Thermodynamics, Stability, and Boundedness of Fluids of Complexity 2 and Fluids of Second Grade

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Contents

1. Introduction

The Navier-Stokes theory of incompressible fluids has undergone theoretical studies unmatched in breadth and depth by any other single theory of liquids. While it is easy to point to the successes of this theory in describing the behavior of certain real fluids, it is equally easy to exhibit its failures in modeling the responses of others. Thus, in the last quarter century many new theories have been proposed and studied in an attempt to understand the phenomena loosely named "non-Newtonian". One such proposal that has gained support from both experimenters and theorists is the Rivlin-Ericksen theory of differential type. In this model not only the stretching tensor but also certain other kinematic tensors—the Rivlin-Ericksen tensors—are identified as important in characterizing the stress response of a deforming fluid body.

A well-known special example of a Rivlin-Ericksen fluid is an incompressible fluid of second grade, for which the stress $\boldsymbol{T}$ is given by

$$\boldsymbol{T}=-p\mathbf{1}+\mu \boldsymbol{A}_1+\alpha_1 \boldsymbol{A}_2+\alpha_2 \boldsymbol{A}_1^2. \tag{A}$$

Here, the spherical stress $-p\mathbf{1}$ is due to incompressibility, $\boldsymbol{A}_1$ and $\boldsymbol{A}_2$ are the first two Rivlin-Ericksen tensors, and the viscosity μ and the two normal stress moduli α_1 and α_2 may depend on temperature. While this model has been applied in numerous articles to describe certain non-Newtonian behavior there has been no apparent attempt to understand it fully as an exact model in its own right.* This is, perhaps, due solely to the prejudice that α_1 is negative – a prejudice that is supported by seemingly plausible arguments and which, if accepted, then enables one to show that so natural a flow as simple shearing between parallel plates not only lacks stability but, for certain plate separations, does not even exist. On such grounds, (A) seems to have been almost completely rejected as an exact model for any real fluid.

However, we are now convinced that the reasons for which (A) has been discredited are insubstantial and that (A) has, therefore, been prematurely discounted. Indeed, a somewhat surprising thermodynamic development in this regard indicates that the modulus α_1 in (A) should be taken non-negative, and, moreover, that if (A) is also compatible** with thermodynamics, then the second grade fluid model has general and pleasant boundedness, stability, and exponential decay properties well known within the classical Navier-Stokes theory. On the other hand, if our thermodynamic basis for non-negative α_1 is set aside and α_1 is taken negative, then we shall show that in quite *arbitrary* flows instability and unboundedness are unavoidable.

In order to expose the nature of our results more fully we first briefly review certain details in the history of (A):

It was eleven years ago that (A) was first applied, by LANGLOIS [1], to a specific problem; namely, to study the slow motion of a "slightly" viscoelastic liquid. While its use in this application was motivated mainly as an approximation within the general class of Rivlin-Ericksen fluids, it also had the desirable feature of avoiding both the ill-fated Weissenberg assertion and the too restrictive Reiner-Rivlin relation. Three years before LANGLOIS' work was published COLEMAN & NOLL [2] had already shown that by retarding the flow of any simple fluid with fading memory the response was approximated by (A) to within second order in the retardation parameter. At that time they also pointed out that for several special but interesting problems the equations of motion for a fluid of second grade are linear and solvable using standard mathematical analysis.

Subsequently, some of these problems were solved by TING [3] and COLEMAN & MARKOVITZ [4, 5]. While TING gave *mathematical* motivation for basing his work on the assumption that μ and α_1 should be taken positive, COLEMAN & MARKOVITZ quoted experimental evidence and gave a *physical* justification of their own which argued for positive μ and negative α_1 – at least when (A) arises out of the time-scale expansion of finite second-order viscoelasticity theory. Shortly thereafter, TRUESDELL [8] also gave an argument in support of negative α_1 by directly relating it to the response time of a fluid of convected elasticity and then observing that this time should be such that only the past, and not future, configurations affect the present value of the stress. Almost coincidentally, COLEMAN, DUFFIN & MIZEL [6] and COLEMAN & MIZEL [7] demonstrated the instability, non-existence, and breakdown of pure shearing motions for second

* However, motivation for expecting (A) to model the general flow behavior of fluids with sufficiently small natural time has been offered by TRUESDELL [40].

** See Section 3.

grade fluids with $\alpha_1 < 0$, thereby casting grave doubts upon the appropriateness of a second grade fluid as a model of *any* real fluid.

However, the issue concerning the sign of α_1 is not as straightforward as the above very brief review might indicate. Specifically, the argument of TRUESDELL [8], which uses purely *mechanical* concepts, is not compelling as a restriction on the class of second grade fluids. Indeed, we shall show at the end of Section 5 that when fluids of convected elasticity are examined within the larger context of *thermodynamics* they may experience only spherical states of stress – a situation that does not provide adequate structure for relating the response time to α_1, as his argument requires. Moreover, while TRUESDELL proves that in viscometric flows a fluid of second grade is indistinguishable from a particular fluid of convected elasticity with response time $t^* = -2\alpha_1/\mu$, we can find no basis for his subsequent interpretation of this *numerical equality* as a statement of *conceptual identity*.* As a numerical equality his work shows only that the viscometric response of a second grade fluid with $\alpha_1 > 0$ cannot be mimed within the class of reasonable (*i.e.*, $t^* \geqq 0$) fluids of convected elasticity. That an unreasonable (*i.e.*, $t^* < 0$) fluid of convected elasticity might nevertheless duplicate, in a very special class of flows, the response of a second grade fluid with $\alpha_1 > 0$, argues neither for nor against $\alpha_1 > 0$.

Further, the result of COLEMAN & MARKOVITZ [4] to the effect that for a retarded history the expansion of the stress functional appropriate to finite second order viscoelasticity produces a term proportional to A_2 for which the constant of proportionality is expected to be negative gives, clearly, no guidance at all for the sign of α_1 in any real second grade fluid.

A similar confusion pervades the interpretation of much experimental data: data purporting to find α_1 of (A) negative, when coupled with the results of [6], [7], and, more generally, Section 7 of the present paper, only establishes that a second grade fluid has not been entrapped within the experimenter's devices.

It is these and related issues that we address and resolve here. After a brief review in Section 2 of the basic equations of thermomechanics, we introduce in Section 3 incompressible fluids of complexity 2, of which second grade fluids are a special sub-class. In a way now standard, [10], [19], [21], we find necessary and sufficient conditions that its constitutive assumptions be compatible with the Clausius-Duhem inequality and material frame indifference. One of the more interesting results obtained in Section 3 is that the Clausius-Duhem inequality requires the free energy to have a stationary point in equilibrium. The character of this stationary point is, however, governed by the particular constitutive choice one makes for the stress (*cf.* $(3.12)_2$). The crucial nature of this equilibrium point begins to emerge in Section 4 where, by assuming the free energy is a minimum in equilibrium and that the specific heat is positive, we are able to apply some ideas of GURTIN [11], COLEMAN & GREENBERG [12], and COLEMAN [13, 14] to obtain certain theorems of uniqueness and boundedness for any fluid of complexity 2 which is both mechanically isolated and immersed in a thermally passive environment.

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* The condition "$t^* = -2\alpha_1/\mu$" does not assert that a second grade fluid possesses a past- (or future-)oriented "response time"; it does not even assert that a second grade fluid has any response time at all in the sense of [8]. Indeed, by (A), only the *present* values of kinematic variables affect the stress.

Our boundedness theorems are of the character that if one computes a certain number, $\omega(t')$, at any instant t', then, for all later times, this number, and possibly the mass of the fluid, provide an upper bound for such quantities as the kinetic energy, the averaged stretching, the departure from an equilibrium entropy, *etc.*, within the fluid.

Second grade fluids are special fluids of complexity 2. In Section 5 we begin our study of them by further refining the thermodynamic analysis of Section 3 to the specific constitutive assumption (A). In addition to the expected result $\mu \geqq 0$, we find, and these are the keys to all subsequent analyses, that (i) $\alpha_2 = -\alpha_1$, and (ii) *the free energy must be a quadratic function of* A_1 (*cf.* Theorem 6). An immediate conclusion from (ii) is that the free energy of any second grade fluid will be a minimum in equilibrium if and only if $\alpha_1 \geqq 0$. In the remainder of Section 5 we show that when fluids of convected elasticity are examined within the context of thermodynamics then the Clausius-Duhem inequality permits these fluids to experience only spherical states of stress* — a result which we feel counteracts TRUESDELL's argument connecting t^* and α_1 and, *a fortiori*, his argument in favor of $\alpha_1 < 0$.

In the remaining four sections of our work we set aside any thermodynamic motivation for the sign of α_1, and proceed to give a fairly detailed analysis of the two complementary situations corresponding to $\alpha_1 \geqq 0$ and $\alpha_1 < 0$ for the special case when both α_1 and μ are constant with $\mu \geqq 0$. This analysis begins in Section 6 for $\alpha_1 \geqq 0$. Here we introduce a positive definite functional of the velocity field, which corresponds, essentially, to the sum of the kinetic energy and the averaged stretching in the fluid, and show that if the fluid body is mechanically isolated after some initial instant then this functional is monotone decreasing in time and bounded *below* by a decreasing exponential. We then show that for two very general types of mechanical isolation this functional is also bounded *above* by a decreasing exponential. In the special case of flow inside a fixed, rigid container these results are then used to reach estimates on $\|v\|^2(t)$ and $\|\operatorname{grad} v\|^2(t)$ of the form

$$0 \leqq \|v\|^2(t) \leqq B_1 e^{-\beta_1 t}, \qquad B_2 e^{-\beta_2 t} \leqq \|\operatorname{grad} v\|^2(t) \leqq B_3 e^{-\beta_3 t},$$

for positive constants B_i, β_i (*cf.* Corollary 2 of Theorem 9).

Section 6 continues with the derivation of a maximum principle for the energy equation for a broad class of second grade fluids which are immersed in a thermally passive environment. We then go on to demonstrate that for two very natural types of immersion every L_p norm, $1 \leqq p < \infty$, of, essentially, the difference between the (constant) environmental temperature and the temperature field of the body must approach zero exponentially in time. In none of these results do we find it necessary to assume a linear relation for either the heat flux — temperature gradient or the energy density — temperature.

In Section 7 we begin our study of the case $\alpha_1 < 0$. Here, we extend the results obtained by COLEMAN, DUFFIN & MIZEL [6] for simple shearing flow to arbitrary flows inside fixed, rigid containers. In particular, we show that unless the initial data for the velocity field is severely restricted, any flow must evolve so as to have its averaged stretching go exponentially to infinity. Moreover, for "small"

* Within a broader context, it thus follows that thermodynamics undercuts the status of this "convected" fluid model as an exemplar of a fluid with "intrinsic elasticity".

canisters we show that this restriction on the initial data can never be satisfied and, consequently, that within such a container every flow of a second grade fluid having $\alpha_1 < 0$ eventually must generate an arbitrarily large averaged stretching. In contrast, we find that for "large" containers decaying flows are possible, though rare and atypical; for them we determine order estimates for the rate of decay of both $\|\boldsymbol{v}\|^2(\cdot)$ and $\|\text{grad}\,\boldsymbol{v}\|^2(\cdot)$. The remainder of Section 7 contains the formulation of an eigenvalue problem that serves not only to interpret these results but also to suggest certain questions which are studied in the next section.

Section 8 begins with the derivation of a new and useful form of the equation of motion for any second grade fluid with constant α_1. Using this and the eigenvalue problem of Section 7 we obtain a functional-differential equation for the projection of the velocity field on the appropriate eigenvectors (*cf.* (8.5)). When $\alpha_1 < 0$ this functional-differential equation implies that for each member of a countable nested sequence of fixed containers no internal flow in which the fluid adheres to the boundary of the container is possible unless the initial data meets a stringent *a priori* orthogonality condition (*cf.* Theorem 21). In addition, this functional-differential equation yields the explicit solution to a certain special class of initial value flow problems more general than those studied by TING [3] and COLEMAN, DUFFIN & MIZEL [6]. A particular consequence of this result is that, for second grade fluids having $\alpha_1 < 0$, the *only* decaying flows in this class are those composed of a finite linear combination of the eigenvectors introduced in Section 7. This considerably generalizes a similar result obtained for plane shearing flows between fixed parallel walls in [6].

For our final results on second grade fluids we return, in Section 9, to the case $\alpha_1 \geqq 0$. Here, our main result is contained in Theorem 23 and generalizes to second grade fluids ($\alpha_1 \geqq 0$) the stability theorems, now classic, obtained by SERRIN [15] for the Navier-Stokes theory (*i.e.*, $\alpha_1 \equiv 0$). In particular, we show that if the stretching and its diffusion are, for a given base flow, sufficiently small, or if the viscosity is sufficiently large, then the base flow is asympototically stable relative to all disturbances $\boldsymbol{u}$ that vanish on the boundary of the fluid domain. That is, we obtain precise sufficient conditions for the estimate

$$\|\boldsymbol{u}\|^2(t) + \frac{\alpha_1}{\rho}\|\text{grad}\,\boldsymbol{u}\|^2(t) \leqq B_4\, e^{-\beta_4 t},$$

where β_4 and B_4 are positive constants. This estimate leads immediately to the generalization of two other results of SERRIN [15] on the uniqueness of the initial-boundary value problem and on the uniqueness of sufficiently mild steady flows.

Finally, in our Appendix, we present a very brief and elementary discussion of the Poincaré and Korn inequalities, which are employed frequently from Section 6 onwards.

The results of our study make it likely that the only second grade fluid to be found in nature is one with $\alpha_1 \geqq 0$. Thus it is relevant to the discovery of such a fluid to note that it will (i) climb* up a rod which is rotating in an open vat [16, 17] and (ii) sustain a depression ** rather than a bulge in its free surface when flowing down an inclined open channel [18].

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* Because for any simple fluid with normal stress viscometric functions (*cf.* [20]) $\sigma_1(\cdot)$ and $\sigma_2(\cdot)$, climbing requires $3\sigma_1''(0) + \sigma_2''(0) > 0$, and here $\sigma_1(\kappa) = \alpha_1 \kappa^2 = -\sigma_2(\kappa)$.

** Because a depression requires $\sigma_1''(0) > 0$.

Finally, we observe that since $\alpha_1+\alpha_2=0$ the constitutive assumption (A) is of the same form as one proposed several years ago by OLDROYD [24] for the study of colloidal suspensions, provided his stress relaxation time constant is set equal to zero and α_1 is such that $\alpha_1 \geq 0$. However, in his work OLDROYD focused on the effects of relaxation and therefore did not expressly consider the possibility of a zero stress relaxation time constant.

2. Preliminary Notions

As is conventional, we identify the material particles of a continuous medium, or *body*, B with the positions $X \in E^3$ they occupy in a fixed reference configuration $\Omega \subseteq E^3$.* The *motion* of B may then be described by a relation of the form $\boldsymbol{x}=\boldsymbol{\chi}(X, t)$ where $\boldsymbol{x} \in E^3$ denotes the point occupied by the particle X at the time t. The domain of $\boldsymbol{\chi}$ is, of course, $\Omega \times \mathbb{R}$ and if D is any subset of Ω we shall denote by D_t the image of D under $\boldsymbol{\chi}$. When convenient, we shall identify Ω_0 with Ω.**

The *deformation gradient* $\boldsymbol{F}(X, t)$ and the *velocity gradient* $\boldsymbol{L}(\boldsymbol{x}, t)$ associated with the motion $\boldsymbol{\chi}$ are defined, respectively, by***

$$\boldsymbol{F}=\nabla\boldsymbol{\chi}, \qquad \boldsymbol{L}=\operatorname{grad}\dot{\boldsymbol{x}}. \tag{2.1}$$

It is easily proven that

$$\boldsymbol{L}=\dot{\boldsymbol{F}}\boldsymbol{F}^{-1}, \tag{2.2}$$

whenever $\boldsymbol{F}$ is non-singular – an assumption we make throughout.

For a given body B, the following collection of eight functions defined on $\Omega \times \mathbb{R}$ will be called a *thermodynamic process* if the balance laws of momentum and energy and the Clausius-Duhem inequality are satisfied:

(1) The *motion* $\boldsymbol{x}=\boldsymbol{\chi}(X, t)$.

(2) The *temperature* $\theta=\theta(X, t)$ which is assumed to be positive, $\theta>0$.

(3) The *specific internal energy* $\varepsilon=\varepsilon(X, t)$ per unit mass.

(4) The *specific entropy* $\eta=\eta(X, t)$ per unit mass.

(5) The *symmetric stress tensor* $\boldsymbol{T}=\boldsymbol{T}(X, t)$.

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* For the most part we employ a standard notation. Thus, E^3 denotes three dimensional Euclidean point space while V denotes its associated vector space. A linear transformation of V into V shall be called a *tensor* and the set of all tensors we denote by T. The subspaces of T which are composed of all traceless tensors or all traceless, symmetric tensors we shall denote by T° and T_s°, respectively. With the exception of material particles of B in the reference configuration Ω, boldface majiscules will denote tensors while the vectors of V and the points of E^3 shall be denoted by boldface miniscules. Inner products, whether among elements of V or elements of T will be denoted by a dot "·", *e.g.*, $\boldsymbol{a}\cdot\boldsymbol{b}$ or, if "tr(·)" denotes the usual trace operator, then $\boldsymbol{A}\cdot\boldsymbol{B}=\operatorname{tr}(\boldsymbol{A}\boldsymbol{B}^T)$. Vertical bars "| |" denote the usual Euclidean norm, e.g., $|\boldsymbol{A}|=(\boldsymbol{A}\cdot\boldsymbol{A})^{\frac{1}{2}}$. Lastly, we employ $\mathbb{R}(\mathbb{R}^+)$ to denote the set of real (positive) numbers.

** For a function defined on either $\Omega\times\mathbb{R}$ or $\Omega_t\times\mathbb{R}$ we use, respectively, ∇ or grad to represent a partial derivative with respect to the points of Ω or Ω_t, and $\dot{\overline{(\ )}}$ or $(\ )_t$ to indicate a partial derivative with respect to the scalars in $\mathbb{R}$. The divergence operators related to ∇ and grad are denoted by $\nabla\cdot$ and div, respectively.

*** We assume $\boldsymbol{\chi}$ to be sufficiently smooth to make these definitions meaningful. We will, in general, abstain from explicit statements concerning the smoothness of the various functions introduced in this paper since the required smoothness will usually be clear from the context.

(6) The *heat flux* $\boldsymbol{q}=\boldsymbol{q}(X, t)$.

(7) The *specific body force* $\boldsymbol{b}=\boldsymbol{b}(X, t)$ per unit mass.

(8) The *radiant heating* $r=r(X, t)$ per unit mass.

While the symmetry of the stress tensor will guarantee the balance of angular momentum, we record here for future reference the *balance of linear momentum*

$$\frac{d}{dt}\int_{D_t}\rho\dot{\boldsymbol{x}}\,dv=\int_{\partial D_t}\boldsymbol{T}\boldsymbol{n}\,da+\int_{D_t}\rho\boldsymbol{b}\,dv, \tag{2.3}$$

the *balance of energy*

$$\frac{d}{dt}\int_{D_t}\rho(\varepsilon+\tfrac{1}{2}\dot{\boldsymbol{x}}\cdot\dot{\boldsymbol{x}})\,dv=\int_{\partial D_t}(\boldsymbol{T}\boldsymbol{n}\cdot\dot{\boldsymbol{x}}-\boldsymbol{q}\cdot\boldsymbol{n})\,da+\int_{D_t}\rho(\boldsymbol{b}\cdot\dot{\boldsymbol{x}}+r)\,dv, \tag{2.4}$$

and the *Clausius-Duhem inequality*

$$\frac{d}{dt}\int_{D_t}\rho\eta\,dv\geqq-\int_{\partial D_t}\frac{\boldsymbol{q}}{\theta}\cdot\boldsymbol{n}\,da+\int_{D_t}\rho\frac{r}{\theta}\,dv, \tag{2.5}$$

all three of which are assumed to hold for every subdomain $D\subseteq\Omega$ and for all $t\in\mathbb{R}$. The quantity $\rho=\rho(X,t)$ denotes the local mass density which, by the conservation of mass, satisfies

$$\rho(X,t)=\frac{\rho_0(X)}{|\det \boldsymbol{F}(X,t)|}, \tag{2.6}$$

where ρ_0 is a positive function given once and for all along with the body B.

When sufficient smoothness is assumed and (2.6) is taken into account it is easily shown that (2.3)–(2.5) are equivalent to the local equations

$$\rho\ddot{\boldsymbol{x}}=\operatorname{div}\boldsymbol{T}+\rho\boldsymbol{b}, \tag{2.7}$$

$$\rho\dot{\varepsilon}=\boldsymbol{T}\cdot\boldsymbol{L}-\operatorname{div}\boldsymbol{q}+\rho r, \tag{2.8}$$

$$\rho(\dot{\varepsilon}-\dot{\eta}\theta)-\boldsymbol{T}\cdot\boldsymbol{L}+\frac{\boldsymbol{q}\cdot\boldsymbol{g}}{\theta}\leqq 0, \tag{2.9}$$

where we have set $\boldsymbol{g}\equiv\operatorname{grad}\theta$.

Finally, if we introduce the *Helmholtz free energy* $\psi=\psi(X,t)$ defined through

$$\psi\equiv\varepsilon-\theta\eta, \tag{2.10}$$

we may write (2.9) in the alternate form

$$\rho(\dot{\psi}+\eta\dot{\theta})-\boldsymbol{T}\cdot\boldsymbol{L}+\frac{\boldsymbol{q}\cdot\boldsymbol{g}}{\theta}\leqq 0, \tag{2.11}$$

which we shall call the *dissipation inequality*.

3. Response Functions and Thermodynamic Compatibility

Let $N_X\subseteq\Omega$ denote some neighborhood of the material particle X. An important class of materials in continuum mechanics is afforded by those in which knowledge of $\chi(\cdot,\cdot)$ and $\theta(\cdot,\cdot)$, alone, on $N_X\times(-\infty, t]$ suffices to determine the respective values $\varepsilon(X, t)$, $\eta(X, t)$, $T(X, t)$, and $q(X, t)$ *up to terms, if any, that reflect any a priori constraints on either* χ *or* θ. A particular example of such a material, and one which we study here, is provided by an *incompressible, homogeneous fluid of complexity* 2.* It is characterized by the existence of four functions, $\hat{\varepsilon}$, $\hat{\eta}$, $\hat{T}$, $\hat{q}$ such that

$$\begin{aligned} \varepsilon &= \hat{\varepsilon}(\theta, g, L, \dot{L}), \\ \eta &= \hat{\eta}(\theta, g, L, \dot{L}), \\ T &= -p\mathbf{1} + \hat{T}(\theta, g, L, \dot{L}), \\ q &= \hat{q}(\theta, g, L, \dot{L}), \end{aligned} \tag{3.1}$$

where we recall that $g \equiv \operatorname{grad} \theta(x, t)$. The four functions $\hat{\varepsilon}$, $\hat{\eta}$, $\hat{T}$, and $\hat{q}$ are called *response functions*, have $\mathbb{R}^+ \times V \times T^\circ \times T^\circ$ as their common domain of definition, and are assumed to be continuously differentiable. The fact that L and $\dot{L}$ must lie in T° is a consequence of incompressibility, which requires

$$\det F = 1 \quad \text{and} \quad \operatorname{tr} L = 0. \tag{3.2}$$

The scalar field $p = p(x, t)$ appearing in $(3.1)_3$ reflects the *a priori* constraint of incompressibility and is not, in general, determined by θ, g, L, and $\dot{L}$ at (x, t).

If we select an arbitrary isochoric motion $\bar{\chi}$, an arbitrary temperature field $\bar{\theta}$, and an arbitrary "pressure" $\bar{p}$, then (3.1) uniquely determines the fields $\bar{\varepsilon}$, $\bar{\eta}$, $\bar{T}$, and $\bar{q}$. Thus, entering these fields along with $\bar{\chi}$ and $\bar{\theta}$ into (2.7) and (2.8), we arrive at *definitions* for $\bar{r}(\cdot,\cdot)$ and $\bar{b}(\cdot,\cdot)$.** The resulting 8-tuple, $[\bar{\chi}, \bar{\theta}; \bar{\varepsilon}, \bar{\eta}, \bar{T}, \bar{q}, \bar{r}, \bar{b}]$, will be said to have been *induced* by $\bar{\chi}$, $\bar{\theta}$, and $\bar{p}$. This 8-tuple will *not* generally be a thermodynamic process: while the balance of momentum (2.7) and the balance of energy (2.8) are trivially satisfied, the Clausius-Duhem inequality (2.9) will generally fail to hold if the functions $\hat{\varepsilon}$, $\hat{\eta}$, $\hat{T}$, and $\hat{q}$ are selected arbitrarily.***

Following COLEMAN & NOLL [10], we seek restrictions on the response functions $\hat{\varepsilon}$, $\hat{\eta}$, $\hat{T}$, $\hat{q}$ such that *every* choice of an isochoric $\bar{\chi}$, a temperature field $\bar{\theta}$, and a "pressure" field $\bar{p}$ will induce an 8-tuple that is also a thermodynamic process. In this regard we make the following definition: *The response functions* (3.1) *will be said to be compatible with thermodynamics if and only if every choice of* (*an isochoric*) $\bar{\chi}$, $\bar{\theta}$ *and* $\bar{p}$ *leads to an induced* 8-*tuple that is a thermodynamic process.* In applying this definition we observe that since $\operatorname{tr} \bar{L} = 0$, $\bar{p}$ will not appear in either of the equivalent inequalities (2.9) or (2.11), and thus (3.1) will be compatible with thermodynamics if and only if for every choice of an isochoric $\bar{\chi}$ and a

* A material is said to be *incompressible* if the only motions it can undergo are isochoric, *i.e.*, $\det F(X, t) = 1$ for every motion. For a full explanation of *homogeneous*, for a justification of the name of *fluid* for (3.1), and for an explanation of the term *complexity 2* see Sections 27, 31, and 35, respectively, of [20]. For convenience we shall henceforth refer to these materials simply as fluids of complexity 2.

** Since the material is incompressible and homogeneous, it is seen from (2.6) that $\rho(=\rho_0)$ is just a fixed number.

*** To see this, it is only necessary to take for example, $\hat{\varepsilon} = \text{const.}$, $\hat{\eta} = \text{const.}$, $\hat{T} = 0$, and $\hat{q} = \operatorname{grad} \theta$.

temperature field $\bar{\theta}$ the inequality

$$\rho(\dot{\bar{\hat{\psi}}}+\bar{\hat{\eta}}\dot{\bar{\theta}})-\bar{\hat{T}}\cdot\bar{L}+\frac{\bar{\hat{q}}\cdot\bar{g}}{\bar{\theta}}\leqq 0 \tag{3.3}$$

is satisfied.* Here, by (2.10) and (3.1), $\hat{\psi}$ is the continuously differentiable function given by

$$\hat{\psi}(\theta,g,L,\dot{L})=\hat{\varepsilon}(\theta,g,L,\dot{L})-\theta\hat{\eta}(\theta,g,L,\dot{L}),$$

and we note, by the chain rule, that for any sufficiently smooth χ and θ one has**

$$\dot{\hat{\psi}}=\hat{\psi}_\theta\dot{\theta}+\hat{\psi}_g\cdot\dot{g}+\hat{\psi}_L\cdot\dot{L}+\hat{\psi}_{\dot{L}}\cdot\ddot{L}. \tag{3.4}$$

In order to find necessary and sufficient conditions that the response functions be compatible with thermodynamics we first require the following

Lemma. *Let a be an arbitrary positive number, and let a' be an arbitrary real number. Let* $\boldsymbol{a}$ *and* $\boldsymbol{a}'$ *be arbitrary vectors. Let* A, A', A'' *be arbitrary tensors in* T°. *Then there exists an isochoric motion* χ^* *and a positive temperature field* θ^* *such that if Y is any fixed particle in* Ω, *then*

$$\begin{aligned} &L^*(Y,0)=A, \quad \dot{L}^*(Y,0)=A', \quad \ddot{L}^*(Y,0)=A''; \\ &\theta^*(Y,0)=a, \quad \dot{\theta}^*(Y,0)=a'; \\ &\operatorname{grad}\theta^*(Y,0)=\boldsymbol{a}, \quad \overline{\operatorname{grad}\theta^*(Y,0)}^{\cdot}=\boldsymbol{a}'. \end{aligned} \tag{3.5}$$

Proof. Defining $L(\tau)$ by

$$L(\tau)=A+\tau A'+\tfrac{1}{2}\tau^2 A'',$$

we know from the theory of ordinary differential equations that

$$\frac{d}{d\tau}F^*(\tau)=L(\tau)F^*(\tau), \qquad F^*(0)=\mathbf{1},$$

has a unique solution on $\mathbb{R}$ and is such that $\det F^*(\tau)=1$ for all $\tau\in\mathbb{R}$. We thus can define $\chi^*(\cdot,\cdot)$ by

$$\chi^*(X,t)=Y+F^*(t)[X-Y]$$

and easily verify that χ^* is isochoric and meets $(3.5)_1$.

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* The symbolism $\bar{\hat{f}}$ is equivalent to the value of $\hat{f}$ for the fields $\bar{\chi}$ and $\bar{\theta}$.

** The subscripts here denote partial differentiation. Since $\hat{\psi}$ is defined on $\mathbb{R}^+\times V\times T^\circ\times T^\circ$, where T° is a *linear manifold* of T, we may regard T° as an inner product space in its own right and define and compute the derivatives $\hat{\psi}_L$ and $\hat{\psi}_{\dot{L}}$ in the usual way: Let $\phi(\cdot)$ map $L\to\mathbb{R}$ where L is any finite dimensional inner product space (here T°). Then $\phi(\cdot)$ is said to be differentiable at $\boldsymbol{u}_0\in L$ if there exists an element $\phi_{\boldsymbol{u}}(\boldsymbol{u}_0)\in L$ such that

$$\phi(\boldsymbol{u}_0+\boldsymbol{h})=\phi(\boldsymbol{u}_0)+\phi_{\boldsymbol{u}}(\boldsymbol{u}_0)\cdot\boldsymbol{h}+o(|\boldsymbol{h}|) \qquad \forall\,\boldsymbol{h}\in L.$$

$\phi_{\boldsymbol{u}}(\boldsymbol{u}_0)$ is called the derivative of $\phi(\cdot)$ at $\boldsymbol{u}_0$.

Having found $\boldsymbol{F}^*(\cdot)$, we then define $\theta^*(\cdot,\cdot)$ by

$$\theta^*(\boldsymbol{X}, t) = a\, e^{\{(a' t + [\boldsymbol{a} + t(\boldsymbol{a}' - \boldsymbol{a}(a'/a))] \cdot \boldsymbol{F}^*(t)[\boldsymbol{X} - \boldsymbol{Y}])/a\}},$$

and readily verify $(3.5)_2$ and $(3.5)_3$. △

We now have the main result of the present section:

Theorem 1. *The response functions* (3.1) *are compatible with thermodynamics if and only if*

(i) *the free energy is independent of* $\boldsymbol{g}$ *and* $\dot{\boldsymbol{L}}$,

$$\psi = \hat{\psi}(\theta, \boldsymbol{L}), \tag{3.6}$$

(ii) *the "entropy relation" holds,*

$$\eta = -\hat{\psi}_\theta(\theta, \boldsymbol{L}), \tag{3.7}$$

(iii) *the response functions* $\hat{\psi}(\cdot,\cdot)$ $\hat{\boldsymbol{T}}(\cdot,\cdot,\cdot,\cdot)$ *and* $\hat{\boldsymbol{q}}(\cdot,\cdot,\cdot,\cdot)$ *must be such that the reduced dissipation inequality,*

$$\rho \hat{\psi}_{\boldsymbol{L}}(\theta, \boldsymbol{L}) \cdot \dot{\boldsymbol{L}} - \hat{\boldsymbol{T}}(\theta, \boldsymbol{g}, \boldsymbol{L}, \dot{\boldsymbol{L}}) \cdot \boldsymbol{L} + \frac{\hat{\boldsymbol{q}}(\theta, \boldsymbol{g}, \boldsymbol{L}, \dot{\boldsymbol{L}}) \cdot \boldsymbol{g}}{\theta} \leqq 0, \tag{3.8}$$

holds in every thermodynamic process.

Proof. Since, as previously remarked, the response functions (3.1) are compatible with thermodynamics if and only if (3.3) is satisfied for every choice of an isochoric $\bar{\chi}$ and a temperature field $\bar{\theta}$, we may take $\bar{\chi} = \chi^*$ and $\theta = \theta^*$ with χ^* and θ^* as in the lemma. Noting that these functions are class C^∞ and that the chain rule (3.4) applies, we see that at the particle Y and at the instant $t = 0$ (3.3) becomes

$$\begin{aligned} \rho \{(\hat{\psi}_\theta(\Gamma) + \hat{\eta}(\Gamma))\, a' + \hat{\psi}_{\boldsymbol{g}}(\Gamma) \cdot \boldsymbol{a}' + \hat{\psi}_{\boldsymbol{L}}(\Gamma) \cdot \boldsymbol{A}' + \hat{\psi}_{\dot{\boldsymbol{L}}}(\Gamma) \cdot \boldsymbol{A}''\} \\ - \hat{\boldsymbol{T}}(\Gamma) \cdot \boldsymbol{A} + \frac{\hat{\boldsymbol{q}}(\Gamma) \cdot \boldsymbol{a}}{a} \leqq 0, \end{aligned} \tag{3.9}$$

where $\Gamma \equiv (a, \boldsymbol{a}, \boldsymbol{A}, \boldsymbol{A}')$. We observe that (3.9) must hold identically for all $(a, a', \boldsymbol{a}, \boldsymbol{a}', \boldsymbol{A}, \boldsymbol{A}', \boldsymbol{A}'')$ provided only that $a > 0$ and $\boldsymbol{A}$, $\boldsymbol{A}'$, and $\boldsymbol{A}''$ be traceless. Moreover, a', $\boldsymbol{a}'$, and $\boldsymbol{A}''$ appear only linearly; hence (3.9) is equivalent to

$$(\hat{\psi}_\theta(\Gamma) + \hat{\eta}(\Gamma))\, a' \equiv \hat{\psi}_{\boldsymbol{g}}(\Gamma) \cdot \boldsymbol{a}' \equiv \hat{\psi}_{\dot{\boldsymbol{L}}}(\Gamma) \cdot \boldsymbol{A}'' \equiv 0, \tag{3.10}$$

$$\rho \hat{\psi}_{\boldsymbol{L}}(\Gamma) \cdot \boldsymbol{A}' - \hat{\boldsymbol{T}}(\Gamma) \cdot \boldsymbol{A} + \frac{\hat{\boldsymbol{q}}(\Gamma) \cdot \boldsymbol{a}}{a} \leqq 0, \tag{3.11}$$

which must hold for all $(a, a', \boldsymbol{a}, \boldsymbol{a}', \boldsymbol{A}, \boldsymbol{A}', \boldsymbol{A}'')$ meeting $a > 0$, and $\operatorname{tr} \boldsymbol{A} = \operatorname{tr} \boldsymbol{A}' = \operatorname{tr} \boldsymbol{A}'' = 0$. Thus (3.6) and (3.7) necessarily follow from (3.10), and with the additional aid of (3.11) we also reach (3.8). To see that these three conclusions are also sufficient for thermodynamic compatibility requires only substitution. △

Theorem 1 has several immediate consequences the first two of which merit the status of corollaries. To set the context for the first of these, we note that while $\hat{\psi}(\cdot,\cdot)$ was only assumed to be class C^1 it is immeditate* that $\hat{\psi}_{\theta\theta}(\cdot,\cdot)$ and $\hat{\psi}_{\theta L}(\cdot,\cdot)$ (and hence, also $\hat{\psi}_{L\theta}(\cdot,\cdot)$) each exist and are continuous. Concerning $\hat{\psi}_{LL}(\cdot,\cdot)$, we have not been able to show quite so much; however, utilizing the assumed C^1 smoothness of $\hat{T}$, we can prove

Corollary 1. *The free energy of a fluid of complexity* 2 *of necessity has a stationary point at equilibrium.*** *Moreover, the character of this stationary point* (i.e., *minimum, maximum,* etc.) *is completely determined by the function* $\hat{T}(\cdot, 0, 0, \cdot)$. *Specifically,* $\hat{\psi}(\theta,\cdot)$ *is* **twice** *differentiable**** *at zero and*

$$\hat{\psi}_L(\theta, 0)=0, \qquad \rho\hat{\psi}_{LL}(\theta,0)\cdot(A\otimes A)=\hat{T}(\theta,0,0,A)\cdot A \tag{3.12}$$

for all traceless A.

Proof. In (3.11), take $A=a=0$ and recall (3.6) to reach

$$\hat{\psi}_L(\theta,0)\cdot A'\leqq 0$$

for all traceless A'. Clearly, only equality can hold and, since $\hat{\psi}_L$ is traceless, we have proven $(3.12)_1$.

To prove $(3.12)_2$****, we begin by showing that $\hat{T}(\theta,0,0,\cdot)\cdot A$ is linear. To establish this, we return to (3.11), take $a=0$, and use (3.6) to find

$$\rho\hat{\psi}_L(\theta,A)\cdot A'\leqq \hat{T}(\theta,0,A,A')\cdot A$$

for all traceless A, A'. Applying this inequality three times and adding, we arrive at

$$\hat{T}(\theta,0,A,-\beta_1 B_1-\beta_2 B_2)\cdot A+\beta_1\hat{T}(\theta,0,A,B_1)\cdot A+\beta_2\hat{T}(\theta,0,A,B_2)\cdot A\geqq 0$$

for all traceless A and B_i and for all non-negative β_i. Replacing A with xA, $x>0$, and dividing by x, one finds, upon letting $x\to 0$, an inequality linear in A; it thus must be an equality and so

$$\hat{T}(\theta,0,0,-\beta_1 B_1-\beta_2 B_2)\cdot A=-\beta_1\hat{T}(\theta,0,0,B_1)\cdot A-\beta_2\hat{T}(\theta,0,0,B_2)\cdot A.$$

By considering different choices for B_i it is straightforward to remove the restriction that β_i be non-negative and thereby establish the linearity of $\hat{T}(\theta,0,0,\cdot)\cdot A$.

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* Apply (3.7) and the assumed C^1 smoothness of $\hat{\eta}$.

** Our use of the term *equilibrium* here is delibrately vague. A more precise term (but one less suggestive of a later usage (*cf.*, Section 4)) would be *locally at rest.*

*** Let $\phi(\cdot)$ and L be as in footnote ** on page 199 and suppose $\phi(\cdot)$ to be differentiable at $u_0\in L$. Then $\phi(\cdot)$ is said to be twice differentiable at u_0 if, in addition to $\phi_u(u_0)\in L$, there exists a symmetric element $\phi_{uu}(u_0)\in L\otimes L$ such that

$$\phi(u_0+h)=\phi(u_0)+\phi_u(u_0)\cdot h+\tfrac{1}{2}\phi_{uu}(u_0)\cdot(h\otimes h)+o(|h|^2) \qquad \forall h\in L.$$

By symmetric we mean $\phi_{uu}(u_0)\cdot(h\otimes k)=\phi_{uu}(u_0)\cdot(k\otimes h)$ for all $h, k\in L$. We also remark that $\phi_{uu}(u_0)$ may, equivalently, be viewed as a symmetric, bilinear form on $L\oplus L$ (HALMOS [39]) and in this case we would write $\phi_{uu}(u_0)(h,k)$ for $\phi_{uu}(u_0)\cdot(h\otimes k)$ and call $\phi_{uu}(u_0)(h,h)$ the quadratic form associated with $\phi_{uu}(u_0)$.

**** If one *assumes* $\hat{\psi}(\theta,\cdot)$ is twice differentiable in a *neighborhood* of zero, then $(3.12)_2$ may be very quickly established by replacing A with xA and taking $a=0$ in (3.11) and then noting that the resulting function of x is non-positive and vanishes at zero; thus its first derivative there is zero and $(3.12)_2$ results.

Using (3.11) and (3.6) yet again, we easily see that

$$\rho\hat{\psi}_L(\theta, sA)\cdot A' \leqq \frac{s}{\varepsilon}\hat{T}(\theta, 0, sA, \varepsilon A')\cdot A$$

for all traceless A and A', any number s, and any positive number ε. Therefore,

$$-\frac{s}{\varepsilon}\hat{T}(\theta, 0, sA, -\varepsilon A')\cdot A \leqq \rho\hat{\psi}_L(\theta, sA)\cdot A' \leqq \frac{s}{\varepsilon}\hat{T}(\theta, 0, sA, \varepsilon A')\cdot A.$$

Taking $A'=A$, we see that the middle term is $\rho\frac{d}{ds}\hat{\psi}(\theta, sA)$ and thus we come to

$$-\frac{1}{\varepsilon}\int_0^1 s\hat{T}(\theta, 0, sA, -\varepsilon A)\cdot A\,ds \leqq \rho\hat{\psi}(\theta, A)-\rho\hat{\psi}(\theta, 0) \leqq \frac{1}{\varepsilon}\int_0^1 s\hat{T}(\theta, 0, sA, \varepsilon A)\cdot A\,ds.$$

Now, we know that $\frac{1}{2}(\hat{T}(\theta, 0, 0, A')\cdot A+\hat{T}(\theta, 0, 0, A)\cdot A')$ is a symmetric bilinear form in (A, A'). If we substract $\frac{1}{2}$ of its associated quadratic form from our last result, we find

$$\begin{aligned}
-\frac{1}{\varepsilon}\int_0^1 & s[\hat{T}(\theta, 0, sA, -\varepsilon A)-\hat{T}(\theta, 0, 0, -\varepsilon A)]\cdot A\,ds \\
&\leqq \rho\hat{\psi}(\theta, A)-\rho\hat{\psi}(\theta, 0)-\tfrac{1}{2}\hat{T}(\theta, 0, 0, A)\cdot A \\
&\leqq \frac{1}{\varepsilon}\int_0^1 s[\hat{T}(\theta, 0, sA, \varepsilon A)-\hat{T}(\theta, 0, 0, \varepsilon A)]\cdot A\,ds,
\end{aligned}$$

or, applying first the integral mean value theorem and then the differential mean value theorem,

$$\begin{aligned}
-\frac{\bar{s}^2}{\varepsilon}\hat{T}_L(\theta, 0, \bar{r}\bar{s}A, -\varepsilon A)[A]\cdot A &\leqq \rho\hat{\psi}(\theta, A)-\rho\hat{\psi}(\theta, 0)-\tfrac{1}{2}\hat{T}(\theta, 0, 0, A)\cdot A \\
&\leqq \frac{\bar{\bar{s}}^2}{\varepsilon}\hat{T}_L(\theta, 0, \bar{\bar{r}}\bar{\bar{s}}A, \varepsilon A)[A]\cdot A,
\end{aligned}$$

where $\bar{s}$, $\bar{\bar{s}}$, $\bar{r}$, and $\bar{\bar{r}}$ all lie in $(0, 1)$. Thus employing the Cauchy-Schwarz inequality and the fact that the linear operator $\hat{T}_L(\theta, 0, A, A')[\cdot]$ is bounded, we come to

$$\begin{aligned}
-\frac{1}{\varepsilon}|\hat{T}_L(\theta, 0, \bar{r}\bar{s}A, -\varepsilon A)| &\leqq \frac{\rho\hat{\psi}(\theta, A)-\rho\hat{\psi}(\theta, 0)-\frac{1}{2}\hat{T}(\theta, 0, 0, A)\cdot A}{|A|^2} \\
&\leqq \frac{1}{\varepsilon}|\hat{T}_L(\theta, 0, \bar{\bar{r}}\bar{\bar{s}}A, \varepsilon A)|,
\end{aligned}$$

and this is easily seen to imply

$$\rho\hat{\psi}(\theta, A)=\rho\hat{\psi}(\theta, 0)+\tfrac{1}{2}\hat{T}(\theta, 0, 0, A)\cdot A+o(|A|^2),$$

that is, $\hat{\psi}(\theta, \cdot)$ is twice differentiable at 0 and $(3.12)_2$ holds. △

As we shall see in Section 5, the Clausius-Duhem inequality, alone, can provide no more specific information on the nature of the stationary point at $(\theta, 0)$ than is indicated in $(3.12)_2$. In particular, it cannot be used to prove that $\hat{\psi}$ has a local minimum at $(\theta, 0)$. However, it will soon become apparent that the character of this stationary point is of signal import for the temporal evolution of the fluid.

Let us now interpret the reduced dissipation inequality, (3.8). We see straightway that in contrast to a Navier-Stokes fluid the stress power in a fluid of complexity 2 need not be positive *even if locally the heat flux or temperature gradient vanishes*. Indeed, when this happens one can only assert that the stress power is bounded below—never being more negative (nor less positive) than the projection of $\rho\hat{\psi}_{L}$ on $\dot{\boldsymbol{L}}$. In particular, the interaction of deformation with the free energy surface provides a possible mechanism for forcing, in certain flows, the stress power to be large and positive—a mechanism not present in the fluid of Navier and Stokes.

Corollary 2. *The response functions $\hat{\boldsymbol{T}}$ and $\hat{\boldsymbol{q}}$ must satisfy the "mechanical dissipation inequality" and the "heat conduction inequality", respectively. That is,*

$$\begin{aligned} \hat{\boldsymbol{T}}(\theta, 0, \boldsymbol{A}, 0) \cdot \boldsymbol{A} &\geqq 0, \\ \hat{\boldsymbol{q}}(\theta, \boldsymbol{a}, 0, \boldsymbol{A}') \cdot \boldsymbol{a} &\leqq 0, \end{aligned} \tag{3.13}$$

for all traceless tensors $\boldsymbol{A}$ and $\boldsymbol{A}'$ and all vectors $\boldsymbol{a}$.

Proof. It is immediate from (3.11) and $(3.12)_1$. △

By replacing $\boldsymbol{A}$ with $x\boldsymbol{A}$ and $\boldsymbol{a}$ with $x\boldsymbol{a}$ in (3.13), then dividing by x and letting $x \to 0$, it is easily seen that $\hat{\boldsymbol{T}}(\theta, 0, 0, 0)$ can only be a multiple of the unit tensor while $\hat{\boldsymbol{q}}(\theta, 0, 0, \boldsymbol{A}')$ must vanish. Thus, at any point in the fluid where $\boldsymbol{g}=\boldsymbol{L}=\dot{\boldsymbol{L}}=0$ the stress system can only be hydrostatic while, if $\boldsymbol{g}=\boldsymbol{L}=0$, the heat flux must be zero. This last is generalized through application of the principle of frame indifference at the end of this section.

As a final direct consequence of Theorem 1 we note that not only must $\hat{\psi}$ and $\hat{\eta}$ depend solely on θ and $\boldsymbol{L}$, but also so must $\hat{\varepsilon}$. To see this, enter (2.10) with (3.6) and (3.7) to find

$$\varepsilon = \hat{\varepsilon}(\theta, \boldsymbol{g}, \boldsymbol{L}, \dot{\boldsymbol{L}}) = \hat{\psi}(\theta, \boldsymbol{L}) - \theta\hat{\psi}_{\theta}(\theta, \boldsymbol{L}). \tag{3.14}$$

We conclude this section by briefly considering the restrictions placed on the response functions by the principle of frame indifference [20]. It is well known that under a change of frame defined by a time dependent orthogonal tensor $\boldsymbol{Q}$ the scalars ψ, η, ε, and θ are unaltered while $\boldsymbol{T}$, $\boldsymbol{q}$, $\boldsymbol{g}$, $\boldsymbol{L}$, and $\dot{\boldsymbol{L}}$ transform according to

$$\begin{aligned} &\boldsymbol{T} \to \boldsymbol{Q}\boldsymbol{T}\boldsymbol{Q}^T, \\ &\boldsymbol{q} \to \boldsymbol{Q}\boldsymbol{q}, \\ &\boldsymbol{g} \to \boldsymbol{Q}\boldsymbol{g}, \\ &\boldsymbol{L} \to \dot{\boldsymbol{Q}}\boldsymbol{Q}^T + \boldsymbol{Q}\boldsymbol{L}\boldsymbol{Q}^T, \\ &\dot{\boldsymbol{L}} \to \overline{\dot{\boldsymbol{Q}}\boldsymbol{Q}^T}^{\boldsymbol{\cdot}} + \dot{\boldsymbol{Q}}\boldsymbol{Q}^T(\boldsymbol{Q}\boldsymbol{L}\boldsymbol{Q}^T) + (\boldsymbol{Q}\boldsymbol{L}\boldsymbol{Q}^T)\dot{\boldsymbol{Q}}\boldsymbol{Q}^T + \boldsymbol{Q}\dot{\boldsymbol{L}}\boldsymbol{Q}^T. \end{aligned}$$

The restrictions which frame indifference imposes on the response functions are readily solvable by use of standard arguments. Thus with the aid of (3.6), (3.7), and

(3.14) we obtain

$$\begin{aligned}
\psi &= \tilde{\psi}(\theta, A_1), \\
\eta &= -\tilde{\psi}_\theta(\theta, A_1), \\
\varepsilon &= \tilde{\psi}(\theta, A_1) - \theta\tilde{\psi}_\theta(\theta, A_1), \\
T &= -p\mathbf{1} + \tilde{T}(\theta, g, A_1, A_2), \\
q &= \tilde{q}(\theta, g, A_1, A_2),
\end{aligned} \tag{3.15}$$

where the symmetric tensors A_1 and A_2 represent the first two Rivlin-Ericksen tensors and are defined by

$$\begin{aligned}
A_1 &\equiv L + L^T, \\
A_2 &\equiv \dot{A}_1 + A_1 L + L^T A_1.
\end{aligned} \tag{3.16}$$

In addition, the functions $\tilde{\psi}$, $\tilde{T}$, and $\tilde{q}$ are defined in terms of $\hat{\psi}$, $\hat{T}$, and $\hat{q}$, respectively, and must be isotropic functions, *i.e.*,

$$\begin{aligned}
\tilde{\psi}(\theta, A_1) &= \tilde{\psi}(\theta, QA_1Q^T), \\
Q\tilde{T}(\theta, g, A_1, A_2)Q^T &= \tilde{T}(\theta, Qg, QA_1Q^T, QA_2Q^T), \\
Q\tilde{q}(\theta, g, A_1, A_2) &= \tilde{q}(\theta, Qg, QA_1Q^T, QA_2Q^T).
\end{aligned} \tag{3.17}$$

While general theorems exist for the specific representation of such isotropic functions (see *e.g.* [20] or the review article of SPENCER [22]), we do not gain any advantage by appealing to them in the present work. It is of interest, however, to observe that by taking $g = A_1 = A_2 = 0$ in $(3.17)_2$ we reach $Q\tilde{T}(\theta, 0, 0, 0)Q^T = \tilde{T}(\theta, 0, 0, 0)$, while $(3.17)_3$ yields, for $Q = -\mathbf{1}$ and $g = 0$, that $\tilde{q}(\theta, 0, A_1, A_2) = -\tilde{q}(\theta, 0, A_1, A_2)$. Thus $\tilde{T}(\theta, 0, 0, 0)$ is isotropic while $\tilde{q}(\theta, 0, A_1, A_2)$ vanishes;

$$\begin{aligned}
\tilde{T}(\theta, 0, 0, 0) &= \tfrac{1}{3}\mathbf{1}\,\mathrm{tr}\,\tilde{T}(\theta, 0, 0, 0), \\
\tilde{q}(\theta, 0, A_1, A_2) &= 0.
\end{aligned} \tag{3.18}$$

In conjunction with $(3.15)_4$, $(3.18)_1$ shows that at a particle in local equilibrium the stress system is hydrostatic. On the other hand, $(3.18)_2$ shows that, regardless of the temperature and the state of motion at a particle, if the temperature gradient vanishes there then so does the heat flux.

4. Boundedness and Mild Uniqueness of Fluids of Complexity 2

We saw in the preceding section that the Clausius-Duhem inequality, while forcing $(\theta, 0)$ to be a stationary point for $\hat{\psi}(\cdot, \cdot)$, gave no explicit guidance as to its character. This is in marked contrast to results obtained by COLEMAN [9], who proved, for a broad class of simple fluids with a certain type of fading memory, that the Clausius-Duhem inequality forces the free energy to have a *minimum* in equilibrium.

While this minimal character of the free energy is trivially satisfied for perfect fluids and the linearly viscous fluids of classical hydrodynamics, it has, more generally, long been a common belief in classical thermodynamics. A second commonly accepted belief in thermodynamics is that the specific heat should be positive. Thus, until further notice (*cf.* Section 6) we make the following two major

Assumptions.

A 1) *The free energy of a fluid of complexity* 2 *is a minimum in equilibrium,*

$$\hat{\psi}(\theta, 0) \leqq \hat{\psi}(\theta, \boldsymbol{L}) \quad or \quad \tilde{\psi}(\theta, 0) \leqq \tilde{\psi}(\theta, \boldsymbol{A}_1). \tag{4.1}$$

A 2) *The specific heat* $c \equiv \hat{\varepsilon}_\theta$ *is positive,*

$$c = \hat{c}(\theta, \boldsymbol{L}) \equiv \hat{\varepsilon}_\theta(\theta, \boldsymbol{L}) = \tilde{\varepsilon}_\theta(\theta, \boldsymbol{A}_1) = \tilde{c}(\theta, \boldsymbol{A}_1) > 0. \tag{4.2}$$

If equality holds in (4.1) only when $\boldsymbol{L}$ (or $\boldsymbol{A}_1$) vanishes then the minimum will be said to be *strict*. With regard to (4.2), we note that by $(3.14)_1$ and $(3.15)_1$, it is equivalent to

$$c = \tilde{c}(\theta, \boldsymbol{A}_1) = -\theta \tilde{\psi}_{\theta\theta}(\theta, \boldsymbol{A}_1) > 0. \tag{4.3}$$

We shall say* that a *body* is *mechanically isolated* at the instant t if

$$\int_{\partial\Omega_t} \dot{\boldsymbol{x}} \cdot \boldsymbol{T}\boldsymbol{n}\, da + \int_{\Omega_t} \rho \dot{\boldsymbol{x}} \cdot \boldsymbol{b}\, dv = 0. \tag{4.4}$$

In particular, (4.4) will be met if (i) $\boldsymbol{b} = 0$ in Ω_t and $\partial\Omega_t$ is traction free and stationary over complementary subsets; or, if (ii) $\boldsymbol{b}$ is conservative and $\partial\Omega_t$ is at rest. For simplicity, we shall often say that a *process* is *mechanically isolated* if (4.4) holds throughout it.

We shall say* that a *body* is *immersed in a thermally passive environment with* (*constant*) *temperature* θ° at time t if

$$\begin{aligned} \{\theta(\boldsymbol{x}, t) - \theta^\circ\}\, \boldsymbol{q}(\boldsymbol{x}, t) \cdot \boldsymbol{n}(\boldsymbol{x}, t) \geqq 0 \quad & \forall\, \boldsymbol{x} \in \partial\Omega_t, \\ \{\theta(\boldsymbol{x}, t) - \theta^\circ\}\, r(\boldsymbol{x}, t) \leqq 0 \quad & \forall\, \boldsymbol{x} \in \Omega_t; \end{aligned} \tag{4.5$_1$}$$

and we shall say* that a *process* is *consistent with a thermally passive environment with* (*constant*) *temperature* θ° if throughout it

$$\int_{\partial\Omega_t} \left(\frac{1}{\theta^\circ} - \frac{1}{\theta}\right) \boldsymbol{q} \cdot \boldsymbol{n}\, da - \int_{\Omega_t} \rho \left(\frac{1}{\theta^\circ} - \frac{1}{\theta}\right) r\, dv \geqq 0. \tag{4.5$_2$}$$

Clearly, $(4.5)_2$ is implied by $(4.5)_1$, whose physical content is evident: at time t heat is neither being conducted (radiated) into B at surface (interior) points having a temperature higher than θ°, nor is heat being conducted (radiated) out of B at surface (interior) points having a temperature lower than θ°. If the radiant heating vanishes, then $(4.5)_1$ is satisfied trivially whenever one of the following three conditions prevail: (i) the boundary of B is insulated (*i.e.*, $\boldsymbol{q} \cdot \boldsymbol{n} \equiv 0$), (ii) the boundary of B is held at the constant temperature θ°, or (iii) the mixed condition that $(\theta - \theta^\circ)$ and $\boldsymbol{q} \cdot \boldsymbol{n}$ are required to vanish over complementary subsets of $\partial\Omega_t$.

* *Cf.* GURTIN [11] and COLEMAN [13].

Consider now the so-called canonical free energy function $\Phi(\cdot)$ given by

$$\Phi(t)=\int_{\Omega_t}\rho(\varepsilon-\theta^\circ\eta+\tfrac{1}{2}\dot{x}\cdot\dot{x})\,dv. \tag{4.6}$$

From the balance of energy (2.4) it readily follows that $\Phi(\cdot)$ satisfies

$$\begin{aligned}\frac{d}{dt}\Phi(t)=&-\theta^\circ\left\{\overline{\int_{\Omega_t}\rho\eta\,dv}^{\,\cdot}+\int_{\partial\Omega_t}\frac{q\cdot n}{\theta}\,da-\int_{\Omega_t}\frac{\rho\,r}{\theta}\,dv\right\}\\&+\left\{\int_{\partial\Omega_t}Tn\cdot\dot{x}\,da+\int_{\Omega_t}\rho b\cdot\dot{x}\,dv\right\}\\&-\theta^\circ\left\{\int_{\partial\Omega_t}\left(\frac{1}{\theta^\circ}-\frac{1}{\theta}\right)q\cdot n\,da-\int_{\Omega_t}\rho\left(\frac{1}{\theta^\circ}-\frac{1}{\theta}\right)r\,dv\right\}\end{aligned} \tag{4.7}$$

in every process. In particular, we see that in any process that is mechanically isolated and also consistent with a thermally passive environment at temperature θ° the Clausius-Duhem inequality (2.5) yields

$$\frac{d}{dt}\Phi(t)\leqq 0. \tag{4.8}$$

As noted by ERICKSEN [25] and GURTIN [11], the term $\varepsilon-\theta^\circ\eta$ appearing in (4.6) has the more convenient form

$$\begin{aligned}\tilde{\varepsilon}(\theta,A_1)-\theta^\circ\tilde{\eta}(\theta,A_1)&=\tilde{\psi}(\theta,A_1)-(\theta-\theta^\circ)\tilde{\psi}_\theta(\theta,A_1)\\&=\tilde{\psi}(\theta^\circ,A_1)-\tfrac{1}{2}\tilde{\psi}_{\theta\theta}(\theta^*,A_1)(\theta-\theta^\circ)^2,\end{aligned}$$

where we have used (3.7), (3.14), $(3.15)_1$, and Taylor's formula. Here, θ^*, which depends on θ, θ°, A_1, and $\tilde{\psi}(\cdot,\cdot)$, lies in the interval (θ,θ°). Recalling $(4.3)_2$ and introducing $K=\tilde{K}(\theta^\circ,\theta,A_1)\equiv\tilde{c}(\theta^*,A_1)/2\theta^*$ we thus obtain

$$\varepsilon-\theta^\circ\eta=\tilde{\psi}(\theta^\circ,A_1)+\tilde{K}(\theta^\circ,\theta,A_1)(\theta-\theta^\circ)^2, \tag{4.9}$$

where, by $(4.3)_3$,

$$\tilde{K}(\theta^\circ,\theta,A_1)>0. \tag{4.10}$$

Hence, $\Phi(\cdot)$ may be written in the equivalent form

$$\Phi(t)=\int_{\Omega_t}\rho[\tilde{\psi}(\theta^\circ,A_1)+\tilde{K}(\theta^\circ,\theta,A_1)(\theta-\theta^\circ)^2+\tfrac{1}{2}\dot{x}\cdot\dot{x}]\,dv. \tag{4.11}$$

We now have the following result concerning the uniqueness of the rest state:*

Theorem 2. *Let a fluid of complexity 2 undergo a process both mechanically isolated and consistent with a thermally passive environment at the constant temperature θ°. Suppose further that*

$$\dot{x}(X,0)=0,\qquad \theta(X,0)=\theta^\circ \tag{4.12}$$

for all $X\in\Omega$. Then

$$\begin{aligned}\dot{x}(X,t)&=0, &\theta(X,t)&=\theta^\circ,\\ T(X,t)&=-\bar{p}\mathbf{1}, & q(X,t)&=0\end{aligned} \tag{4.13}$$

for all (X,t), where $\bar{p}=\bar{p}(X,t)\equiv p(X,t)-\frac{1}{3}\operatorname{tr}\hat{T}(\theta,0,0,0)$.

* See GURTIN [11] who also gives an argument that would suffice for the proof of Theorem 2 provided that (4.1) hold merely in a neighborhood of 0.

Proof. By hypotheses (4.8) holds, so that

$$\Phi(t) \leqq \Phi(0).$$

Equivalently, by use of (4.12), (4.11) yields

$$\int_{\Omega_t} \rho[K(\theta-\theta^\circ)^2 + \tfrac{1}{2}\dot{x}\cdot\dot{x}]\,dv \leqq \int_{\Omega_0} \rho[\tilde{\psi}(\theta^\circ, 0) - \tilde{\psi}(\theta^\circ, A_1)]\,dv \leqq 0, \tag{4.14}$$

where we have used the conservation of mass and the inequality (4.1). But, since $K>0$, we immediately obtain $(4.13)_{1,2}$ which, when coupled with $(2.1)_1$, (3.16), (3.17), and (3.18), gives $(4.13)_{3,4}$. △

We next note that the hypotheses of this theorem are considerably less general than they might seem. Specifically, we have the following

Corollary 1. *The only processes compatible with both the hypotheses and conclusions of Theorem 2 are those in which the radiant heating vanishes and the body force is derivable from a potential,* i.e.,

$$r(X, t) \equiv 0, \qquad \rho\, b(x, t) = \operatorname{grad} \bar{p}(x, t).$$

Proof. This follows by entering (4.13) into (2.7) and (2.8). △

In the remainder of this section we obtain certain boundedness theorems concerning any process that is mechanically isolated and consistent with a thermally passive environment at a temperature θ°. As a first result of this type* we have the following

Theorem 3. *Suppose a fluid of complexity 2 is experiencing a thermodynamic process which is both mechanically isolated and consistent with a thermally passive environment at the constant temperature θ°. Let t' be any particular instant during this process and consider the positive number $\omega(t')$ given by*

$$\omega(t') \equiv \int_{\Omega_{t'}} \rho[(\tilde{\psi}(\theta^\circ, A_1(t')) - \tilde{\psi}(\theta^\circ, 0)) + \tilde{K}(t')(\theta(t') - \theta^\circ)^2 + \tfrac{1}{2}|\dot{x}|^2(t')]\,dv. \tag{4.15}$$

Then, for all $t \geqq t'$ one has

$$\begin{aligned} 0 &\leqq \int_{\Omega_t} \rho[\tilde{\psi}(\theta^\circ, A_1(t)) - \tilde{\psi}(\theta^\circ, 0)]\,dv \leqq \omega(t'), \\ 0 &\leqq \int_{\Omega_t} \rho \tilde{K}(t)(\theta(t) - \theta^\circ)^2\,dv \leqq \omega(t'), \\ 0 &\leqq \int_{\Omega_t} \tfrac{1}{2}\rho|\dot{x}|^2(t)\,dv \leqq \omega(t'), \end{aligned} \tag{4.16}$$

where $\tilde{K}(\tau) \equiv \tilde{K}(\theta^\circ, \theta(x, \tau), A_1(x, \tau))$, and where explicit dependence on x has been suppressed.

* See Gurtin [11] for a related theorem within a different context in which he emphasizes the notion of stability.

Proof. Integration of (4.8) on $[t', t]$, and use of (4.11), (4.15), and the conservation of mass results in

$$\begin{aligned}\int_{\Omega_t} \rho[\tilde{\psi}(\theta^\circ, A_1(t)) - \tilde{\psi}(\theta^\circ, 0)]\,dv + \int_{\Omega_t} \rho\tilde{K}(t)(\theta(t)-\theta^\circ)^2\,dv \\ + \int_{\Omega_t} \tfrac{1}{2}\rho|\dot{x}|^2(t)\,dv \leqq \omega(t').\end{aligned} \tag{4.17}$$

Appealing to (4.1) and (4.10), we see that each of the three integrals on the left is non-negative and (4.16) then follows. △

The bound $(4.16)_1$ is empty for Navier-Stokes fluids and, indeed, for any fluid that has $\tilde{\psi}(\theta, A) = \tilde{\psi}(\theta, 0)$ for all $A \in T_s^\circ$.* Moreover, even for those fluids of complexity 2 for which the bound $(4.16)_1$ is *not* empty its significance may not be particularly transparent since its structure is dependent upon $\tilde{\psi}(\cdot,\cdot)$. Thus, a more explicit bound than $(4.16)_1$ would be desirable and, for some fluids of complexity 2, we shall now show that such an improvement is, indeed, possible.

Recall that by $(4.1)_2$ we have $\tilde{\psi}_{A_1A_1}(\theta, 0)\cdot(A\otimes A) \geqq 0$ for all $A \in T_s^\circ$. That is, $\tilde{\psi}(\theta,\cdot)$ is convex at 0. Let us now suppose that $\tilde{\psi}(\theta,\cdot)$ is, in fact, everywhere convex, *i.e.*,

$$\tilde{\psi}_{A_1A_1}(\theta, A_1)\cdot(A\otimes A) \geqq 0 \tag{4.18}$$

for all A_1 and A in T_s°.** For this class of fluids of complexity 2 the following more explicit form of $(4.16)_1$ is available:***

Corollary 1. *Let the hypotheses of Theorem* 3 *hold, suppose* 0 *is a strict minimum for* $\tilde{\psi}(\theta,\cdot)$, *and also assume* (4.18). *Define* $\omega(t')$ *as in* (4.15). *Then for any* $\delta > 0$ *there exists a positive number,* $N(\delta)$, *depending only on* δ *and the structure of* $\tilde{\psi}(\cdot,\cdot)$, *such that for all* $t \geq t'$

$$\begin{aligned} 0 &\leqq \int_{\Omega_t} \rho|A_1|(t)\,dv \leqq M\delta + N(\delta)\,\omega(t'),\\ 0 &\leqq \int_{\Omega_t} \rho\tilde{K}(t)(\theta(t)-\theta^\circ)^2\,dv \leqq \omega(t'),\\ 0 &\leqq \int_{\Omega_t} \tfrac{1}{2}\rho|\dot{x}|^2(t)\,dv \leqq \omega(t'),\end{aligned} \tag{4.19}$$

where $M \equiv \int_{\Omega_t} \rho\,dv$ *is the mass of the fluid.*

Proof. It is clear from (4.16) that we need establish only that $(4.16)_1$ implies $(4.19)_1$. To do this, consider the function $f(\cdot) \equiv \tilde{\psi}(\theta^\circ,\cdot) - \tilde{\psi}(\theta^\circ, 0)$. By our hypotheses, $f(\cdot)$ is convex and has $f(A) \geqq 0$ for all $A \in T_s^\circ$ with equality holding only

* While we have not shown that this situation prevails for a Navier-Stokes fluid, it is an easy consequence of (3.11) if $\hat{T}$ is assumed independent of $\dot{L}$.

** We now assume $\tilde{\psi}(\theta,\cdot)$ is twice continuously differentiable on all of T_s°.

*** The footnote on page 207 also applies here.

if $A=0$. Therefore, by a result essentially due to COLEMAN [13],* we have that given any $\delta>0$, there exists a $\gamma(\cdot)$ such that

$$\int_{\Omega_t} \rho\,|A_1|(x,t)\,dv \leqq \frac{\delta}{2} + \frac{1}{\gamma\left(\frac{\delta}{2M}\right)} \int_{\Omega_t} \rho f(A_1(x,t))\,dv,$$

and replacing δ with $2M\delta$ and using $(4.16)_1$ we achieve $(4.19)_1$ with $N(\delta)\equiv\frac{1}{\gamma(\delta)}$.** △

We remark that while $(4.19)_1$ is more explicit than $(4.16)_1$ it may, for a particular fluid, be weaker than $(4.16)_1$. Fluids of second grade provide an example of this as will be apparent from the form of $\tilde{\psi}(\cdot,\cdot)$ in Section 5.

We now obtain a boundedness theorem for the internal energy and entropy functions based essentially on the conditions (4.3) and (4.18) which require, respectively, that $\tilde{\psi}(\cdot, A)$ be *concave* on $\mathbb{R}^+$ for each $A\in T_s^\circ$ and that $\tilde{\psi}(\theta,\cdot)$ be *convex* on T_s° for each $\theta\in\mathbb{R}^+$. First, however, applying (4.3) we observe that $(3.15)_2$ has a smooth solution $\theta=\bar{\theta}(\eta, A_1)$. Thus,

$$\eta+\tilde{\psi}_\theta(\bar{\theta}(\eta,A_1),A_1)=0 \tag{4.20}$$

is an identity in (η, A_1) and we may introduce the function $\bar{\varepsilon}(\cdot,\cdot)$ given by

$$\bar{\varepsilon}(\eta,A_1)\equiv\tilde{\varepsilon}(\bar{\theta}(\eta,A_1),A_1)=\tilde{\psi}(\bar{\theta}(\eta,A_1),A_1)+\bar{\theta}(\eta,A_1)\eta, \tag{4.21}$$

where the last holds by $(3.15)_2$ and $(3.15)_3$.

Theorem 4. *Suppose* (4.3) *holds. Then the function* $\bar{\varepsilon}(\cdot,\cdot)$ *defined in* (4.21) *is a minimum in equilibrium,*

$$\bar{\varepsilon}(\eta,A_1)\geqq\bar{\varepsilon}(\eta,0),$$

and the "temperature relation" holds,

$$\bar{\varepsilon}_\eta(\eta,A_1)=\theta.$$

In addition, if (4.18) *holds, then* $\bar{\varepsilon}(\cdot,\cdot)$ *is convex on* $\mathbb{R}^+\times T_s^\circ$.***

Proof. The conclusion that $\bar{\varepsilon}$ is a minimum in equilibrium is apparent from entering $(4.21)_2$ into (4.9) and using $(4.1)_2$:

$$\bar{\varepsilon}(\eta,A_1)-\bar{\varepsilon}(\eta,0)-(\theta^\circ-\bar{\theta}(\eta,0))\eta=\tilde{\psi}(\theta^\circ,A_1)-\tilde{\psi}(\bar{\theta}(\eta,0),0)+\tilde{K}(\theta-\theta^\circ)^2.$$

Thus by taking $\theta^\circ\equiv\bar{\theta}(\eta,0)$ and using (4.1) and (4.10), we obtain the desired result.

To establish the temperature relation we need only differentiate (4.21)

$$\bar{\varepsilon}_\eta=\tilde{\psi}_\theta\bar{\theta}_\eta+\eta\bar{\theta}_\eta+\bar{\theta}=\bar{\theta},$$

where the last holds by (4.20).

* See COLEMAN's Lemmas A.1 and A.2. Either these lemmas may be modified so as to apply to the present situation where $f(\cdot)$ is defined only on the subset T_s° of T, or the function $f(\cdot)$ may be extended to all of T so as to preserve its positive definiteness and convexity, for which the lemmas apply without modification.

** It can be shown that $\gamma(d)\to0$ as $d\to0$, so that $N(\delta)\to\infty$ as $\delta\to0$.

*** We adopt the usual convention $\eta>0$.

Finally, to show that $\bar{\varepsilon}(\cdot,\cdot)$ is convex, we first observe, using (4.20) and (4.21), that

$$\bar{\varepsilon}_{\eta\eta}=\frac{-1}{\tilde{\psi}_{\theta\theta}}, \quad \bar{\varepsilon}_{\eta A_1}=-\frac{\tilde{\psi}_{\theta A_1}}{\tilde{\psi}_{\theta\theta}}, \quad \bar{\varepsilon}_{A_1 A_1}=\tilde{\psi}_{A_1 A_1}-\frac{\tilde{\psi}_{\theta A_1}\otimes\tilde{\psi}_{\theta A_1}}{\tilde{\psi}_{\theta\theta}}. \tag{4.22}$$

Thus, for any $(\alpha, A)\in\mathbb{R}\times T_s^0$ the characteristic quadratic form, Q, for $\bar{\varepsilon}$ is given by

$$\begin{aligned}
Q&=\bar{\varepsilon}_{\eta\eta}\alpha^2+2\alpha\bar{\varepsilon}_{\eta A_1}\cdot A+\bar{\varepsilon}_{A_1A_1}\cdot(A\otimes A)\\
&=\left(-\frac{1}{\tilde{\psi}_{\theta\theta}}\right)(\alpha^2+2\alpha\tilde{\psi}_{\theta A_1}\cdot A+\tilde{\psi}_{\theta A_1}\otimes\tilde{\psi}_{\theta A_1}\cdot(A\otimes A))+\tilde{\psi}_{A_1A_1}\cdot(A\otimes A)\\
&=(-1/\tilde{\psi}_{\theta\theta})(\alpha+\tilde{\psi}_{\theta A_1}\cdot A)^2+\tilde{\psi}_{A_1A_1}\cdot(A\otimes A),
\end{aligned}$$

and we see, by (4.3) and (4.18), that $Q\geqq 0$, which is both necessary and sufficient for the convexity of $\bar{\varepsilon}(\cdot,\cdot)$. $\triangle$

If we now return to (4.9) and use $(3.15)_{2,3}$, (4.1), (4.10), and $(4.21)_2$ we find

$$\bar{\varepsilon}(\eta, A_1)-\varepsilon^\circ-\theta^\circ(\eta-\eta^\circ)=\tilde{\psi}(\theta^\circ, A_1)-\tilde{\psi}(\theta^\circ, 0)+\tilde{K}(\theta^\circ, \theta, A_1)(\theta-\theta^\circ)^2\geqq 0, \tag{4.23}$$

where we have defined $\eta^\circ\equiv-\tilde{\psi}_\theta(\theta^\circ, 0)$ and $\varepsilon^\circ\equiv\bar{\varepsilon}(\eta^\circ, 0)$. Moreover, assuming $(\theta^\circ, 0)$ is a strict minimum for $\tilde{\psi}(\cdot,\cdot)$ it is clear that equality holds in $(4.23)_2$ if and only if $A_1=0$, $\theta=\theta^\circ$, *i.e.*, if and only if $(\eta, A_1)=(\eta^\circ, 0)$. Thus, if, for purposes of normalization, we let $y\equiv\theta^\circ(\eta-\eta^\circ)$ and define

$$f(y, A_1)\equiv\bar{\varepsilon}\left(\frac{y}{\theta^\circ}+\eta^\circ, A_1\right)-\varepsilon^\circ-y,$$

then it follows that $f(y, A_1)\geq 0$ for all $(y, A_1)\in(-\theta^\circ\eta^\circ, \infty)\times T_s^\circ$, and equality holds only at (0, 0). Further, since $f(\cdot,\cdot)$ and $\bar{\varepsilon}(\cdot,\cdot)$ have essentially the same characteristic quadratic form, then by Theorem 4, $f(\cdot,\cdot)$ is convex. Finally, we see that, in addition to $(4.16)_3$, (4.17) also implies

$$\int_{\Omega_t}\rho f(y(\boldsymbol{x}, t), A_1(\boldsymbol{x}, t))\,dv\leqq\omega(t') \tag{4.24}$$

for all $t\geq t'$. Applying the results of COLEMAN mentioned earlier* we therefore find that for any $\delta>0$,

$$\int_{\Omega_t}\rho\,|A_1|(\boldsymbol{x}, t)\,dv+\theta^\circ\int_{\Omega_t}\rho\,|\eta-\eta^\circ|(\boldsymbol{x}, t)\,dv\leqq M\delta+N(\delta)\,\omega(t').$$

Here, as before, M is the mass of the fluid and $N(\delta)$ is a number depending only on δ and the structure of $\bar{\varepsilon}$ (and, hence, of $\tilde{\psi}$). Thus we have

Theorem 5.** *Suppose a fluid of complexity* 2 *is experiencing a thermodynamic process which is both mechanically isolated and consistent with a thermally passive environment at the constant temperature* θ°. *Suppose also that* $\bar{\varepsilon}(\cdot,\cdot)$ *is convex and*

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* See footnote * on page 209.

** This theorem is analogous to Theorem 3.2 of COLEMAN [13].

has a strict minimum in equilibrium. Finally, let t' be any particular instant during this process and consider the positive number $\omega(t')$ given by

$$\omega(t') \equiv \int_{\Omega_{t'}} \rho[\bar{\varepsilon}(\eta(t'), A_1(t')) - \varepsilon^\circ - \theta^\circ(\eta(t') - \eta^\circ) + \tfrac{1}{2}|\dot{x}|^2(t')]\, dv,$$

where $\eta^\circ \equiv -\tilde{\psi}_\theta(\theta^\circ, 0)$ and $\varepsilon^\circ \equiv \bar{\varepsilon}(\eta^\circ, 0)$. Then for all $t \geqq t'$ and for any $\delta > 0$ one has

$$\begin{aligned} 0 &\leqq \int_{\Omega_t} \rho\,|\varepsilon(x,t) - \varepsilon^\circ|\, dv \leqq M\delta + (N(\delta)+1)\,\omega(t'), \\ 0 &\leqq \theta^\circ \int_{\Omega_t} \rho\,|\eta(x,t) - \eta^\circ|\, dv \leqq M\delta + N(\delta)\,\omega(t'), \\ 0 &\leqq \int_{\Omega_t} \rho\,|A_1|(x,t)\, dv \leqq M\delta + N(\delta)\,\omega(t'), \\ 0 &\leqq \int_{\Omega_t} \tfrac{1}{2}\rho\,|\dot{x}|^2(x,t)\, dv \leqq \omega(t'), \end{aligned} \tag{4.25}$$

where M is the mass of the fluid and $N(\delta)$ is a positive number depending only on δ and the structure of $\bar{\varepsilon}(\cdot,\cdot)$.

Proof. We have already proven $(4.25)_{2,3,4}$. To prove $(4.25)_1$, we first observe that

$$\varepsilon - \varepsilon^\circ = f(y, A_1) + \theta^\circ(\eta - \eta^\circ).$$

Thus by the triangle inequality and the fact that $f \geqq 0$ we find

$$|\varepsilon - \varepsilon^\circ| \leqq f + \theta^\circ\,|\eta - \eta^\circ|,$$

from which $(4.25)_1$ is immediate upon using (4.24) and $(4.25)_2$. △

As was the case for the result $(4.19)_1$ in Corollary 1 of Theorem 3, we note that the proof of $(4.25)_3$ does not apply to a fluid that has $\tilde{\psi}(\theta, A) = \tilde{\psi}(\theta, 0)$. However, in this case the results $(4.25)_{1,2,4}$ will still hold.

5. Fluids of Second Grade: Thermodynamic Compatibility

An incompressible fluid of second grade is a specific example of a fluid of complexity 2. For it the response function $\tilde{T}(\cdot,\cdot,\cdot,\cdot)$ of $(3.15)_4$ is required to be of the special form

$$\begin{aligned} \tilde{T}(\theta, g, A_1, A_2) &= \bar{\mu}(\theta)A_1 + \bar{\alpha}_1(\theta)A_2 + \bar{\alpha}_2(\theta)A_1^2 \\ &= \bar{\mu}(\theta)(L + L^T) + \bar{\alpha}_1(\theta)(\dot{L} + \dot{L}^T + L^2 + L^{T^2} + 2L^T L) \\ &\quad + \bar{\alpha}_2(\theta)(L^2 + L^{T^2} + LL^T + L^T L) \\ &= \hat{T}(\theta, g, L, \dot{L}), \end{aligned} \tag{5.1}$$

where (3.16) has been applied. Thus, with some rearrangement of terms, the reduced dissipation inequality (3.8) requires

$$\begin{aligned} &\rho\hat{\psi}_L(a, A)\cdot A' - \bar{\alpha}_1(a)[A' + A'^T]\cdot A - \bar{\mu}(a)[A + A^T]\cdot A \\ &\quad - (\bar{\alpha}_1(a) + \bar{\alpha}_2(a))[A^2 + A^{2^T} + 2AA^T]\cdot A + \frac{\hat{q}(a, a, A, A')\cdot a}{a} \leqq 0 \end{aligned} \tag{5.2}$$

for all $(a, \boldsymbol{a}, A, A')$ such that $a>0$ and $A, A' \in T^{\circ}$. Taking $\boldsymbol{a}=0$, we reach an inequality that must hold identically for all traceless A and A'. However, since A' appears linearly, the term involving it must vanish identically and we have

$$\hat{\psi}_{L}(a, A) \cdot A' = \frac{\bar{\alpha}_1(a)}{\rho} [A' + A'^{T}] \cdot A. \tag{5.3}$$

Again, since the remaining inequality must hold identically for every traceless A, the terms cubic in A must vanish identically, and we obtain

$$(\bar{\alpha}_1(a) + \bar{\alpha}_2(a))[A^2 + A^{2^T} + 2AA^T] \cdot A = 0. \tag{5.4}$$

This leaves the residual inequality

$$\bar{\mu}(a)[A + A^T] \cdot A \geqq 0. \tag{5.5}$$

Finally, since (5.3) and (5.4) are identities, it follows that (5.2) has the general form

$$\frac{\hat{\boldsymbol{q}}(a, \boldsymbol{a}, A, A') \cdot \boldsymbol{a}}{a} - \bar{\mu}(a)[A + A^T] \cdot A \leqq 0. \tag{5.6}$$

It is clear that (5.4) and (5.5) can hold for all traceless A if and only if

$$\bar{\alpha}_1(a) + \bar{\alpha}_2(a) = 0 \quad \text{and} \quad \bar{\mu}(a) \geqq 0, \tag{5.7}$$

respectively. It is equally clear by the definition of $\tilde{\psi}_L$ that (5.3) is equivalent to

$$\frac{d}{ds} \hat{\psi}(a, A + sA')|_{s=0} = \frac{\bar{\alpha}_1(a)}{\rho} [A' + A'^T] \cdot A$$

for all traceless A and A'. Taking, in particular, $A = rA'$ and setting $\lambda = s + r$, we arrive at

$$\frac{d}{d\lambda} \hat{\psi}(a, \lambda A')|_{\lambda=r} = \frac{\bar{\alpha}_1(a)}{\rho} [A' + A'^T] \cdot A' r,$$

which, when integrated over the interval (0, 1), gives

$$\hat{\psi}(a, A') = \hat{\psi}(a, 0) + \frac{\bar{\alpha}_1(a)}{2\rho} [A' + A'^T] \cdot A', \tag{5.8}$$

for all $a>0$ and all traceless A'.

Thus, we have established

Theorem 6. *The response functions $\hat{\psi}$, $\hat{T}$, and $\hat{\boldsymbol{q}}$ of a second grade fluid are compatible with thermodynamics if and only if*

(i) *the viscosity is non-negative,*

$$\bar{\mu}(\theta) \geqq 0, \tag{5.9}$$

(ii) *the normal stress coefficients are related by*

$$\bar{\alpha}_2(\theta) = -\bar{\alpha}_1(\theta), \tag{5.10}$$

(iii) *the free energy is quadratic in* **L** *and has the explicit form**

$$\hat{\psi}(\theta, L) = \bar{\psi}(\theta) + \frac{\bar{\alpha}_1(\theta)}{4\rho} |L + L^T|^2, \tag{5.11}$$

(iv) *the response functions* $\hat{q}$ *and* $\bar{\mu}$ *must be such that the reduced dissipation inequality,*

$$\frac{\hat{q}(\theta, g, L, \dot{L}) \cdot g}{\theta} \leqq \bar{\mu}(\theta)(L + L^T) \cdot L, \tag{5.12}$$

holds in every thermodynamic process.

Proof. Necessity follows at once from (5.6), (5.7), and (5.8). Sufficiency is immediate. $\triangle$

In terms of (θ, A_1) we see that (5.11) and (5.12) are equivalent to

$$\begin{aligned} \tilde{\psi}(\theta, A_1) &= \bar{\psi}(\theta) + \frac{\bar{\alpha}_1(\theta)}{4\rho} |A_1|^2, \\ \frac{\tilde{q}(\theta, g, A_1, A_2) \cdot g}{\theta} &\leqq \frac{1}{2} \bar{\mu}(\theta) |A_1|^2, \end{aligned} \tag{5.13}$$

respectively. Moreover, by substituting (5.10) into (5.1) we obtain

$$\begin{aligned} \tilde{T}(\theta, g, A_1, A_2) &= \bar{\mu}(\theta) A_1 + \bar{\alpha}_1(\theta)(A_2 - A_1^2) \\ &= \bar{\mu}(\theta)(L + L^T) + \bar{\alpha}_1(\theta)(\dot{L} + \dot{L}^T + L^T L - L L^T) \\ &= \hat{T}(\theta, g, L, \dot{L}) \end{aligned} \tag{5.14}$$

as the only possible response function for the stress in a second grade fluid.

We see from Theorem 6 that, beyond forcing $\bar{\alpha}_2(\cdot) = -\bar{\alpha}_1(\cdot)$, the Clausius-Duhem inequality imposes no restrictions at all on the sign of $\bar{\alpha}_1(\cdot)$. However, from $(5.13)_1$ we easily obtain the following

Corollary 1. *The free energy function* $\tilde{\psi}(\cdot, \cdot)$ *of a second grade fluid has a minimum in equilibrium if and only if*

$$\bar{\alpha}_1(\theta) \geqq 0. \quad \triangle \tag{5.15}$$

Exempting the trivial case when $\bar{\alpha}_1(\theta) \equiv 0$, we see that this minimum will be strict, and that $\tilde{\psi}(\theta, \cdot)$ is convex. Thus the results of Section 4 are all applicable. Indeed, in Sections 6 and 9 we shall obtain decay theorems far stronger than the boundedness results of Section 4 for the special case when $\bar{\alpha}_1(\cdot) = \text{constant} \geqq 0$.

On the other hand, if (5.15) does *not* hold, so that $\bar{\alpha}_1(\theta) < 0$, we are forced into the position that $\tilde{\psi}(\theta, \cdot)$ has a strict maximum (!) in equilibrium. In this circumstance not only do none of the results of Section 4 hold but, indeed, *unboundedness*, *non-uniqueness*, and *non-existence* theorems can be demonstrated. Some of these are reported in Sections 7 and 8 for the special case $\bar{\alpha}_1(\cdot) = \text{const.} < 0$.

* Here we set $\bar{\psi}(\theta) \equiv \hat{\psi}(\theta, 0)$.

On the basis of the above remarks, and in anticipation of the theorems to be presented in the remainder of this paper, it seems reasonable to assert, then, that a second grade fluid must have $\bar{\alpha}_1 \geqq 0$, equality corresponding to the case of Navier-Stokes fluids.

The concept of fluids of convected elasticity has been used by TRUESDELL [8] to support the inequality $\alpha_1 < 0$. Inasmuch as this result is in disagreement with the above simple remarks, we now briefly discuss the thermodynamics of these materials and, in particular, show that for all non-spherical states of stress they are themselves in conflict with the Clausius-Duhem inequality.

While TRUESDELL does not consider thermal effects, we may account for them by including in his stress response function a dependence on $\theta(t)$ and $g(t)$, and by introducing similar constitutive assumptions for ψ, η, and q. Thus an incompressible material of convected elasticity is characterized by the following constitutive assumptions:

$$\begin{aligned} \psi &= \check{\psi}(F(t), F(t-t^*), \theta(t), g(t)), \\ \eta &= \check{\eta}(F(t), F(t-t^*), \theta(t), g(t)), \\ T &= -p\mathbf{1} + \check{T}(F(t), F(t-t^*), \theta(t), g(t)), \\ q &= \check{q}(F(t), F(t-t^*), \theta(t), g(t)), \end{aligned} \tag{1}$$

where t^* is some fixed, positive constant.

Letting T^+ denote that subset of T consisting in all tensors with positive determinant, and defining T_1 as that subset of T^+ containing those tensors with unit determinant, we see from $(3.2)_1$ that the natural, common domain of $\check{\psi}$, $\check{\eta}$, $\check{T}$, and $\check{q}$ is $T_1 \times T_1 \times \mathbb{R}^+ \times V$. However, since T_1 is not open in T we shall, for the purpose of partial differentiation, extend the domain of definition of these response functions to $D \times D \times \mathbb{R}^+ \times V$, where D is an open subset of T such that $T_1 \subseteq D \subseteq T^+$. On this extended domain we assume that the response functions are continuously differentiable. Of course, since such an extension is not unique the values of $\check{\psi}$, $\check{\eta}$, $\check{T}$, and $\check{q}$ will be non-arbitrary only when evaluated on $T_1 \times T_1 \times \mathbb{R}^+ \times V \subseteq D \times D \times \mathbb{R}^+ \times V$; likewise the only components of the partial derivatives of $\check{\psi}$, $\check{\eta}$, $\check{T}$, and $\check{q}$ that will have intrinsic significance will be those that lie in the tangent manifold of $T_1 \times T_1 \times \mathbb{R}^+ \times V$.

Since (2.11) must be satisfied in any thermodynamic process, then for those of sufficient smoothness we have

$$(\rho\check{\psi}_F - \check{T}F^{-1^T}) \cdot \dot{F} + \rho\check{\psi}_{F^*} \cdot \dot{F}^* + \rho(\check{\psi}_\theta + \check{\eta})\dot{\theta} + \rho\check{\psi}_g \cdot \dot{g} + \frac{\check{q} \cdot g}{\theta} \leqq 0, \tag{2}$$

where (2.2) has been used, and where a quantity with a superposed star denotes that quantity evaluated at $(t - t^*)$.

We now show that *the stress system in a fluid of convected elasticity must be spherical.* Once this is established, the equations (5), (7) and (10) of TRUESDELL [8] are rendered powerless* as a means of motivating $\alpha_1 < 0$.

Toward this end, let A and B be any two elements of T_1 and let A' and B' be any two elements of T. Then, since $t^* \neq 0$, we know that there exists a function $G(\cdot)\colon \mathbb{R} \to T^+$ such that**

$$G(0) = A, \dot{G}(0) = A'; \quad G(-t^*) = B, \dot{G}(-t^*) = B'.$$

It then follows that the function $F^*(\cdot)\, \mathbb{R} \to T_1$ given by

$$F^*(\tau) \equiv \frac{G(\tau)}{(\det G(\tau))^{\frac{1}{3}}}$$

* Of course, it is only within the larger context of thermodynamics that we have been able to do this.

** Here we use the fact that T^+ is not only open but also arcwise connected; *cf.* DAY [38].

is well defined and thus $\chi^*(X,t)\equiv Y+F^*(t)[X-Y]$ is, for fixed Y, an isochoric motion. Having $F^*(\cdot)$, we now construct a temperature field $\theta^*(\cdot,\cdot)$ exactly as in the lemma of Section 3 for any given a, a', $\boldsymbol{a}$, and $\boldsymbol{a}'$ with $a>0$.

Thus, entering χ^* and θ^* into (2) and evaluating at $(X,t)=(Y,0)$, we find, as a necessary condition for $\check{\psi}$, $\check{\eta}$, $\check{T}$, and $\check{q}$ to be compatible with thermodynamics, that

$$\{\rho\check{\psi}_F(\Gamma)-\check{T}(\Gamma)A^{-1^T}\}\cdot\{A'-\tfrac{1}{3}(A^{-1^T}\cdot A')A\}+\rho\check{\psi}_{F^*}(\Gamma)\cdot\{B'-\tfrac{1}{3}(B^{-1^T}\cdot B')B\}$$
$$+\rho\big(\check{\psi}_\theta(\Gamma)+\check{\eta}(\Gamma)\big)a'+\rho\check{\psi}_g(\Gamma)\cdot \boldsymbol{a}'+\frac{\check{q}(\Gamma)\cdot\boldsymbol{a}}{a}\leqq 0,$$

for all $(a,a',\boldsymbol{a},\boldsymbol{a}',A,A',B,B')$ with $a>0$, and A and B in T_1. Here $\Gamma\equiv(a,\boldsymbol{a},A,B)$. Now, since A', B', a', and $\boldsymbol{a}'$ appear only linearly, we conclude

$$\begin{aligned}\{\rho\check{\psi}_F(\Gamma)-\check{T}(\Gamma)A^{-1^T}\}\cdot\{A'-\tfrac{1}{3}(A^{-1^T}\cdot A')A\}&=0,\\ \check{\psi}_{F^*}(\Gamma)\cdot\{B'-\tfrac{1}{3}(B^{-1^T}\cdot B')B\}&=0,\\ \big(\check{\psi}_\theta(\Gamma)+\check{\eta}(\Gamma)\big)a'&=0,\\ \check{\psi}_g(\Gamma)\cdot\boldsymbol{a}'&=0.\end{aligned}\tag{3}$$

Thus from $(3)_3$ and $(3)_4$ it follows, respectively, that the entropy relation holds and that ψ is independent of $\boldsymbol{g}$. On the other hand, by removing A' from $(3)_1$ we find

$$\check{T}(\Gamma)=\rho\check{\psi}_F(\Gamma)A^T-\tfrac{1}{3}(\rho\check{\psi}_F\cdot A-\operatorname{tr}\check{T})\mathbf{1}.\tag{4}$$

Hence the deviatoric part of $\check{T}(\cdot)$ is determined by that component of $\check{\psi}_F(\cdot)$ which lies in the tangent plane at the point A of the surface $T_1\subseteq T$.

To study the consequences of $(3)_2$ we first remove B' and find that

$$\check{\psi}_{F^*}(A,B,a,\boldsymbol{a})=\tfrac{1}{3}[\check{\psi}_{F^*}(A,B,a,\boldsymbol{a})\cdot B]B^{-1^T}.\tag{5}$$

Now if $S(\cdot)$ denotes any path on T_1, then $S(\tau)^{-1^T}\cdot\dfrac{dS(\tau)}{d\tau}=0$ and, with the aid of (5), the function $\phi(\tau)\equiv\check{\psi}(A,S(\tau),a,\boldsymbol{a})$ is seen to satisfy

$$\frac{d\phi}{d\tau}=\check{\psi}_{F^*}(A,S(\tau),a,\boldsymbol{a})\cdot\frac{dS}{d\tau}=0.$$

That is, $\check{\psi}(A,S(\tau),a,\boldsymbol{a})$ is unchanged for all paths on T_1, and we conclude that on T_1 $\check{\psi}$ is independent of $F(t-t^*)$. Thus, noting $(3)_4$, we have

$$\psi=\check{\psi}\big(F(t),\theta(t)\big).\tag{6}$$

Now, by (4) and (6), we see that $(1)_3$ reduces to

$$T=-p\mathbf{1}+\rho\check{\psi}_F(F,\theta)F^T,\tag{7}$$

where p is an indeterminate pressure.

If we now suppose that the material is a fluid in the sense of NOLL [36], then it follows that $\check{\psi}(\cdot,\theta)$ must meet

$$\check{\psi}(F,\theta)=\check{\psi}(FH,\theta)$$

for all unimodular H. Thus, since $F\in T_1$, we take $H=F^{-1}$ and find $\check{\psi}(F,\theta)=\check{\psi}(\mathbf{1},\theta)$. Hence, $\check{\psi}(\cdot,\theta)$ is constant on T_1 and (7) becomes

$$T=-p\mathbf{1}.$$

That is, the stress system in a fluid of convected elasticity is spherical, as was to be shown.

6. Asymptotic Mechanical and Thermodynamical Stability

In the present and the following two sections we seek to describe the temporal evolution of the velocity and temperature fields in a second grade fluid which, having been shaken and heated in an arbitrary fashion during $(-\infty, 0]$, is suddenly at $t=0$ and forever after mechanically isolated and immersed in a thermally passive environment with temperature θ°. While the boundedness results of Section 4 provide a partial description of this evolution when $\bar{\alpha}_1(\theta) \geqq 0$, we focus here on certain time dependent estimates and asymptotic analyses.

In Corollary 1 of Theorem 6 the question of whether or not the free energy of a second grade fluid has a minimum in equilibrium was observed to be governed by the sign of $\bar{\alpha}_1(\theta)$. In the remainder of this paper we shall drop the assumption A1 in Section 4 concerning the equilibrium minimal character of the free energy and will record in the statement of all results an explicit assumption concerning the sign of $\bar{\alpha}_1(\theta)$ whenever it is necessary to do so. Also, for the remainder of this paper, we shall suppose that $\bar{\mu}(\cdot)$ and $\bar{\alpha}_1(\cdot)$ are constants, μ and α, respectively. By (5.9), μ must be non-negative, while α may be of any sign. However, for ease of proving theorems we shall usually have in mind the situation $\mu > 0$ and $\alpha \neq 0$.*

With the above conventions, we may use $(3.15)_{3,4}$, $(5.13)_1$, and (5.14) to obtain the following expressions for the stress, the stress power, and the internal energy:

$$
\begin{aligned}
\boldsymbol{T} &= -p\mathbf{1} + \mu \boldsymbol{A}_1 + \alpha(\dot{\boldsymbol{A}}_1 + \boldsymbol{A}_1 \boldsymbol{W} - \boldsymbol{W}\boldsymbol{A}_1),\\
\boldsymbol{T}\cdot\boldsymbol{L} &= \tfrac{1}{2}\boldsymbol{T}\cdot\boldsymbol{A}_1 = \tfrac{1}{2}(\mu|\boldsymbol{A}_1|^2 + \tfrac{1}{2}\alpha\,\dot{\overline{|\boldsymbol{A}_1|^2}}),\\
\varepsilon &= \tilde{\varepsilon}(\theta, \boldsymbol{A}_1) = \bar{\varepsilon}(\theta) + \frac{\alpha}{4\rho}|\boldsymbol{A}_1|^2,
\end{aligned} \tag{6.1}
$$

where

$$
\bar{\varepsilon}(\theta) \equiv \bar{\psi}(\theta) - \theta\bar{\psi}_\theta(\theta), \qquad \boldsymbol{W} \equiv \tfrac{1}{2}(\boldsymbol{L} - \boldsymbol{L}^T).
$$

It follows from the assumption A2 of Section 4 concerning the positivity of the specific heat, $c = \tilde{\varepsilon}_\theta(\theta, \boldsymbol{A}_1)$, that $\bar{\varepsilon}(\cdot)$ is a monotone increasing function, *i.e.*,

$$
\bar{\varepsilon}_\theta(\theta) > 0 \tag{6.2}
$$

for all θ.

Entering $(6.1)_{2,3}$ into the balance of energy (2.8), we find

$$
\rho\dot{\bar{\varepsilon}}(\theta) = \tfrac{1}{2}\mu|\boldsymbol{A}_1|^2 - \operatorname{div}\tilde{\boldsymbol{q}} + \rho r, \tag{6.3}
$$

where $\tilde{\boldsymbol{q}}$ is as in $(3.15)_5$. Thus except for a possible dependence of $\tilde{\boldsymbol{q}}$ on $\boldsymbol{A}_2$, (6.3) is of exactly the same form as the energy equation for a Navier-Stokes fluid. Here, however, unless $\alpha = 0$, $\dot{\bar{\varepsilon}}(\theta)$ is not the rate of change of internal energy, nor is $\tfrac{1}{2}\mu|\boldsymbol{A}_1|^2$ the stress power.

Setting $\boldsymbol{v} = \dot{\boldsymbol{x}}$ we re-record the balance of momentum (2.7) in the form

$$
\operatorname{div}\boldsymbol{T} + \rho\boldsymbol{b} = \rho\dot{\boldsymbol{v}}, \tag{6.4}
$$

where $\boldsymbol{T}$ is as in $(6.1)_1$.

---

* Nevertheless, since simple limiting forms of our conclusions will apply when α is zero we shall often allow the value $\alpha = 0$ in the *statement* of our theorems.

Finally, to complete the system of field equations for incompressible fluids of the second grade, we record the condition that all possible motions must be isochoric:

$$\operatorname{div} \boldsymbol{v} = 0, \tag{6.5}$$

which is clearly equivalent to $(3.2)_2$.

We note that (6.3), (6.4), and (6.5), when coupled with the response function $\tilde{\boldsymbol{q}}$ and the expression in $(6.1)_1$ for $\boldsymbol{T}$, form a set of 5 equations for the 5 unknowns $\boldsymbol{v}$, θ, and p once $\boldsymbol{b}$ and r are specified.

If $\boldsymbol{f}(\cdot)$ and $\boldsymbol{k}(\cdot)$ map Ω_t into V, we shall find it helpful to use the notation

$$\langle \boldsymbol{f} \cdot \boldsymbol{k} \rangle \equiv \int_{\Omega_t} \boldsymbol{f} \cdot \boldsymbol{k} \, dv, \qquad \|\boldsymbol{f}\|^2 \equiv \langle \boldsymbol{f} \cdot \boldsymbol{f} \rangle.$$

Likewise, if $\boldsymbol{F}$ and $\boldsymbol{K}$ map Ω_t into T, we shall write $\langle \boldsymbol{F} \cdot \boldsymbol{K} \rangle$ and $\|\boldsymbol{F}\|^2$, the meaning here being clear.

We now have

Theorem 7. *In any fluid of grade* 2 *which is mechanically isolated for* $t \geqq 0$ *one has*

$$\dot{\overline{\|\boldsymbol{v}\|^2}}(t) + \frac{\mu}{\rho} \|\boldsymbol{A}_1\|^2(t) + \frac{\alpha}{2\rho} \dot{\overline{\|\boldsymbol{A}_1\|^2}}(t) = 0. \tag{6.6}$$

Proof. If we form the scalar product of (2.7) with $\boldsymbol{v}$, integrate over Ω_t, use the divergence theorem and the conservation of mass, we establish the familiar *power theorem*

$$\frac{d}{dt} \int_{\Omega_t} \tfrac{1}{2} \rho |\boldsymbol{v}|^2 \, dv + \int_{\Omega_t} \boldsymbol{T} \cdot \boldsymbol{L} \, dv = \int_{\partial\Omega_t} \boldsymbol{v} \cdot \boldsymbol{T} \boldsymbol{n} \, da + \int_{\Omega_t} \rho \boldsymbol{v} \cdot \boldsymbol{b} \, dv.$$

By the definition (4.4) of mechanical isolation, $(6.1)_2$ and the fact that ρ is constant, we therefore find, for a second grade fluid,

$$\frac{d}{dt} \int_{\Omega_t} |\boldsymbol{v}|^2 \, dv + \frac{\mu}{\rho} \int_{\Omega_t} |\boldsymbol{A}_1|^2 \, dv + \frac{\alpha}{2\rho} \frac{d}{dt} \int_{\Omega_t} |\boldsymbol{A}_1|^2 \, dv = 0,$$

which is (6.6). △

For the remainder of this section we assume $\alpha \geqq 0$. With this agreement, the quantity $E(\cdot)$, given by

$$E(t) \equiv \|\boldsymbol{v}\|^2(t) + \frac{\alpha}{2\rho} \|\boldsymbol{A}_1\|^2(t), \tag{6.7}$$

will then be a positive definite functional of the velocity field $\boldsymbol{v}$. We interpret $E(t)$ as the sum of the averaged stretching and the kinetic energy in the fluid. In terms of $E(\cdot)$, Theorem 7 has the following

Corollary 1. *Let the hypotheses of Theorem* 7 *hold and, in addition, assume* $\alpha \geqq 0$. *Then* $E(\cdot)$, *given by* (6.7), *is a non-increasing function of time,*

$$\dot{E}(t) \leqq 0, \tag{6.8}$$

and is bounded below according to

$$E(0)\, e^{-\frac{2\mu}{\alpha} t} \leqq E(t). \tag{6.9}$$

Proof. To prove (6.8), we need merely appeal to (6.6), use the definition of $E(\cdot)$, and note that $\|\cdot\|^2 \geqq 0$. The proof of (6.9) follows by adding $\frac{2\mu}{\alpha}\|\boldsymbol{v}\|^2$ to (6.6) to obtain $\dot{E}+\frac{2\mu}{\alpha}E=\frac{2\mu}{\alpha}\|\boldsymbol{v}\|^2 \geqq 0$, which, by integration, yields (6.9). △

We note in passing that the inequality (6.8) is strict if and only if B is deforming (*i.e.*, when $A_1(\cdot, t) \not\equiv 0$). The inequality (6.9) is strict if $\boldsymbol{v}(\cdot, 0) \not\equiv 0$.

The inequalities (6.8) and (6.9) can be considerably augmented if we specify in finer detail the character of the mechanical isolation of B. Toward this end, we first introduce the position $\boldsymbol{x}_0(t)$ of the center of mass of B,

$$\boldsymbol{x}_0(t) \equiv \frac{1}{M}\int_{\Omega} \rho \chi(X, t)\, dv = \frac{1}{M}\int_{\Omega_t} \rho \boldsymbol{x}\, dv. \tag{6.10}$$

Then (2.3) and (2.6) give

$$M\ddot{\boldsymbol{x}}_0(t) = \int_{\partial\Omega_t} \boldsymbol{T}\boldsymbol{n}\, da + \int_{\Omega_t} \rho \boldsymbol{b}\, dv, \tag{6.11}$$

and with (2.7) and the symmetry of the stress tensor, we obtain

$$\frac{d}{dt}\int_{\Omega_t} \rho(\boldsymbol{x}-\boldsymbol{x}_0)\times(\boldsymbol{v}-\dot{\boldsymbol{x}}_0)\, dv = \int_{\partial\Omega_t} (\boldsymbol{x}-\boldsymbol{x}_0)\times \boldsymbol{T}\boldsymbol{n}\, da + \int_{\Omega_t} \rho(\boldsymbol{x}-\boldsymbol{x}_0)\times \boldsymbol{b}\, dv. \tag{6.12}$$

This is, of course, the expression of the balance of angular momentum with respect to the center of mass. The quantities appearing on the right hand side of (6.11) and (6.12) are called, respectively, the *net load* and the *net moment about* $\boldsymbol{x}_0$ which act on B. The vector

$$\boldsymbol{h}_0(t) \equiv \int_{\Omega_t} \rho(\boldsymbol{x}-\boldsymbol{x}_0)\times(\boldsymbol{v}-\dot{\boldsymbol{x}}_0)\, dv \tag{6.13}$$

is called the *angular momentum relative to the center of mass.*

It is convenient to introduce the following particular decomposition of the velocity field, $\boldsymbol{v}(\cdot,\cdot)$, by defining a vector field $\boldsymbol{d}(\cdot,\cdot)$ according to

$$\boldsymbol{d}(\boldsymbol{x}, t) \equiv \boldsymbol{v}(\boldsymbol{x}, t) - \dot{\boldsymbol{x}}_0(t) - \boldsymbol{\omega}(t)\times(\boldsymbol{x}-\boldsymbol{x}_0), \tag{6.14}$$

where $\boldsymbol{\omega}(t)$ is the axial vector of the skew tensor

$$\bar{W}(t) \equiv \frac{1}{2M}\int_{\Omega_t} \rho[\operatorname{grad}\boldsymbol{v} - (\operatorname{grad}\boldsymbol{v})^T]\, dv. \tag{6.15}$$

To interpret $\boldsymbol{d}(\cdot,\cdot)$ and to set down some of its properties for later use we have the following

Lemma. *The vector field* $\boldsymbol{d}(\cdot,\cdot)$, *defined by* (6.14) *and* (6.15), *satisfies*

(i) $\operatorname{grad}\boldsymbol{d} + (\operatorname{grad}\boldsymbol{d})^T = A_1$,

(ii) $A_1(\cdot, t) \equiv 0$ *if and only if* $\boldsymbol{d}(\cdot, t) \equiv 0$,

(iii) $\int_{\Omega_t} \rho \boldsymbol{d}\, dv = 0,$ (6.16)

(iv) $\int_{\Omega_t} \rho[\operatorname{grad}\boldsymbol{d} - (\operatorname{grad}\boldsymbol{d})^T]\, dv = 0.$

Proof. Use the definitions and the conservation of mass (2.6). $\triangle$

In words, (6.16) implies that the body can be moving rigidly at time t when and only when $\boldsymbol{d}(\cdot, t)$ vanishes. Moreover, if $\boldsymbol{d}$ is viewed as a velocity field, then (iii) and (iv) of (6.16) state that both its averaged translation and its averaged rotation vanish.

With the above prerequisites established, we have

Theorem 8. *Let B be any second grade fluid with $\alpha \geq 0$, and suppose B is mechanically isolated for $t \geqq 0$. Further, suppose that the net load, net moment about $\boldsymbol{x}_0$, and the initial angular momentum relative to the center of mass are all zero. Then there exists a positive domain dependent function $\kappa(t) = \kappa(\Omega_t)$ such that, in addition to* (6.8) *and* (6.9), *one has*

$$
\begin{aligned}
0 \leqq \|\boldsymbol{v} - \dot{\boldsymbol{x}}_0\|^2(t) &\leqq \frac{\rho\kappa(t) E_0(0)}{\alpha + \rho\kappa(t)} e^{-\int_0^t \frac{2\mu\, ds}{\alpha + \rho\kappa(s)}}, \\
\frac{2\rho E_0(0)}{\alpha + \rho\kappa(t)} e^{-\frac{2\mu}{\alpha} t} \leqq \|\boldsymbol{A}_1\|^2(t) &\leqq \frac{2\rho E_0(0)}{\alpha} e^{-\int_0^t \frac{2\mu\, ds}{\alpha + \rho\kappa(s)}},
\end{aligned}
\tag{6.17}
$$

where

$$E_0(0) \equiv \|\boldsymbol{v} - \dot{\boldsymbol{x}}_0\|^2(0) + \frac{\alpha}{2\rho} \|\boldsymbol{A}_1\|^2(0).$$

Proof. By (6.14) and the fact that ρ is constant, we have

$$
\begin{aligned}
\|\boldsymbol{d}\|^2 &= \|\boldsymbol{v} - \dot{\boldsymbol{x}}_0\|^2 + \|\boldsymbol{\omega} \times (\boldsymbol{x} - \boldsymbol{x}_0)\|^2 - 2\boldsymbol{\omega} \cdot \int_{\Omega_t} (\boldsymbol{x} - \boldsymbol{x}_0) \times (\boldsymbol{v} - \dot{\boldsymbol{x}}_0)\, dv, \\
&= \|\boldsymbol{v} - \dot{\boldsymbol{x}}_0\|^2 + \|\boldsymbol{\omega} \times (\boldsymbol{x} - \boldsymbol{x}_0)\|^2 - \frac{2}{\rho} \boldsymbol{\omega}(t) \cdot \boldsymbol{h}_0(t),
\end{aligned}
$$

where we have also used (6.13). But since the net moment about $\boldsymbol{x}_0$ is required to vanish, then (6.12) yields $\boldsymbol{h}_0(t) = \boldsymbol{h}_0(0)$, which vanishes by hypothesis. Thus we have

$$\|\boldsymbol{d}\|^2 = \|\boldsymbol{v} - \dot{\boldsymbol{x}}_0\|^2 + \|\boldsymbol{\omega} \times (\boldsymbol{x} - \boldsymbol{x}_0)\|^2 \geqq \|\boldsymbol{v} - \dot{\boldsymbol{x}}_0\|^2. \tag{6.18}$$

Now since $\rho = \text{const.} \neq 0$, (iii) and (iv) of (6.16) are sufficient for the validity of both the Poincaré inequality* and the Korn inequality* for $\boldsymbol{d}$, so we may assert the existence of positive numbers $c_P(\Omega_t)$ and $c_K(\Omega_t)$ such that

$$\|\boldsymbol{d}\|^2 \leqq c_P \|\operatorname{grad} \boldsymbol{d}\|^2 \leqq c_P c_K \|\boldsymbol{A}_1\|^2, \tag{6.19}$$

where we have also used (i) of (6.16). Combining (6.18) and (6.19) we readily reach

$$\|\boldsymbol{v} - \dot{\boldsymbol{x}}_0\|^2 \leqq c_P c_K \|\boldsymbol{A}_1\|^2$$

and thus obtain

$$\left(1 + \frac{\alpha}{2\rho c_P c_K}\right) \|\boldsymbol{v} - \dot{\boldsymbol{x}}_0\|^2 \leqq E_0(t) \leqq \left(c_P c_K + \frac{\alpha}{2\rho}\right) \|\boldsymbol{A}_1\|^2, \tag{6.20}$$

where

$$E_0(t) \equiv \|\boldsymbol{v} - \dot{\boldsymbol{x}}_0\|^2(t) + \frac{\alpha}{2\rho} \|\boldsymbol{A}_1\|^2(t).$$

* See (A.5), (A.6) and (A.7) of the Appendix.

Next, we observe from (6.10) that $\|v-\dot{x}_0\|^2=\|v\|^2-\frac{M}{\rho}|\dot{x}_0|^2$, and, since the net load is, by hypothesis, zero, we see from (6.11) that $\ddot{x}_0=0$ so that $\overline{\|v-x_0\|^2}^{\bullet}=\overline{\|v\|^2}^{\bullet}$. Thus upon entering into (6.6), using the definition of $E_0(\cdot)$, and using the estimate $(6.20)_1$, we obtain

$$0\leqq\frac{2\mu}{\alpha}\|v-\dot{x}_0\|^2=\dot{E}_0+\frac{2\mu}{\alpha}E_0\leqq\frac{2\mu}{\alpha}\frac{\rho c_P c_K}{\alpha+2\rho c_P c_K}E_0.$$

Integrating each of these two differential inequalities, we find

$$E_0(0)\,e^{-\frac{2\mu}{\alpha}t}\leqq E_0(t)\leqq E_0(0)\,e^{-\int_0^t\frac{2\mu\,ds}{\alpha+\rho\kappa(s)}},\tag{6.21}$$

where we have set $\kappa(s)\equiv 2c_P(\Omega_s)c_K(\Omega_s)$. But (6.21), (6.20), and the fact that $\frac{\alpha}{2\rho}\|A_1\|^2\leqq E_0(t)$ yields (6.17). △

Corollary 1. *Under the hypotheses of Theorem* 8 *the "angular velocity tensor" $\overline{W}(t)$ of* (6.15) *meets*

$$0\leqq\overline{W}^T(t)\,\overline{W}(t)\cdot J(t)\leqq\frac{\rho^2\kappa(t)}{\alpha}E_0(0)\,e^{-\int_0^t\frac{2\mu\,ds}{\alpha+\rho\kappa(s)}},\tag{6.22}$$

while its axial vector $\omega(t)$ satisfies

$$I(t)\,\omega(t)=\int_{\Omega_t}\rho\,d\times(x-x_0)\,dv.\tag{6.23}$$

Here, J is Euler's tensor and I is the inertia tensor, i.e.,

$$J(t)\equiv\int_{\Omega_t}\rho(x-x_0)\otimes(x-x_0)\,dv;\qquad I(t)\equiv 1\int_{\Omega_t}\rho\,|x-x_0|^2\,dv-J(t).$$

Proof. By $(6.18)_1$, (6.19), and the definition of $\kappa(t)$ we have

$$\|\omega\times(x-x_0)\|^2\leqq\frac{\kappa(t)}{2}\|A_1\|^2,$$

which is clearly equivalent to

$$\overline{W}^T\,\overline{W}\cdot J\leqq\frac{\rho\kappa}{2}\|A_1\|^2.$$

Thus using (6.17), we obtain (6.22).

To prove (6.23) we recall that $h_0(t)=0$, and by (6.13) and (6.14) observe that this is equivalent to

$$\int_{\Omega_t}\rho(x-x_0)\times(\omega\times(x-x_0)+d)\,dv=0,$$

which is (6.23). △

Corollary 2. *Under the hypotheses of Theorem* 8, *if $A_1(\cdot,0)$ vanishes then $E_0(0)=0$ and the velocity field is a pure translation for all future times,* i.e. $v(x,t)=\dot{x}_0(t)$ *for all* $(x,t)\in\Omega_t\times[0,\infty)$.

Proof. By (6.20), we see that $\|A_1\|^2(0)=0$ forces $E_0(0)=0$, and this, with (6.17), gives the result. Δ

Considering now a rather different way in which B could be mechanically isolated, we have

Theorem 9. *Let B be any second grade fluid with $\alpha \geqq 0$, and suppose B is mechanically isolated for $t \geqq 0$. Further, suppose there exists a subsurface $S \subseteq \partial\Omega$ of non-zero area measure such that $X \in S \Rightarrow \boldsymbol{v}(X, t)=0\ \forall\ t \geqq 0$,* i.e., *$S_t = S_0$ and $\boldsymbol{v}(\boldsymbol{x}, t)=0\ \forall\ (\boldsymbol{x}, t) \in S_t \times [0, \infty)$. Then, there exist two positive domain dependent functions $\kappa_i(t)=\kappa_i(\Omega_t)$, $i=1$ or 2, such that, in addition to* (6.8) *and* (6.9), *one has*

$$0 \leqq \|\boldsymbol{v}\|^2(t) \leqq \frac{\rho\kappa_1(t)}{\alpha+\rho\kappa_1(t)} E(0)\, e^{-\int_0^t \frac{2\mu\, ds}{\alpha+\rho\kappa_1(s)}}, \tag{6.24}$$

$$\begin{aligned} 0 &\leqq \frac{2\rho E(0)}{\alpha+\rho\kappa_1(t)} e^{-\frac{2\mu}{\alpha}t} \leqq \|A_1\|^2(t) \leqq 4\|\operatorname{grad}\boldsymbol{v}\|^2(t) \leqq 4\kappa_2(t)\|A_1\|^2(t) \\ &\leqq \frac{8\rho\kappa_2(t)E(0)}{\alpha} e^{-\int_0^t \frac{2\mu\, ds}{\alpha+\rho\kappa_1(s)}}, \end{aligned} \tag{6.25}$$

where

$$E(0)=\|\boldsymbol{v}\|^2(0)+\frac{\alpha}{2\rho}\|A_1\|^2(0).$$

Proof. Since $\boldsymbol{v}(\cdot, t)$ vanishes on $S_t=S_0$, we may use both the Poincaré and the Korn inequalities* to write

$$\|\boldsymbol{v}\|^2 \leqq c_P\|\operatorname{grad}\boldsymbol{v}\|^2 \leqq c_P c_K\|A_1\|^2. \tag{6.26}$$

Thus, in view of the definition (6.7) of $E(t)$,

$$\left(1+\frac{\alpha}{2\rho c_P c_K}\right)\|\boldsymbol{v}\|^2 \leqq E(t) \leqq \left(c_P c_K+\frac{\alpha}{2\rho}\right)\|A_1\|^2. \tag{6.27}$$

Turning to (6.6) and using $(6.27)_2$ we have

$$\dot{E}=-\frac{\mu}{\rho}\|A_1\|^2 \leqq -\frac{2\mu}{\alpha+2\rho c_P c_K}E,$$

which, upon integration, leads to

$$E(t) \leqq E(0)\, e^{-\int_0^t \frac{2\mu\, ds}{\alpha+\rho\kappa_1(s)}}, \tag{6.28}$$

where $\kappa_1(s) \equiv 2c_P(\Omega_s)c_K(\Omega_s)$.

Now (6.28), (6.27), and (6.9) give $(6.24)_2$ and $(6.25)_2$. In addition, recalling that one always has $\|A_1\|^2 \leqq 4\|\operatorname{grad}\boldsymbol{v}\|$ and using $(6.26)_2$, we reach $(6.25)_{3,4}$, where $\kappa_2(s) \equiv c_K(\Omega_s)$. Finally, $(6.25)_5$ follows from (6.28) and the fact that $\frac{\alpha}{2\rho}\|A_1\|^2 \leqq E(t)$. Δ

Corollary 1. *Under the hypotheses of Theorem 9, if $A_1(\cdot, 0)$ vanishes, then $\boldsymbol{v}(\cdot, t)$ vanishes for all $t \geqq 0$.*

* See (A.5) and (A.6) of the Appendix.

Proof. Since $A_1(\cdot, 0)=0$ implies that $v(\cdot, 0)$ must be instantaneously rigid, and since $v(\cdot, 0)$ vanishes on S_0, then $v(\cdot, 0)=0$. But this in turn yields $E(0)=0$, and we now apply (6.24) to complete the proof.* △

Corollary 2. *Any flow of a second grade fluid having $\alpha \geqq 0$ which takes place in a stationary rigid container under a conservative body force field meets the estimates*

$$\begin{aligned} 0 \leqq \|v\|^2(t) \leqq \frac{\rho\kappa}{\alpha+\rho\kappa} E(0)\, e^{-\frac{2\mu t}{\alpha+\rho\kappa}}, \\ \frac{\rho E(0)}{\alpha+\rho\kappa} e^{-\frac{2\mu}{\alpha}t} \leqq \|\operatorname{grad} v\|^2(t) \leqq \frac{\rho}{\alpha} E(0)\, e^{-\frac{2\mu t}{\alpha+\rho\kappa}}, \end{aligned} \tag{6.29}$$

for a fixed number κ which does not exceed $\left(\frac{1}{\pi}\right)^2 \left(\frac{3}{4\pi}\right)^{\frac{2}{3}} V(\Omega_0)^{\frac{2}{3}}$, where $V(\Omega_0)$ is the volume of the container Ω_0.

Proof. Here we assume the classical adherence condition: $v(x, t)=0$ on $\partial\Omega_t=\partial\Omega_0$. Therefore, since the body force field is conservative and $\operatorname{div} v=0$, we see that B is mechanically isolated and that Theorem 9 holds with $S=\partial\Omega$. We now observe that since $v=0$ on $\partial\Omega_t$ and $\operatorname{div} v=0$, then

$$\|\operatorname{grad} v\|^2 = \tfrac{1}{2}\|A_1\|^2,$$

which implies that Korn's constant c_K of $(6.26)_2$ may be taken to be $\frac{1}{2}$. We also note that the Poincaré inequality of $(6.26)_1$ may be written with a time independent constant, $c_P=c_P(\Omega_t)=c_P(\Omega_0)$. Turning now to (6.24) and (6.25) and setting $\kappa \equiv \kappa_1 = 2c_P \cdot \frac{1}{2} = c_P$, $\kappa_2=\frac{1}{2}$, we may explicitly evaluate the integrals in the exponential terms and (6.29) then follows. The fact that we have

$$\kappa = c_P(\Omega_0) \leqq \left(\frac{1}{\pi}\right)^2 \left(\frac{3}{4\pi}\right)^{\frac{2}{3}} V(\Omega_0)^{\frac{2}{3}},$$

follows from (A.2) of the Appendix. △

We remark that Theorems 8 and 9 have in common the result that *no initial disturbance in any second grade fluid having $\alpha>0$ can ever decay away in a finite time.* That is, $A_1(\cdot, 0) \neq 0$ implies $A_1(\cdot, t) \neq 0$ for all later time. This result has been shown (using entirely different methods) for Navier-Stokes fluids ($\alpha=0$) by OGAWA [23] and DYER & EDMUNDS [26], both of whom give *lower* bound for $\|v\|^2(t)$ but only under special regularity assumptions on the flow.** In our

* It is interesting to note that the argument used here can be applied to (6.9) to prove that $A_1(\cdot, 0) \neq 0 \Rightarrow A_1(\cdot, t) \neq 0\ \forall\ t>0$ if v is merely required to vanish at three non-collinear points. For if $A_1(\cdot, t)=0$ for some $t>0$, then $v(\cdot, t)$ would be instantaneously rigid, and being required to vanish at three points, it then must vanish everywhere. By (6.9) we would then have $E(0)=0$, contradicting $A_1(\cdot, 0) \neq 0$. Thus, any second grade fluid with $\alpha>0$ which is initially deforming and mechanically isolated will, if it has at least three fixed non-collinear points, never cease deforming in finite time. Of course, this simple argument does not yield the estimates of (6.24) and (6.25), though it is consistent with them.

** OGAWA requires that $\sup_{x\in\Omega_0}|v(x, t)| \in L_2[0, \infty)$ while DYER & EDMUNDS require that either $\sup_{x\in\Omega_0}|v_t(x, t)| \in L_p[0, \infty)$ or $\sup_{x\in\Omega_0}|\operatorname{grad} v_t(x, t)| \in L_p[0, \infty)$ for some $p\in[1, \infty)$. For both, $\Omega_t=\Omega_0$ and $v(\cdot, t)$ vanishes on $\partial\Omega_t$.

work such materials are specifically excluded and, in fact, the estimates (6.9), $(6.17)_2$, (6.22), (6.25) and (6.29) are rendered either empty or trivial when $\alpha=0$. However, it is worth noting that one role possessed by viscosity in a Navier-Stokes fluid is also preserved in the theory of second grade fluids: *The presence of viscosity* (i.e., $\mu\neq 0$) *is mandatory if disturbances are ever to subside.* Indeed, from (6.6), we see that $\mu=0$ forces $E(t)=E(0)$ for all t no matter what α may be.

Turning now to the behavior of the temperature field, we may expect that the sharpness and, indeed, the very possibility of decay theorems for the temperature field similar to those just obtained for $\boldsymbol{v}(\cdot,\cdot)$ will be closely connected to the specific properties of $\tilde{\boldsymbol{q}}(\cdot,\cdot,\cdot,\cdot)$, the response function for the heat flux vector. From (5.13), we recall that $\tilde{\boldsymbol{q}}$ must be such as at least to satisfy the reduced dissipation inequality

$$\frac{\tilde{\boldsymbol{q}}(\theta,\boldsymbol{g},\boldsymbol{A}_1,\boldsymbol{A}_2)\cdot\boldsymbol{g}}{\theta}\leqq\frac{1}{2}\mu|\boldsymbol{A}_1|^2,$$

in order to be compatible with thermodynamics.

With these remarks in mind we make the following two

Definitions. (i) *The heat flux vector is strictly compatible with thermodynamics if*

$$\tilde{\boldsymbol{q}}(a,\boldsymbol{a},\boldsymbol{A},\boldsymbol{A}')\cdot\boldsymbol{a}\leqq 0, \tag{6.30}$$

for all $a>0$, $\boldsymbol{a}\in V$, *and* $\boldsymbol{A},\boldsymbol{A}'\in T_s^\circ$.

(ii) *The heat flux vector is strongly conductive if there exists a number* $\lambda>0$ *such that*

$$\tilde{\boldsymbol{q}}(a,\boldsymbol{a},\boldsymbol{A},\boldsymbol{A}')\cdot\boldsymbol{a}\leqq-\lambda\boldsymbol{a}\cdot\boldsymbol{a}, \tag{6.31}$$

for all $a>0$, $\boldsymbol{a}\in V$, *and* $\boldsymbol{A},\boldsymbol{A}'\in T_s^\circ$.

It is obvious that strongly conductive $\Rightarrow$ strict compatibility with thermodynamics $\Rightarrow$ satisfaction of the reduced dissipation inequality (since $\mu\geqq 0$).

We now have

Theorem 10. *Let B be any second grade fluid whose heat flux vector is strictly compatible with thermodynamics. Further, suppose that, for $t\geqq 0$, B is immersed in a thermally passive environment with constant temperature θ°. Then for any $t'\geqq 0$ and any $t\geqq t'$ the following* ***maximum principle*** *holds*:

$$\max_{\boldsymbol{X}\in\Omega}|\bar{\varepsilon}(\theta(\boldsymbol{X},t))-\varepsilon^\circ|\leqq\max_{\boldsymbol{X}\in\Omega}|\bar{\varepsilon}(\theta(\boldsymbol{X},t'))-\varepsilon^\circ|+\frac{\mu}{2\rho}\int_{t'}^{t}\max_{\boldsymbol{X}\in\Omega}|\boldsymbol{A}_1|^2(\boldsymbol{X},\tau)\,d\tau, \tag{6.32}$$

where $\varepsilon^\circ\equiv\bar{\varepsilon}(\theta^\circ)$.

Proof. We begin by deriving an important inequality which is valid for any second grade fluid immersed in a thermally passive environment. To achieve it, let p be any positive integer and define

$$I_p(t)\equiv\int_\Omega\rho\big(\bar{\varepsilon}(\theta(\boldsymbol{X},t))-\varepsilon^\circ\big)^p\,dv.$$

Using conservation of mass, the energy equation (6.3), and the divergence theorem we obtain

$$\begin{aligned}\dot{I}_{2n}(t)=2n\Big\{&\frac{\mu}{2}\int_{\Omega_t}\big(\bar{\varepsilon}(\theta)-\varepsilon^\circ\big)^{2n-1}|A_1|^2\,dv\\&+(2n-1)\int_{\Omega_t}\big(\bar{\varepsilon}(\theta)-\varepsilon^\circ\big)^{2n-2}\bar{\varepsilon}_\theta(\theta)\,\tilde{\boldsymbol{q}}\cdot\boldsymbol{g}\,dv\\&-\int_{\partial\Omega_t}\big(\bar{\varepsilon}(\theta)-\varepsilon^\circ\big)^{2n-1}\tilde{\boldsymbol{q}}\cdot\boldsymbol{n}\,da+\int_{\Omega_t}\rho\big(\bar{\varepsilon}(\theta)-\varepsilon^\circ\big)^{2n-1}r\,dv\Big\},\end{aligned}\tag{6.33}$$

for positive integer n. Now since, by $(4.5)_1$, immersion in a thermally passive environment means $(\theta-\theta^\circ)\tilde{\boldsymbol{q}}\cdot\boldsymbol{n}\geqq 0$ and $(\theta-\theta^\circ)r\leqq 0$, and since, by (6.2), $\bar{\varepsilon}(\cdot)$ is monotone increasing, then we have

$$\big(\bar{\varepsilon}(\theta)-\varepsilon^\circ\big)^{2n-1}\tilde{\boldsymbol{q}}\cdot\boldsymbol{n}\geqq 0,\qquad \big(\bar{\varepsilon}(\theta)-\varepsilon\ \big)^{2n-1}r\leqq 0$$

for positive integer n. Therefore, by (6.33), we find

$$\begin{aligned}\dot{I}_{2n}(t)\leqq 2n\Big\{&\frac{\mu}{2}\int_{\Omega_t}\big(\bar{\varepsilon}(\theta)-\varepsilon^\circ\big)^{2n-1}|A_1|^2\,dv\\&+(2n-1)\int_{\Omega_t}\big(\bar{\varepsilon}(\theta)-\varepsilon^\circ\big)^{2n-2}\bar{\varepsilon}_\theta(\theta)\,\tilde{\boldsymbol{q}}\cdot\boldsymbol{g}\,dv\Big\}\end{aligned}\tag{6.34}$$

whenever any second grade fluid is immersed in a thermally passive environment with constant temperature θ°.

Now $\bar{\varepsilon}_\theta(\cdot)>0$, $2n-2$ is even, and, by hypothesis $\tilde{\boldsymbol{q}}\cdot\boldsymbol{g}\leqq 0$, so we obtain

$$\dot{I}_{2n}(t)\leqq\frac{n\mu}{\rho}\int_{\Omega_t}\rho\big(\bar{\varepsilon}(\theta)-\varepsilon^\circ\big)^{2n-1}|A_1|^2\,dv,$$

which, upon applying Hölder's inequality, implies

$$\dot{I}_{2n}(t)\leqq\frac{n\mu}{\rho}I_{2n}^{1-\frac{1}{2n}}(t)\Big(\int_{\Omega_t}\rho|A_1|^{4n}\,dv\Big)^{\frac{1}{2n}}.\tag{6.35}$$

Let $t'\geqq 0$ and $t\geqq t'$ be given. In addition, we may suppose $t>t'$ and $I_{2n}(t)\neq 0$, for otherwise there is nothing to prove. Let (t_1,t_2) be the largest interval containing t and on which $I_{2n}(\cdot)$ does not vanish. By continuity, this interval is not null, and on it we may use the chain rule to compute $\dot{\overline{I_{2n}^{\frac{1}{2n}}}}(t)$. We then find, using (6.35), that

$$\dot{\overline{I_{2n}^{\frac{1}{2n}}}}(t)=\frac{1}{2n}\dot{I}_{2n}(t)I_{2n}^{\frac{1}{2n}-1}(t)\leqq\frac{\mu}{2\rho}\Big(\int_{\Omega_t}\rho|A_1|^{4n}\,dv\Big)^{\frac{1}{2n}},$$

and so, for any $\hat{t}\in(t_1,t)$,

$$I_{2n}^{\frac{1}{2n}}(t)\leqq I_{2n}^{\frac{1}{2n}}(\hat{t})+\frac{\mu}{2\rho}\int_{\hat{t}}^{t}\Big(\int_{\Omega_\tau}\rho|A_1|^{4n}\,dv\Big)^{\frac{1}{2n}}d\tau.$$

Writing this last as an integral over Ω and letting $n\to\infty$ we find*

$$\max_{X\in\Omega}|\bar{\varepsilon}(\theta(X,t))-\varepsilon^\circ|\leqq\max_{X\in\Omega}|\bar{\varepsilon}(\theta(X,\hat{t}))-\varepsilon^\circ|+\frac{\mu}{2\rho}\int_{\hat{t}}^{t}\max_{X\in\Omega}|A_1|^2(X,\tau)\,d\tau. \tag{6.36}$$

Now, if $t'\in(t_1,t_2)$, we merely take $\hat{t}=t'$ in (6.36) and the proof is completed. If t' is not in the interval (t_1,t_2), then since $I_{2n}(t_1)$ must vanish, we have $\theta(X,t_1)=\theta^\circ\ \forall\ X\in\Omega$. If we now let $\hat{t}\to t_1$ in (6.36), then by continuity,

$$\max_{X\in\Omega}|\varepsilon(\theta(X,t))-\varepsilon^\circ|\leqq\frac{\mu}{2\rho}\int_{t_1}^{t}\max_{X\in\Omega}|A_1|^2(X,\tau)\,d\tau,$$

and, since we are supposing $t'\leq t_1$, the right hand side here is certainly smaller than the right hand side of (6.32). △

Corollary 1. *Let the hypothesis of Theorem* 10 *hold and, in addition, suppose that the monotone increasing function* $\bar{\varepsilon}(\cdot)$ *of* $(6.1)_3$ *is such that*

$$\bar{\varepsilon}(p)\to\infty \quad as \quad p\to\infty. \tag{6.37}$$

Then the temperature at any particle Y *and time* t *is bounded according to*

$$\theta(Y,t)\leqq\bar{\varepsilon}^{-1}\left(\max_{X\in\Omega}|\bar{\varepsilon}(\theta(X,t'))-\varepsilon^\circ|+|\varepsilon^\circ|+\int_{t'}^{t}s^2(\tau)\,d\tau\right), \tag{6.38}$$

where t' *is any given instant preceding* t, *and where*

$$s(\tau)\equiv\max_{X\in\Omega}|A_1|(X,\tau). \tag{6.39}$$

In particular, if the motion is such that $s(\cdot)$ *is square integrable on* $[t',\infty)$ *then* $\theta(\cdot,\cdot)$ *is bounded on* $\Omega\times[t',\infty)$.

Proof. Writing $\bar{\varepsilon}(\theta(Y,t))=(\bar{\varepsilon}(\theta(Y,t))-\varepsilon^\circ)+\varepsilon^\circ$, using the triangle inequality and the definition (6.39), and applying Theorem 10, we find

$$\begin{aligned}\bar{\varepsilon}(\theta(Y,t))&\leqq|\bar{\varepsilon}(\theta(Y,t))-\varepsilon^\circ|+|\varepsilon^\circ|\\&\leqq\max_{X\in\Omega}|\bar{\varepsilon}(\theta(X,t))-\varepsilon^\circ|+|\varepsilon^\circ|\\&\leqq\max_{X\in\Omega}|\bar{\varepsilon}(\theta(X,t'))-\varepsilon^\circ|+|\varepsilon^\circ|+\frac{\mu}{2\rho}\int_{t'}^{t}s^2(\tau)\,d\tau.\end{aligned}$$

Since (6.2) assures the existence of $\bar{\varepsilon}^{-1}(\cdot)$, and since (6.37) implies that the term on the right hand side above is in its domain, we readily obtain (6.38). △

Recalling that A_1 is twice the stretching tensor, we give (6.38) the following physical interpretation: The temperature field in any second grade fluid can, at an

* Here we use the fact that if $h(\cdot)$ is any continuous function defined on Ω then

$$\max_{X\in\Omega}|h(X)|=\lim_{n\to\infty}\Big(\int_\Omega h(X)^{2n}\Big)^{\frac{1}{2n}}.$$

A proof may be found in [35].

instant t, be no larger than a number determined, jointly, by the temperature field at any prior instant t' and the maximum local stretching that took place during $[t', t]$.

For a rigid heat conductor, $A_1 \equiv 0$ and Theorem 10 implies that if a state of uniform temperature θ° is ever attained, then the conductor must remain in that state for all subsequent times. For a moving fluid the presence of A_1 in (6.32) as well as in the energy balance (6.3) serves to illustrate a "generative" mechanism which often rules out the possibility that the constant uniform temperature θ° could ever be maintained after a finite time. For example, suppose we assume the conditions of either Theorem 8 or Theorem 9, and, in addition, neglect all radiant heating effects (*i.e.*, $r \equiv 0$). Then at any instant t of uniform temperature θ° we have $g = \operatorname{grad} \theta^\circ = 0$ so that $(3.18)_2$ yields $q = 0$ in Ω_t. Thus the energy balance (6.3) reduces to $\rho \bar{\varepsilon}_\theta(\theta^\circ)\dot{\theta} = \mu |A_1|^2/2$ in Ω_t, where, since $\|A_1\|^2$ is bounded away from zero at any finite time, we may conclude that the right hand side does not vanish at some point in Ω_t. At that point, since $\bar{\varepsilon}_\theta(\cdot) > 0$, we obtain $\dot{\theta} > 0$ which implies that the local temperature θ° would not be maintained.

For, say, steady motions we observe that $s(\cdot)$ is not square integrable and, therefore, the bounds of (6.32) and (6.39) become increasingly poor as time goes on. However, if we require a bit more of the heat flux vector we then have

Theorem 11. *Let B be any second grade fluid whose heat flux vector is strongly conductive, and suppose that B is immersed in a thermally passive environment with temperature θ° for $t \geq 0$. Further, let there exist a subsurface $S \subseteq \partial\Omega$ of non-zero area measure such that $X \in S \Rightarrow \theta(X, t) = \theta^\circ \; \forall \, t \geq 0$, i.e., $\theta(x, t) = \theta^\circ \; \forall \, (x, t) \in S_t \times [0, \infty)$. Then there exists a domain dependent function $\kappa(t) = \kappa(\Omega_t)$ such that*

$$\overline{I_{2n}^{\frac{1}{2n}}}^{\bullet}(t) + \frac{\lambda(2n-1)}{n^2 \rho\, m(t)\, \kappa(t)} I_{2n}^{\frac{1}{2n}}(t) \leqq \frac{\mu}{2\rho} \Big(\int_{\Omega_t} \rho\, |A_1|^{4n} \, dv \Big)^{\frac{1}{2n}}, \tag{6.40}$$

for positive integer n and where $m(t)$ is a mean value of $\bar{\varepsilon}_\theta(\theta(\cdot, t))$.

Proof. Recalling the fundamental inequality (6.34), using the definition (6.31) of strongly conductive, and observing that $\bar{\varepsilon}_\theta(\cdot) > 0$ and $2n-2$ is even, we may write

$$\begin{aligned} \dot{I}_{2n}(t) \leqq 2n \Big\{ & \frac{\mu}{2} \int_{\Omega_t} (\bar{\varepsilon}(\theta) - \varepsilon^\circ)^{2n-1} |A_1|^2 \, dv \\ & - \lambda(2n-1) \int_{\Omega_t} \bar{\varepsilon}_\theta(\theta) \big(\bar{\varepsilon}(\theta) - \varepsilon^\circ\big)^{2n-2} g \cdot g \, dv \Big\}, \end{aligned} \tag{6.41a}$$

where λ is a fixed number. Now,

$$\begin{aligned} \int_{\Omega_t} \bar{\varepsilon}_\theta(\theta) \big(\bar{\varepsilon}(\theta) - \varepsilon^\circ\big)^{2n-2} g \cdot g \, dv &= \frac{1}{n^2} \int_{\Omega_t} \frac{1}{\bar{\varepsilon}_\theta} |\operatorname{grad} (\bar{\varepsilon}(\theta) - \varepsilon^\circ)^n|^2 \, dv \\ &= \frac{1}{n^2 m(t)} \int_{\Omega_t} |\operatorname{grad} (\bar{\varepsilon}(\theta) - \varepsilon^\circ)^n|^2 \, dv, \end{aligned} \tag{6.41b}$$

for some mean value $m(t)$ of $\bar{\varepsilon}_\theta(\theta(\cdot, t))$. But since, by hypothesis, the function $(\bar{\varepsilon}(\theta(\cdot, t)) - \varepsilon^\circ)^n$ vanishes on a subsurface S_t of $\partial\Omega_t$, we may apply the Poincaré inequality* to find

$$\int_{\Omega_t} \bar{\varepsilon}_\theta(\theta)\left(\bar{\varepsilon}(\theta)-\varepsilon^\circ\right)^{2n-2} g\cdot g\, dv \geqq \frac{1}{n^2 m(t)\, c_P(t)} \int_{\Omega_t} \left(\bar{\varepsilon}(\theta)-\varepsilon\;\right)^{2n} dv = \frac{1}{n^2 \rho m \kappa} I_{2n}(t),$$

where $\kappa(t) \equiv c_P(\Omega_t)$.

Applying this estimate to the second term on the right of (6.41 a) and using, again, Hölder's inequality to estimate the first term on the right of (6.41 a), we find

$$\dot{I}_{2n}(t) \leqq 2n \left\{ \frac{\mu}{2\rho} I_{2n}^{1-\frac{1}{2n}}(t) \left(\int_{\Omega_t} \rho\, |A_1|^{4n} dv\right)^{\frac{1}{2n}} - \frac{\lambda(2n-1)}{n^2 \rho m \kappa} I_{2n}(t) \right\},$$

which gives (6.40) whenever $I_{2n}(t) \neq 0$. If $I_{2n}(\hat{t})=0$ for some $\hat{t}$, then (6.40) is trivially satisfied; *i.e.*, $I_{2n}(\hat{t})=0 \Rightarrow I_{2n}^{\frac{1}{2n}}(\hat{t})=0 \Rightarrow \overset{\cdot}{\overline{I_{2n}^{\frac{1}{2n}}}}(\hat{t})=0$ since $I_{2n}^{\frac{1}{2}}(\cdot)$ has a minimum at $\hat{t}$. △

Before stating our next theorem, we first define a deformation as being *substantially decadent* if there exist two positive numbers, B_1 and B_2, such that

$$\|A_1\|^2(t) \leqq B_1\, e^{-B_2 t}. \tag{6.42}$$

Thus, by Corollary 2 of Theorem 9, *every* deformation of a second grade fluid having $\alpha > 0$ that takes place in a fixed, rigid container will be substantially decadent. Moreover, since we expect that many of the flows contained in the hypothesis of Theorems 8 and 9 will be such as to have bounded Korn and Poincaré coefficients, they too will correspond to substantially decadent deformations. For any such motion we have

Theorem 12. *Let B be any second grade fluid having $\alpha > 0$ and possessing a strongly conductive heat flux vector. Suppose, for $t \geq 0$, B is undergoing a substantially decadent deformation and is immersed in a thermally passive environment with temperature θ°. In addition, let $\theta(X,t)=\theta^\circ$ for all $(X,t)\in S\times[0,\infty)$ for some subsurface of non-zero area measure $S \subseteq \partial\Omega$. Finally, suppose that the Poincaré coefficient $\kappa(\cdot)$ of* (6.40) *is bounded*** *and that either*

(i) *the temperature field is bounded,*

or

(ii) *the function $\bar{\varepsilon}(\cdot)$ is such that*

$$\bar{\varepsilon}_\theta(p) < M \qquad \forall\, p > 0,$$

for some $M > 0$.

*Then, there exist positive constants m_i, $i=1,2,3$, such that for every positive integer n****

$$\begin{aligned} |\bar{\varepsilon}(\theta(\cdot,t))-\varepsilon^\circ|_{2n} \leqq{}& |\bar{\varepsilon}(\theta(\cdot,0))-\varepsilon^\circ|_{2n}\, e^{-\frac{m_1(2n-1)}{n^2}t} \\ &+ \frac{m_2 S^{2-\frac{1}{n}}(t)}{\left(m_1 \frac{2n-1}{n^2} - \frac{m_3}{n}\right)} \left(e^{-\frac{m_3}{n}t} - e^{-\frac{m_1(2n-1)}{n^2}t}\right), \end{aligned} \tag{6.43}$$

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* See (A.1) of the Appendix.

** This is always the case if $S=\partial\Omega$. See (A.2) of the Appendix.

*** Here, we have assumed $\frac{m_1(2n-1)}{n^2} \neq \frac{m_3}{n}$. If this fails, then (6.43) is replaced by an obvious limit formula which we do not record.

where $\varepsilon^\circ \equiv \bar{\varepsilon}(\theta^\circ)$, *where* $|\cdot|_p$ *denotes the* L_p *norm on* Ω *relative to the mass measure, and where*

$$S(t) \equiv \max_{\tau \in [0,t]} s(\tau). \tag{6.44}$$

Proof. Either (i) or (ii) of the hypotheses tell us that $\bar{\varepsilon}_\theta(\theta(\boldsymbol{x},t)) \leqq c_1$ for some $c_1 > 0$. Thus, $m(t)$ of (6.40) meets $m(t) \leqq c_1$, while, by hypothesis, $\kappa(t)$ of (6.40) satisfies $\kappa(t) < c_2$ for some c_2. Thus, we may replace (6.40) by

$$\begin{aligned}\dot{\overline{I_{2n}^{\frac{1}{2n}}}} + \frac{\lambda(2n-1)}{n^2 \rho c_1 c_2} I_{2n}^{\frac{1}{2n}} &\leqq \frac{\mu}{2\rho} \Big(\int_{\Omega_t} \rho |A_1|^{4n} dv\Big)^{\frac{1}{2n}} \\ &\leqq \frac{\mu}{2\rho} \Big(\int_{\Omega} \rho |A_1|^2 s^{4n-2} dv\Big)^{\frac{1}{2n}} \\ &\leqq \frac{\mu}{2\rho} s^{2-\frac{1}{n}}(t) \Big(\int_{\Omega_t} \rho |A_1|^2 dv\Big)^{\frac{1}{2n}} \\ &\leqq \frac{\mu}{2\rho} s^{2-\frac{1}{n}}(t) (\rho B_1)^{\frac{1}{2n}} e^{-\frac{B_2}{2n} t},\end{aligned}$$

where we have used (6.39) and (6.42). Upon integration, we therefore obtain

$$I_{2n}^{\frac{1}{2n}}(t) \leqq I_{2n}^{\frac{1}{2n}}(0)\, e^{-\frac{\lambda(2n-1)}{n^2 \rho c_1 c_2} t} + \frac{\mu S^{2-\frac{1}{n}}(t) (\rho B_1)^{\frac{1}{2n}}}{2\rho \left(\frac{\lambda(2n-1)}{n^2 \rho c_1 c_2} - \frac{B_2}{n}\right)} \Big(e^{-\frac{B_2}{n} t} - e^{-\frac{\lambda(2n-1)}{n^2 \rho c_1 c_2} t}\Big), \tag{6.45}$$

where we have used definition (6.44) and assumed $\frac{\lambda(2n-1)}{n^2 \rho c_1 c_2} \neq \frac{B_2}{n}$. Finally, noting that $I_{2n}^{\frac{1}{2n}}(t) = |\bar{\varepsilon}(\theta(\cdot,t)) - \varepsilon^\circ|_{2n}$, we see that (6.45) is equivalent to (6.43). $\triangle$

Corollary 1. *Let the hypothesis of Theorem* 12 *hold and suppose* $S(\cdot)$ *of* (6.44) *is such that* $S(t) = o(e^{\delta t})$ *for every* $\delta > 0$. *Then*

$$|\bar{\varepsilon}(\theta(\cdot,t)) - \varepsilon^\circ|_p \to 0 \quad \text{as } t \to \infty,$$

for every finite $p \geq 1$. *If* $S(\cdot)$ *is bounded, the convergence is exponential.*

Proof. Since Ω is bounded, an easy application of Hölder's inequality gives $|\cdot|_p \leqq M(\Omega)^{1/p - 1/2n} |\cdot|_{2n}$ for $p < 2n$. If we now use (6.43), the corollary is proven. $\triangle$

When the boundary is insulated, *i.e.*, $\boldsymbol{q} \cdot \boldsymbol{n} = 0$ on $\partial\Omega_t$, the situation is more complicated, principally because we require only that $\tilde{\boldsymbol{q}}(\cdot,\cdot,\cdot,\cdot)$ be strongly conductive. We begin by generalizing Theorem 11.

Theorem 13. *Let B be any second grade fluid whose heat flux vector is strongly conductive, and suppose that B is immersed in a thermally passive environment with temperature* θ° *for* $t \leqq 0$. *Then there exists a domain dependent function*

$\kappa(t)=\kappa(\Omega_t)$ such that

$$\dot I_{2n}(t)+\frac{(2n-1)}{\rho}\left\{\frac{2\lambda}{n\,m(t)\,\kappa(t)}-\frac{\mu}{2\sigma^r}\right\}I_{2n}(t)\leqq\frac{\mu\sigma^s}{2}\int_{\Omega_t}|A_1|^{4n}\,dv+\left\{\frac{2\lambda(2n-1)}{M\,n\,m(t)\,\kappa(t)}\right\}I_n^2(t), \tag{6.46}$$

where $r=\dfrac{2n}{2n-1}$, $s=2n$, $m(t)$ is a mean value of $\bar\varepsilon_\theta(\theta(\cdot,t))$, $\sigma>0$ is arbitrary, and n is a positive integer.

Proof. We begin, as in the proof of Theorem 11, by noting that (6.41 a, b) also apply here. In the present case, however, we do not know the boundary behavior of $\bar\varepsilon(\theta(\cdot,t))-\varepsilon^\circ$, and, thus, may only apply the Poincaré inequality to (6.41 b) in the form (A.1), Case (2),* to find

$$\begin{aligned}\int_{\Omega_t}\bar\varepsilon_\theta(\theta)\big(\bar\varepsilon(\theta)-\varepsilon^\circ\big)^{2n-2}g\cdot g\,dv&\geqq\frac{1}{n^2 m(t)\,c_P(t)}\left\{\int_{\Omega_t}(\bar\varepsilon(\theta)-\varepsilon^\circ)^{2n}\,dv\right.\\&\left.-\frac{1}{V(\Omega_t)}\Big(\int_{\Omega_t}(\bar\varepsilon(\theta)-\varepsilon^\circ)^n\,dv\Big)^2\right\}\geqq\frac{1}{n^2 m\kappa}\left\{\frac{1}{\rho}I_{2n}(t)-\frac{1}{M}I_n^2(t)\right\},\end{aligned} \tag{6.47}$$

where we have set $\kappa(t)\equiv c_P(\Omega_t)$ and used the earlier definition of I_p.

To estimate the first term on the right in (6.41 a), we use Young's inequality in the form $AB\leqq\left(\frac{1}{r}\right)\left(\frac{A}{\sigma}\right)^r+\left(\frac{1}{s}\right)(B\sigma)^s$ for all non-negative A, B, and σ, and for all r and s meeting $r>1$, $\frac{1}{s}=1-\frac{1}{r}$. Thus, for positive integer n we may write

$$\begin{aligned}\int_{\Omega_t}(\bar\varepsilon(\theta)-\varepsilon^\circ)^{2n-1}|A_1|^2\,dv&\leqq\int_{\Omega_t}|\bar\varepsilon(\theta)-\varepsilon^\circ|^{2n-1}|A_1|^2\,dv\\&\leqq\frac{(2n-1)}{2n\sigma^r}\int_{\Omega_t}(\bar\varepsilon(\theta)-\varepsilon^\circ)^{2n}\,dv+\frac{\sigma^s}{2n}\int_{\Omega_t}|A_1|^{4n}\,dv,\\&\leqq\frac{(2n-1)}{2n\rho\sigma^r}I_{2n}(t)+\frac{\sigma^s}{2n}\int_{\Omega_t}|A_1|^{4n}\,dv,\end{aligned} \tag{6.48}$$

where $r=\dfrac{2n}{2n-1}$, $s=2n$, and σ is arbitrary.

Finally, entering (6.47) and (6.48) into (6.41 a) we reach

$$\dot I_{2n}(t)+\frac{(2n-1)}{\rho}\left\{\frac{2\lambda}{n\,m(t)\,\kappa(t)}-\frac{\mu}{2\sigma^r}\right\}I_{2n}(t)\leqq\frac{\mu\sigma^s}{2}\int_{\Omega_t}|A_1|^{4n}\,dv+\frac{2\lambda(2n-1)}{M\,n\,m(t)\,\kappa(t)}I_n^2(t),$$

which is equivalent to (6.46). △

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* In order to meet the condition (A.3) of the Appendix we take $w(\cdot)$ to be the function

$$(\bar\varepsilon(\theta)-\varepsilon^\circ)^n-\frac{1}{V(\Omega_t)}\int_{\Omega_t}(\bar\varepsilon(\theta)-\varepsilon^\circ)^n\,dv.$$

Before stating our next theorem we first introduce the following definition: Given a constant vector $\boldsymbol{c}$, a deformation is *substantially decadent with terminal velocity* $\boldsymbol{c}$ if, in addition to (6.42), there exists a positive constant B_3 such that

$$0 \leqq \|\boldsymbol{v}-\boldsymbol{c}\|^2(t) \leqq B_3 \|\boldsymbol{A}_1\|^2(t). \tag{6.49}$$

Again, we remark that many of the flows covered by the hypothesis of Theorems 8 and 9 may be expected to be substantially decadent with the terminal velocity $\boldsymbol{c}(=\dot{\boldsymbol{x}}_0$ for Theorem 8* and $=0$ for Theorem 9). In any case, Corollary 2 of Theorem 9 provides many such motions corresponding to the choice $\boldsymbol{c}=0$.

Now suppose that a body, isolated against any energy gain or loss, reaches a terminal velocity $\boldsymbol{c}$ having undergone a substantially decadent deformation. In this case the number

$$\tfrac{1}{2}\int_{\Omega_0} \rho |\boldsymbol{v}|^2(\cdot,0)\,dv + \int_{\Omega_0} \rho\tilde{\varepsilon}(\theta(\cdot,0), \boldsymbol{A}_1(\cdot,0))\,dv - \tfrac{1}{2}M|\boldsymbol{c}|^2$$

would correspond to the amount of energy that would go toward heating the body, and, indeed, balance of energy (2.4) yields

$$\begin{aligned}\int_{\Omega_\infty} \rho\tilde{\varepsilon}(\theta(\cdot,\infty),0)\,dv = &\int_{\Omega_0} \rho\tilde{\varepsilon}(\theta(\cdot,0),\boldsymbol{A}_1(\cdot,0))\,dv \\ &+\tfrac{1}{2}\int_{\Omega_0}\rho|\boldsymbol{v}|^2(\cdot,0)\,dv - \tfrac{1}{2}M|\boldsymbol{c}|^2\end{aligned} \tag{6.50$_1$}$$

assuming both mechanical isolation and $\tilde{\boldsymbol{q}}\cdot\boldsymbol{n}=r=0$. Thus, if the body reaches a uniform temperature at $t=\infty$, we expect it to be the number

$$\bar{\varepsilon}^{-1}\left(\frac{1}{M}\int_{\Omega_\infty}\rho\tilde{\varepsilon}(\theta(\cdot,\infty),0)\,dv\right), \tag{6.50$_2$}$$

since $\tilde{\varepsilon}(\theta,0)=\bar{\varepsilon}(\theta)$.

The above remarks should serve to motivate

Theorem 14. *Let B be any second grade fluid with $\alpha>0$ which possesses a strongly conductive heat flux vector, and which satisfies*

$$\bar{\varepsilon}(p)\to\infty \quad \text{as } p\to\infty. \tag{6.51}$$

Suppose, in addition, that for $t\geqq 0$ B is (1) *mechanically isolated,* (2) *undergoing a substantially decadent deformation with a terminal velocity $\boldsymbol{c}$ which is either zero, or equal to $\dot{\boldsymbol{x}}_0$ if $\dot{\boldsymbol{x}}_0$ is known to be constant, and* (3), *satisfies $\boldsymbol{q}\cdot\boldsymbol{n}=0$ on $\partial\Omega_t\times[0,\infty)$ and $r=0$ in $\Omega_t\times[0,\infty)$. Finally, suppose there exist constants c_1, c_2 such that $0<c_1\leqq m(t)\kappa(t)\leqq c_2$ with $m(t)\kappa(t)$ as in* (6.46).*** *Then, if $S(\cdot)$ of* (6.44) *is such that $S(t)=o(e^{\delta t})$ for every $\delta>0$, one has*

$$|\bar{\varepsilon}(\theta(\cdot,t))-\varepsilon^{\infty}|_p\to 0 \quad \text{as } t\to\infty, \tag{6.52}$$

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* The hypothesis of zero net load in Theorem 8 is equivalent to $\dot{x}_0=$const.

** These hypotheses on $m(\cdot)$ and $\kappa(\cdot)$ are slightly stronger than those required in the statement of Theorem 12.

for every finite $p \geqq 1$. If $S(\cdot)$ is bounded, then the convergence is exponential and, in particular, if $p=2$,

$$|\bar{\varepsilon}(\theta(\cdot,t))-\varepsilon^\circ|_2^2 \leqq |\bar{\varepsilon}(\theta(\cdot,0))-\varepsilon^\circ|_2^2 e^{-m_1 t}+m_2 S(t)|e^{-m_3 t}-e^{-m_1 t}| + m_4|e^{-2m_3 t}-e^{-m_1 t}|, \tag{6.53}$$

where m_i, $i=1,2,3,4$, are positive constants. Here $\varepsilon^\circ \equiv \bar{\varepsilon}(\theta^\circ)$ and θ° is the number of $(6.50)_2$.

Proof. We shall establish (6.53), which then gives (6.52) when $p=2$. The general case for arbitrary p will then follow by induction and by use of an estimate equivalent to that used in the proof of Corollary 1 to Theorem 12.

The hypotheses here are sufficient to guarantee the validity of Theorem 13. Thus using the fact that $m(t)\kappa(t)$ in (6.46) satisfies $0<c_1 \leqq m(t)\kappa(t) \leqq c_2$, we may, taking $n=1$, replace (6.46) by

$$\dot{I}_2(t)+\frac{1}{\rho}\left\{\frac{2\lambda}{c_2}-\frac{\mu}{2\sigma^2}\right\} I_2(t) \leqq \frac{\mu\sigma^2}{2}\int_{\Omega_t}|A_1|^4\,dv+\frac{2\lambda}{Mc_1} I_1^2(t),$$

where $\sigma>0$ is an arbitrary number. So as to insure that the coefficient of $I_2(t)$ is positive, we take $\sigma \equiv \sqrt{\dfrac{\mu c_2}{2\lambda}}$. Thus, recalling the definition of $s(\cdot)$ in (6.39), we may write

$$\dot{I}_2(t)+\frac{\lambda}{\rho c_2} I_2(t) \leqq \frac{\mu^2 c_2}{4\lambda} s^2(t)\,\|A_1\|^2(t)+\frac{2\lambda}{Mc_1} I_1^2(t). \tag{6.54}$$

We now estimate $I_1^2(\cdot)$. To do this we note that, since $\boldsymbol{q}\cdot\boldsymbol{n}=0$ on $\partial\Omega_t$ and $r=0$ in Ω_t, the fluid is, trivially, immersed in a thermally passive environment with constant temperature θ° *for any number* θ°. That is, (6.54) remains true independent of the value of θ°. We shall select this number so as to have $I_1^2(\cdot)$ bounded above by a decreasing exponential. Toward this end, we turn to the energy equation (2.4) and observe that since B is mechanically isolated and $\boldsymbol{q}\cdot\boldsymbol{n}=r=0$, then use of $(6.1)_3$ yields

$$\int_{\Omega_t}\rho\left(\bar{\varepsilon}(\theta)+\frac{\alpha}{4\rho}|A_1|^2+\frac{1}{2}|\boldsymbol{v}|^2\right)dv = \int_{\Omega_0}\rho\left(\bar{\varepsilon}(\theta(\cdot,0))+\frac{\alpha}{4\rho}|A_1|^2(\cdot,0)+\frac{1}{2}|\boldsymbol{v}|^2(\cdot,0)\right)dv.$$

Thus it follows that

$$\begin{aligned}
&\int_{\Omega_t}\rho\left\{\bar{\varepsilon}(\theta(\cdot,t))-\left[\frac{1}{M}\int_{\Omega_0}\rho\left(\bar{\varepsilon}(\theta(\cdot,0))+\frac{\alpha}{4\rho}|A_1|^2(\cdot,0)+\frac{1}{2}|\boldsymbol{v}|^2(\cdot,0)\right)dv-\frac{1}{2}|\boldsymbol{c}|^2\right]\right\}dv\\
&\quad=-\frac{\rho}{2}\left\{\int_{\Omega_t}|\boldsymbol{v}|^2\,dv+\frac{\alpha}{2\rho}\int_{\Omega_t}|A_1|^2\,dv\right\}+\frac{1}{2}M|\boldsymbol{c}|^2\\
&\quad=-\frac{\rho}{2}\left\{\|\boldsymbol{v}-\boldsymbol{c}\|^2(t)+\frac{\alpha}{2\rho}\|A_1\|^2(t)\right\}
\end{aligned} \tag{6.55}$$

for either $\boldsymbol{c}=\dot{\boldsymbol{x}}_0$ or $\boldsymbol{c}=0$.

Now, by $(6.50)_1$ and (6.51) the number ζ given by

$$\zeta \equiv \frac{1}{M} \int_{\Omega_0} \rho \left(\bar{\varepsilon}(\theta(\cdot,0)) + \frac{\alpha}{4\rho} |A_1|^2(\cdot,0) + \frac{1}{2} |v|^2(\cdot,0) \right) dv - \frac{1}{2} |c|^2 \qquad (6.56)_1$$

is in the domain of $\bar{\varepsilon}^{-1}(\cdot)$, and so the number

$$\theta^\circ \equiv \bar{\varepsilon}^{-1}(\zeta) \qquad (6.56)_2$$

is well defined and allows us to write (6.55) as

$$I_1(t) \equiv \int_{\Omega_t} \rho(\bar{\varepsilon}(\theta(\cdot,t)) - \varepsilon(\theta^\circ))\, dv = -\frac{\rho}{2} \left\{ \|v - c\|^2 + \frac{\alpha}{2\rho} \|A_1\|^2 \right\}.$$

However, by hypothesis the deformation is substantially decadent with terminal velocity $\dot{x}_0$ or zero (*i.e.*, $c = \text{const.}$), and thus, with the aid of (6.42) and (6.49), we reach

$$I_1^2(t) = \left(\frac{\rho}{2}\right)^2 \left\{ \|v - c\|^2 + \frac{\alpha}{2\rho} \|A_1\|^2 \right\}^2 \leqq B_4 e^{-2B_2 t}, \qquad (6.57)$$

for some $B_4 \geqq 0$. This is the desired estimate for $I_1^2(\cdot)$.

Entering (6.57) into (6.54) and again using the estimate (6.42), we find

$$\dot{I}_2(t) + \left(\frac{\lambda}{\rho c_2}\right) I_2(t) \leqq \left(\frac{\mu^2 c_2}{4\lambda}\right) s^2(t) B_1 e^{-B_2 t} + \frac{2\lambda}{M c_1} B_4 e^{-2B_2 t},$$

and, upon integration,

$$\begin{aligned} I_2(t) \leqq I_2(0) e^{-\frac{\lambda}{\rho c_2} t} &+ \frac{\rho c_2^2 B_1 \mu^2}{4\lambda(\lambda - \rho c_2 B_2)} S^2(t) \{ e^{-B_2 t} - e^{-\frac{\lambda}{\rho c_2} t} \} \\ &+ \frac{2\lambda B_4 \rho c_2}{M c_1 (\lambda - 2\rho c_2 B_2)} \{ e^{-2B_2 t} - e^{-\frac{\lambda}{\rho c_2} t} \}, \end{aligned} \qquad (6.58)$$

where we have used (6.44), and have assumed without loss of generality that $\rho c_2 B_2 \neq \lambda \neq 2\rho c_2 B_2$. This establishes (6.53).

Finally, to reach (6.52), we need only note that (6.46) is a family of differential inequalities which provides a bound on I_{4n} in terms of I_{2n} for positive integer n and, with (6.53), we have started the induction. The function $S(\cdot)$, being monotone can always be brought out of any finite time integrals and thus only powers of it will multiply differences of exponentials. △

It is worth noting that while the hypotheses of Theorems 12 and 14 included $\alpha > 0$, such an assumption was at no time used in their proofs. Thus, these two theorems apply to any second grade fluid provided the remaining hypotheses are satisfied. In particular, they apply to a Navier-Stokes fluid if we know that the flows are substantially decadent (with terminal velocity $\dot{x}_0$ or 0). However, we emphasize that in the present work we have only been able to prove that substantially decadent flows can occur in those second grade fluids having $\alpha > 0$.

7. Unboundedness and Instability: $\alpha<0$

In this section we give an analysis of the velocity fields that arise out of (6.4) and (6.5) when $\alpha<0$. Although we shall deal first with quite general flows, our most concrete results will concern flows that take place in a fixed, rigid container and meet the usual adherence condition, i.e., $\Omega_t=\Omega_0=\Omega$ and $\boldsymbol{v}(\cdot,t)\equiv 0$ on $\partial\Omega$; for the sake of brevity we call this *canister flow*. Thus, if we suppose B to be mechanically isolated*, we shall be concerned with velocity fields $\boldsymbol{v}(\cdot,\cdot)$ that, by Theorem 7, satisfy

$$\overline{\|\boldsymbol{v}\|^2}^{\bullet}(t)+\frac{\mu}{\rho}\|A_1\|^2(t)-\frac{|\alpha|}{2\rho}\overline{\|A_1\|^2}^{\bullet}(t)=0, \tag{7.1$_1$}$$

where we have written $-|\alpha|$ for α. In the following analysis it is convenient to introduce the number $N(t)$ defined according to

$$N(t)\equiv\frac{|\alpha|}{2\rho}\|A_1\|^2(t)-\|\boldsymbol{v}\|^2(t), \tag{7.2$_1$}$$

and having the interpretation of the excess of the averaged stretching over the kinetic energy in the fluid.

In the special case of canister flow the equations $(7.1)_1$ and $(7.2)_1$ have the equivalent forms**

$$\overline{\|\boldsymbol{v}\|^2}^{\bullet}(t)+\frac{2\mu}{\rho}\|\operatorname{grad}\boldsymbol{v}\|^2(t)-\frac{|\alpha|}{\rho}\overline{\|\operatorname{grad}\boldsymbol{v}\|^2}^{\bullet}(t)=0 \tag{7.1$_2$}$$

and

$$N(t)\equiv\frac{|\alpha|}{\rho}\|\operatorname{grad}\boldsymbol{v}\|^2(t)-\|\boldsymbol{v}\|^2(t). \tag{7.2$_2$}$$

The striking difference between fluids for which $\alpha<0$ and those for which $\alpha\geqq 0$ can begin to be seen from

Theorem 15. *Let $\boldsymbol{v}(\cdot,\cdot)$ be a velocity field satisfying* $(7.1)_1$ *and let $N(\cdot)$ be as in* $(7.2)_1$. *Then $N(\cdot)$ is a non-decreasing function of time,*

$$\dot N(t)\geqq 0, \tag{7.3}$$

bounded below according to

$$N(t)\geqq N(0)\,e^{\frac{2\mu}{|\alpha|}t}. \tag{7.4}$$

Proof. Using the definition $(7.2)_1$, we may write $(7.1)_1$ as

$$\dot N(t)-\frac{\mu}{\rho}\|A_1\|^2(t)=0, \tag{7.5}$$

and so establish (7.3). If we now add $\frac{2\mu}{|\alpha|}\|\boldsymbol{v}\|^2$ to (7.5) and again use $(7.2)_1$, we find

$$\dot N(t)-\frac{2\mu}{|\alpha|}N(t)=\frac{2\mu}{|\alpha|}\|\boldsymbol{v}\|^2(t)\geqq 0, \tag{7.6}$$

which establishes (7.4). $\triangle$

---

* As observed earlier, if the fluid occupies a fixed, rigid container to the walls of which it adheres, then it will always be mechanically isolated if the body force is conservative.

** To see this it suffices to note that $2\|\operatorname{grad}\boldsymbol{v}\|^2=\|A_1\|^2$, which follows from the divergence theorem, the condition $\operatorname{div}\boldsymbol{v}=0$, and the fact that $\boldsymbol{v}(\cdot,t)$ vanishes on $\partial\Omega$.

Corollary 1. *If $N(0)>0$ or, indeed, if $N(t')>0$ for any t', then both $N(t)$ and the averaged stretching, $\|A_1\|^2(t)$ must grow exponentially as $t\to\infty$. In particular, for $N(0)>0$, one has*

$$\frac{|\alpha|}{2\rho}\|A_1\|^2(t)\geqq\|v\|^2(t)+N(0)\,e^{\frac{2\mu}{|\alpha|}t}\geqq N(0)\,e^{\frac{2\mu}{|\alpha|}t}. \tag{7.7}$$

Proof. The only item not a direct consequence of (7.4), $(7.2)_1$, and the hypothesis is the observation concerning $N(t')$; this follows by integrating (7.6) from t' to t. △

Actually, Corollary 1 can be strengthened slightly:

Corollary 2. *If $N(0)\geqq 0$ and $v(\cdot,0)\not\equiv 0$ or, indeed, if $N(t')\geqq 0$ and $v(\cdot,t')\not\equiv 0$, then both $N(t)$ and the averaged stretching $\|A_1\|^2(t)$ grow exponentially as $t\to\infty$. In particular, for $N(0)=0$ and $v(\cdot\,0)\not\equiv 0$ one has that for any $\delta>0$ there exists an $N_\delta>0$ such that*

$$N(t)\geqq N_\delta\, e^{\frac{2\mu}{|\alpha|}t}\qquad\forall\, t>\delta. \tag{7.8}$$

Proof. It suffices to prove (7.8). Thus, let $\delta>0$ be given and observe that since $v(\cdot,0)\not\equiv 0$ and $N(0)=0$ we have $\|A_1\|^2(0)>0$ and so, by (7.5), $\dot N(0)>0$. Therefore, there exists a $\hat\delta\in(0,\delta)$ such that $N(\hat\delta)>0$. If we now integrate (7.6) from $\hat\delta$ to t we have (7.8) with $N_\delta\equiv N(\hat\delta)\,e^{-\frac{2\mu}{|\alpha|}\hat\delta}$. △

In words, Theorem 15 and its corollaries state that both the averaged stretching and its excess over the kinetic energy will increase without bound if either (i) the averaged stretching exceeds, by any amount whatsoever, the kinetic energy at an instant or, (ii) the averaged stretching merely equals the kinetic energy at an instant when there is any internal motion.

Since the "unpleasant" conclusions of Theorem 15 and its corollaries require that $N(0)\geqq 0$, it might be hoped that one could show that $N(0)$ must be negative. However, this is not generally true and, indeed, for certain canister flows it can never be true. In this regard, we have

Theorem 16. *Let $v(\cdot,\cdot)$ be a velocity field satisfying $(7.1)_2$ inside a fixed rigid container Ω, let $N(\cdot)$ be as in $(7.2)_2$, and let Ω be any container whose Poincaré coefficient, $\kappa(\Omega)\equiv c_P(\Omega)$ satisfies* $\kappa\leqq\dfrac{|\alpha|}{\rho}$. Then,*

(i) *$N(0)\geqq 0$ for every field $v(\cdot,0)$.*

Moreover, if Ω is such that $\kappa(\Omega)<\dfrac{|\alpha|}{\rho}$, then

(ii) $v(\cdot,0)\not\equiv 0\Rightarrow N(0)>0$.

* Since $c_P(\Omega)\leqq\left(\frac{1}{\pi}\right)^2\left(\frac{3}{4\pi}\right)^{\frac{2}{3}}V(\Omega)^{\frac{2}{3}}$ (see (A.2) of the Appendix) where $V(\Omega)$ is the volume of Ω, the condition $\kappa\leqq|\alpha|/\rho$ will always be satisfied for "small enough" containers (which may be quite large!).

Proof. The Poincaré inequality asserts that $\kappa\|\operatorname{grad} \boldsymbol{w}\|^2 - \|\boldsymbol{w}\|^2 \geqq 0$ for all smooth vector fields $\boldsymbol{w}(\cdot)$ on Ω that vanish on $\partial\Omega$. Thus, for any $t \geqq 0$, we have

$$\begin{aligned} N(t) &= \frac{|\alpha|}{\rho}\|\operatorname{grad} \boldsymbol{v}\|^2(t) - \|\boldsymbol{v}\|^2(t), \\ &= \left(\frac{|\alpha|}{\rho} - \kappa\right)\|\operatorname{grad} \boldsymbol{v}\|^2(t) + \kappa\|\operatorname{grad} \boldsymbol{v}\|^2(t) - \|\boldsymbol{v}\|^2(t) \geqq 0, \end{aligned} \tag{7.9}$$

since $\kappa \leqq \frac{|\alpha|}{\rho}$. Moreover, if $\kappa < \frac{|\alpha|}{\rho}$, it is clear that equality can hold only if $\boldsymbol{v}(\cdot, t) \equiv 0$. The proof is complete. △

We remark that it is possible to generalize Theorem 16 so as to apply to general flows occurring under conditions of isolation but not necessarily in a fixed, rigid container. To do this we must restrict the initial size and shape of the fluid mass and subtract out the rigid motion which would now be permitted of the fluid. We omit carrying out the details of this generalization but observe that they are similar to those encountered in Theorems 8 and 9.

We now consider another method of insuring that $N(0)$ is non-negative, and so obtain an instability theorem relative to the rest state for *any* canister flow.

Theorem 17. *For any $\varepsilon > 0$ there exists $\boldsymbol{v}_0(\cdot)$: $\Omega \to V$ with $|\boldsymbol{v}_0(\boldsymbol{x})| \leqq \varepsilon$ for all $\boldsymbol{x} \in \Omega$ and such that any velocity field $\boldsymbol{v}(\cdot, \cdot)$, possessing $\boldsymbol{v}_0(\cdot)$ as initial data and satisfying $(7.1)_2$ inside the fixed, rigid container Ω, has $N(t)$ and $\|\operatorname{grad} \boldsymbol{v}\|^2(t)$ growing exponentially as $t \to \infty$.**

Proof.** Theorem 15 and its corollaries tell us that we need only exhibit a $\boldsymbol{v}_0(\cdot)$ satisfying both $|\boldsymbol{v}_0(\boldsymbol{x})| < \varepsilon$ and

$$N(0) \equiv \frac{|\alpha|}{\rho}\|\operatorname{grad} \boldsymbol{v}_0\|^2 - \|\boldsymbol{v}_0\|^2 \geqq 0.$$

Since $\frac{|\alpha|}{\rho} > 0$, it is easy to find a smooth $\mathring{\boldsymbol{w}}(\cdot)$ which vanishes on $\partial\Omega$ and which is such that***

$$\frac{\|\mathring{\boldsymbol{w}}\|^2}{\|\operatorname{grad} \mathring{\boldsymbol{w}}\|^2} < \frac{|\alpha|}{\rho}.$$

Thus for given $\varepsilon > 0$, we define $\boldsymbol{v}_0(\cdot)$ through

$$\boldsymbol{v}_0(\boldsymbol{x}) \equiv \frac{\varepsilon\mathring{\boldsymbol{w}}(\boldsymbol{x})}{\max\limits_{\boldsymbol{x} \in \Omega} |\mathring{\boldsymbol{w}}|(\boldsymbol{x})}$$

* We assume the existence of $\boldsymbol{v}(\cdot, \cdot)$ corresponding to the initial data $\boldsymbol{v}_0(\cdot)$. That such solutions do exist, at least in the special case of flow between parallel plates, is exhibited in the work of Coleman, Duffin & Mizel [6]. See also our Theorem 21.

** Of course, for canisters whose Poincaré coefficient meets $\kappa(\Omega) \leqq \frac{|\alpha|}{\rho}$, we have already established the far stronger result of Theorem 16.

*** See equation (7.17) and the two paragraphs following it as to how this could be done. In the notation used there, we could in fact select for $\mathring{\boldsymbol{w}}(\cdot)$ any function in S_n where n is such that $\kappa_n < \frac{|\alpha|}{\rho}$.

and observe that $|\boldsymbol{v}_0(\boldsymbol{x})| \leq \varepsilon$, and that

$$\frac{\|\boldsymbol{v}_0\|^2}{\|\operatorname{grad} \boldsymbol{v}_0\|^2} < \frac{|\alpha|}{\rho}.$$

Therefore, $N(0) \equiv \frac{|\alpha|}{\rho} \|\operatorname{grad} \boldsymbol{v}_0\|^2 - \|\boldsymbol{v}_0\|^2 > 0$ and the theorem is proven. △

We have shown in Theorems 15 and 16 that in a "small enough" canister $\left(i.e.,\ \kappa(\Omega) \leqq \frac{|\alpha|}{\rho}\right)$ *every* flow of a second grade fluid having $\alpha < 0$ will be such that both its averaged stretching and the excess of its averaged stretching over its kinetic energy grow unbounded, *at least* exponentially, with time. The fact that this growth can be *at most* exponential is the content of

Theorem 18. *Let $\boldsymbol{v}(\cdot,\cdot)$ be a velocity field satisfying $(7.1)_2$ inside a fixed, rigid container Ω. Let $N(\cdot)$ be as in $(7.2)_2$, and let Ω be such that the Poincaré coefficient $\kappa(\Omega) \equiv c_P(\Omega)$ satisfies $\kappa(\Omega) < \frac{|\alpha|}{\rho}$. Then*

$$0 \leqq N(0)\, e^{\frac{2\mu}{|\alpha|} t} \leqq N(t) \leqq N(0)\, e^{\frac{2\mu}{|\alpha| - \rho\kappa} t}, \tag{7.10}$$

$$0 \leqq \frac{\rho}{|\alpha|} N(0)\, e^{\frac{2\mu}{|\alpha|} t} \leqq \|\operatorname{grad} \boldsymbol{v}\|^2(t) \leqq \frac{\rho}{|\alpha| - \rho\kappa} N(0)\, e^{\frac{2\mu}{|\alpha| - \rho\kappa} t}, \tag{7.11}$$

where equality holds in $(7.10)_1$ and $(7.11)_1$ only if $\boldsymbol{v}(\cdot, 0) \equiv 0$.

Proof. By the Poincaré inequality we have

$$(|\alpha| - \rho\kappa) \|\boldsymbol{v}\|^2 \leqq \rho\kappa \left(\frac{|\alpha|}{\rho} \|\operatorname{grad} \boldsymbol{v}\|^2 - \|\boldsymbol{v}\|^2\right),$$

and since $\kappa < \frac{|\alpha|}{\rho}$ this yields

$$\|\boldsymbol{v}\|^2(t) \leqq \frac{\rho\kappa}{|\alpha| - \rho\kappa} N(t).$$

Thus (7.6) yields

$$\dot{N}(t) - \frac{2\mu}{|\alpha|} N(t) \leqq \frac{2\mu}{|\alpha|} \frac{\rho\kappa}{|\alpha| - \rho\kappa} N(t),$$

or, equivalently,

$$\dot{N}(t) - \frac{2\mu}{|\alpha| - \rho\kappa} N(t) \leqq 0,$$

which, by integration, establishes $(7.10)_3$. To establish $(7.11)_3$, we need merely note that, by (7.9),

$$\|\operatorname{grad} \boldsymbol{v}\|^2 \leqq \frac{\rho}{|\alpha| - \rho\kappa} N.$$

Finally, since for canister flow $2\, \|\operatorname{grad} \boldsymbol{v}\|^2 = \|A_1\|^2$, we see that $(7.10)_{1,2}$ and $(7.11)_{1,2}$ are direct consequences of Theorem 15, its corollaries, and Theorem 16. △

As a result of the above work we know that when $\alpha<0$, canister flows that decay in time are to be found only in those canisters having $\kappa(\Omega)>\frac{|\alpha|}{\rho}$ and, in such canisters, only among those flows that have initial data satisfying $N(0)<0$. Indeed, by Corollary 1 of Theorem 15, we know a little more: any flows in these canisters that decay must not only have $N(0)<0$ but also $N(t)\leqq 0$ for all t with $N(t')=0$ only for those times t' when $\boldsymbol{v}(\cdot,t')\equiv 0$.

However, it might appear that the flow need not decay even when $\kappa>\frac{|\alpha|}{\rho}$ and $N(\cdot)<0$; that this is impossible is established in

Theorem 19. *Let $\boldsymbol{v}(\cdot,\cdot)$ be a velocity field satisfying* (7.1) *inside a fixed, rigid container Ω. Let $N(\cdot)$ be as in* $(7.2)_2$, *and let Ω be such that its Poincaré coefficient satisfies $\kappa(\Omega)>\frac{|\alpha|}{\rho}$. Then, $N(\cdot)$ is bounded below according to*

$$N(t)\geqq N(0)\,e^{\frac{-2\mu}{\rho\kappa-|\alpha|}t}. \tag{7.12}$$

Proof. It is similar to Theorem 18. △

Of course, Theorem 19 is superfluous if $N(0)\geqq 0$ since then Theorem 15 gives a far stronger result. If $N(0)<0$, however, Theorem 19 implies that any such solution must either (i) decay (*i.e.*, $\inf_{t=[0,\infty)}|N(t)|=0=\lim_{t\to\infty}N(t)$, $N(\cdot)\leqq 0$), or (ii) become positive at some time and, as we know, then grow unbounded. When decaying solutions exist, their rate of decay is described by

Theorem 20. *Let $\boldsymbol{v}(\cdot,\cdot)$ be a velocity field for which $N(\cdot)$ as given in* $(7.2)_2$ *decays. Then*

$$\begin{gathered}\|\boldsymbol{v}\|^2(t)=o\left(e^{\left(\delta-\frac{2\mu}{\rho\kappa-|\alpha|}\right)t}\right),\\ \|\operatorname{grad}\boldsymbol{v}\|^2(t)=o\left(e^{\left(\delta-\frac{2\mu}{\rho\kappa-|\alpha|}\right)t}\right),\end{gathered} \tag{7.13}$$

for any $\delta\in\left(0,\frac{2\mu}{\rho\kappa-|\alpha|}\right)$.

Proof. From (7.6) we have

$$N(t)\,e^{-\frac{2\mu}{|\alpha|}t}=N(0)+\frac{2\mu}{|\alpha|}\int_0^t e^{-\frac{2\mu}{|\alpha|}s}\|\boldsymbol{v}\|^2(s)\,ds. \tag{7.14}$$

Recalling from (7.3) that $N(\cdot)$ is monotone, and using the assumption that $N(\cdot)$ decays, we may take the limit as $t\to\infty$ to find

$$N(0)=-\frac{2\mu}{|\alpha|}\int_0^\infty e^{-\frac{2\mu}{|\alpha|}s}\|\boldsymbol{v}\|^2(s)\,ds.$$

Thus (7.14) may be rewritten as

$$N(t)=-\frac{2\mu}{|\alpha|}e^{\frac{2\mu}{|\alpha|}t}\int_t^\infty e^{-\frac{2\mu}{|\alpha|}s}\|\boldsymbol{v}\|^2(s)\,ds\geqq N(0)\,e^{-\frac{2\mu t}{\rho\kappa-|\alpha|}},$$

where the inequality follows from (7.12). Consequently,

$$0 \geqq -\frac{\frac{2\mu}{|\alpha|}\int_t^\infty e^{-\frac{2\mu}{|\alpha|}s}\|v\|^2(s)\,ds}{e^{-\frac{2\mu}{|\alpha|}t}\,e^{\left(\delta-\frac{2\mu}{\rho\kappa-|\alpha|}\right)t}} \geqq N(0)\,e^{-\delta t},$$

where we restrict δ to be any number in the domain $\left(0, \frac{2\mu}{\rho\kappa-|\alpha|}\right)$. It is seen that not only must the above quotient have the limit zero as $t\to\infty$, but also both its numerator and denominator must take on zero in this limit. Thus, L'Hospital's rule yields

$$\frac{\|v\|^2(t)}{e^{\left(\delta-\frac{2\mu}{\rho\kappa-|\alpha|}\right)t}} \to 0 \quad \text{as } t\to\infty,$$

which proves $(7.13)_1$. To obtain $(7.13)_2$, we recall that

$$N(t)\leqq 0 \Rightarrow \frac{|\alpha|}{\rho}\,\|\operatorname{grad} v\|^2(t) \leqq \|v\|^2(t),$$

and then apply $(7.13)_1$. $\triangle$

The last two theorems gave some of the properties which decaying velocity fields for canister flows will possess, provided they exist. In the work of COLEMAN, DUFFIN & MIZEL [6, §4] the complete structure of such a decaying velocity field is exhibited explicitly for the case of simple shearing flow between two fixed parallel plates.* The structure of their solution also serves to indicate that the results of (7.13) for arbitrary containers cannot, in general, be improved. In particular, one cannot set $\delta=0$ in (7.13).

We have seen that the initial condition $N(0)\geqq 0$ is critical for flows of those second grade fluids which have $\alpha<0$. In the remainder of this section we give an interpretation of this condition using methods that will be essential in the analysis of Section 8.

From (A.1) and (A.5) of the Appendix we know that $\kappa(\Omega)(\equiv c_P(\Omega))$ may be defined by

$$\kappa(\Omega) \equiv \sup_{w\in S_\Omega} \frac{\|w\|^2}{\|\operatorname{grad} w\|^2}, \tag{7.15}$$

where S_Ω is the set of all smooth vector fields $w(\cdot)\colon \Omega\to V$, vanishing on $\partial\Omega$, not identically zero in Ω, and having $\operatorname{div} w=0$ throughout Ω. Using standard methods from the calculus of variations we obtain the following eigenvalue problem associated with (7.15):

$$\kappa_1 \Delta w^1 + w^1 = \operatorname{grad}\lambda^1 \quad \text{in } \Omega, \tag{7.16}$$

where we have written κ_1 for $\kappa(\Omega)$, and where $w^1(\cdot)\in S_\Omega$, the solution of (7.16), maximizes the ratio appearing in (7.15) in the sense that

$$\sup_{w\in S_\Omega} \frac{\|w\|^2}{\|\operatorname{grad} w\|^2} = \frac{\|w^1\|^2}{\|\operatorname{grad} w^1\|^2} = \kappa_1.$$

* In our Theorem 22, we obtain the complete and explicit structure of decaying solutions for a problem class of sufficient generality to include the work of COLEMAN, DUFFIN & MIZEL.

We assume, without loss of generality, that w^1 is normalized according to $\|w^1\|=1$. The scalar field $\lambda^1(\cdot)$ is introduced by the divergence constraint, and Δ is the spatial Laplacian operator.

Having so defined the pair (κ_1, w^1), we may analogously and inductively define for every integer n the pair (κ_n, w^n) according to

$$\begin{gathered}\kappa_n \equiv \sup_{w\in S_n} \frac{\|w\|^2}{\|\operatorname{grad} w\|^2} = \frac{\|w^n\|^2}{\|\operatorname{grad} w^n\|^2},\\ \kappa_n \Delta w^n + w^n = \operatorname{grad} \lambda^n \quad \text{in } \Omega, \quad w^n \in S_n,\\ S_n \equiv \{w(\cdot) \mid w(\cdot)\in S_\Omega;\ \langle w\cdot w^i\rangle = 0 \text{ for } i=1,2,\dots n-1\},\end{gathered} \tag{7.17}$$

where w^n is assumed to be normalized by $\|w^n\|=1$. We note that $S_1 = S_\Omega$.

It is well known and easily seen from $(7.17)_{1,3}$ that $\kappa_{n+1}\leqq\kappa_n$. Further, by use of the min-max characterization [27] for κ_n, it follows that $\kappa_n(\Omega')\leqq\kappa_n(\Omega'')$ for any two canisters (*i.e.*, domains) Ω' and Ω'' which satisfy $\Omega'\subseteqq\Omega''$.*

Finally, we remark that the sequence $\{\kappa_n\}$ can be shown to converge to zero as $n\to\infty$, and that the set of functions w^1, w^2... is complete and orthonormal on S_Ω; *i.e.*, for any $w\in S_\Omega$ one has

$$w = \sum_{i=1}^{\infty} \langle w\cdot w^i\rangle\, w^i. \tag{7.18}$$

Turning to the interpretation of the sign of $N(0)$, we let $v_0(\cdot)\in S_\Omega$ be the initial data for a canister flow $v(\cdot,\cdot)$ and write

$$v_0 = \sum_{i=1}^{N} c_i w^i + p,$$

where $p\in S_{N+1}$, $c_i = \langle w^i\cdot v_0\rangle$, and where N is some, as yet arbitrary, positive integer. Thus, using (7.17) and the orthogonality of the set $\{w^i, p\}$, we obtain

$$\begin{aligned} N(0) &= \frac{|\alpha|}{\rho}\|\operatorname{grad} v_0\|^2 - \|v_0\|^2,\\ &= \sum_{i=1}^{N}\left(\frac{|\alpha|}{\rho} c_i^2 \|\operatorname{grad} w^i\|^2 - c_i^2\|w^i\|^2\right) + \frac{|\alpha|}{\rho}\|\operatorname{grad} p\|^2 - \|p\|^2\\ &= \sum_{i=1}^{N}\left(\frac{|\alpha|}{\rho}\frac{1}{\kappa_i} - 1\right)c_i^2 + \frac{|\alpha|}{\rho}\|\operatorname{grad} p\|^2 - \|p\|^2. \end{aligned} \tag{7.19}$$

But $p\in S_{N+1}$, so by $(7.17)_1$ we have

$$N(0) \geqq \sum_{i=1}^{N}\frac{1}{\kappa_i}\left(\frac{|\alpha|}{\rho} - \kappa_i\right)c_i^2 + \frac{1}{\kappa_{N+1}}\left(\frac{|\alpha|}{\rho} - \kappa_{N+1}\right)\|p\|^2. \tag{7.20}$$

Now, if we assume that $\kappa_1 \leqq \frac{|\alpha|}{\rho}$, then $\kappa_j \leqq \frac{|\alpha|}{\rho}$ for all j, and we see from (7.20) that $N(0)\leqq 0$, a result also obtained earlier in Theorem 16. Next, suppose $\kappa_1 > \frac{|\alpha|}{\rho}$: since $\kappa_n\to 0$ as $n\to\infty$, and since $\frac{|\alpha|}{\rho} > 0$, it is clear that there exists an integer

* See also [16, §3] for a proof of this.

$N \geqq 1$ such that $\kappa_{N+1} \leqq \frac{|\alpha|}{\rho} < \kappa_N$. In this case the summation on the right of (7.20) will be non-positive while the term involving $\|\boldsymbol{p}\|^2$ will be non-negative. Thus we have $N(0) \geqq 0$ provided the projection $\boldsymbol{p}$ of the initial data $\boldsymbol{v}_0$ on S_{N+1} is large in comparison to the projection of $\boldsymbol{v}_0$ on $S_1 \oplus S_2 \oplus \cdots \oplus S_N$. In particular, if $\boldsymbol{v}_0 (\not\equiv 0)$ has no projection on any of the first N eigenvectors, *i.e.*, if $\boldsymbol{v}_0 = \boldsymbol{p} (\not\equiv 0)$, then $N(0) \geqq 0$, and, as we concluded earlier, both $N(\cdot)$ and $\|\text{grad}\, \boldsymbol{v}\|^2(\cdot)$ will grow unbounded in time. Moreover, in this case, $N(0)=0$ only if $\kappa_{N+1} = \frac{|\alpha|}{\rho}$ and $\boldsymbol{p} = \boldsymbol{w}^{N+1}$.

We now observe that since any canister flow $\boldsymbol{v}(\cdot,\cdot)$ may be written as

$$\boldsymbol{v} = \sum_{i=1}^{N} c_i \boldsymbol{w}^i + \boldsymbol{p},$$

where $\boldsymbol{p}(\cdot, t) \in S_{N+1}$ for all time, and $c_i(\cdot) = \langle \boldsymbol{w}^i \cdot \boldsymbol{v} \rangle (\cdot)$, then, analogously to (7.19), it follows that

$$N(t) \leqq \sum_{i=1}^{N} \frac{1}{\kappa_i} \left(\frac{|\alpha|}{\rho} - \kappa_i \right) c_i^2(t) + \frac{|\alpha|}{\rho} \|\text{grad}\, \boldsymbol{p}\|^2(t).$$

Thus if $\kappa_1 > \frac{|\alpha|}{\rho}$, then the sum is always non-positive provided N is such that

$$\kappa_{N+1} \leqq \frac{|\alpha|}{\rho} < \kappa_N,$$

and we may conclude from Theorem 19 that if there exists a flow whose projection on $S_1 \oplus S_2 \oplus \cdots \oplus S_N$ is, for all time, large in comparison to the gradient of its projection on S_{N+1}, then that flow must decay. A related, but far stronger, result was established *for simple shearing flow* between fixed parallel plates by COLEMAN, DUFFIN & MIZEL [6]. For $\kappa_1 > \frac{|\alpha|}{\rho}$ and for N determined as above, they proved that bounded simple shearing velocity fields were possible only if the initial data was a linear combination of the eigenfunctions $\boldsymbol{w}^1, \boldsymbol{w}^2, \ldots \boldsymbol{w}^N$. Moreover, they showed that such initial data led to decaying solution fields that remained a (time-dependent) linear combination of the eigenfunctions $\boldsymbol{w}^1, \boldsymbol{w}^2, \ldots \boldsymbol{w}^N$. In other words, bounded solutions $\Leftrightarrow \boldsymbol{p}(\cdot, 0) \equiv 0 \Rightarrow$ decaying solutions and $\boldsymbol{p}(\cdot, t) \equiv 0$. In Theorem 22 of the present work we shall derive such a result within a more general context—here we only note that these remarks concerning the sign of $N(t)$ point toward such a possibility.

8. Non-Existence and Projection Results

In this section we develop a functional-differential equation for the projection $\langle \boldsymbol{v} \cdot \boldsymbol{w}^i \rangle (\cdot)$, of the velocity field $\boldsymbol{v}(\cdot,\cdot)$ in canister flow onto the eigenvectors $\boldsymbol{w}^i(\cdot)$ of (7.17). Using this result, we then show that (i) for $\alpha < 0$ and for a certain set of fluid domains the initial data $\boldsymbol{v}_0(\cdot) = \boldsymbol{v}(\cdot, 0)$ must necessarily satisfy an *a priori* orthogonality condition if any flow (bounded or unbounded) is to exist;

and that (ii) regardless of the sign of α, for all other canisters and for certain types of flow problems the projections $\langle \boldsymbol{v} \cdot \boldsymbol{w}^i \rangle(\cdot)$ satisfy explicit growth or decay formulae.

Toward these ends, we first obtain a reduced and particularly convenient form of the equation of motion (6.4) and $(6.1)_1$. Indeed, by (6.1), (6.5) and the observations

$$\dot{\boldsymbol{A}}_1 = (\boldsymbol{A}_1)_t + (\operatorname{grad} \boldsymbol{A}_1)\boldsymbol{v}, \qquad \operatorname{div} \boldsymbol{A}_1 = \Delta \boldsymbol{v},$$

we find

$$\operatorname{div} \boldsymbol{T} = -\operatorname{grad} p + \mu \Delta \boldsymbol{v} + \alpha \Delta \boldsymbol{v}_t + \alpha \operatorname{div}\{(\operatorname{grad} \boldsymbol{A}_1)\boldsymbol{v} + \boldsymbol{A}_1 \boldsymbol{W} - \boldsymbol{W}\boldsymbol{A}_1\}. \tag{8.1}$$

But it follows by routine manipulation that

$$\begin{aligned}\operatorname{div}\{(\operatorname{grad} \boldsymbol{A}_1)\boldsymbol{v} + \boldsymbol{A}_1 \boldsymbol{W} - \boldsymbol{W}\boldsymbol{A}_1\} &= 2(\Delta \boldsymbol{W})\boldsymbol{v} + \operatorname{grad}(\boldsymbol{v} \cdot \Delta \boldsymbol{v}) + \tfrac{1}{4} \operatorname{grad}(|\boldsymbol{A}_1|^2),\\ &= \Delta \boldsymbol{\omega} \times \boldsymbol{v} + \operatorname{grad}\left((\boldsymbol{v} \cdot \Delta \boldsymbol{v}) + \tfrac{1}{4}|\boldsymbol{A}_1|^2\right),\end{aligned}$$

where $\boldsymbol{\omega}$ denotes curl $\boldsymbol{v}$, the vorticity vector, that is, twice the axial vector of $\boldsymbol{W}$. Whence (8.1) may be written as

$$\operatorname{div} \boldsymbol{T} = -\operatorname{grad} p + \mu \Delta \boldsymbol{v} + \alpha(\Delta \boldsymbol{v}_t + \Delta \boldsymbol{\omega} \times \boldsymbol{v}) + \alpha \operatorname{grad}(\boldsymbol{v} \cdot \Delta \boldsymbol{v} + \tfrac{1}{4}|\boldsymbol{A}_1|^2),$$

and by taking into account the well known representation

$$\dot{\boldsymbol{v}} = \boldsymbol{v}_t + \boldsymbol{\omega} \times \boldsymbol{v} + \tfrac{1}{2} \operatorname{grad} |\boldsymbol{v}|^2,$$

we may write the equation of motion (6.4) as

$$\mu \Delta \boldsymbol{v} + (\alpha \Delta \boldsymbol{v} - \rho \boldsymbol{v})_t + (\alpha \Delta \boldsymbol{\omega} - \rho \boldsymbol{\omega}) \times \boldsymbol{v} = \operatorname{grad} P, \tag{8.2}$$

where

$$P \equiv p - \alpha(\boldsymbol{v} \cdot \Delta \boldsymbol{v} + \tfrac{1}{4}|\boldsymbol{A}_1|^2) + \tfrac{1}{2}\rho|\boldsymbol{v}|^2 + \rho\phi. \tag{8.3}$$

Here, we have assumed that $\boldsymbol{b}$ is conservative, that is, $\boldsymbol{b} = -\operatorname{grad} \phi$ for some scalar field $\phi(\cdot,\cdot)$

Now let $(\kappa_n, \boldsymbol{w}^n)$ be an eigenvalue–eigenvector pair as defined in (7.17), and suppose that $\boldsymbol{v}(\cdot,\cdot)$ is a canister flow field inside Ω; *i.e.*, Ω is a fixed, rigid container in which $\boldsymbol{v}(\cdot,\cdot)$ satisfies (8.2), (8.3), (6.5), and vanishes on $\partial\Omega$. We write these last two conditions:

$$\operatorname{div} \boldsymbol{v}(\cdot, t) = 0 \quad \text{in } \Omega, \qquad \boldsymbol{v}(\cdot, t) = 0 \quad \text{on } \partial\Omega. \tag{8.4}$$

By forming the scalar product of (8.2) with $\boldsymbol{w}^n$, making use of $(7.17)_{2,3}$, and applying (8.4), we find

$$\left(\kappa_n + \frac{\alpha}{\rho}\right) \overline{\langle \boldsymbol{v} \cdot \boldsymbol{w}^n \rangle}^{\boldsymbol{\cdot}}(t) + \frac{\mu}{\rho} \langle \boldsymbol{v} \cdot \boldsymbol{w}^n \rangle(t) = \frac{\kappa_n}{\rho} \langle \boldsymbol{w}^n \cdot (\alpha \Delta \boldsymbol{\omega} - \rho \boldsymbol{\omega}) \times \boldsymbol{v} \rangle(t), \tag{8.5}$$

which is the desired functional-differential equation for $\langle \boldsymbol{v} \cdot \boldsymbol{w}^n \rangle(\cdot)$.

While in this work we do not analyze (8.5) in complete detail, we do note some of its simpler consequences which have particular bearing on questions raised earlier. The first such consequence is

Theorem 21. *Let B be a second grade fluid with $\alpha<0$ undergoing canister flow in Ω, and let $v(\cdot,\cdot)$ be its velocity field with initial data $v(\cdot,0)=v_0(\cdot)$. Further, assume that the container Ω is such that $\kappa_j(\Omega)=\frac{|\alpha|}{\rho}$ for some integer j. Then if $v(\cdot,\cdot)$ is to exist, it must satisfy*

$$\mu\langle v\cdot w^j\rangle(t)=\frac{|\alpha|}{\rho}\langle w^j\cdot(\alpha\Delta\,\omega-\rho\,\omega)\times v\rangle(t)$$

*for all $t\geqq 0$, and, in particular, the initial data must meet**

$$\left\langle w^j\cdot\left(\mu v_0+|\alpha|\left(\frac{|\alpha|}{\rho}\Delta\,\omega_0+\omega_0\right)\times v_0\right)\right\rangle=0. \tag{8.6}$$

Proof. The proof is an immediate consequence of (8.5) and the hypotheses. △

This theorem has no analog for the case $\alpha\geqq 0$, for then, since $\kappa_n>0$, it is impossible to have $\left(\kappa_n+\frac{\alpha}{\rho}\right)=0$ for any n.

Even when $\alpha<0$, this theorem is empty for canisters that are "small enough" to have $\kappa_1(\Omega)<\frac{|\alpha|}{\rho}$ since we know that $\kappa_{i+1}\leqq\kappa_i$ for all i. Of course, it is exactly these canisters for which Theorems 16 and 15 guarantee the unbounded growth of every solution $v(\cdot,\cdot)$. Indeed, by those theorems, even if $\kappa_1(\Omega)=\frac{|\alpha|}{\rho}$ we will have unbounded growth of $v(\cdot,\cdot)$ provided $v_0(\cdot)\not\equiv 0$. Moreover, $v(\cdot,\cdot)$ will not even exist unless (8.6) is met for $j=1$.

Further, since the eigenvalues are an increasing function of the domain size, and since, for any one canister, the eigenvalues form a decreasing sequence with limit zero, it is clear that there is an infinite set of canisters $\{\Omega_n\}$ with $\Omega_n\subseteq\Omega_{n+1}$ such that $\kappa_n(\Omega_n)=\frac{|\alpha|}{\rho}$, $n=1,2\ldots$. For each member Ω_n of this set Theorem 21 will apply and the orthogonality condition (8.6) for $j=n$ will be necessary for the existence of a canister flow in Ω_n. Nevertheless, any solution will grow unbounded in time if the initial data is such that $N(0)\geqq 0$.

We now return to (8.5) and specialize it in yet another way by supposing the flow $v(\cdot,\cdot)$ and/or the flow domain (container) Ω is such that

$$\langle w^n\cdot(\alpha\Delta\,\omega-\rho\,\omega)\times v\rangle(t)=0 \tag{8.7}$$

for all $t>0$ and for all eigenvectors $w^n(\cdot)$. Since $\{w^n\}$ is complete in S_Ω, this condition requires $(\alpha\Delta\omega-\rho\omega)\times v$ to lie in the orthogonal complement of S_Ω, and while stringent, it nevertheless can be satisfied in at least one of the following three ways: (i) $v(\cdot,\cdot)$ is irrotational (*i.e.*, $\omega=\text{curl}\,v=0$), (ii) the fluid is Newtonian

* Here $\omega_0(\cdot)=\text{curl}\,v_0(\cdot)$ denotes the initial vorticity vector.

and $\boldsymbol{v}(\cdot,\cdot)$ is a Beltrami field (*i.e.*, $\alpha=0$ and $\boldsymbol{\omega}\times\boldsymbol{v}=0$), and (iii) $\boldsymbol{w}^i(\cdot)\times\boldsymbol{v}(\cdot,\cdot)=0$ for all integers $i=1, 2, 3\ldots$.*

Now, by assuming (8.7), we reduce (8.5) to an ordinary differential equation for the projections which yields

$$\langle \boldsymbol{v}\cdot\boldsymbol{w}^n\rangle(t)=\langle \boldsymbol{v}_0\cdot\boldsymbol{w}^n\rangle\, e^{-\frac{\mu}{\rho\kappa_n+\alpha}t}, \tag{8.8}$$

where $\boldsymbol{v}_0(\cdot)$ is the initial data $\boldsymbol{v}(\cdot,0)$, and where we have assumed $\kappa_n \neq -\dfrac{\alpha}{\rho}$. It is clear from earlier remarks that this inequality is always true if $\alpha\geqq 0$, and, in fact, can be violated for only one value of n if $\alpha<0$, in which case Theorem 21 applies.

When $\alpha\geqq 0$, the exponent appearing in (8.8) is negative for all n, and so the projection of $\boldsymbol{v}(\cdot,\cdot)$ on every $\boldsymbol{w}^n(\cdot)$ must decay, a result that is contained in Corollary 2 of Theorem 9. Moreover, since the orthonormal set $\{\boldsymbol{w}^i\}$ is complete on S_Ω**, and since $\boldsymbol{v}(\cdot,\cdot)\in S_\Omega$ for all time, we have, *under the strong assumption* (8.7), the *exact* results

$$\begin{aligned}
\boldsymbol{v}(\boldsymbol{x},t)&=\sum_{i=1}^{\infty}\langle \boldsymbol{v}_0\cdot\boldsymbol{w}^i\rangle\,\boldsymbol{w}^i(\boldsymbol{x})\, e^{-\frac{\mu}{\rho\kappa_i+\alpha}t},\\
\|\boldsymbol{v}\|^2(t)&=\sum_{i=1}^{\infty}|\langle \boldsymbol{v}_0\cdot\boldsymbol{w}^i\rangle|^2\, e^{-\frac{2\mu}{\rho\kappa_i+\alpha}t},\\
\|\operatorname{grad}\boldsymbol{v}\|^2(t)&=\sum_{i=1}^{\infty}\frac{|\langle \boldsymbol{v}_0\cdot\boldsymbol{w}^i\rangle|^2}{\kappa_i}\, e^{-\frac{2\mu}{\rho\kappa_i+\alpha}t}.
\end{aligned}\tag{8.9}$$

We observe that (8.9) is also valid for Newtonian fluids where $\alpha=0$.

Now suppose $\alpha<0$ and rewrite (8.8) for $\kappa_n\neq\dfrac{|\alpha|}{\rho}$ with $\alpha=-|\alpha|$:

$$\langle \boldsymbol{v}\cdot\boldsymbol{w}^n\rangle(t)=\langle \boldsymbol{v}_0\cdot\boldsymbol{w}^n\rangle\, e^{-\frac{\mu}{\rho\left(\kappa_n-\frac{|\alpha|}{\rho}\right)}t}. \tag{8.10}$$

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* Condition (iii) is frequently satisfied in applications; *e.g.*, simple shearing flow between parallel plates, Poiseuille flow down an infinite pipe, and Couette flow between two infinite concentric cylinders. Although the flow domain in these flows is *infinite* one may take for Ω an appropriately selected *finite* subdomain and, by exploiting the assumed periodicity of the motion, obtain null *net* effect from boundary integrals—thus making all our analysis applicable. Of course, we suppose that any physical boundaries are brought to rest at $t=0$ and held fixed for all future time. In addition, we assume that whatever driving force was present on $(-\infty, 0]$ is removed on $[0,\infty)$, for otherwise $\langle \boldsymbol{w}^n\cdot\operatorname{grad}P\rangle\neq 0$. Under these conditions a usual argument for $\boldsymbol{w}^i(\cdot)\times\boldsymbol{v}(\cdot,\cdot)=0$ is as follows: One *assumes* $\boldsymbol{v}(\cdot,\cdot)$ to have a given direction in terms of the geometry of the flow domain, *i.e.*, $\boldsymbol{v}(\boldsymbol{x},t)=v_1(x_2,x_3,t)\boldsymbol{e}_1$ for some fixed $\boldsymbol{e}_1$ in both the simple shearing and the Poiseuille flow problems, or $\boldsymbol{v}(\boldsymbol{x},t)=v_\theta(r,t)\boldsymbol{e}_\theta$ for the Couette flow problem. Once this is done, it is then recognized that the analysis from Section 6 onward does not require the Poincaré inequality for all functions defined on Ω, but only the analogous special inequality which is appropriate to those fields that are parallel (pointwise) to the assumed direction of $\boldsymbol{v}(\cdot,\cdot)$. That is to say, the eigenvalue problem (7.17) need be solved for $(\kappa_i,\boldsymbol{w}^i)$ only on that submanifold of S_Ω which is composed of fields that are parallel to the assumed direction of $\boldsymbol{v}(\cdot,\cdot)$, thus yielding $\boldsymbol{w}^i\times\boldsymbol{v}=0$.

** Depending upon how (8.7) is in fact satisfied, we note that the set $\{\boldsymbol{w}^i\}$ may be complete only on that submanifold of S_Ω described in the previous footnote. Ting [3] has recorded several explicit solutions of the form (8.9) for which (8.7) is satisfied according to the condition (iii).

Thus, since $\kappa_i < \kappa_1$ for all i, we see once again* that $\kappa_1(\Omega) < \frac{|\alpha|}{\rho}$ permits no bounded solutions for the special flows of (8.7): the exponent in (8.10) is *positive* for every κ_n, and the projection of $\boldsymbol{v}(\cdot,\cdot)$ on every eigenvector $\boldsymbol{w}^i(\cdot)$ that $\boldsymbol{v}_0(\cdot)$ had a projection upon will grow unbounded exponentially. The exact results of (8.9) also apply here with α replaced by $-|\alpha|$. On the other hand if $\kappa_1 > \frac{|\alpha|}{\rho}$ and $\kappa_i \neq \frac{|\alpha|}{\rho}$ for any i, we have the following generalization of the Theorem 4.1 of COLEMAN, DUFFIN & MIZEL [6]:

Theorem 22. *Let $\boldsymbol{v}(\cdot,\cdot)$ be a velocity field satisfying* (8.5) *and* (8.7) *with $\alpha<0$ and with $\kappa_1 > \frac{|\alpha|}{\rho}$, $\kappa_j \neq \frac{|\alpha|}{\rho}$ for any integer j. Then, except for a finite-dimensional family of the form*

$$\boldsymbol{v}(\boldsymbol{x},t) = \sum_{i=1}^{N} \langle \boldsymbol{v}_0 \cdot \boldsymbol{w}^i \rangle \, e^{-\frac{\mu}{\rho\left(\kappa_i - \frac{|\alpha|}{\rho}\right)} t} \, \boldsymbol{w}^i(\boldsymbol{x}), \tag{8.11}$$

where N is such that $\kappa_{N+1} < \frac{|\alpha|}{\rho} < \kappa_N$, every solution field $\boldsymbol{v}(\cdot,\cdot)$ is unbounded of **exactly** *exponential order as $t\to\infty$.*

Proof. The proof follows by use of the exact results (8.9) with α replaced by $-|\alpha|$. $\triangle$

Theorem 22 takes care of the situation $\alpha<0$ completely unless it happens that $\kappa_j(\Omega) = \frac{|\alpha|}{\rho}$ for some j. However, if this occurs, then Theorem 21, specialized by assumption (8.7), implies that $\langle \boldsymbol{w}^j \cdot \boldsymbol{v}\rangle(t) \equiv 0$ for this particular j. Thus, to ensure the existence of $\boldsymbol{v}(\cdot,\cdot)$, neither it nor its initial data may have any projection on $\boldsymbol{w}^j(\cdot)$. In addition, the result (8.11) of Theorem 22 again applies with N now equal to the largest integer less than the particular j.

We conclude that for $\alpha<0$ the formula (8.11) gives the most general bounded solution for canister flow that meets the condition (8.7). In particular, bounded solutions will decay and can occur only if the initial data is a finite linear combination of eigenvectors $\boldsymbol{w}^i(\cdot)$. In contrast, if the initial data differs by even the slightest amount from a finite linear combination of eigenvectors, the solution will experience unbounded growth in time.

9. The Asymptotic Stability of a Base Flow: $\alpha \geqq 0$

In the previous three sections of this paper we have been concerned almost exclusively with the temporal behavior of a single solution of the field equations (6.3), (6.4), and (6.5) subject to the constitutive equations $(6.1)_1$ and $(3.15)_5$. Though most of our results may be interpreted as statements about the *asymptotic stability* of the rest state with uniform temperature, we have not generally emphasized such an interpretation. However, as in the theory of Navier-Stokes fluids, it is of fundamental interest to know when a given base motion is stable with respect

* Recall Theorem 16.

to arbitrary disturbances. Thus, it becomes necessary to study the temporal evolution of the *difference* $\boldsymbol{v}'-\boldsymbol{v}$, between a given base motion $\boldsymbol{v}$ and a second motion $\boldsymbol{v}'$.

To carry out the details of such a study, let $(p, \boldsymbol{v})$ denote the pressure-velocity pair associated with the supposed given base flow, the stability of which we wish to examine, and let $(p', \boldsymbol{v}')$ denote another pressure-velocity pair with different initial data. It is convenient to introduce the difference field $\boldsymbol{u}(\cdot,\cdot)$ defined by

$$\boldsymbol{u}(\boldsymbol{x}, t) \equiv \boldsymbol{v}'(\boldsymbol{x}, t) - \boldsymbol{v}(\boldsymbol{x}, t). \tag{9.1$_1$}$$

The domain of definition of $\boldsymbol{u}(\cdot, t)$ is necessarily the intersection, $\Omega_t \cap \Omega_t'$, of the respective domains in which $\boldsymbol{v}(\cdot, t)$ and $\boldsymbol{v}'(\cdot, t)$ are defined. We study the situation here where $\Omega_t = \Omega_t'$, or, equivalently, $\boldsymbol{v}=\boldsymbol{v}'$ on $\partial\Omega$ so that $\Omega_t \cap \Omega_t' = \Omega_t = \Omega_t'$ and

$$\boldsymbol{u}(\cdot, t) = 0 \quad \text{on } \partial\Omega_t. \tag{9.1$_2$}$$

Further, since $\boldsymbol{v}$ and $\boldsymbol{v}'$ are both divergence-free, then

$$\operatorname{div} \boldsymbol{u} = 0 \quad \text{in } \Omega_t. \tag{9.1$_3$}$$

We now develop the main field equation for $\boldsymbol{u}$. To do this we note that since both $(p, \boldsymbol{v})$ and $(p', \boldsymbol{v}')$ must satisfy the linear momentum balance (8.2), we obtain by subtraction

$$\mu\Delta\boldsymbol{u} + (\alpha\Delta\boldsymbol{u} - \rho\boldsymbol{u})_t + (\alpha\Delta\bar{\boldsymbol{\omega}} - \rho\bar{\boldsymbol{\omega}}) \times \boldsymbol{v} + (\alpha\Delta\boldsymbol{\omega}' - \rho\boldsymbol{\omega}') \times \boldsymbol{u} = \operatorname{grad}(P' - P), \tag{9.2}$$

where P and P' are as in (8.3) for $\boldsymbol{v}$ and $\boldsymbol{v}'$, respectively,* and where we have introduced $\bar{\boldsymbol{\omega}} \equiv \operatorname{curl} \boldsymbol{u}$ and $\boldsymbol{\omega}' = \operatorname{curl} \boldsymbol{v}'$.

If we now form the scalar product of (9.2) with $\boldsymbol{u}$ and use $(9.1)_{2,3}$ we find

$$\mu\langle\boldsymbol{u}\cdot\Delta\boldsymbol{u}\rangle + \alpha\langle\boldsymbol{u}\cdot\Delta\boldsymbol{u}_t\rangle - \rho\langle\boldsymbol{u}\cdot\boldsymbol{u}_t\rangle = -\langle\boldsymbol{u}\cdot(\alpha\Delta\bar{\boldsymbol{\omega}} - \rho\bar{\boldsymbol{\omega}}) \times \boldsymbol{v}\rangle,$$

and, after integrating by parts and once again using $(9.1)_2$, we obtain

$$\mu\|\boldsymbol{U}\|^2 + \frac{\alpha}{2}\dot{\overline{\|\boldsymbol{U}\|^2}} + \frac{\rho}{2}\dot{\overline{\|\boldsymbol{u}\|^2}} = \langle\boldsymbol{u}\cdot(\alpha\Delta\bar{\boldsymbol{\omega}} - \rho\bar{\boldsymbol{\omega}}) \times \boldsymbol{v}\rangle, \tag{9.3}$$

where $\boldsymbol{U} \equiv \operatorname{grad} \boldsymbol{u}$. Now, since $\bar{\boldsymbol{\omega}}$ is twice the axial vector of the skew part of $\boldsymbol{U}$, we may write the right-hand side of (9.3) as

$$\langle\boldsymbol{u}\cdot(\alpha\Delta(\boldsymbol{U} - \boldsymbol{U}^T) - \rho(\boldsymbol{U} - \boldsymbol{U}^T))\boldsymbol{v}\rangle,$$

which, after integrating by parts, using $(9.1)_{2,3}$, and symmetrizing, may be written as

$$-\frac{\rho}{2}\langle\boldsymbol{u}\cdot\boldsymbol{A}_1\boldsymbol{u}\rangle + \frac{\alpha}{2}\{\langle\boldsymbol{u}\cdot\Delta\boldsymbol{A}_1\boldsymbol{u}\rangle - \langle\boldsymbol{U}\boldsymbol{U}^T\cdot\boldsymbol{A}_1\rangle + \langle\boldsymbol{U}^T\boldsymbol{U}\cdot\boldsymbol{A}_1\rangle - 2\langle\boldsymbol{U}\boldsymbol{U}\cdot\boldsymbol{A}_1\rangle\},$$

---

* While (8.2) was derived for conservative body force fields, the result (9.2) holds for any $\boldsymbol{b}$ as long as we assume $\boldsymbol{b}=\boldsymbol{b}'$ and remove ϕ from the definition (8.3).

where A_1 is the first Rivlin-Ericksen tensor for the base flow v. Thus, (9.3) may be rewritten as

$$\begin{aligned}\dot{\overline{\|u\|^2}}+\frac{\alpha}{\rho}\dot{\overline{\|U\|^2}} &= -\frac{2\mu}{\rho}\|U\|^2-\langle u\cdot A_1 u\rangle+\frac{\alpha}{\rho}\{\langle u\cdot \Delta A_1 u\rangle \\ &\quad -\langle UU^T\cdot A_1\rangle+\langle U^T U\cdot A_1\rangle-2\langle UU\cdot A_1\rangle\} \\ &= -\frac{2\mu}{\rho}\|U\|^2-\langle u\cdot A_1 u\rangle+\frac{\alpha}{\rho}\{\langle u\cdot \Delta A_1 u\rangle \\ &\quad +2\langle U^T U\cdot A_1\rangle-\langle (U+U^T)^2\cdot A_1\rangle\}.\end{aligned} \tag{9.4}$$

The fact that A_1 and ΔA_1 are both symmetric allows us to introduce their respective (real) proper numbers λ_i and Λ_i, which we order according to $\lambda_1 \leqq \lambda_2 \leqq \lambda_3$ and $\Lambda_1 \leqq \Lambda_2 \leqq \Lambda_3$. Since A_1 and ΔA_1 are also traceless (v being divergence-free), we have $\lambda_1+\lambda_2+\lambda_3=\Lambda_1+\Lambda_2+\Lambda_3=0$, which, in particular, implies

$$\lambda_1 \leqq 0, \quad \lambda_3 \geqq 0, \quad \lambda_3 \leqq 2|\lambda_1| \leqq 4\lambda_3.$$

Thus, when $|\lambda_1|$ (or λ_3) is small, then λ_3 (or $|\lambda_1|$) is small, and likewise, so is $|\lambda_2|$. If we now let $\lambda_{i\,m}(t)$ and $\lambda_{iM}(t)$ denote the respective minimum and maximum of $\lambda_i(\cdot, t)$ over Ω_t, and similarly define $\Lambda_{i\,m}(t)$ and $\Lambda_{iM}(t)$, then the following estimates are immediate:

$$\begin{aligned}-\langle u\cdot A_1 u\rangle &\leqq -\lambda_{1\,m}\|u\|^2, \\ \langle u\cdot \Delta A_1 u\rangle &\leqq \Lambda_{3M}\|u\|^2, \\ \langle U^T U\cdot A_1\rangle &\leqq \lambda_{3M}\|U\|^2, \\ -\langle (U+U^T)^2\cdot A_1\rangle &\leqq -\lambda_{1\,m}\|U+U^T\|^2=-2\lambda_{1\,m}\|U\|^2,\end{aligned}$$

where the last equality holds by virtue of $(9.1)_{2,3}$. Thus, when $\alpha \geqq 0$, these estimates along with (9.4) yield

$$\dot{\overline{\|u\|^2}}+\frac{\alpha}{\rho}\dot{\overline{\|U\|^2}} \leqq \frac{2}{\rho}(\alpha(\lambda_{3M}+|\lambda_{1\,m}|)-\mu)\|U\|^2+\left(\frac{\alpha}{\rho}\Lambda_{3M}+|\lambda_{1\,m}|\right)\|u\|^2, \tag{9.5}$$

where we have written $-|\lambda_{1\,m}|$ for $\lambda_{1\,m}$ since $\lambda_{1\,m}$ is non-positive. Recall that λ_{3M} and Λ_{3M} are both non-negative.

We now have

Theorem 23. *Let* (p, v) *and* (p', v') *denote two pressure-velocity pairs satisfying the equations of motion for an incompressible second grade fluid having* $\alpha \geqq 0$ (i.e., (p, v) *and* (p', v') *satisfy* (6.4) *and* (6.5) *subject to* $(6.1)_1$). *Let* $-m$ *and* M *denote, respectively, the minimum and maximum of the proper numbers of the tensor* A_1 *of the base flow* v *on* $\Omega \times [0, T)$, $T \leqq \infty$, *and let* N *denote the maximum of the proper numbers of* ΔA_1 *on* $\Omega \times [0, T)$.* *Then,*

$$\|u\|^2(t)+\frac{\alpha}{\rho}\|U\|^2(t) \leqq \left(\|u\|^2(0)+\frac{\alpha}{\rho}\|U\|^2(0)\right)e^{\delta t} \tag{9.6}$$

* The numbers m, M, and N are necessarily non-negative.

for all $t \in [0, T)$, *where* $\boldsymbol{u} = \boldsymbol{v}' - \boldsymbol{v}$, $\boldsymbol{U} = \operatorname{grad} \boldsymbol{u}$, *and where* δ *is the number*

$$\delta \equiv \begin{cases} \dfrac{\rho}{\rho\bar{\kappa}+\alpha}\left\{\dfrac{2\alpha}{\rho}(M+m)+\bar{\kappa}\left(\dfrac{\alpha}{\rho}N+m\right)-\dfrac{2\mu}{\rho}\right\}, \\ \qquad\qquad \text{if } \alpha\left(\dfrac{\alpha}{\rho}N+m\right) \geqq 2(\alpha(M+m)-\mu); \\ \dfrac{\rho}{\alpha}\left\{\dfrac{2\alpha}{\rho}(M+m)+\bar{\kappa}\left(\dfrac{\alpha}{\rho}N+m\right)-\dfrac{2\mu}{\rho}\right\}, \\ \qquad\qquad \text{if } \alpha\left(\dfrac{\alpha}{\rho}N+m\right) < 2(\alpha(M+m)-\mu). \end{cases} \tag{9.7}$$

Here, $\bar{\kappa}$ *is any upper bound for the Poincaré coefficient* $c_P(\Omega_t)$ *on* $[0, T)$.*

Proof. Since $\alpha \geqq 0$ we may apply the definitions of $-m$, M, and N to (9.5) and obtain

$$\dot{\overline{\|\boldsymbol{u}\|^2}}(t)+\frac{\alpha}{\rho}\dot{\overline{\|\boldsymbol{U}\|^2}}(t) \leqq \frac{2}{\rho}(\alpha(M+m)-\mu)\|\boldsymbol{U}\|^2(t)+\left(\frac{\alpha}{\rho}N+m\right)\|\boldsymbol{u}\|^2(t)$$

for all $t \in [0, T)$.

If we now suppose $\alpha\left(\frac{\alpha}{\rho}N+m\right) \geqq 2(\alpha(M+m)-\mu)$ and use the Poincaré inequality and the definition of $\bar{\kappa}$, it follows that

$$\frac{2}{\rho}(\alpha(M+m)-\mu)\|\boldsymbol{U}\|^2+\left(\frac{\alpha}{\rho}N+m\right)\|\boldsymbol{u}\|^2 \leqq \delta\left(\|\boldsymbol{u}\|^2+\frac{\alpha}{\rho}\|\boldsymbol{U}\|^2\right),$$

where δ is as in $(9.7)_1$. When the resulting differential inequality is integrated, we arrive at (9.6) for this case.

Next we note that by direct application of the Poincaré inequality we may always write

$$\frac{2}{\rho}(\alpha(M+m)-\mu)\|\boldsymbol{U}\|^2+\left(\frac{\alpha}{\rho}N+m\right)\|\boldsymbol{u}\|^2 \leqq \frac{\alpha}{\rho}\delta\|\boldsymbol{U}\|^2$$

with δ as in $(9.7)_2$. Moreover, if $\alpha\left(\frac{\alpha}{\rho}N+m\right) < 2(\alpha(M+m)-\mu)$,** then this choice of δ is positive and we need only note that

$$\|\boldsymbol{U}\|^2 \leqq \frac{\rho}{\alpha}\left(\|\boldsymbol{u}\|^2+\frac{\alpha}{\rho}\|\boldsymbol{U}\|^2\right)$$

to reach (9.6). △

The exponential estimate in (9.6) is of a decaying type only when $\delta < 0$. While this is possible only when the inequality $(9.7)_1$ is satisfied, even then we must

* By (A.2) of the Appendix we may choose the value $\left(\frac{1}{\pi}\right)^2\left(\frac{3}{4\pi}\right)^{\frac{2}{3}} V(\Omega)^{\frac{2}{3}}$ for $\bar{\kappa}$.

** When $\alpha = 0$ this inequality cannot be satisfied, but then $(9.7)_1$ will apply.

satisfy the additional requirement that

$$\frac{2\mu}{\rho} > \frac{2\alpha}{\rho}(M+m) + \bar{\kappa}\left(\frac{\alpha}{\rho}N + m\right), \tag{9.8}$$

on the interval $[0, T)$, $T \leqq \infty$. Thus, if the viscosity μ is "large enough", or if the stretching and its diffusion in the base flow are both sufficiently small (*i.e.* the proper numbers of A_1 and ΔA_1 are small), the flow will be asymptotically stable* relative to all disturbances, $\boldsymbol{u}$, that vanish on the boundary of Ω_t.

Theorem 23 also yields the following two corollaries concerning uniqueness:

Corollary 1. *Let B be a second grade fluid having $\alpha \geqq 0$ with Ω a fixed reference configuration. Then any two flows of B, agreeing on $\partial\Omega$ for all t and having the same velocity distribution at $t=0$, are identical.*

Proof. If we denote the two flows by $\boldsymbol{v}$ and $\boldsymbol{v}'$, then the estimate (9.6) applies with $\boldsymbol{u}$ as defined in $(9.1)_1$. However, since $\boldsymbol{u}(\cdot, 0)=0$ and $\alpha \geqq 0$, then (9.6) implies that $\|\boldsymbol{u}\|^2(t) = \|\boldsymbol{U}\|^2(t) = 0$ for all t. △

Corollary 2. *Let B be a second grade fluid having $\alpha \geqq 0$ with Ω a fixed reference configuration. Let $\boldsymbol{v}$ and $\boldsymbol{v}'$ be any two* **steady** *flows of B having the same steady velocity distribution on $\partial\Omega$. Let $-m$, M, and N be as in Theorem* 23 *and suppose $\boldsymbol{v}$ meets* (9.8). *Then, the two flows are identical.*

Proof. We follow SERRIN [15] and observe that since both $\boldsymbol{v}$ and $\boldsymbol{v}'$ must be steady, then so too will be their difference $\boldsymbol{u}$. But then, $\|\boldsymbol{u}\|^2(\cdot)$ and $\|\boldsymbol{U}\|^2(\cdot)$ must be independent of time, while (9.6) requires their sum to be bounded above by a decaying exponential. Therefore, we conclude that $\|\boldsymbol{u}\|^2(\cdot) = \|\boldsymbol{u}\|^2(0) = 0$. △

Appendix

We collect here some known results concerning the inequalities of POINCARÉ and KORN. While no formal proofs of the results quoted here will be presented, we, instead, offer an appropriate list of references or give a brief sketch of how a proof may be constructed. For simplicity, we shall assume that the region of space under investigation, $R \subseteq E^3$, is the closure of a bounded open set and homeomorphic to a ball. Usually, however, such smoothness and connectedness are not, in fact, essential.

Let $T(R)$ denote a subset (to be defined in various ways below) of the set of all non-constant smooth, scalar-valued functions defined on R. The Poincaré inequality is the assertion that there exists a constant $c_P(R, T) \in (0, \infty)$ such that

$$\sup_{w(\cdot)\in T} \left\{ \frac{\int_R w^2\, dv}{\int_R |\operatorname{grad} w|^2\, dv} \right\} = c_P(R, T), \tag{A.1$_1$}$$

and therefore

$$\int_R w^2\, dv \leqq c_P \int_R |\operatorname{grad} w|^2\, dv \qquad \forall\, w(\cdot) \in T. \tag{A.1$_2$}$$

In the following we record three main choices of $T(R)$ for which (A.1) holds.

* Taking $T = \infty$.

Case (1). *Let $T(R)$ be the set of all smooth, scalar valued functions defined on R, vanishing on ∂R, and not identically zero.* The existence of c_P in this case is well known [28]; in fact, an elementary upper bound for it is given by d^2, where d is the width of any 3-dimensional strip containing R (see, for example, Section 1.1 of [28]). A refinement of this bound is provided by the Faber-Krahn inequality

$$c_P \leqq \left(\frac{1}{\pi}\right)^2 \left(\frac{3}{4\pi}\right)^{\frac{2}{3}} V(R)^{\frac{2}{3}}, \tag{A.2}$$

which is particularly well suited for our purposes since the volume of an incompressible fluid is an invariant under any of its motions while d is not. We caution, however, that while equality actually holds in (A.2) if R is a ball, the estimate may be highly inaccurate if R departs severely from the general shape of ball; indeed, for some domains d^2 may actually be a better bound for c_P. For a proof of (A.2) we refer to [27].

Case (2). *Let $T(R)$ be the set of all smooth, not identically zero scalar-valued functions on R that vanish in mean,*

$$\int_R w\, dv = 0. \tag{A.3}$$

Here we see from PAYNE & WEINBERGER [29] that if R is *convex*, then

$$c_P \leqq D^2/\pi^2, \tag{A.4}$$

where D is the diameter of R.

Case (3). *Let $T(R)$ be the set of all smooth not identically zero scalar-valued functions on R that vanish on a portion S of ∂R having positive area measure.*

In both Cases (2) and (3) the fact that c_P exists satisfying $(A.1)_1$ can be shown by first proving the result for a ball and then mapping R onto this ball. Thus in Case (2) we may pick the ball to be of the same volume as R and then use a mapping that has a unit Jacobian everywhere in order to preserve the condition (A.3). The result for a ball may be obtained in a way similar to that employed by FRIEDMAN in [30]. For Case (3), the result may be shown by utilizing a coordinate transformation that sends R to a ball R_B while "stretching" S sufficiently to cover a hemisphere S_B of R_B. Then, the method of proof which we referenced above for establishing the Poincaré inequality for Case (1) will again apply to this ball.

Finally, we note that since the conditions defining Cases (1), (2), and (3) are linear, we may, for any smooth vector-valued function $w(\cdot)$ defined on R, write

$$\int_R |w|^2\, dv \leqq c_P(R, T) \int_R |\operatorname{grad} w|^2\, dv \tag{A.5}$$

for all $w(\cdot) \in T^3(R) \equiv T(R) \times T(R) \times T(R)$. Here, $c_P(R, T)$ is as in our above discussion, and the direct product set $T^3(R)$ is structured from the set $T(R)$ appropriate to each of the cases.

Now let $w(\cdot) \in T^3(R)$ with $T(R)$ as in either Case (1) or Case (3). Then the Korn inequality asserts that there exists a constant $c_K(R, T) \in (0, \infty)$ such that

$$\sup_{w(\cdot) \in T^3} \left\{ \frac{\int_R |\operatorname{grad} w|^2\, dv}{\int_R |\operatorname{grad} w + (\operatorname{grad} w)^T|^2\, dv} \right\} = c_K(R, T), \tag{A.6$_1$}$$

and therefore

$$\int_R |\operatorname{grad} w|^2\, dv \leqq c_K \int_R |\operatorname{grad} w + (\operatorname{grad} w)^T|^2\, dv \qquad \forall\, w(\cdot) \in T^3. \qquad (A.6)_2$$

While in Case (1) the proof that c_K exists is trivial (see *e.g.* [37, p. 38]), Case (3) presents a far more difficult task [31]. In the applications in the body of this paper $w(\cdot)$ is divergence-free, which implies that for Case (1) equality always holds in $(A.6)_2$ with $c_K = \frac{1}{2}$. On the other hand, in Case (3) nothing beyond the existence of c_K seems to be known.

One other important case in which the Korn inequality, (A.6), is known to hold may be described by letting $T^3(R)$ now denote the set of all smooth vector-valued functions on R that meet the normalization*

$$\int_R [\operatorname{grad} w - (\operatorname{grad} w)^T]\, dv = 0. \qquad (A.7)$$

For this set, a proof of (A.6) may be found in either [31] or [32]. Indeed, in [32], PAYNE & WEINBERGER are able to show that for a ball $c_K = 14/13$. It is also worth noting that in the present case BERNSTEIN & TOUPIN [33] have shown that c_K may be made arbitrarily large by taking for R a sufficiently long and thin circular cylinder. In addition, for this case DAFERMOS [34] has shown that if R_1 and R_2 are two sets in E^3 with intersection of positive volume measure, and if the Korn inequality holds for each of these domains with constants c_{K_1} and c_{K_2}, respectively, then the Korn inequality holds for $R_1 \cup R_2$ with

$$c_K(R_1 \cup R_2) \leqq c_{K_1} + c_{K_2} + \frac{\min(V(R_1), V(R_2))}{V(R_1 \cap R_2)} (\sqrt{c_{K_1}} + \sqrt{c_{K_2}})^2.$$

Similar "chaining" formulas are well known for c_P.

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References

1. LANGLOIS, W. E., Steady flow of a slightly viscoelastic fluid between rotating spheres. Quart. Appl. Math. **21**, 61–71 (1963).
2. COLEMAN, B. D., & W. NOLL, An approximation theorem for functionals, with applications in continuum mechanics. Arch. Rational Mech. Anal. **6**, 355–370 (1960).
3. TING, T.-W., Certain non-steady flows of second order fluids. Arch. Rational Mech. Anal. **14**, 1–26 (1963).
4. COLEMAN, B. D., & H. MARKOVITZ, Normal stress effects in second-order fluids. J. Appl. Physics **35**, 1–9 (1964).
5. MARKOVITZ, H., & B. D. COLEMAN, Nonsteady helical flows of second-order fluids. Physics of Fluids **7**, 833–841 (1964).

* The sup in $(A.6)_1$ is then calculated omitting constant fields. $(A.6)_2$ remains valid for all fields in $T^3(R)$.

6. COLEMAN, B. D., R. J. DUFFIN, & V. MIZEL, Instability, uniqueness, and non-existence theorems for the equation $u_t = u_{xx} - u_{xtx}$ on a strip. Arch. Rational Mech. Anal. **19**, 100–116 (1965).
7. COLEMAN, B. D., & V. MIZEL, Breakdown of laminar shearing flows for second-order fluids in channels of critical width. ZAMM **46**, 445–448 (1966).
8. TRUESDELL, C., Fluids of second grade regarded as fluids of convected elasticity. Physics of Fluids **8**, 1936–1938 (1965).
9. COLEMAN, B. D., Thermodynamics of materials with memory. Arch. Rational Mech. Anal. **17**, 1–46 (1964).
10. COLEMAN, B. D., & W. NOLL, The thermodynamics of elastic materials with heat conduction and viscosity. Arch. Rational Mech. Anal. **13**, 167–178 (1963).
11. GURTIN, M. E., Modern Continuum Thermodynamics. Notas De Matemática Física, Vol. II. Instituto de Matemática, Universidade Federal do Rio de Janeiro, 1972.
12. COLEMAN, BERNARD D., & JAMES M. GREENBERG, Thermodynamics and the stability of fluid motion. Arch. Rational Mech. Anal. **25**, 321–341 (1967).
13. COLEMAN, BERNARD D., On the stability of equilibrium states of general fluids. Arch. Rational Mech. Anal. **36**, 1–32 (1970).
14. COLEMAN, B. D., On the dynamical stability of fluid phases. IUTAM Symposium on Instability of Continuous Systems. Berlin-Heidelberg-New York: Springer, 1971.
15. SERRIN, JAMES, On the stability of viscous fluid motions. Arch. Rational Mech. Anal. **3**, 1–13 (1959).
16. JOSEPH, DANIEL D., & ROGER L. FOSDICK, The free surface on a liquid between cylinders rotating at different speeds, Part I. Arch. Rational Mech. Anal. **49**, 321–380 (1973).
17. JOSEPH, DANIEL D., GORDON S. BEAVERS, & ROGER L. FOSDICK, The free surface on a liquid between cylinders rotating at different speeds, Part II. Arch. Rational Mech. Anal. **49**, 381–401 (1973).
18. TANNER, R. I., Some methods for estimating the normal stress functions in viscometric flows. Trans. Soc. Rheology **14**, 483–507 (1970).
19. GURTIN, MORTON E., On the thermodynamics of materials with memory. Arch. Rational Mech. Anal. **28**, 40–50 (1968).
20. TRUESDELL, C., & W. NOLL, The Non-Linear Field Theories of Mechanics. Flügge's Handbuch der Physik, III/3. Berlin-Heidelberg-New York: Springer 1965.
21. TRUESDELL, C., Rational Thermodynamics. New York: McGraw-Hill, 1969.
22. SPENCER, A. J. M., Theory of invariants. Continuum Physics, Vol. I, Ed. A. CEMAL ERINGEN. New York: Academic Press 1971.
23. OGAWA, HAJIMU, On lower bounds and uniqueness for solutions of the Navier-Stokes equations. J. Math. Mech. **18**, 445–452 (1968).
24. OLDROYD, J. G., The motion of an elastico-viscous liquid contained between coaxial cylinders, I. Quart. J. Mech. Appl. Math. **4**, 271–282 (1951).
25. ERICKSEN, J. L., Thermoelastic stability. Proc. 5th U.S. National Congr. Appl. Mech., 187–193 (1966).
26. DYER, R. H., & D. E. EDMUNDS, Lower bounds for solutions of the Navier-Stokes equations. Proc. London Math. Soc. **3**, 169–178 (1968).
27. GARABEDIAN, P., Partial Differential Equations. New York: Wiley, 1964.
28. LADYZHENSKAYA, O. A., The Mathematical Theory of Viscous Incompressible Flow. New York: Gordon & Breach 1969.
29. PAYNE, L. E., & H. F. WEINBERGER, An optimal Poincaré inequality for convex domains. Arch. Rational Mech. Anal. **5**, 286–292 (1960).
30. FRIEDMAN, AVNER, Partial Differential Equations of Parabolic Type. Prentice Hall: Englewood Cliffs, N.J. 1964.
31. FICHERA, G., Existence theorems in elasticity. Handbuch der Physik, VIa/2. Berlin-Heidelberg-New York: Springer 1972.
32. PAYNE, L. E., & H. F. WEINBERGER, On Korn's inequality. Arch. Rational Mech. Anal. **8**, 89–98 (1961).
33. BERNSTEIN, B., & R. TOUPIN, Korn inequalities for the sphere and for the circle. Arch. Rational Mech. Anal. **6**, 51–64 (1960).

34. Dafermos, C. M., Some remarks on Korn's inequality. Z. Angew. Math. Phys. **19**, 913–920 (1968).
35. Hardy, G. H., J. E. Littlewood, & G. Pólya, Inequalities. Cambridge: University Press 1964.
36. Noll, W., A mathematical theory of the mechanical behavior of continuous media. Arch. Rational Mech. Anal. **2**, 197–226 (1958).
37. Gurtin, Morton E., The linear theory of elasticity. Handbuch der Physik, VIa/2. Berlin-Heidelberg-New York: Springer 1972.
38. Day, William Alan, The Thermodynamics of Simple Materials with Fading Memory. Springer Tracts in Natural Philosophy, Vol. 22. Berlin-Heidelberg-New York: Springer 1972.
39. Halmos, Paul Richard, Finite-Dimensional Vector Spaces. Princeton, N. J.: D. Van Nostrand 1958.
40. Truesdell, C., The natural time of a viscoelastic fluid: its significance and measurement. Physics of Fluids **7**, 1134–1142 (1964).

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