of $B(\eta)$. Thus

$$P(\eta, B(\eta))=0$$

by the Hamilton-Cayley theorem. Now

$$Q(\eta, s)=Q_1(\eta, s)\, Q_2(\eta, s)\ldots Q_l(\eta, s)$$

by definition. Thus if $m_1=m_2=\cdots=m_l=1$, this result is the same as (4.26). If $P(\eta, s)$ has a repeated irreducible factor, say $m_1>1$, then

$$P(\eta, s)=Q_1^2(\eta, s)\, R(\eta, s).$$

To derive (4.26) in this case, let $u(\eta)$, $v(\eta)$ be m-component vectors whose components depend on η and write

$$\big(u(\eta), v(\eta)\big)=u_1(\eta)\, v_1(\eta)+\cdots+u_m(\eta)\, v_m(\eta).$$

Then

$$0=(P(\eta, B(\eta))\,u(\eta), v(\eta))=(Q_1^2(\eta, B(\eta))\,R(\eta, B(\eta))\,u(\eta), v(\eta))$$
$$=(Q_1(\eta, B(\eta))\,R(\eta, B(\eta))\,u(\eta), Q_1(\eta, B(\eta))\,v(\eta)).$$

The last step follows from the symmetry of $B(\eta)$. Taking $u(\eta)=\xi$, a constant, and $v(\eta)=R(\eta, B(\eta))\,\xi$ gives

$$\| Q_1(\eta, B(\eta))\,R(\eta, B(\eta))\,\xi \|^2=0 \qquad \text{for all } \xi,$$

which implies

$$Q_1(n, B(\eta))\,R(\eta, B(\eta))=Q_1^{m-1}(\eta, B(\eta))\,Q_2^{m_2}(\eta, B(\eta)) \dots Q_l^{m_l}(\eta, B(\eta))=0.$$

If $m_1=2, m_2=1, \dots, m_l=1$, this result is the same as (4.26). If $m_1>2$, *etc.*, the argument may be repeated until each exponent m_j is reduced to 1 which proves (4.26).

$Q(\eta, s)$ is the minimal polynomial for $B(\eta)$, by (4.26), because each root of $P(\eta, s)$ is a root of the minimal polynomial, and the roots of $Q(\eta, s)$ are simple and are the distinct roots of $P(\eta, s)$.

The wave cone is considered next.

Definition. The wave cone W for a system (4.1) is the envelope of the set of characteristic hyperplanes for (4.1) which pass through $(x, t)=(0, 0)$.

Each such hyperplane has an equation

$$\varphi(x, t, \eta) \equiv s_k(\eta)\,t-\eta \cdot x=0 \tag{4.27}$$

where $s_k(\eta)$ is one of the normal speeds for (4.1). Thus there is a family of such planes for each of the analytic functions $s_k(\eta)$. The envelope of such a family is determined by (4.27) and the equations $\nabla_\eta\, \varphi(x, t, \eta)=0$, or

$$x=t\,\nabla_\eta\, s_k(\eta). \tag{4.28}$$

This is obviously a cone in space-time. There is a sheet of the wave cone W for each of the distinct roots $s_k(\eta)$. Note that in Case 2, $s_{p+1}(\eta)\equiv 0$, W also includes the t-axis, $x=0$.

If $W_t=\{x: (x, t) \in W\}$, then (4.28) implies that the locus W_1 is the "polar reciprocal" or "dual" of the slowness surface S. This property could be used to define W. It is also clear from (4.28) that W is just the set of all bicharacteristics for the system (4.1) passing through $(x, t)=(0, 0)$.

The following property of uniformly propagative systems is needed in § 5.

Theorem 4.4. *For uniformly propagative systems, in Case 1 W contains a cone $|x| \leqq \gamma t$ with $\gamma>0$. In Case 2, $W-\{(x, t): x=0\}$ contains a cone $|x| \leqq \gamma t$ with $\gamma>0$.*

Proof. W is the envelope of the planes (4.27). For $t>0$ fixed, (4.27) defines a plane in space whose distance for the origin is $s_k(\eta)\,t$ if $|\eta|=1$. Thus for points (x, t) on W, $|x|/t$ is not less than

$$\gamma=\inf_{\substack{k=1, 2, \dots, r\\ |\eta|=1}} |s_k(\eta)|.$$

In Case 1, Corollary 4.5 implies that

$$\gamma = \inf_{|\eta|=1} s_\rho(\eta), \tag{4.29}$$

which is positive because $s_\rho(\eta)$ is continuous and never vanishes for $|\eta|=1$. In Case 2 $\gamma=0$, but if the t-axis (corresponding to the root $s_{\rho+1}(\eta)\equiv 0$) is removed from W, then for the remaining points (x,t) on W, $|x|/t$ is not less than the $\gamma(>0)$ defined by (4.29).

§ 5. The Riemann Matrix and the Asymptotic Behavior of Waves in Homogeneous Media

In this section the Riemann matrix $R(x,t)$ for a homogeneous medium is defined and several of its properties are described. Then asymptotic estimates for large time are derived, using $R(x,t)$, for waves in uniformly propagative media. Stronger estimates are derived for a special class of uniformly propagative systems, the systems of Maxwell type, which include MAXWELL's equations and the equations of acoustics. Concepts from SCHWARTZ's theory of distributions are used in this section. A concise discussion of the relevant concepts may be found in [*10*, Ch. 1].

The class $\mathscr{H}_0$ of initial values φ^0 with finite energy includes the class $C_0^\infty(R^n)$ of testing functions of distribution theory; *i.e.*, functions $\varphi^0(x)=(\varphi_1^0(x), \varphi_2^0(x), \dots, \varphi_m^0(x))$ whose components $\varphi_\alpha^0(x)$ have continuous derivatives of all orders and vanish outside a bounded set. The solution formulas for the propagation problem provided by the Fourier transform method imply that

$$u^0(x,t)=(e^{itH_0}\varphi^0)(x)\in C^\infty(R^{n+1}) \qquad \text{for } \varphi^0\in C_0^\infty(R^n),$$

and, moreover, for each fixed $(x,t)\in R^{n+1}$ the number $u_\alpha^0(x,t)$ are continuous linear functionals of $\varphi^0\in C_0^\infty(R^n)$ in the sense of distribution theory; *i.e.*, $u_\alpha^0(x,t)$ is a distribution on R^n for each (x,t). Moreover, examination of the solution formulas reveals that this distribution has the form of a convolution* of φ^0 with a one-parameter family of distributions $R(x,t)$ on R^n [*7*, *10*]:

$$u^0(x,t)=\big(R(\cdot,t)*\varphi^0\big)(x). \tag{5.1}$$

Thus the solution operator e^{-itH_0} is characterized by an ($m\times m$ matrix-valued) distribution $R(x,t)$ on R^n. $R(x,t)$ is itself characterized as the (unique) distribution solution of the initial value problem

$$E^0\frac{\partial R}{\partial t}=\sum_{j=1}^n A^j\frac{\partial R}{\partial x_j}, \qquad R(x,0)=\delta(x)\,I \tag{5.2}$$

---

* If $F=F(x)$ is a locally integrable function on R^n and $\varphi(x)\in C_0^\infty(R^n)$, the convolution $F*\varphi$ is defined by

$$(F*\varphi)(x)=\int_{R^n} F(y)\,\varphi(x-y)\,dy.$$

If F is a distribution on R^n, this definition is extended by defining

$$(F*\varphi)(x)=F_y(\varphi(x-y))$$

where F_y indicates that F is applied to $\varphi(x-y)$ as a function of y with x fixed; see [*10*].

where $\delta(x)$ is the n-dimensional Dirac delta function and I is the $m\times m$ unit matrix. $R(x,t)$ has been studied from this point of view by LUDWIG [13] and DUFF [8] who called it the Riemann matrix for the system

$$E^0\frac{\partial u^0}{\partial t}=\sum_{j=1}^{n}A^j\frac{\partial u^0}{\partial x_j}. \tag{5.3}$$

The representation (5.1) is used below to obtain estimates for large t of $u^0(x,t)$ and its derivatives. To see how this can be done, note that $\delta(x)$ is a homogeneous distribution of degree $-n$; *i.e.*,

$$\delta(kx)=k^{-n}\delta(x),\qquad k>0.$$

It follows that $R(x,t)$ is homogeneous of degree $-n$ in x and t together;

$$R(kx,kt)=k^{-n}R(x,t),\qquad k>0, \tag{5.4}$$

a fact that has been noted by DUFF [8] and other authors. Indeed, the distribution $v(x,t)=k^n R(kx,kt)$ satisfies conditions (5.2), so that identity (5.4) follows from the uniqueness theorem for (5.2). Identity (5.4) implies that

$$R(x,t)=t^{-n}R\left(\frac{x}{t},1\right),\qquad t>0.$$

This identity yields an estimate for $R(x,t)$ when combined with the well-known fact that $R(x,t)$ is an analytic function in the interior of its wave cone W [6, p. 733; 3].

If W contains a solid cone $|x|\leqq\gamma t\ (\gamma>0)$, it follows that the components $R_{\alpha\beta}(x,t)$ satisfy

$$|R_{\alpha\beta}(x,t)|\leqq K t^{-n}\qquad \text{for } |x|\leqq\gamma t,\ 1\leqq\alpha,\beta\leqq m, \tag{5.5}$$

where K is a suitable constant. Combining (5.5) with (5.1) gives a similar estimate for $u^0(x,t)$. The same technique also yields estimates for the derivatives of $u^0(x,t)$.

If (5.3) is a uniformly propagative system with no static solutions (*i.e.*, no normal speeds which are zero), then its wave cone W does contain a cone $|x|\leqq\gamma t$, by Theorem 4.4 (Case 1). Moreover, $\partial R(x,t)/\partial t$ is homogeneous of degree $-n-1$, and the same argument implies that

$$\left|\frac{\partial R_{\alpha\beta}(x,t)}{\partial t}\right|\leqq K_1 t^{-n-1}\qquad \text{for } |x|\leqq\gamma t,\ 1\leqq\alpha,\beta\leqq m, \tag{5.6}$$

where K_1 is another constant.

If (5.3) has static solutions (Case 2), then W contains the t-axis and (5.5) fails. However, it will be shown that (5.6) still is valid. To this end, consider the Riemann matrix $R'(x,t)$ defined by

$$\frac{\partial R'}{\partial t}=-\sum_{j=1}^{n}B^j\frac{\partial R'}{\partial x_j}\quad\text{and}\quad R'(x,0)=\delta(x)I, \tag{5.7}$$

where

(5.8) $$T^* E^0 T = I, \quad T^* A^j T = -B^j \quad \text{and} \quad \det T \neq 0.$$

Lemma 5.1. *$R(x,t)$ and $R'(x,t)$ are related by*

(5.9) $$R(x,t) = T R'(x,t) T^{-1}.$$

Proof. If $R(x,t)$ is defined by (5.7) and (5.9), then direct computation, using (5.8), gives

$$E^0 \frac{\partial R}{\partial t} = E^0 T \frac{\partial R'}{\partial t} T^{-1} = T^{*-1} \frac{\partial R'}{\partial t} T^{-1} = -\sum_{j=1}^{n} T^{*-1} B^j \frac{\partial R'}{\partial x_j} T^{-1}$$

$$= \sum_{j=1}^{n} A^j T \frac{\partial R'}{\partial x_j} T^{-1} = \sum_{j=1}^{n} A^j \frac{\partial R}{\partial x_j}$$

and

$$R(x,0) = T\,\delta(x)\, I\, T^{-1} = \delta(x)\, I.$$

Thus $R(x,t)$, given by (5.7) and (5.9), coincides with the (unique) Riemann matrix for (5.3).

Next, notice that Theorem 4.3 implies

Theorem 5.1. *Let $D_j = \partial/\partial x_j$ and $D = (D_1, \ldots, D_n)$. Then the matrix differential operator*

(5.10) $$B(D) = \sum_{j=1}^{n} B^j D_j$$

satisfies the identity

(5.11) $$Q(D, B(D)) \equiv B(D)^r + Q_1(D) B(D)^{r-1} + \cdots + Q_{r-1}(D) B(D) + Q_r(D) I \equiv 0.$$

Proof. $Q(\eta, B(\eta))$ is an $m \times m$ matrix whose entries are polynomials in $\eta_1, \eta_2, \ldots, \eta_n$. By Theorem 4.3 these polynomials are identically zero; *i.e.* all their coefficients are zero. Hence, replacing η by D in (4.26) gives (5.11)

By (5.7) the Riemann matrix $R'(x,t)$ satisfies

$$\frac{\partial R'}{\partial t} = -B(D) R'$$

where $B(D)$ is defined by (5.10). Hence

(5.12) $$B(D)^j R' = (-1)^j \frac{\partial^j R'}{\partial t^j}, \qquad j = 0, 1, 2, \ldots.$$

Combining (5.11) and (5.12) gives

(5.13) $$\frac{\partial^r R'}{\partial t^r} - Q_1(D) \frac{\partial^{r-1} R'}{\partial t^{r-1}} + \cdots + (-1)^{r-1} Q_{r-1}(D) \frac{\partial R'}{\partial t} + (-1)^r Q_r(D) R' = 0.$$

Note that this is a *scalar* equation; *i.e.*, $\partial/\partial t$ and $Q_1(D)$, $Q_2(D)$, ... are scalar partial differential operators. In Case 1, $r = 2\rho$ and R' solves

(5.14) $$\frac{\partial^{2\rho} R'}{\partial t^{2\rho}} - Q_1(D) \frac{\partial^{2\rho-1} R'}{\partial t^{2\rho-1}} + \cdots - Q_{2\rho-1}(D) \frac{\partial R'}{\partial t} + Q_{2\rho}(D) R' = 0.$$

In Case 2, $r=2\rho+1$, $Q_r(D)\equiv 0$ and (5.13) becomes

$$\frac{\partial^{2\rho+1} R'}{\partial t^{2\rho+1}}-Q_1(D)\frac{\partial^{2\rho} R'}{\partial t^{2\rho}}+\cdots-Q_{2\rho-1}(D)\frac{\partial^2 R'}{\partial t^2}+Q_{2\rho}(D)\frac{\partial R'}{\partial t}=0. \tag{5.15}$$

Differentiating (5.14) with respect to t gives (5.15). This proves

Theorem 5.2. *For uniformly propagative systems the matrix*

$$S(x,t)=\frac{\partial R'(x,t)}{\partial t} \tag{5.16}$$

always satisfies the scalar equation

$$\frac{\partial^{2\rho} S}{\partial t^{2\rho}}-Q_1(D)\frac{\partial^{2\rho-1} S}{\partial t^{2\rho-1}}+\cdots-Q_{2\rho-1}(D)\frac{\partial S}{\partial t}+Q_{2\rho}(D)S=0. \tag{5.17}$$

Equation (5.17) is used below to construct S. The construction is based on

Lemma 5.2. *For uniformly propagative systems*

$$L\left(D,\frac{\partial}{\partial t}\right)\equiv\frac{\partial^{2\rho}}{\partial t^{2\rho}}-Q_1(D)\frac{\partial^{2\rho-1}}{\partial t^{2\rho-1}}+\cdots-Q_{2\rho-1}(D)\frac{\partial}{\partial t}+Q_{2\rho}(D)$$

is a strictly hyperbolic operator.

Proof. By definition $L(D,\partial/\partial t)$ is strictly hyperbolic if its characteristic polynomial $L(\eta,s)$ is a strictly hyperbolic polynomial. But

$$L(\eta,s)=s^{2\rho}-Q_1(\eta)s^{2\rho-1}+\cdots-Q_{2\rho-1}(\eta)s+Q_{2\rho}(\eta),$$

whence $L(\eta,s)=Q(\eta,-s)$ in Case 1 and $L(\eta,s)=-s^{-1}Q(\eta,-s)$ in Case 2. Thus Lemma 5.2 follows from Corollary 4.3.

Theorem 5.2 and Lemma 5.2 imply that the components $S_{\alpha\beta}(x,t)$ of the matrix $S(x,t)$ solve the scalar hyperbolic equation $L(D,\partial/\partial t)\,S_{\alpha\beta}=0$. It follows that $S(x,t)$ is uniquely determined by the initial values of its time derivatives of orders $0,1,\dots,2\rho-1$. These may be obtained from (5.12) which implies

$$\frac{\partial^j S}{\partial t^j}=(-1)^{j+1}B(D)^{j+1}R', \qquad j=0,1,2,\dots.$$

Corollary 5.1. $S(x,t)=(S_{\alpha\beta}(x,t))$ *is uniquely determined by* (5.17) *and the initial conditions*

$$\begin{aligned} &S(x,0)=-B(D)\delta(X)I,\ \frac{\partial S(x,0)}{\partial t}=B(D)^2\delta(X)I,\dots \\ &\dots,\ \frac{\partial^{2\rho-1}S(x,0)}{\partial t^{2\rho-1}}=B(D)^{2\rho}\delta(X)I. \end{aligned} \tag{5.18}$$

This result makes it possible to express $S(x,t)$ in terms of the scalar Riemann function $R^0(x,t)$ for the operator $L(D,\partial/\partial t)$ which is defined by

$$\begin{aligned} &\frac{\partial^{2\rho}R^0}{\partial t^{2\rho}}-Q_1(D)\frac{\partial^{2\rho-1}R^0}{\partial t^{2\rho-1}}+\cdots-Q_{2\rho-1}(D)\frac{\partial R^0}{\partial t}+Q_{2\rho}(D)R^0=0, \\ &R^0(x,0)=0,\ \frac{\partial R^0(x,0)}{\partial t}=0,\dots,\ \frac{\partial^{2\rho-2}R^0(x,0)}{\partial t^{2\rho-2}}=0,\ \frac{\partial^{2\rho-1}R^0(x,0)}{\partial t^{2\rho-1}}=\delta(x). \end{aligned} \tag{5.19}$$

To derive a relation between S and R, write

$$S(x,t)=\sum_{j=0}^{2\rho-1}\frac{\partial^j u_{2\rho-j-1}(x,t)}{\partial t^j} \tag{5.20}$$

where $u_0, u_1, \ldots, u_{2\rho-1}$ are defined by

$$\begin{gathered}\frac{\partial^{2\rho}u_j}{\partial t^{2\rho}}-Q_1(D)\frac{\partial^{2\rho-1}u_j}{\partial t^{2\rho-1}}+\cdots-Q_{2\rho-1}(D)\frac{\partial u_j}{\partial t}+Q_{2\rho}(D)u_j=0\\ u_j(x,0)=0,\ \frac{\partial u_j(x,0)}{\partial t}=0,\ldots,\ \frac{\partial^{2\rho-2}u_j(x,0)}{\partial t^{2\rho-2}}=0,\ \frac{\partial^{2\rho-1}u_j(x,0)}{\partial t^{2\rho-1}}=f_j(x)\end{gathered} \tag{5.21}$$

and $f_0, f_1, \ldots, f_{2\rho-1}$ are to be determined. (5.20) and (5.21) imply

$$\frac{\partial^k S(x,0)}{\partial t^k}=\sum_{j=2\rho-1-k}^{2\rho-1}\frac{\partial^{j+k}u_{2\rho-1-k}(x,0)}{\partial t^{j+k}}.$$

This, with (5.18) and (5.21), gives

$$\begin{aligned}f_k(x)&=\frac{\partial^{2\rho-1}u_k(x,0)}{\partial t^{2\rho-1}}=\sum_{j=0}^{k}(-1)^j Q_j(D)\frac{\partial^{k-j}S(x,0)}{\partial t^{k-j}}\\ &=\sum_{j=0}^{k}(-1)^{k+1}B(D)^{k-j+1}Q_j(D)\,\delta(x)\,I.\end{aligned}$$

Comparison of (5.19) and (5.21) with this expression for $f_k(x)$ shows that

$$u_k(x,t)=\sum_{j=0}^{k}(-1)^{k+1}B(D)^{k-j+1}Q_j(D)\,R^0(x,t)\,I,$$

and substitution in (5.20) gives

Theorem 5.3.

$$S(x,t)=\sum_{k=0}^{2\rho-1}\sum_{j=0}^{2\rho-1-k}(-1)^k B(D)^{2\rho-k-j}Q_j(D)\frac{\partial^k R^0(x,t)}{\partial t^k}I. \tag{5.22}$$

The correctness of this formula may also be verified directly.

Theorem 5.4. *For uniformly propagative systems, the wave cone W^0 for the scalar Riemann function $R^0(x,t)$ always contains a cone $|x|\leqq\gamma t$ with $\gamma>0$.*

Proof. By the proof of Lemma 5.2, the characteristic polynomial for the operator $L(D,\partial/\partial t)$ which defines $R^0(x,t)$ is $Q(\eta,-s)$ in Case 1 and $-s^{-1}Q(\eta,-s)$ in Case 2. Thus in both cases its roots $s(\eta)$ are precisely the non-zero roots of $Q(\eta,-s)$. The proof that W^0 contains a cone $|x|\leqq\gamma t$ is therefore identical with the proof of Theorem 4.4, Case 1.

Corollary 5.2. *For uniformly propagative systems, there are positive constants γ and K such that the components $S_{\alpha\beta}(x,t)$ satisfy*

$$|S_{\alpha\beta}(x,t)|\leqq K t^{-n-1}\qquad \textit{for } |x|\leqq\gamma t,\ 1\leqq\alpha,\beta\leqq m. \tag{5.23}$$

Proof. Theorem 5.4 implies that $R^0(x, t)$ is analytic in $|x| \leqq \gamma t$. Hence $S(x, t)$ is analytic there, by Theorem 5.3. Since $S(x, t) = \partial R'(x, t)/\partial t$ is homogeneous of degree $-n-1$, (5.23) follows.

Lemma 5.1 implies that

$$\frac{\partial R(x,t)}{\partial t} = T S(x, t) T^{-1}$$

where T is a constant matrix. Hence Corollary 5.2 implies

Theorem 5.5. *For uniformly propagative systems, there are positive constants γ and K' such that*

$$(5.24) \qquad \left| \frac{\partial R_{\alpha\beta}(x,t)}{\partial t} \right| \leqq K' t^{-n-1} \qquad \text{for } |x| \leqq \gamma t, \ 1 \leqq \alpha, \beta \leqq m.$$

Finally, combining this result with the convolution formula (5.1) gives

Theorem 5.6. *Let $u^0(x, t)$ solve the propagation problem for a uniformly propagative system (5.3), with initial values $u^0(x, 0) = \varphi^0(x) \in C_0^\infty(R^n)$. Moreover, assume that*

$$\varphi^0(x) = 0 \qquad \text{for } |x| \geqq a.$$

Then there are constants γ and K such that

$$(5.25) \qquad \left| \frac{\partial u_\alpha^0(x,t)}{\partial t} \right| \leqq K_0 t^{-n-1} \qquad \text{for } |x| \leqq \gamma t - a, \ 1 \leqq \alpha \leqq m.$$

Proof. $u^0(x, t)$ is given by (5.1), where $R(x, t)$ denotes the Riemann matrix for (5.3). It follows that

$$\frac{\partial u_\alpha^0(x,t)}{\partial t} = \left(\frac{\partial R(\cdot, t)}{\partial t} * \varphi^0 \right)(x).$$

This may be written as an ordinary convolution, *i.e.*,

$$(5.26) \qquad \frac{\partial u_\alpha^0(x,t)}{\partial t} = \int_{|x'| \leqq a} \frac{\partial R_{\alpha\beta}(x-x', t)}{\partial t} \varphi_\beta^0(x')\, dx'$$

provided (x, t) is chosen so that $|x'| \leqq a$ is contained in a set on which $\partial R(x-x', t)/\partial t$ is analytic. Now

$$|x| \leqq \gamma t - a \quad \text{and} \quad |x'| \leqq a \quad \text{imply} \quad |x-x'| \leqq \gamma t,$$

and $\partial R(x-x')/\partial t$ is analytic on $|x-x'| \leqq \gamma t$ if γ is the constant in Theorem 5.5. Thus (5.26) is valid for $|x| \leqq \gamma t - a$. Taking absolute values in (5.26) and using (5.24) gives (5.25) with

$$K_0 = K' \int_{|x'| \leqq a} \left(|\varphi_1^0(x')| + \cdots + |\varphi_m^0(x)| \right) dx'.$$

Stronger estimates than (5.25) may hold for special classes of uniformly propagative systems. An important example of such a class is described by the

Definition. A uniformly propagative system (5.3) is said to be of Maxwell type if its minimal polynomial has the form

$$Q(\eta, s) = s^3 - c^2 |\eta|^2 s$$

where c is a positive constant.

In the Appendix it is shown that this class includes MAXWELL's equations for a homogeneous, isotropic medium and the equations of acoustics. By Theorem 5.1 the operator $B(D)$ for a system of Maxwell type satisfies the identity

$$B(D)^3 - c^2 \Delta B(D) \equiv 0 \tag{5.27}$$

where

$$\Delta = |D|^2 = D_1^2 + \cdots + D_n^2$$

is the Laplace operator. This implies

Lemma 5.3. *Let $u^0(x, t)$ be a solution of a system* (5.3) *of Maxwell type. Then the time derivatives*

$$\psi_\alpha(x, t) = \frac{\partial u_\alpha^0(x, t)}{\partial t}$$

solve the wave equation:

$$\frac{\partial^2 \psi_\alpha}{\partial t^2} - c^2 \Delta \psi_\alpha = 0. \tag{5.28}$$

Proof. Put $v(x, t) = T^{-1} u^0(x, t)$ where T is the constant matrix defined by (5.8). Then substituting $u^0 = Tv$ in (5.3) gives

$$\frac{\partial v}{\partial t} = -B(D) v.$$

Combining this with (5.27) gives

$$\frac{\partial^3 v}{\partial t^3} - c^2 \Delta \frac{\partial v}{\partial t} = 0;$$

i.e., the components $\frac{\partial v_\alpha}{\partial t}$ satisfy the wave equation. This implies (5.28), since

$$\psi_\alpha(x, t) = \frac{\partial u_\alpha^0(x, t)}{\partial t} = \sum_{\beta=1}^{m} T_{\alpha\beta} \frac{\partial v_\beta(x, t)}{\partial t}$$

where the $T_{\alpha\beta}$ are constant.

Lemma 5.3 implies the following estimate.

Theorem 5.7. *Let $u^0(x, t)$ solve the propagation problem for a system* (5.3) *of Maxwell type with $n = 3$, and let the initial values $u^0(x, 0) = \varphi^0(x) \in C_0^\infty(R^3)$. Moreover, assume that $\varphi^0(x) \equiv 0$ for $|x| \geqq a$. Then*

$$\frac{\partial u^0(x, t)}{\partial t} \equiv 0 \qquad \text{for } |x| \geqq ct + a \text{ and } |x| \leqq ct - a,$$

and

$$\left| \frac{\partial u_\alpha^0(x, t)}{\partial t} \right| \leqq K t^{-1} \qquad \text{for } ct - a \leqq |x| \leqq ct + a,\ 1 \leqq \alpha \leqq m,$$

where K is a constant which depends on φ^0 only.

Proof. Lemma 5.3 implies that

$$\psi(x,t)=\frac{\partial u^0(x,t)}{\partial t}$$

solves the following initial value problem for the wave equation:

$$\frac{\partial^2\psi}{\partial t^2}-c^2\Delta\psi=0 \qquad \text{for } x\in R^3,\ -\infty<t<\infty,$$

$$\psi(x,0)=\left((E^0)^{-1}\sum_{j=1}^{3}A^j D_j\right)\varphi^0(x), \qquad \frac{\partial\psi(x,0)}{\partial t}=\left((E^0)\sum_{j=1}^{3}A^j D_j\right)^2\varphi^0(x)$$

$$\text{for } x\in R^3.$$

Thus each component $\psi_\alpha(x,t)$ solves an initial value problem for the wave equation with three space dimension and initial values in $C_0^\infty(R^3)$ which vanish for $|x|\geqq a$. Hence, Theorem 5.7 follows directly from

Lemma 5.4. *Let $w(x,t)$ solve the initial value problem*

$$\frac{\partial^2 w}{\partial t^2}-c^2\Delta w=0 \qquad \textit{for } x\in R^3,\ -\infty<t<\infty,$$

$$w(x,0)=f(x), \qquad \frac{\partial w(x,0)}{\partial t}=g(x) \qquad \textit{for } x\in R^3,$$

where f and g are in $C_0^\infty(R^3)$ and vanish for $|x|\geqq a$. Then for $ct>2a$

$$w(x,t)\equiv 0 \qquad \textit{for } |x|\geqq ct+a \textit{ and } |x|\leqq ct-a, \tag{5.29}$$

and

$$|w(x,t)|\leqq K t^{-1} \qquad \textit{for } ct-a\leqq|x|\leqq ct+a, \tag{5.30}$$

where K depends on f and g only.

Proof. $w(x,t)$ may be expressed in terms of its initial values by the classical Poisson formula

$$w(x,t)=t\,M_{x,ct}[g]+\frac{\partial}{\partial t}\left(t\,M_{x,ct}[f]\right) \tag{5.31}$$

where $M_{x,r}[f]$ denotes the spherical mean of f over the surface of the sphere $S(x,r)$ with center x and radius r. Thus if dS denotes the element of area on $S(x,r)$, ω denotes a unit vector, Ω denotes the unit sphere and $d\Omega$ the element of area on Ω,

$$M_{x,r}[f]=\frac{1}{4\pi r^2}\int_{S(x,r)} f(x')\,dS=\frac{1}{4\pi}\int_{\Omega} f(x+r\omega)\,d\Omega.$$

Property (5.29) follows immediately from (5.31), since $S(x,ct)$ does not intersect the set $|x|\leqq a$ in this case. For $ct>2a$ it follows from $ct-a\leqq|x|\leqq ct+a$ that x lies outside $|x|\leqq a$, and

$$|t^2 M_{x,ct}[g]|=\frac{1}{4\pi c^2}\int_{S(x,ct)}|g(x')|\,dS\leqq\frac{C}{4\pi c^2}\int_{S(x,ct)\cap|x'|\leqq a} dS$$

where C is a bound for g. It follows that $t^2 M_{x,ct}[g]$ is bounded, since the area of the portion of $S(x, ct)$ lying inside $|x| \leqq a$ is obviously bounded, also

$$\left| t \frac{\partial}{\partial t}(t M_{x,ct}[f]) \right| = \left| t M_{x,ct}[f] + \frac{c t^2}{4\pi} \int_{\Omega} \nabla f(x + c t \omega) \cdot \omega \, d\Omega \right|$$

$$\leqq |t M_{x,ct}[f]| + \frac{c}{4\pi} \int_{S(x,ct)} |\nabla f(x')| \, dS \leqq |t M_{x,ct}[f]| + \frac{c C_1}{4\pi} \int_{S(x,ct) \cap |x| \leqq a} dS$$

where C_1 is a bound for $|\nabla f|$. This completes the proof of (5.30), since each of the last two terms is bounded.

§ 6. The Existence of Wave Operators for Wave Propagation Problems of Classical Physics

In this section the results of the preceding sections are used to derive criteria for the existence of wave operators, and therefore asymptotic solutions, for propagation problems involving inhomogeneous media which are perturbations of uniformly propagative homogeneous media. Two criteria are given. The first is applicable to perturbations of any uniformly propagative medium. The second is applicable to perturbations of media of Maxwell type. Both criteria require that $E(x) - E^0$, the difference between the energy forms for the two media, be "small at ∞" in a certain sense. The perturbation may be arbitrarily large on bounded sets of points. Finally, the wave operator $\Omega: \mathscr{H}_0 \to \mathscr{H}$ is shown to be isometric if the homogeneous medium has no static solutions (Case 1), and a generalization of this result is proved for Case 2.

The criteria for the existence of wave operators are derived from Theorem 3.2. The spaces $\mathscr{L}_0, \mathscr{H}_0$ and $\mathscr{H}$ and operators H_0 and H are defined as in § 2, and the set $C_0^\infty(R^n)$ (*cf.* § 5) is selected as the subset D of Theorem 3.2. With these choices conditions (3.17) and (3.18) of the theorem are satisfied and there remains the problem of finding criteria which ensure the convergence of the integral (3.19). To see how this can be done, consider the integrand

$$I(t) = \|(H - H_0) e^{-itH_0} \varphi\|, \qquad \varphi \in C_0^\infty(R^n). \tag{6.1}$$

The operators H and H_0 have the same domain and satisfy the identity

$$H - H_0 = E^{-1} E^0 H_0 - H_0 = (E^{-1} E^0 - I) H_0.$$

Moreover, $u^0(x, t) = e^{-itH_0} \varphi(x)$ represents the solution of the propagation problem for the homogeneous medium and satisfies

$$H_0 u^0(x, t) = i \frac{\partial u^0(x, t)}{\partial t}.$$

Thus

$$I(t) = \|(E^{-1} E^0 - I) H_0 u^0(\cdot, t)\| = \left\| (E^{-1} E^0 - I) \frac{\partial u^0(\cdot, t)}{\partial t} \right\|.$$

Applying the definition of the energy norm gives

$$I(t)^2 = \int_{R^n} \left((E^{-1} E^0 - I) \frac{\partial u^0(\cdot, t)}{\partial t} \right)^* E (E^{-1} E^0 - I) \frac{\partial u^0(\cdot, t)}{\partial t} \, dx.$$

Using the symmetry of the matrices E and E^0, we may rewrite this as follows:

$$
(6.2)\quad
\begin{aligned}
I(t)^2 &= \int_{R^n} \left(\frac{\partial u^0(\cdot,t)}{\partial t}\right)^* (E^0 E^{-1} - I)\, E(E^{-1} E^0 - I) \frac{\partial u^0(\cdot,t)}{\partial t}\, dx \\
&= \int_{R^n} \left(\frac{\partial u^0(\cdot,t)}{\partial t}\right)^* (E^0 - E)\, E^{-1}(E^0 - E) \frac{\partial u^0(\cdot,t)}{\partial t}\, dx \\
&= \int_{R^n} \left((E^0 - E)\frac{\partial u^0(\cdot,t)}{\partial t}\right)^* E^{-1}(E^0 - E) \frac{\partial u^0(\cdot,t)}{\partial t}\, dx \\
&= \int_{R^n} v^*(x,t)\, E^{-1}(x)\, v(x,t)\, dx
\end{aligned}
$$

where

$$
(6.3)\qquad v(x,t) = \big(E(x) - E^0\big) \frac{\partial u^0(x,t)}{\partial t}.
$$

Now there is a constant μ such that

$$
\xi^* E^{-1}(x)\, \xi \leqq \frac{1}{\mu^2}\, \xi^* \xi \qquad \text{for } x \in R^n,\ \xi \in R^m,
$$

by Lemma 2.1. Thus (6.2) implies

$$
I(t)^2 \leqq \frac{1}{\mu^2} \int_{R^n} v^*(x,t)\, v(x,t)\, dx = \frac{1}{\mu^2} \int_{R^n} \sum_{\alpha=1}^{m} \left\{ \sum_{\beta=1}^{m} (E_{\alpha\beta}(x) - E^0_{\alpha\beta}) \frac{\partial u^0_\beta(x,t)}{\partial t} \right\}^2 dx.
$$

Applying CAUCHY's inequality to the β-summation in the last integral gives the estimate

$$
(6.4)\qquad I(t)^2 \leqq \frac{1}{\mu^2} \int_{R^n} \left\{ \sum_{\alpha=1}^{m} \sum_{\beta=1}^{m} (E_{\alpha\beta}(x) - E^0_{\alpha\beta})^2 \right\} \left\{ \sum_{\beta=1}^{m} \left(\frac{\partial u^0_\beta(x,t)}{\partial t} \right)^2 \right\} dx.
$$

This estimate is used below to prove the integrability of $I(t)$ on $T \leqq t < \infty$ for every $\varphi^0(x) = u^0(x,0) \in C_0^\infty(R^n)$, and thus demonstrate the existence of wave operators. The principal result is

Theorem 6.1. *Let the matrices $E(x)$, E^0 and A^j $(j=1, \ldots, n)$ have the following properties.*

(6.5) $\displaystyle E^0 \frac{\partial u^0}{\partial t} = \sum_{j=1}^{n} A^j \frac{\partial u^0}{\partial x_j}$ *is a uniformly propagative system.*

(6.6) *$E(x)$ is Lebesgue-measurable, bounded, and uniformly positive definite;* i.e., *there are positive constants μ and μ' such that*

$$
\mu^2\, \xi^* \xi \leqq \xi^* E(x)\, \xi \leqq \mu'^2\, \xi^* \xi \qquad \textit{for every } x \in R^n \textit{ and } \xi \in R^m.
$$

(6.7) *There are constants $K>0$, $R>0$ and $p>1$ such that*

$$
|E_{\alpha\beta}(x) - E^0_{\alpha\beta}| \leqq K\, |x|^{-p} \qquad \textit{for } |x| \geqq R \textit{ and } 1 \leqq \alpha, \beta \leqq m.
$$

Then the wave operator $\Omega(H, H_0)$ exists for the operators H and H_0 defined in §2 by E, E^0 and A^j $(j=1, \ldots, n)$.

Proof. The estimate (6.4) is used to show that $I(t)$, defined by (6.1), is integrable on $T \leqq t < \infty$. The result then follows from Theorem 3.2. To estimate the right-hand side of (6.4), the integral is split into two parts, corresponding to the domains of integration $|x| \leqq \gamma t - a$ and $|x|\ t \geqq \gamma t - a$, and these are estimated separately. If a is chosen so that $\varphi^0(x) = u^0(x,0) \equiv 0$ for $|x| \geqq a$, then Theorem 5.6 is applicable and implies the estimate

$$
\begin{aligned}
I_1(t) &\equiv \int_{|x| \leqq \gamma t - a} \left\{ \sum_{\alpha=1}^{m} \sum_{\beta=1}^{m} (E_{\alpha\beta}(x) - E^0_{\alpha\beta})^2 \right\} \left\{ \sum_{\beta=1}^{m} \left(\frac{\partial u^0_\beta(x,t)}{\partial t} \right)^2 \right\} dx \\
&\leqq \frac{m K_0^2}{t^{2n+2}} \int_{|x| \leqq \gamma t - a} \sum_{\alpha=1}^{m} \sum_{\beta=1}^{m} (E_{\alpha\beta}(x) - E^0_{\alpha\beta})^2\, dx.
\end{aligned}
\tag{6.8}
$$

Now (6.6) implies that the components $E_{\alpha\beta}(x)$ are bounded. Indeed, $\xi^* E(x)\,\eta$ is an inner product, for each fixed $x \in R^n$, and Schwarz's inequality gives

$$
|\xi^* E(x)\,\eta| \leqq (\xi^* E(x)\,\xi)^{\frac{1}{2}} (\eta^* E(x)\,\eta)^{\frac{1}{2}} \leqq \mu'^2 (\xi^* \xi)^{\frac{1}{2}} (\eta^* \eta)^{\frac{1}{2}}.
$$

Taking $\xi_\gamma = \delta_{\alpha\gamma}$, $\eta_\gamma = \delta_{\beta\gamma}$ (α and β fixed) gives

$$
|E_{\alpha\beta}(x)| \leqq \mu'^2, \qquad x \in R^n.
$$

Thus

$$
\begin{aligned}
\int_{|x| \leqq \gamma t - a} (E_{\alpha\beta}(x) - E^0_{\alpha\beta})^2\, dx &= \left(\int_{|x| \leqq R} + \int_{R \leqq |x| \leqq \gamma t - a} \right) (E_{\alpha\beta}(x) - E^0)^2\, dx \\
&\leqq K_1 + K^2 \int_{R \leqq |x| \leqq \gamma t - a} |x|^{-p}\, dx \\
&= K_1 + K^2 \omega_n \int_R^{\gamma t - a} r^{-2p+n-1}\, dr \\
&= K_1 + K^2 \omega_n \{(\gamma t - a)^{n-2p} - R^{n-2p}\} \leqq K_2\, t^{n-2p}
\end{aligned}
$$

where K_1 and K_2 are constants and ω_n is the area of the unit sphere in n-dimensional space. Combining this with (6.8) gives an estimate

$$
I_1(t) \leqq K_3\, t^{-n-2p-2} \tag{6.9}
$$

where K_3 is a constant. Next

$$
\begin{aligned}
I_2(t) &\equiv \int_{|x| \geqq \gamma t - a} \left\{ \sum_{\alpha=1}^{m} \sum_{\beta=1}^{m} (E_{\alpha\beta}(x) - E^0_{\alpha\beta})^2 \right\} \left\{ \sum_{\beta=1}^{m} \left(\frac{\partial u^0_\beta(x,t)}{\partial t} \right)^2 \right\} dx \\
&\leqq \left(\sup_{|x| \geqq \gamma t - a} \sum_{\alpha=1}^{m} \sum_{\beta=1}^{m} (E_{\alpha\beta}(x) - E_{\alpha\beta})^2 \right) \int_{|x| \geqq \gamma t - a} \sum_{\beta=1}^{m} \left(\frac{\partial u^0_\beta(x,t)}{\partial t} \right)^2 dx.
\end{aligned}
\tag{6.10}
$$

Now, by (6.7)

$$
\sup_{|x| \geqq \gamma t - a} (E_{\alpha\beta}(x) - E^0_{\alpha\beta})^2 \leqq K^2 (\gamma t - a)^{-2p} \leqq K_4\, t^{-2p}
$$

where K_4 is a constant. Thus

$$
\sup_{|x| \geqq \gamma t - a} \sum_{\alpha=1}^{m} \sum_{\beta=1}^{m} (E_{\alpha\beta}(x) - E^0_{\alpha\beta})^2 \leqq m^2 K_4\, t^{-2p}. \tag{6.11}
$$

Moreover

$$(6.12)\qquad \int_{|x|\geq \gamma t-a}\sum_{\beta=1}^{m}\left(\frac{\partial u_\beta^0(x,t)}{\partial t}\right)^2 dx \leq \frac{1}{\lambda^2}\int_{R^n}\left(\frac{\partial u^0(x,t)}{\partial t}\right)^* E^0\,\frac{\partial u^0(x,t)}{\partial t}\,dx = K_5$$

by the conservation of energy law for $u^0(x,t)$. Here λ is the smallest eigenvalue of E^0 and K_5 is a constant. Combining (6.10), (6.11) and (6.12) gives

$$(6.13)\qquad I_2(t)\leqq K_6\, t^{-2p}.$$

Combining (6.4), (6.9) and (6.13) gives

$$I(t)^2 \leqq \frac{1}{\mu^2} I_1(t)+\frac{1}{\mu^2} I_2(t) \leqq \frac{1}{\mu^2} K_3\, t^{-n-2p-2}+\frac{1}{\mu^2} K_6\, t^{-2p} \leqq K_7^2\, t^{-2p}$$

where K_7 is a constant. Thus

$$I(t)=\|(H-H_0)\,e^{-itH_0}\varphi\|\leqq K_7\, t^{-p},\qquad p>1,$$

which proves that $I(t)$ is integrable and completes the proof of Theorem 6.1.

For perturbations of systems of Maxwell type, Theorem 6.1 can be strengthened as follows.

Theorem 6.2. *Let the matrices $E(x)$, E^0 and A^j $(j=1,\ldots,n)$ have the following properties.*

(6.14) $E^0\dfrac{\partial u}{\partial t}=\sum_{j=1}^{n} A^j\dfrac{\partial u^0}{\partial x_j}$ *is a system of Maxwell type.*

(6.15) *$E(x)$ is Lebesgue-measurable, bounded and uniformly positive definite;*

$$\mu^2\,\xi^*\xi\leqq \xi^* E(x)\,\xi\leqq \mu'^2\,\xi^*\xi \qquad \textit{for every } x\in R^n \textit{ and } \xi\in R^m.$$

(6.16) *There are constants τ_0 and $\delta>0$ such that*

$$\frac{1}{\tau}\left(\int_{\tau\leqq|x|\leqq\tau+\delta}(E_{\alpha\beta}(x)-E_{\alpha\beta}^0)^2\,dx\right)^{\frac{1}{2}}$$

is integrable on $0<\tau_0\leqq\tau<\infty$ for $1\leqq\alpha,\beta\leqq m$.

Then the wave operator $\Omega(H,H_0)$ exists for the operators H and H_0 defined in § 2 by E, E^0 and A^j $(j=1,\ldots,n)$.

Proof. It suffices to show that $I(t)$ is integrable on $T\leqq t<\infty$. By (6.4) and Theorem 5.7, if $\varphi^0(x)=u^0(x,0)\equiv 0$ for $|x|\geqq a$, then

$$(6.17)\qquad I(t)^2\leqq\frac{m^2}{\mu^2}\,\frac{K^2}{t^2}\sum_{\alpha=1}^{m}\sum_{\beta=1}^{m}\int_{ct-a\leqq|x|\leqq ct+a}(E_{\alpha\beta}(x)-E_{\alpha\beta}^0)^2\,dx.$$

But (6.16) implies that

$$\frac{1}{\tau}\left(\int_{\tau-\delta_1\leqq|x|\leqq\tau+\delta_2}(E_{\alpha\beta}(x)-E_{\alpha\beta}^0)^2\,dx\right)^{\frac{1}{2}}$$

is integrable on $0<\tau_1\leqq\tau<\infty$ for any δ_1, δ_2 and $\tau_1=\tau_1(\delta_1)$. Thus (6.17) implies that $I(t)$ is integrable on $\tau(a)\leqq\tau<\infty$, which completes the proof.

Corollary 6.1. *Under hypotheses* (6.14) *and* (6.15) *a sufficient condition for the wave operator* $\Omega=\Omega(H, H_0)$ *to exist is*

$$(6.18) \qquad \int\limits_{\tau\leqq|x|\leqq\tau+\delta} (E_{\alpha\beta}(x)-E^0_{\alpha\beta})^2\,dx \leqq K_8\,\tau^{-\varepsilon}, \qquad \tau\geqq\tau_0>0,\ 1\leqq\alpha,\beta\leqq m,$$

where δ, K_8, ε *and* τ_0 *are positive constants.*

This is immediate because (6.18) implies (6.16).

The wave operator $\Omega=\Omega(H, H_0)$, when it exists, is a bounded operator from H_0 to H. In fact, it was shown in § 3, Theorem 3.1, that if

$$c^2\,\xi^*\,E^0\,\xi\leqq\xi^*\,E(x)\,\xi\leqq c'^2\,\xi^*\,E^0\,\xi \qquad \text{for all } x\in R^n \text{ and } \xi\in R^m,$$

then

$$c\,\|\varphi\|_0\leqq\|\Omega\,\varphi\|\leqq c'\,\|\varphi\|_0 \qquad \text{for all } \varphi\in\mathscr{H}_0.$$

It will be shown that under the hypotheses of Theorem 6.1 this result can be strengthened. The following lemma is needed for the proof.

Lemma 6.1. *Under the hypotheses of Theorem* 6.1, *if* $u^0(x,t)=(e^{-itH_0})\,\varphi^0(x)$ *where* $\varphi^0\in C_0^\infty(R^n)$ *and* $\varphi^0(x)\equiv 0$ *for* $|x|\geqq a$, *then the following statements hold.*

(6.19) $\varphi^\infty(x)=\lim\limits_{t\to\infty} u^0(x,t)$ *exists for each* x, *uniformly on bounded sets in* R^n.

(6.20) *There is a constant* K_9 *such that*

$$|u^0_\alpha(x,t)-\varphi^\infty_\alpha(x)|\leqq K_9\,t^{-n} \qquad \text{for } |x|\leqq\gamma\,t-a,\ 1\leqq\alpha\leqq m.$$

(6.21) $\varphi^\infty(x)$ *is a (weak) static solution;* i.e.,

$$\sum_{j=1}^n A^j\,\frac{\partial\varphi^\infty(x)}{\partial x_j}=0.$$

Proof. Note that since $\varphi^0\in C_0^\infty(R^n)$, $u^0(x,t)\in C^\infty(R^{n+1})$ and

$$(6.22) \qquad u^0(x,t)-u^0(x,\tau)=\int_\tau^t \frac{\partial u^0(x,t')}{\partial t'}\,dt' \qquad \text{for all } x, t \text{ and } \tau.$$

Moreover, by Theorem 5.6,

$$(6.23) \qquad \left|\frac{\partial u^0_\alpha(x,t')}{\partial t'}\right|\leqq K_0\,t'^{\,-n-1} \qquad \text{for } |x|\leqq\gamma\,t'-a,\ 1\leqq\alpha\leqq m.$$

(6.22) and (6.23) imply (6.19). Also, making $\tau\to\infty$ in (6.22) and using (6.23) gives (6.20). Finally, since $u^0(x,t)$ solves

$$(6.24) \qquad E^0\,\frac{\partial u^0}{\partial t}=\sum_{j=1}^n A^j\,\frac{\partial u^0}{\partial x_j},$$

it follows immediately from (6.19) and (6.23) that φ^∞ is a weak (distribution) solution of (6.24) which proves (6.21).

Theorem 6.3*. *Under the hypotheses of Theorem* 6.1, *if*

$$\varphi^0 \in C_0^\infty(R^n) \quad \textit{and} \quad \varphi^\infty(x) = \lim_{t\to\infty} u^0(x,t) = \lim_{t\to\infty} (e^{-itH_0})\,\varphi^0(x),$$

then

$$\|\Omega\varphi^0\|^2 = \|\varphi^0\|_0^2 + \int_{R^n} \varphi^{\infty *}(x)\big(E(x)-E^0\big)\varphi^\infty(x)\,dx. \tag{6.25}$$

In particular, in Case 1 where there are no static solutions, $\Omega\colon \mathscr{H}_0 \to \mathscr{H}$ is an isometry:

$$\|\Omega\varphi^0\| = \|\varphi^0\|_0. \tag{6.26}$$

In Case 2, if $\varphi^\infty \in \mathscr{H}$ (and therefore $\varphi^\infty \in \mathscr{H}_0$), then

$$\|\Omega\varphi^0\|^2 - \|\varphi^\infty\|^2 = \|\varphi^0\|_0^2 - \|\varphi^\infty\|_0^2. \tag{6.27}$$

Proof. Note that

$$\|\Omega\varphi^0\|^2 = \lim_{t\to\infty} \|e^{itH} e^{-itH_0}\varphi^0\|^2 = \lim_{t\to\infty} \|u^0(\cdot,t)\|^2 \tag{6.28}$$

where $u^0(x,t) = (e^{-itH_0})\,\varphi^0(x)$. Moreover,

$$\begin{aligned}
\|u^0(\cdot,t)\|^2 &= \int_{R^n} u^{0*}(x,t)\,E(x)\,u^0(x,t)\,dx \\
&= \int_{R^n} u^{0*}(x,t)\,E^0 u^0(x,t)\,dx + \int_{R^n} u^{0*}(x,t)\big(E(x)-E^0\big)u^0(x,t)\,dx \\
&= \|u(\cdot,t)\|_0^2 + J(t) = \|e^{-itH_0}\varphi^0\|_0^2 + J(t) \\
&= \|\varphi^0\|_0^2 + J(t).
\end{aligned} \tag{6.29}$$

(6.28) and (6.29) imply that

$$\lim_{t\to\infty} J(t)$$

exists. To prove (6.25), it must be shown that the value of the limit is

$$\lim_{t\to\infty} J(t) = \int_{R^n} \varphi^{\infty *}(x)\big(E(x)-E^0\big)\varphi^\infty(x)\,dx.$$

Now

$$\begin{aligned}
J(t) &= \Big(\int_{|x|\le\gamma t-a} + \int_{|x|\ge\gamma t-a}\Big)\, u^{0*}(x,t)\big(E(x)-E^0\big)u^0(x,t)\,dx \\
&= J_1(t) + J_2(t).
\end{aligned}$$

Moreover,

$$\begin{aligned}
J_1(t) = &\int_{|x|\le\gamma t-a} \varphi^{\infty *}(x)\big(E(x)-E^0\big)\varphi^\infty(x)\,dx + \\
&+ 2\int_{|x|\le\gamma t-a} u^{0*}(x,t)\big(E(x)-E^0\big)\big(u^0(x,t)-\varphi^\infty(x)\big)\,dx + \\
&+ \int_{|x|\le\gamma t-a} \big(u^0(x,t)-\varphi^\infty(x)\big)^*\big(E(x)-E^0\big)\big(u^0(x,t)-\varphi^\infty(x)\big)\,dx.
\end{aligned}$$

* This result was suggested by Professor DALE THOE.

The last two integrals above tend to zero when $t\to\infty$. This follows by a simple argument using the energy law for $u^0(x,t)$, (6.7) for $E(x)-E^0$, and (6.20) for $u^0(x,t)-\varphi^\infty(x)$. Next.

$$J_2(t)\leqq \int_{|x|\geqq\gamma t-a} \sum_{\alpha=1}^m |u_\alpha^0(x,t)| \sum_{\beta=1}^m |E_{\alpha\beta}(x)-E_{\alpha\beta}^0|\,|u_\beta^0(x,t)|\,dx$$

$$\leqq \int_{|x|\geqq\gamma t-a} \sum_{\alpha=1}^m |u_\alpha^0(x,t)| \left\{\sum_{\beta=1}^m (E_{\alpha\beta}(x)-E_{\alpha\beta}^0)^2\right\}^{\frac{1}{2}} \left\{\sum_{\beta=1}^m (u_\beta^0(x,t))^2\right\}^{\frac{1}{2}} dx$$

$$\leqq \int_{|x|\geqq\gamma t-a} \sum_{\alpha=1}^m (u_\alpha^0(x,t))^2 \left\{\sum_{\alpha=1}^m \sum_{\beta=1}^m (E_{\alpha\beta}(x)-E_{\alpha\beta}^0)^2\right\}^{\frac{1}{2}} dx$$

by two applications of CAUCHY's inequality. Thus

$$J_2(t)\leqq\left\{\sup_{|x|\geqq\gamma t-a} \sum_{\alpha=1}^m \sum_{\beta=1}^m (E_{\alpha\beta}(x)-E_{\alpha\beta}^0)^2\right\}^{\frac{1}{2}} \int_{R^n} \sum_{\alpha=1}^m (u_\alpha^0(x,t))^2\,dx$$

$$\leqq \{m^2 K_4 t^{-2p}\}^{\frac{1}{2}} \frac{1}{\lambda^2} \int_{R^n} u^{0\,*}(x,t)\,E^0 u^0(x,t)\,dx \leqq K_{10}\,t^{-p}$$

where

$$K_{10}=m\sqrt{K_4}\,\lambda^{-1}\,\|u^0(\cdot,t)\|_0^2=\text{constant}.$$

In particular

$$\lim_{t\to\infty} J_2(t)=0.$$

Combining the estimate for $J_1(t)$ and $J_2(t)$ gives

$$\lim_{t\to\infty} J(t)=\lim_{t\to\infty} \int_{|x|\leqq\gamma t-a} \varphi^{\infty\,*}(x)\big(E(x)-E^0\big)\varphi^\infty(x)\,dx$$

$$=\int_{R^n} \varphi^{\infty\,*}(x)\big(E(x)-E^0\big)\varphi^\infty(x)\,dx.$$

In particular, the last integral exists. This proves equation (6.25) of Theorem 6.3. Equations (6.26) and (6.27) follow immediately from (6.25).

§ 7. Concluding Remarks

The existence of the wave operator $\Omega(H,H_0)$ implies the existence of asymptotic solutions $u^0(x,t)$ which approximate true solutions $u(x,t)$ in the energy norm (or mean square) sense when $t\to\infty$. It is also desirable to find conditions which guarantee that $u^0(x,t)$ approximates $u(x,t)$ point-wise. This can be done by showing that the partial derivatives up to a prescribed order of $u^0(x,t)$ approximate those of $u(x,t)$ in the mean square sense, provided that $E(x)$ and $\varphi(x)=u(x,0)$ have a suitable number of derivatives. Point-wise estimates for $u(x,t)-u^0(x,t)$ can then be obtained from Sobolev's lemma. This program has been carried out by the author for the classical non-relativistic Schrödinger equation [*19*] and an extension of the results to the problems studied in this paper is planned.

It was shown in § 6 that under certain conditions $\Omega\colon \mathscr{H}_0\to\mathscr{H}$ is an isometry; *i.e.*, $\|\Omega\varphi^0\|=\|\varphi^0\|_0$. It is of considerable interest to find additional conditions which ensure that Ω is unitary, *i.e.*, the range of Ω equals $\mathscr{H}$. In this case (Theorem 3.1) $\Omega=\Omega(H, H_0)$ has an inverse, $\Omega(H_0, H)$ exists and is unitary and

$$\Omega(H, H_0)^{-1}=\Omega(H_0, H).$$

In studying the same question for the classical Schrödinger equation, IKEBE [*11*] has shown that the unitarity of Ω is closely related to the completeness of the generalized eigenfunction expansions associated with H and H_0. Moreover, he has given an explicit construction for Ω in terms of these expansions. It is of considerable interest to develop analogous results for the wave propagation problems of classical physics. In a special case, the transmission line equations, this has been done by BROWN [*4*].

Media governed by systems of the form

$$E^0 \frac{\partial u^0}{\partial t}=\sum_{j=1}^{n} A^j \frac{\partial u^0}{\partial x_j}$$

are non-dispersive; *i.e.*, their phase and group velocities coincide, whereas systems of the form

$$E^0 \frac{\partial u^0}{\partial t}=\sum_{j=1}^{n} A^j \frac{\partial u^0}{\partial x_j}+B u^0, \qquad B\neq 0, \tag{7.1}$$

are dispersive. It is of interest to develop a theory of wave operators and asymptotic solutions for perturbations of dispersive systems such as (7.1). In this connection, LUDWIG [*14*] has shown that the Riemann matrix of such a system may decrease much more slowly than t^{-n} when $t\to\infty$. In fact he has given examples with $n=2$ and $n=3$ where $R(x, t)$ decreases no faster than $t^{-\frac{1}{2}}$. The existence of wave operators for such systems can be proved by the techniques of this paper, provided the rate of decrease of $E(x)-E^0$(for $|x|\to\infty$) is raised to compensate for the slow decrease of $R(x, t)$.

Appendix. Some Wave Equations of Classical Physics in Matrix Form

Many of the wave equations of classical physics can be written as systems of first order linear partial differential equations of the form

$$M^0(x) \frac{\partial u}{\partial t}=\sum_{j=1}^{n} M^j(x) \frac{\partial u}{\partial x_j}+N(x) u+f(x, t) \tag{A.1}$$

where $u(x, t)=(u_1(x, t), \ldots, u_m(x, t))^*$ and f are $m\times 1$ (column) matrices and $M^0, M^1, \ldots, M^n$ are $m\times m$ matrices. The wave equations are distinguished among the general first order systems by possessing a quadratic energy density and corresponding energy conservation law in the sense of the following definition [*cf. 9*].

Definition. Let $E(x)$ represent a symmetric, positive definite $m\times m$ matrix. System (A.1) is said to admit the energy density $\eta=u^* E u$ if and only if there exist symmetric matrices $P^1(x), \ldots, P^n(x)$ and $Q(x)$ and a matrix $R(x)$ such that

$$\frac{\partial \eta}{\partial t}\equiv\sum_{j=1}^{n} \frac{\partial}{\partial x_j}(u^* P^j u)+u^* Q u+u^* R f \tag{A.2}$$

for all $u \in C^1$, where by definition

$$f = M^0 \frac{\partial u}{\partial t} - \sum_{j=1}^{n} M^j \frac{\partial u}{\partial x_j} - N u .$$

Note that if (A.1) admits an energy density, then solutions of the homogeneous equation ($f=0$) satisfy the energy conservation law

$$\frac{\partial \eta}{\partial t} = \sum_{j=1}^{n} \frac{\partial}{\partial x_j} (u^* P^j u) + u^* Q u .$$

It is assumed here that the matrix $M^0(x)$ is non-singular. (Otherwise, the process described by (A.1) is indeterminate.) Hence (A.1) can be rewritten as

$$E(x) \frac{\partial u}{\partial t} = \sum_{j=1}^{n} A^j(x) \frac{\partial u}{\partial x_j} + B(x) u + g(x, t) \tag{A.3}$$

where

$$A^j = E(M^0)^{-1} M^j, \quad B = E(M^0)^{-1} N \quad \text{and} \quad g = E(M^0)^{-1} f.$$

(A.3) is called the canonical form of (A.1), relative to the energy density $\eta = u^* Eu$. It is not difficult to verify the following theorem.

Theorem. *A system* (A.1) *admits the energy density $\eta = u^* Eu$ if and only if, when it is written in canonical form* (A.3) *relative to η, the matrices A^j ($j = 1, 2, \dots, n$) are symmetric. If* (A.1) *admits the energy density η, then the matrices P^j, Q and R of* (A.2) *are*

$$P^j = A^j \quad (j = 1, 2, \dots, n), \qquad Q = B + B^* - \sum_{j=1}^{n} \frac{\partial A^j}{\partial x_j}, \qquad R = 2E(M^0)^{-1}.$$

The purpose of this Appendix is to exhibit a number of wave equations of classical physics in the canonical form (A.3).

In the examples, the physics provides both the basic equations (A.1) and an appropriate energy density. In each case it will be seen that the equations for inhomogeneous media can be written in the form (1.1) of this paper; *i.e.*, when they are written in canonical form the matrices A^j are constant and B is zero, so that the inhomogeneity is described entirely by $E(x)$. Of course, the equations can be written in the form (A.3) in a number of different ways, corresponding to different choices of dependent and independent variables. In most cases they will assume the special form (1.1) only after a judicious choice of variables.

The Transmission Line Equations. In a conventional notation these equations are

$$\begin{aligned} L(x) \frac{\partial i}{\partial t} + \frac{\partial e}{\partial x} &= 0, \\ C(x) \frac{\partial e}{\partial t} + \frac{\partial i}{\partial x} &= 0 \end{aligned} \tag{A.4}$$

where I and e are the current and voltage in the line and L and C are the inductance and capacitance per unit length. They can be written as a 2×2 matrix equation for $u=(u_1, u_2)^*=(i, e)^*$. The appropriate energy density is $\eta=Li^2+Ce^2$. Hence, (A.4) can be put into canonical form with

$$E(x)=\begin{pmatrix} L(x) & 0 \\ 0 & C(x) \end{pmatrix}, \quad A^1=\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}.$$

Maxwell's Equations. The equations for an inhomogeneous, anisotropic dielectric medium can be written

$$(\nabla\times\vec{H})_j-\sum_{k=1}^{3}\varepsilon_{jk}(x)\frac{\partial E_k}{\partial t}=0$$

$$(\nabla\times\vec{E})_j+\sum_{j=1}^{3}\mu_{jk}(x)\frac{\partial H_k}{\partial t}=0.$$

Here $\vec{E}$ and $\vec{H}$ are the electric and magnetic field vectors, ε_{jk} and μ_{jk} are the dielectric and magnetic permeability tensors and the subscripts denote components in a rectangular coordinate system. They can be written as a 6×6 matrix equation for $u=(E_1, E_2, E_3, H_1, H_2, H_3)^*$. The energy density is (apart from a constant factor)

$$\eta=\sum_{i,j=1}^{3}(\varepsilon_{ij}E_iE_j+\mu_{ij}H_iH_j).$$

Hence, MAXWELL's equations can be put into canonical form with

$$E(x)=\begin{pmatrix}
\varepsilon_{11}(x) & \varepsilon_{12}(x) & \varepsilon_{13}(x) & 0 & 0 & 0 \\
\varepsilon_{21}(x) & \varepsilon_{22}(x) & \varepsilon_{23}(x) & 0 & 0 & 0 \\
\varepsilon_{31}(x) & \varepsilon_{32}(x) & \varepsilon_{33}(x) & 0 & 0 & 0 \\
0 & 0 & 0 & \mu_{11}(x) & \mu_{12}(x) & \mu_{13}(x) \\
0 & 0 & 0 & \mu_{21}(x) & \mu_{22}(x) & \mu_{23}(x) \\
0 & 0 & 0 & \mu_{31}(x) & \mu_{32}(x) & \mu_{33}(x)
\end{pmatrix}$$

and

$$\sum_{j=1}^{3}A^jD_j=\begin{pmatrix}
0 & 0 & 0 & 0 & -D_3 & D_2 \\
0 & 0 & 0 & D_3 & 0 & -D_1 \\
0 & 0 & 0 & -D_2 & D_1 & 0 \\
0 & D_3 & -D_2 & 0 & 0 & 0 \\
-D_3 & 0 & D_1 & 0 & 0 & 0 \\
D_2 & -D_1 & 0 & 0 & 0 & 0
\end{pmatrix}.$$

The Equations of Acoustics. The equation for acoustic waves in an inhomogeneous fluid at rest can be written

$$\text{(A.5)} \qquad \frac{1}{c^2(x)}\frac{\partial^2 p}{\partial t^2}=\rho(x)\nabla\cdot\left(\frac{1}{\rho(x)}\nabla p\right).$$

Here p represents the difference between the instantaneous pressure and the equilibrium pressure, $\rho(x)$ is the equilibrium density and $c(x)$ is the local speed of sound. This can be rewritten as a 4×4 matrix system for

$$u=\left(\frac{1}{\rho(x)}\frac{\partial p}{\partial x_1},\frac{1}{\rho(x)}\frac{\partial p}{\partial x_2},\frac{1}{\rho(x)}\frac{\partial p}{\partial x_3},\frac{\partial p}{\partial t}\right)^*.$$

Then energy density is

$$\eta=\frac{1}{\rho(x)}\left\{(\nabla p)^2+\frac{1}{c^2(x)}\left(\frac{\partial p}{\partial t}\right)^2\right\}$$

and the equations have the canonical form with

$$E(x)=\begin{pmatrix}\rho(x) & 0 & 0 & 0\\ 0 & \rho(x) & 0 & 0\\ 0 & 0 & \rho(x) & 0\\ 0 & 0 & 0 & \dfrac{1}{\rho(x)c^2(x)}\end{pmatrix},\qquad \sum_{j=1}^{3}A^j D_j=\begin{pmatrix}0&0&0&D_1\\0&0&0&D_2\\0&0&0&D_3\\D_1&D_2&D_3&0\end{pmatrix}.$$

The Equations of Elasticity. Elastic waves in an inhomogeneous anisotropic medium satisfy equations of the form

$$\frac{\partial^2 w_i}{\partial t^2}=\sum_{j,m,n=1}^{3}\frac{\partial}{\partial x_j}\left(c^{ij}_{mn}(x)\frac{\partial w_m}{\partial x_n}\right),\qquad i=1,2,3.$$

Here w_i is the i^{th} component of the displacement vector and c^{ij}_{mn} is the tensor which relates the stress and strain tensors in the medium. The tensor c^{ij}_{mn} has the symmetries

$$c^{ij}_{mn}=c^{ji}_{mn}=c^{ji}_{nm}=c^{nm}_{ji}$$

and hence has 21 independent components. The energy density is

$$\eta=\sum_{i=1}^{3}\left(\frac{\partial w_i}{\partial t}\right)^2+\sum_{i,j,m,n=1}^{3}c^{ij}_{mn}\frac{\partial w_m}{\partial x_n}\frac{\partial w_i}{\partial x_j}.$$

Define the velocity vector

$$v_i=\frac{\partial w_i}{\partial t}$$

and the stress tensor

$$\Sigma_{ij}=c^{ij}_{mn}\frac{\partial w_m}{\partial x_n}$$

(symmetric in i,j), and put

$$u=(\Sigma_{11},\Sigma_{22},\Sigma_{33},\Sigma_{12},\Sigma_{23},\Sigma_{31}\ v_1,v_2,v_3)^*.$$

Moreover, define a symmetric positive definite 6×6 matrix $\Gamma(x)$ by

$$\Gamma(x)^{-1}=\begin{pmatrix}c^{11}_{11} & c^{11}_{22} & c^{11}_{33} & c^{11}_{12} & c^{11}_{23} & c^{11}_{31}\\ c^{22}_{11} & c^{22}_{22} & c^{22}_{33} & c^{22}_{12} & c^{22}_{23} & c^{22}_{31}\\ \vdots & & & & & \\ c^{31}_{11} & c^{31}_{22} & \cdots & \cdots & \cdots & c^{31}_{31}\end{pmatrix}.$$

Then the elasticity equations can be written in the canonical form with

$$E(x)=\begin{pmatrix} \Gamma_{11}(x) & \Gamma_{12}(x) \dots \Gamma_{16}(x) & 0 & 0 & 0 \\ \Gamma_{21}(x) & \Gamma_{22}(x) \dots \Gamma_{26}(x) & 0 & 0 & 0 \\ \vdots & & & & \\ \Gamma_{61}(x) & \Gamma_{62}(x) \dots \Gamma_{66}(x) & 0 & 0 & 0 \\ 0 & 0 \quad \dots \quad 0 & 1 & 0 & 0 \\ 0 & 0 \quad \dots \quad 0 & 0 & 1 & 0 \\ 0 & 0 \quad \dots \quad 0 & 0 & 0 & 1 \end{pmatrix}$$

and

$$\sum_{j=1}^{3} A^j D_j=\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & D_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & D_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & D_3 \\ 0 & 0 & 0 & 0 & 0 & 0 & D_2 & D_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & D_3 & D_2 \\ 0 & 0 & 0 & 0 & 0 & 0 & D_3 & 0 & D_1 \\ D_1 & 0 & 0 & D_2 & 0 & D_3 & 0 & 0 & 0 \\ 0 & D_2 & 0 & D_1 & D_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & D_3 & 0 & D_2 & D_1 & 0 & 0 & 0 \end{pmatrix}.$$

Each of the four examples from classical physics given above has the form (1.1). Moreover, for homogeneous media ($E(x)=E^0$ constant) each example is uniformly propagative except for certain special values of the parameters. For example, in crystal optics the sheets of the slowness surface may intersect for certain values of the dielectric constants ε_{ij} but this behavior disappears if the values of the ε_{ij} are changed slightly (see also [7]). Finally, the equations of acoustics and MAXWELL's equations for an isotropic medium ($\varepsilon_{ij}=\varepsilon_0\,\delta_{ij}$, $\mu_{ij}=\mu_0\,\delta_{ij}$) are systems of Maxwell type. This is easy to verify by finding the polynomial

$$\det\left(E^0 s+\sum_{j=1}^{n} A^j \eta_j\right).$$

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University of Wisconsin
Madison
and
Argonne National Laboratory
Argonne, Illinois

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