Homogenization and Optimal Bounds in Linear Elasticity

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0. Introduction

In their celebrated paper of 1963, HASHIN & SHTRIKMAN [1] addressed the problem of determining optimal bounds for the bulk and shear moduli of a statistically isotropic elastic composite material with arbitrary isotropic phase geometry. They derived a set of bounds for these moduli in the physically meaningful case of three-dimensional elasticity.

In the case of a two phase composite, let K^1, μ^1 and K^2, μ^2 respectively denote the bulk and shear moduli for the first and the second phase, let K, μ denote their analogues for the composite and let θ stand for the volume fraction of the first phase in the composite. Under the ordering restriction that both

$$K^1 < K^2 \quad \text{and} \quad \mu^1 < \mu^2, \tag{0.1}$$

HASHIN & SHTRIKMAN's calculations lead to (*cf.* their equations (4.1)–(4.4))

$$K^l \leqq K \leqq K^u, \quad \mu^l \leqq \mu \leqq \mu^u, \tag{0.2}$$

with

$$K^l = K^1 + \frac{1-\theta}{\dfrac{1}{K^2 - K^1} + \dfrac{3\theta}{3K^1 + 4\mu^1}}, \quad K^u = K^2 + \frac{\theta}{\dfrac{1}{K^1 - K^2} + \dfrac{3(1-\theta)}{3K^2 + 4\mu^2}},$$

(0.3)

$$\mu^l = \mu^1 + \frac{1-\theta}{\dfrac{1}{\mu^2 - \mu^1} + \dfrac{6\theta(K^1 + 2\mu^1)}{5\mu^1(3K^1 + 4\mu^1)}},$$

$$\mu^u = \mu^2 + \frac{\theta}{\dfrac{1}{\mu^1 - \mu^2} + \dfrac{6(1-\theta)(K^2 + 2\mu^2)}{5\mu^2(3K^2 + 4\mu^2)}}.$$

According to HASHIN himself, "the original derivation of the bounds included some mathematical liberties" (HASHIN [1], p. 486), and many authors then tried to improve it. Of course, a mathematical definition of the notion of effective properties of a mixture is a prerequisite for a rigorous derivation. In recent years, such an attempt was made *e.g.* by WILLIS [1], GOLDEN & PAPANICOLAOU [1]. Their approach is probabilistic, and effective properties are understood as expectations.

HASHIN & SHTRIKMAN also examined the optimality of the bounds (0.3). With the help of coated spheres, they established the optimality of the bounds on the bulk modulus. The optimality of the bounds on the shear modulus was until recently an open problem (CHRISTENSEN [1], p. 147, HASHIN [1], p. 486), but towards the end of 1984, several authors (*e.g.* MILTON [1], NORRIS [1]) proposed composites that attain the bounds on the shear modulus. Their procedure is incremental and it requires an infinite number of mixing processes. It does not specifically describe the underlying microscopic structure of the composite.

In this work we reconsider the derivation of bounds on the bulk and shear moduli and their optimality in the context of homogenization. From our standpoint homogenization coincides with the mathematical notion of G-convergence introduced by SPAGNOLO [1] or with that of H-convergence introduced by MURAT and TARTAR (MURAT [1], TARTAR [2]). In the theory of H-convergence, sequences of composites with a heat conductivity tensor a^ε satisfying for every ξ and almost every x

$$\alpha\,|\xi|^2 \leqq a^\varepsilon_{ij}(x)\,\xi_i\xi_j \leqq \beta\,|\xi|^2 \qquad (0 < \alpha < \beta < +\infty \text{ fixed})$$

are examined. The most basic fact is the existence of a subsequence a^{ε_j} of a^ε and of a heat conductivity tensor a^0 such that the solution of any heat conduction problem for a^{ε_j} converges to the solution of the corresponding problem for a^0. This result was first proved by SPAGNOLO [1], then reexamined by MURAT and TARTAR (*cf. e.g.* MURAT [1], TARTAR [2]). The theory was subsequently developed by many authors (*cf. e.g.* MURAT [1, 3], TARTAR [1–3]; *cf.* also BENSOUSSAN, LIONS & PAPANICOLAOU [1], SANCHEZ-PALENCIA [1] for other points of view and for general references).

The question of characterizing all possible anisotropic tensors a^0 for the composites made by mixing two isotropic conducting components in prescribed volume fractions θ, $1 - \theta$ was later investigated in the framework of homogenization. The principle of the method was first described by TARTAR in 1977 (TARTAR [3]). It uses in an essential manner the theory of compensated compactness developed in MURAT [3] and TARTAR [4]. In 1980 a set of bounds was obtained and their optimality was established whenever both components are isotropic; the results were later published in MURAT [2], TARTAR [5]. See also the independent work of LURIÉ & CHERKAEV [1] in the two-dimensional case. For an isotropic mixture of two isotropic conducting components, the bounds coincide with HASHIN & SHTRIKMAN's (equation (6.27), p. 497 in HASHIN [1]).

We attempt in this work to obtain the same type of results in the realm of elasticity. Our mathematical definition of the effective properties of a mixture of elastic components is given in the context of homogenization. We then restrict our attention to mixtures of two isotropic materials in prescribed volume fractions, and strive to characterize all possible macroscopically isotropic composites. Our analysis relies in a decisive manner on LUC TARTAR's ideas; it uses the techniques of H-convergence and compensated compactness.

The first section of the paper is very short, and is entirely devoted to notation and basic definitions. In the second section the problem to be addressed is formulated in the mathematical framework of homogenization. After recalling a few results about compensated compactness, we proceed in the third section to construct and prove necessary bounds on the bulk and on the shear moduli. The bounds obtained on the bulk modulus coincide with those announced in HASHIN & SHTRIKMAN [1]. They are not restricted to the case of well-ordered phases (*i.e.* satisfying (0.1)). We point out that the removal of restriction (0.1) was first studied by WALPOLE [1], when he improved HASHIN & SHTRIKMAN's original derivation. Our bounds on the shear modulus, however, are not as tight as HASHIN & SHTRIKMAN's, but they are better than the classical ones, sometimes referred to as PAUL's bounds (PAUL [1]):

$$\mu^h \leqq \mu \leqq \mu^a, \tag{0.4}$$

where

$$\frac{1}{\mu^h} = \frac{\theta}{\mu^1} + \frac{1-\theta}{\mu^2}, \qquad \mu^a = \theta\mu^1 + (1-\theta)\,\mu^2. \tag{0.5}$$

In the fourth section, we address the question of optimality. Extending to the case of elasticity a formula devised by TARTAR, we prove that HASHIN & SHTRIKMAN's bounds on the bulk and shear moduli can *both* be achieved and even *simultaneously* achieved by multiple layering. We thus give a positive answer to HASHIN's conjecture (HASHIN [1], p. 486): "it has never been shown that [the bounds on the shear modulus] are also best possible in terms of volume fractions but they well may be". We produce a multi-layered composite (with a *finite* number of layering directions) independently of the phase ordering restriction (0.1). Its macroscopic behavior is isotropic and its macroscopic bulk and shear moduli are respectively K^l and μ^l (or K^u and μ^u) defined in (0.3). This constrasts with the recent incremental procedures introduced in MILTON [1], or NORRIS [1], which require an uncountable infinity of layering directions.

In our opinion, the interest of the techniques used in the present paper is threefold. The method developed for proving bounds differs from the variational approach of HASHIN & SHTRIKMAN and the analytical approach of BERGMAN & KANTOR. It includes a mathematical definition of the effective properties of a composite without any kind of periodicity or statistical assumptions. Finally the method for constructing the composite that attains the HASHIN & SHTRIKMAN bounds is an elementary calculation; as such it could have been done a long time ago, but apparently was not.

Many questions concerning the macroscopic behavior of the mixture of two isotropic elastic materials have yet to be answered, even under the assumption of macroscopic isotropy. The first priority in our programme is to recover the

HASHIN & SHTRIKMAN bounds on the shear modulus in the mathematical context of homogenization. Then, one should strive to characterize the admissible region in a K, μ plane, since MILTON's computations (MILTON [2]) suggest a kind of lens-shaped domain.

1. Notation and Basic Definitions

Throughout the paper, Einstein's summation convention is used.

Small greek letters denote vectors in $\mathbb{R}^N$, except for α, β (*strictly* positive real numbers), ε (a small strictly positive real number), θ (the volume fraction, a real number between 0 and 1), λ, μ (the Lamé constants of isotropic elasticity), χ (a characteristic function), and φ, ψ (test functions in $\mathscr{C}_0^\infty(\mathbb{R}^N)$). Small latin letters denote second order tensors on $\mathbb{R}^N$, except for x (the position vector in $\mathbb{R}^N$) or when used as indices. Capital latin letters denote fourth-order tensors on $\mathbb{R}^N$, except for K (the bulk modulus), and N (the space dimension).

The following definitions are designed to simplify reading the text:

- $\mathscr{M}_s$ is the space of all symmetric linear mappings from $\mathbb{R}^N$ into itself; i is the identity element, tr the trace operator and pq the inner product, *i.e.*

$$pq = \operatorname{tr}({}^t p \circ q) = p_{ij} q_{ij},$$

  where p_{ij} (respectively q_{ij}) is the matrix of p with respect to a given orthonormal basis on $\mathbb{R}^N$,
- $\mathscr{L}(\mathscr{M}_s)$ is the space of all linear mappings from $\mathscr{M}_s$ into itself; I is the identity element, Tr the trace operator,
- $\mathscr{L}_s(\mathscr{M}_s)$ is the subspace of all symmetric elements of $\mathscr{L}(\mathscr{M}_s)$,
- $\mathscr{M}(\alpha, \beta)$ is the set of all A in $L_\infty(\mathbb{R}^N; \mathscr{L}_s(\mathscr{M}_s))$ with $\|A\|_{L_\infty} \leqq \beta$ and such that, for almost every x in $\mathbb{R}^N$, $A(x) \geqq \alpha I$ as an element of $\mathscr{L}(\mathscr{M}_s)$ ("positive-definiteness", KNOPS & PAYNE [1]),
- if Ω is an arbitrary domain of $\mathbb{R}^N$, $L_p^s(\Omega)$ denotes $L_p(\Omega; \mathscr{M}_s)$, $1 \leqq p \leqq +\infty$, whereas $\mathbb{H}^q(\Omega)$ (respectively $\mathbb{H}^q_{\text{loc}}, \ldots$) denotes $H^q(\Omega; \mathbb{R}^N)$ (respectively $H^q_{\text{loc}}, \ldots$), $q \in \mathbb{R}$.

Juxtaposed tensorial quantities are to be either contracted, composed or "tensorialized". Contraction is automatically assumed unless otherwise indicated. Composition is denoted by $\circ$ and "tensorialization" by $\otimes$. Thus, if ζ_i, ξ_i, p_{ij}, A_{ijkh}, B_{ijkh} are the representatives (with respect to a given orthonormal basis of $\mathbb{R}^N$) of ζ, ξ, p, A, B,

$\xi\zeta$ is a scalar with value $\xi_i \zeta_i$,
$p\xi$ is a vector with coordinates $(p\xi)_i = p_{ij}\xi_j$,
Ap is a second order tensor with coefficients $(Ap)_{ij} = A_{ijkh} p_{kh}$,
$A \circ B$ is a fourth order tensor with coefficients $(A \circ B)_{ijkh} = A_{ijpq} B_{pqkh}$,
$\zeta \otimes \xi$ is a linear mapping with coefficients $(\zeta \otimes \xi)_{ij} = \zeta_i \xi_j$,
etc.

The action of an element A of $\mathscr{L}(\mathscr{M}_s)$ on any symmetrized rank-one tensor $\frac{1}{2}(\xi \otimes \eta + \eta \otimes \xi)$ is occasionally denoted by $A(\xi \otimes \eta)$.

Finally ∇ stands for the gradient of a *vector* field, div for the divergence of a *second-order* tensor field and ^{t} denotes the adjoint of a linear mapping, *i.e.* in components,

$$(\nabla \xi)_{ij} = \frac{\partial \xi_i}{\partial x_j},$$

$$(\operatorname{div} p)_i = \frac{\partial p_{ij}}{\partial x_j},$$

$$({}^t p)_{ij} = p_{ji}, \qquad ({}^t A)_{ijkh} = A_{khij}.$$

A final word of caution concerns the position of certain indices: to avoid notations such as $e_{\varepsilon ij}$, indices 0, 1, 2 and ε are always used as superscripts; they should not be confused with powers, which are always used with parentheses, *i.e.* λ^2 reads as lambda index two, whereas $(\lambda)^2$ reads as lambda squared.

2. Setting of the Problem

2.1. Preliminaries

Let us consider two *homogeneous* and *isotropic* elastic materials respectively referred to as material 1 and material 2. The corresponding elasticity tensors A^1 and A^2 are

$$\begin{aligned} A^1 &= \lambda^1 i \otimes i + 2\mu^1 I, \\ A^2 &= \lambda^2 i \otimes i + 2\mu^2 I, \end{aligned} \tag{2.1}$$

where λ^1, μ^1 and λ^2, μ^2 are given constants. These tensors are assumed to belong to $\mathcal{M}(\alpha, \beta)$. The positive definiteness conditions which are part of the definition of the set $\mathcal{M}(\alpha, \beta)$ become

$$K^i > 0, \quad \mu^i > 0, \quad i = 1, 2, \tag{2.2}$$

where the bulk moduli K^i are defined as

$$NK^i = N\lambda^i + 2\mu^i, \quad i = 1, 2. \tag{2.3}$$

We assume with no loss of generality that $\mu_1 \leqq \mu_2$.

An arbitrary mixture of material 1 and material 2 is characterized by the characteristic function $\chi(x)$ of material 1 in $\mathbb{R}^N$. At any point x of $\mathbb{R}^N$ the elasticity tensor of the mixture is

$$A(x) = A^1 \chi(x) + A^2(1 - \chi(x)). \tag{2.4}$$

If we are to investigate the macroscopic properties of all such mixtures in the setting of homogenization theory, we have to consider a family of characteristic functions $\chi^\varepsilon(x)$ and the corresponding family of elastic tensors

$$A^\varepsilon(x) = A^1 \chi^\varepsilon(x) + A^2(1 - \chi^\varepsilon(x)). \tag{2.5}$$

Since both A^1 and A^2 are in $\mathcal{M}(\alpha, \beta)$,

$$A^\varepsilon \text{ belongs to } \mathcal{M}(\alpha, \beta). \tag{2.6}$$

An elastic material, with tensor A^0, macroscopically represents the mixture if there is a sequence of A^ε such that, on any domain Ω of $\mathbb{R}^N$, the solution of an arbitrary elastic boundary-value problem with A^ε as elasticity tensor yields a strain tensor, a stress tensor and an elastic energy which are close to the strain tensor, stress tensor and elastic energy associated with the solution of the same elastic boundary-value problem with A^0 as elasticity tensor. The mathematical translation of these statements is the notion of H-convergence (MURAT [1], TARTAR [2]).

The following definition and theorem can be applied to *an arbitrary sequence* of elasticity tensors A^ε in $\mathcal{M}(\alpha, \beta)$; ε should be understood as a scaling parameter which describes the size of the heterogeneities.

Definition 2.1. *A sequence A^ε of elements of $\mathcal{M}(\alpha, \beta)$ H-converges to an element A^0 of $\mathcal{M}(\alpha, \beta)$ if and only if for any bounded domain Ω of $\mathbb{R}^N$ and for any sequence $(v^\varepsilon, \omega^\varepsilon)$ of elements of $\mathbb{H}^1(\Omega) \times \mathbb{H}^{-1}(\Omega)$ such that as ε tends to zero,*

$$\begin{gathered} e^\varepsilon \rightharpoonup e^0 \text{ weakly in } L_2^s(\Omega), \\ \omega^\varepsilon \to \omega^0 \text{ strongly in } \mathbb{H}^{-1}(\Omega), \\ e^\varepsilon = \tfrac{1}{2}(\nabla v^\varepsilon + {}^t\nabla v^\varepsilon), \quad e^0 = \tfrac{1}{2}(\nabla v^0 + {}^t\nabla v^0), \\ \operatorname{div}(A^\varepsilon e^\varepsilon) + \omega^\varepsilon = 0, \end{gathered} \tag{2.7}$$

where v^0 lies in $\mathbb{H}^1(\Omega)$, then

$$A^\varepsilon e^\varepsilon \rightharpoonup A^0 e^0 \text{ weakly in } L_2^s(\Omega). \tag{2.8}$$

Remark 2.1. The following equivalent definition of H-limits is the exact transposition to elasticity of a result originally established in the case of a second-order elliptic scalar equation (*cf. e.g.* MURAT [1]): A^ε H-converges to A^0 if and only if for any bounded domain Ω of $\mathbb{R}^N$ and for any ω in $\mathbb{H}^{-1}(\Omega)$, the unique solution $v^\varepsilon \in \mathbb{H}_0^1(\Omega)$ of

$$\begin{gathered} e^\varepsilon = \tfrac{1}{2}(\nabla v^\varepsilon + {}^t\nabla v^\varepsilon), \\ s^\varepsilon = A^\varepsilon e^\varepsilon, \\ \operatorname{div} s^\varepsilon + \omega = 0, \end{gathered} \tag{2.9}$$

satisfies

$$\begin{gathered} v^\varepsilon \rightharpoonup v^0 \text{ weakly in } \mathbb{H}_0^1(\Omega), \\ s^\varepsilon \rightharpoonup s^0 \text{ weakly in } L_2^s(\Omega), \end{gathered} \tag{2.10}$$

as ε tends to zero, where $v^0 \in \mathbb{H}_0^1(\Omega)$ is the unique solution of

$$\begin{gathered} e^0 = \tfrac{1}{2}(\nabla v^0 + {}^t\nabla v^0), \\ s^0 = A^0 e^0, \\ \operatorname{div} s^0 + \omega = 0. \end{gathered} \tag{2.11}$$

Our preference is for Definition 2.1, because it stresses the local character of the notion of H-limit, and its independence from the boundary conditions.

Remark 2.2. As a consequence of (2.7), (2.8), it can be proved that, as ε tends to zero,

$$s^\varepsilon e^\varepsilon \rightharpoonup s^0 e^0 \text{ weak-* in } \mathscr{D}'(\Omega). \tag{2.12}$$

This convergence is a typical result of compensated compactness (*cf.* Theorem 3.1 in Section 3).

Remark 2.3. The characterization of Remark 2.1 is at the root of the proof of the existence of H-limits. The uniform positive-definiteness of A^ε, which is part of the definition of $\mathscr{M}(\alpha, \beta)$, suffices to ensure existence and uniqueness of v^ε together with the boundedness of v^ε in $\mathbb{H}_0^1(\Omega)$ and of s^ε in $\mathrm{L}_2^s(\Omega)$.

In fact, the existence of H-limits is guaranteed through the following result:

Theorem 2.1. *Consider a family A^ε of elements of $\mathscr{M}(\alpha, \beta)$. There exists a subsequence of A^ε which H-converges to an element A^0 of $\mathscr{M}(\alpha, \beta)$.*

The proof of the theorem is the exact transposition to the elastic case of the proof given by TARTAR for the case of a second order elliptic equation (*cf. e.g.* MURAT [1], SIMON [1], ZHIKOV, KOZLOV, OLEINIK & KHA T'EN NGOAN [1]). It will not be repeated here.

This theorem asserts the existence of elastic materials which adequately represent the macroscopic behavior of any kind of microscopically heterogeneous materials.

Our goal in the present study is to obtain as much information as possible about A^0, provided some information is given about the A^ε's, *namely their weak-* limit*, as ε tends to zero.

2.2. Statement of the Problem

From now on the A^ε's under consideration are of the form described at the beginning of subsection 2.1. In that context, the weak-* limit of the A^ε's is determined as soon as the weak-* limit of the χ^ε's, denoted by θ, is given. Theorem 2.1 ensures the existence of an H-limit, at least for a subsequence of the A^ε's.

After repeated extractions of subsequences, we are led to the following set of hypotheses, *referred to as the* (H) *set*:

$$A^\varepsilon = A^1\chi^\varepsilon + A^2(1-\chi^\varepsilon), \tag{2.13}$$

where

$$A^1, A^2 \text{ are } \textit{homogeneous} \text{ elasticity tensors in } \mathscr{M}(\alpha\ \beta), \tag{2.14}$$

$$\chi^\varepsilon \text{ converges weak-* in } L_\infty(\mathbb{R}^N) \text{ to an element } \theta \text{ of } L_\infty(\mathbb{R}^N), \tag{2.15}$$

$$A^\varepsilon\ H\text{-converges to an element } A^0 \text{ of } \mathscr{M}(\alpha, \beta). \tag{2.16}$$

If, as was assumed at the beginning of subsection 2.1, A^1 and A^2 are isotropic (*cf.* (2.1), (2.2), (2.3)) the (H) set of hypotheses is labelled *the* (HI) *set*. Note that the associated bulk and shear moduli do not have to satisfy (0.1).

In the following sections, we mainly attempt to characterize all possible isotropic H-limits of a sequence of A^ε's satisfying (HI). Specifically we consider the case when A^0 is isotropic, *i.e.*

$$\begin{aligned} A^0(x) &= \lambda^0(x)\, i \otimes i + 2\mu^0(x)\, I, \\ K^0(x) &= \frac{1}{N}\left(N\lambda^0(x) + 2\mu^0(x)\right), \end{aligned} \tag{2.17}$$

for almost every x in $\mathbb{R}^N$, and seek necessary and sufficient conditions on K^0 and μ^0 for A^0 to be the H-limit of a sequence of A^ε's satisfying (HI).

3. Necessary Conditions

The derivation of bounds on the elastic coefficients is performed with the help of the theory of compensated compactness. In Subsection 3.1 we adapt the method developed in MURAT [2] and TARTAR [4] to the setting of linear elasticity. The results of Subsection 3.1 are used in Subsections 3.2 and 3.3 to obtain bounds on the bulk moduli and the shear moduli.

3.1. Compensated Compactness and Linear Elasticity

The theory of compensated compactness is concerned with necessary and sufficient conditions for weak lower semi-continuity of functionals.

Definition 3.1.

$$\Lambda = \{(e, s) \in \mathcal{M}_s \times \mathcal{M}_s \mid \textit{there exists a non-zero element } \xi \textit{ of } \mathbb{R}^N \textit{ and an element } \tau \textit{ of } \mathbb{R}^N \textit{ such that } e = \tau \otimes \xi + \xi \otimes \tau,\ s\xi = 0\}. \tag{3.1}$$

In the context of quadratic forms, the following result holds:

Theorem 3.1. *Let $B(e, s)$ be a quadratic form on $\mathcal{M}_s \times \mathcal{M}_s$ which is positive on Λ. Let Ω be an arbitrary domain of $\mathbb{R}^N$, let s^ε be a sequence in $L_2^s(\Omega)$ and v^ε be a sequence in $\mathbb{H}^1_{\mathrm{loc}}(\Omega)$ such that*

$$e^\varepsilon = \tfrac{1}{2}\left(\nabla v^\varepsilon + {}^t\nabla v^\varepsilon\right) \textit{ is in } L_2^s(\Omega). \tag{3.2}$$

Assume that, as ε tends to zero,

(3.3) *e^ε and s^ε converge weakly in $L_2^s(\Omega)$ to e^0 and s^0 respectively, and that* div s^ε *is in a compact set of $\mathbb{H}^{-1}_{\mathrm{loc}}(\Omega)$.*

Then $B(e^\varepsilon, s^\varepsilon)$ is weakly lower semi-continuous in $\mathcal{D}'(\Omega)$, i.e., *for any φ in $\mathcal{C}_0^\infty(\Omega)$ with $\varphi \geqq 0$,*

$$\varliminf_{\varepsilon \to 0} \int_\Omega \varphi B(e^\varepsilon, s^\varepsilon)\, dx \geqq \int_\Omega \varphi B(e^0, s^0)\, dx. \tag{3.4}$$

Remark 3.1. Theorem 3.1 can be generalized to the case of an arbitrary sequence e^ε of $L_2^s(\Omega)$ such that, for all i, k, l, m in $\{1, \dots, N\}$

$$(3.5)\quad \frac{\partial^2 e^\varepsilon_{mk}}{\partial x_i\,\partial x_l} + \frac{\partial^2 e^\varepsilon_{il}}{\partial x_m\,\partial x_k} - \frac{\partial^2 e^\varepsilon_{kl}}{\partial x_i\,\partial x_m} - \frac{\partial^2 e^\varepsilon_{im}}{\partial x_k\,\partial x_l} \text{ is in a compact set of } H^{-2}_{\mathrm{loc}}(\Omega).$$

Under some regularity assumptions, a tensor field $e(x)$ in $\mathcal{M}_s$ is a linearized strain tensor, *i.e.* it is of the form

$$(3.6)\qquad e = \tfrac{1}{2}(\nabla v + {}^t\nabla v),$$

for some vector field $v(x)$, if and only if the following compatibility conditions hold true (*cf.* GERMAIN [1]):

$$(3.7)\qquad \frac{\partial^2 e_{mk}}{\partial x_i\,\partial x_l} + \frac{\partial^2 e_{il}}{\partial x_m\,\partial x_k} - \frac{\partial^2 e_{kl}}{\partial x_i\,\partial x_m} - \frac{\partial^2 e_{im}}{\partial x_k\,\partial x_l} = 0.$$

The condition (3.5) can thus be interpreted as a control on the deviation of the fields e^ε from strain tensors.

Remark 3.2. In the context of Remark 3.1, an equivalent definition for the set Λ can be given as

$$(3.8)\qquad \Lambda = \{(e, s) \in \mathcal{M}_s \times \mathcal{M}_s \mid \text{there exists a non-zero element } \xi \text{ of } \mathbb{R}^N$$
$$\text{such that, for all } i, k, l, m \text{ in } \{1, \dots, N\},$$
$$s_{ij}\xi_j = 0,\ e_{mk}\xi_i\xi_l + e_{il}\xi_m\xi_k - e_{kl}\xi_i\xi_m - e_{im}\xi_k\xi_l = 0\}.$$

The result of Theorem 3.1 can be obtained through application of the general result of compensated compactness (TARTAR [4], Thorem 11). To this effect, the quadratic form B must be dissymmetrized and written as a quadratic form acting on ∇v^ε, not on e^ε. Its generalization in Remark 3.1 cannot however be obtained through direct application of that result which only considers the case of linear relations involving first order derivatives of weakly converging sequences, and is thus not concerned with relations of the form (3.5). Nevertheless, the proof of the general result can be faithfully adapted to our setting, provided that the definition of the set Λ given in TARTAR [4] is replaced by Definition 3.1. It will not be reproduced in this study.

Remark 3.3. In the context of Theorem 3.1, the quadratic form associated with the elastic energy, *i.e.*

$$(3.9)\qquad E(e, s) = es,$$

is weakly continuous (*cf.* Remark 2.2).

Following the method devised in TARTAR [3] and developed in TARTAR [5], we seek a quadratic form which satisfies the hypothesis of Theorem 3.1 in the hope that inequality (3.4) will yield the desired bounds. In a study of heat conductivity it was observed in TARTAR [5] that sequences of vector fields (the analogues of our s^ε's and e^ε's) only yielded elementary bounds and that sequences of second order tensors had to be used to derive better bounds. This remark applies equally here, at least as far as shear moduli are concerned. It results in the following extensions of Definition 3.1 and of Theorem 3.1:

Definition 3.2.

(3.10) $\Lambda = \{(P, Q) \in \mathscr{L}(\mathscr{M}_s) \times \mathscr{L}(\mathscr{M}_s) \mid$ *there exists a non-zero element* ξ *of* $\mathbb{R}^N$ *such that for any* (ζ, η) *in* $\mathbb{R}^N \times \mathbb{R}^N$, *there also exists an element* $\tau^{\zeta\eta}$ *in* $\mathbb{R}^N$ *satisfying*

$$(Q(\zeta \otimes \eta))\, \xi = 0,$$

$$P(\zeta \otimes \eta) = \tau^{\zeta\eta} \otimes \xi + \xi \otimes \tau^{\zeta\eta}\}.$$

Remark 3.4. In the light of Remark 3.2 an equivalent definition for Λ is

(3.11) $\Lambda = \{P, Q \in \mathscr{L}(\mathscr{M}_s) \times \mathscr{L}(\mathscr{M}_s) \mid$ there exists a non-zero element ξ of $\mathbb{R}^N$ such that for all i, k, l, m, p, q in $\{1, \ldots, N\}$,

$$Q_{ijpq}\xi_j = 0,$$

$$P_{mkpq}\xi_i\xi_l + P_{ilpq}\xi_m\xi_k - P_{klpq}\xi_i\xi_m - P_{impq}\xi_k\xi_l = 0\}.$$

Theorem 3.2. *Let* $\mathbb{B}(P, Q)$ *be a quadratic form on* $\mathscr{L}(\mathscr{M}_s) \times \mathscr{L}(\mathscr{M}_s)$ *which is positive on* Λ. *Let* Ω *be an arbitrary domain of* $\mathbb{R}^N$ *and* P^ε, Q^ε *be two sequences of elements of* $L_2(\Omega; \mathscr{L}(\mathscr{M}_s))$ *such that, as* ε *tends to zero,*

(3.12) P^ε *and* Q^ε *converge weakly in* $L_2(\Omega; \mathscr{L}(\mathscr{M}_s))$ *to* P^0 *and* Q^0;

for any (ζ, η) *in* $\mathbb{R}^N \times \mathbb{R}^N$, $P^\varepsilon(\zeta \otimes \eta)$ *is a linearized strain tensor,* i.e. *there exists* ${}^{\zeta\eta}v^\varepsilon$ *in* $\mathbb{H}^1_{\mathrm{loc}}(\Omega)$ *such that* $P^\varepsilon(\zeta \otimes \eta) = \frac{1}{2}(\nabla^{\zeta\eta}v^\varepsilon + {}^t\nabla^{\zeta\eta}v^\varepsilon)$, $\operatorname{div}(Q^\varepsilon(\zeta \otimes \eta))$ *is in a compact set of* $\mathbb{H}^{-1}_{\mathrm{loc}}(\Omega)$.

Then $\mathbb{B}(P^\varepsilon, Q^\varepsilon)$ *is weakly lower semi-continuous in* $\mathscr{D}'(\Omega)$, i.e., *for any* φ *in* $\mathscr{C}_0^\infty(\Omega)$ *with* $\varphi \geqq 0$,

(3.13) $$\underline{\lim}_{\varepsilon \to 0} \int_\Omega \varphi \mathbb{B}(P^\varepsilon, Q^\varepsilon)\, dx \geqq \int_\Omega \varphi \mathbb{B}(P^0, Q^0)\, dx.$$

The proof of Theorem 3.2 is a mere repetition of the proof of Theorem 3.1 in the extended context of Definition 3.2.

We are now in a position to prove the

Theorem 3.3. *Let* $\mathbb{B}(P, Q)$ *be a quadratic form on* $\mathscr{L}(\mathscr{M}_s) \times \mathscr{L}(\mathscr{M}_s)$ *which is positive on* Λ *and let* $\mathbb{L}(P, Q)$ *be any linear form on* $\mathscr{L}(\mathscr{M}_s) \times \mathscr{L}(\mathscr{M}_s)$. *Define, for any given elasticity tensor* A,

(3.14) $$\mathscr{G}(A) = \sup_{P \in \mathscr{L}(\mathscr{M}_s)} [\mathbb{B}(P, A \circ P) + \mathbb{L}(P, A \circ P)].$$

Let A^ε be a sequence of elements of $\mathcal{M}(\alpha, \beta)$ which H-converges to an element A^0 of $\mathcal{M}(\alpha, \beta)$. Assume that $\mathcal{G}(A^\varepsilon)$ lies in $L_\infty(\mathbb{R}^N)$ and that

$$\mathcal{G}(A^\varepsilon) \text{ converges weak-* in } L_\infty(\mathbb{R}^N) \text{ to } \mathcal{G}^0. \tag{3.15}$$

Then, for almost every x in $\mathbb{R}^N$,

$$\mathcal{G}(A^0(x)) \leqq \mathcal{G}^0(x). \tag{3.16}$$

Remark 3.5. The elasticity tensors A^ε and A^0 considered in Theorem 3.3 do not have to satisfy (H) or (HI).

Proof of Theorem 3.3. For any positive φ in $\mathscr{C}_0^\infty(\mathbb{R}^N)$, we define Ω^φ to be a bounded open set of $\mathbb{R}^N$ which contains the support of φ, and we take ψ to be an element of $\mathscr{C}_0^\infty(\Omega^\varphi)$ with value 1 on the support of φ.

If P^0 is an arbitrary element of $\mathscr{L}(\mathcal{M}_s)$ we define the displacement field ${}^{pq}v^0$ as

$${}^{pq}v_i^0(x) = \psi\, P_{ijpq}^0 x_j. \tag{3.17}$$

The strain tensor ${}^{pq}e^0$ associated with ${}^{pq}v^0$ satisfies the relation

$${}^{pq}e_{ij}^0 = P_{ijpq}^0 \text{ on } \operatorname{supp}\varphi. \tag{3.18}$$

Since A^ε H-converges to A^0, the solution ${}^{pq}v^\varepsilon$ of

$$\begin{aligned} &{}^{pq}v^\varepsilon \in \mathbb{H}_0^1(\Omega^\varphi),\\ &{}^{pq}e^\varepsilon = \tfrac{1}{2}(\nabla^{pq}v^\varepsilon + {}^t\nabla^{pq}v^\varepsilon),\\ &{}^{pq}s^\varepsilon = A^{\varepsilon\, pq}e^\varepsilon,\\ &\operatorname{div}{}^{pq}s^\varepsilon = \operatorname{div}(A^{0\, pq}e^0), \end{aligned} \tag{3.19}$$

converges weakly in $\mathbb{H}_0^1(\Omega^\varphi)$ to ${}^{pq}v^0$ (*cf.* Remark 2.1) as ε tends to zero. Thus the fourth-order tensors E^ε, S^ε defined by

$$E_{ijpq}^\varepsilon = {}^{pq}e_{ij}^\varepsilon, \qquad S_{ijpq}^\varepsilon = {}^{pq}s_{ij}^\varepsilon = A_{ijkh}^\varepsilon\, {}^{pq}e_{kh}^\varepsilon, \tag{3.20}$$

converge weakly in $L_2^s(\Omega^\varphi)$ to the fourth-order tensors E^0 and S^0 defined by

$$E_{ijpq}^0 = {}^{pq}e_{ij}^0, \qquad S_{ijpq}^0 = A_{ijkh}^0\, {}^{pq}e_{kh}^0. \tag{3.21}$$

In view of (3.19), E^ε and S^ε satisfy the hypothesis (3.12) of Theorem 3.2 and the conclusion of that theorem applies to $\mathbb{B}(E^\varepsilon, S^\varepsilon)$. We obtain

$$\int_{\Omega^\varphi} \varphi \mathbb{B}(E^0, A^0 \circ E^0)\, dx \leqq \varliminf_{\varepsilon\to 0} \int_{\Omega^\varphi} \varphi \mathbb{B}(E^\varepsilon, A^\varepsilon \circ E^\varepsilon)\, dx. \tag{3.22}$$

Since linear forms are weakly continuous,

$$\int_{\Omega^\varphi} \varphi \mathbb{L}(E^0, A^0 \circ E^0)\, dx = \lim_{\varepsilon\to 0} \int_{\Omega^\varphi} \varphi \mathbb{L}(E^\varepsilon, A^\varepsilon \circ E^\varepsilon)\, dx. \tag{3.23}$$

Thus,

$$\int_{\Omega^\varphi} \varphi(\mathbb{B}(E^0, A^0 \circ E^0) + \mathbb{L}(E^0, A^0 \circ E^0))\, dx \le \underline{\lim}_{\varepsilon\to 0} \int_{\Omega^\varphi} \varphi(\mathbb{B}(E^\varepsilon, A^\varepsilon \circ E^\varepsilon) + \mathbb{L}(E^\varepsilon, A^\varepsilon \circ E^\varepsilon))\, dx. \tag{3.24}$$

On the support of φ, $E^0 = P^0$ and $\operatorname{supp}\varphi \subset \Omega^\varphi$; so that (3.24) reads

$$\int_{\mathbb{R}^N} \varphi(\mathbb{B}(P^0, A^0 \circ P^0) + \mathbb{L}(P^0, A^0 \circ P^0))\, dx \le \underline{\lim}_{\varepsilon\to 0} \int_{\mathbb{R}^N} \varphi(\mathbb{B}(E^\varepsilon, A^\varepsilon \circ E^\varepsilon) + \mathbb{L}(E^\varepsilon, A^\varepsilon \circ E^\varepsilon))\, dx. \tag{3.25}$$

By definition of $\mathscr{G}(A^\varepsilon)$,

$$\mathbb{B}(E^\varepsilon, A^\varepsilon \circ E^\varepsilon) + \mathbb{L}(E^\varepsilon, A^\varepsilon \circ E^\varepsilon) \le \mathscr{G}(A^\varepsilon). \tag{3.26}$$

We thus obtain

$$\int_{\mathbb{R}^N} \varphi(\mathbb{B}(P^0, A^0 \circ P^0) + \mathbb{L}(P^0, A^0 \circ P^0))\, dx \le \underline{\lim}_{\varepsilon\to 0} \int_{\mathbb{R}^N} \varphi\mathscr{G}(A^\varepsilon)\, dx, \tag{3.27}$$

and this holds for any P^0 in $\mathscr{L}(\mathscr{M}_s)$.

Since by hypothesis $\mathscr{G}(A^\varepsilon)$ converges weak-* in $L_\infty(\mathbb{R}^N)$ to $\mathscr{G}^0$, we deduce from (3.27) that for almost every x in $\mathbb{R}^N$,

$$\mathbb{B}(P^0, A^0(x) \circ P^0) + \mathbb{L}(P^0, A^0(x) \circ P^0) \le \mathscr{G}^0(x). \tag{3.28}$$

Taking the supremum of the left-hand side of (3.28) over all P^0's in $\mathscr{L}(\mathscr{M}_s)$ yields the result.

3.2. Necessary Conditions on the Bulk Modulus

An H-converging sequence A^ε of elements of $\mathscr{M}(\alpha, \beta)$ is considered. Our goal is to devise suitable quadratic and linear forms on $\mathscr{L}(\mathscr{M}_s) \times \mathscr{L}(\mathscr{M}_s)$ and to apply Theorem 3.3 in the hope of deriving valuable bounds from inequality (3.16).

Lemma 3.1. *For any positive real numbers $\bar\alpha$, $\bar\beta$, the quadratic forms*

$$\mathbb{K}^l(P, Q) = \bar\alpha[\operatorname{tr}({}^t(Pi) \circ Pi) - (\operatorname{tr} Pi)^2] - \operatorname{tr}({}^t(Qi) \circ Pi), \tag{3.29}$$

$$\mathbb{K}^u(P, Q) = [(N-1) \operatorname{tr}({}^t(Qi) \circ Qi) - (\operatorname{tr} Qi)^2] - \bar\beta(N-1) \operatorname{tr}({}^t(Qi) \circ Qi), \tag{3.30}$$

are positive on Λ.

Proof of Lemma 3.1. Let P, Q belong to Λ; ξ is the associated element of $\mathbb{R}^N$ (*cf.* (3.11)) which has been normalized.

All mappings are written in an orthonormal basis of $\mathbb{R}^N$ with ξ as first basis vector. For any p, q in $\{1, \ldots, N\}$,

$$(3.31)\qquad \begin{gathered} P_{ijpq} = 0, \quad 1 < i \leqq N, \quad 1 < j \leqq N, \quad P_{1kpq} = P_{k1pq}, \quad 1 \leqq k \leqq N, \\ Q_{1ipq} = Q_{i1pq} = 0, \quad 1 \leqq i \leqq N. \end{gathered}$$

The proof of the lemma reduces to a simple but lengthy computation of the quantities $\mathbb{K}^l(P, Q)$ and $\mathbb{K}^u(P, Q)$. The term $\operatorname{tr}({}^t(Qi) \circ Pi)$ is found to be null on Λ; the remaining terms are seen to be positive on Λ by simple inspection for the bracket in (3.29) and by application of the Cauchy-Schwarz inequality for the bracket in (3.30).

Lemma 3.2. *For any elasticity tensor A, we define*

$$(3.32)\qquad \mathscr{G}^l(A) = \sup_{P \in \mathscr{L}(\mathscr{M}_s)} [\mathbb{K}^l(P, A \circ P) + 2 \operatorname{tr}(Pi))],$$

$$(3.33)\qquad \mathscr{G}^u(A) = \sup_{P \in \mathscr{L}(\mathscr{M}_s)} [\mathbb{K}^u(P, A \circ P) + 2 \operatorname{tr}(A(Pi))].$$

If A is isotropic, with bulk modulus K and shear modulus μ ($K > 0$, $\mu > 0$), $\mathscr{G}^l(A)$ and $\mathscr{G}^u(A)$ are finite if and only if $\bar{\alpha} \leqq 2\mu \leqq \bar{\beta}$. Furthermore

$$(3.34)\qquad \mathscr{G}^l(A) = \frac{N}{NK + (N-1)\,\bar{\alpha}}, \quad \mathscr{G}^u(A) = N - \frac{N(N-1)\,\bar{\beta}}{NK + (N-1)\,\bar{\beta}}.$$

Proof of Lemma 3.2. The proof is sketched for $\mathscr{G}^u(A)$; an analogous proof applies to $\mathscr{G}^l(A)$.

Let P be an arbitrary element of $\mathscr{L}(\mathscr{M}_s)$, represented by P_{ijkh} in a given orthonormal basis of $\mathbb{R}^N$. We obtain

$$(3.35)\qquad \begin{aligned} &\mathbb{K}^u(P, A \circ P) + 2 \operatorname{tr}(A(Pi)) \\ &= [(N-1)(N\lambda^2 + 4\lambda\mu) - (N\lambda + 2\mu)^2 - \bar{\beta}(N-1)\,\lambda]\,(P_{kkpp})^2 \\ &\quad + 2(N-1)\,\mu\,(2\mu - \bar{\beta}) \sum_{i,j=1}^{N} (P_{ijpp})^2 + 2(N\lambda + 2\mu)\,(P_{kkpp}). \end{aligned}$$

With the help of the positivity conditions on K and μ the second-degree polynomial (3.35) is tediously seen to be bounded above if and only if $2\mu \leqq \bar{\beta}$. It reaches a maximal value at the point $\tilde{P}$ such that

$$(3.36)\qquad \begin{gathered} \tilde{P}_{ijpp} = 0, \quad i \neq j, \\ \sum_{p=1}^{N} \tilde{P}_{jjpp} = \frac{1}{N} \operatorname{tr}(\tilde{P}i), \quad j = 1, \ldots, N, ^{\dagger} \\ \operatorname{tr}(\tilde{P}i) = \frac{N}{NK + (N-1)\,\bar{\beta}}. \end{gathered}$$

† The summation convention is suppressed in this formula.

$\mathscr{G}^u(A)$ is the value of (3.35) at the point $\tilde{P}$,

$$\mathscr{G}^u(A) = (N\lambda + 2\mu)\operatorname{tr}(\tilde{P}i) = \frac{N^2K}{NK + (N-1)\bar{\beta}} = N - \frac{N(N-1)\bar{\beta}}{NK + (N-1)\bar{\beta}}. \tag{3.37}$$

We are now in a position to apply Theorem 3.3 to a sequence A^ε satisfying (HI) (*cf.* Subsection 2.2), *i.e.* to a sequence A^ε describing the mixture in prescribed volume fractions of two isotropic homogeneous materials. Recall that $\mu^1 \leqq \mu^2$.

Theorem 3.4. *Let A^ε be a sequence of elasticity tensors satisfying* (HI). *Assume that A^ε H-converges to an isotropic, not necessarily homogeneous, elasticity tensor A^0. Set, for almost every x in $\mathbb{R}^N$,*

$$A^0(x) = \lambda^0(x)\, i \otimes i + 2\mu^0(x)\, I. \tag{3.38}$$

The bulk modulus K^0 associated with A^0 satisfies

$$K^l(x) \leqq K^0(x) \leqq K^u(x), \text{ for almost every } x \text{ of } \mathbb{R}^N, \tag{3.39}$$

where, by definition,

$$\begin{aligned} \frac{1}{NK^l(x) + 2(N-1)\mu^1} &= \frac{\theta(x)}{NK^1 + 2(N-1)\mu^1} + \frac{1-\theta(x)}{NK^2 + 2(N-1)\mu^1}, \\ \frac{1}{NK^u(x) + 2(N-1)\mu^2} &= \frac{\theta(x)}{NK^1 + 2(N-1)\mu^2} + \frac{1-\theta(x)}{NK^2 + 2(N-1)\mu^2}, \end{aligned} \tag{3.40}$$

and where $\theta(x)$ given in (2.15) *is the local volume fraction of material* 1.

Remark 3.6. The bounds (3.39) on the bulk modulus hold true whether the moduli of the phases of the mixture satisfy (0.1) or not. They coincide with the bounds given by HASHIN & SHTRIKMAN [1] for the case $N = 3$ (*cf.* (0.3)); this results from a simple algebraic manipulation of (3.40). The algebraic form of (3.40) renders obvious the following remark due to HILL (HASHIN & SHTRIKMAN [1], p. 135 or HILL [1], p. 369): if both materials have identical shear modulus, the bulk modulus of the mixture is uniquely determined.

Proof of Theorem 3.4. Since the A^ε's satisfy (HI), they satisfy the hypotheses of Lemma 3.2 whenever

$$0 \leqq \bar{\alpha} \leqq 2\mu_1 \leqq 2\mu_2 \leqq \bar{\beta}. \tag{3.41}$$

We obtain, for almost every x in $\mathbb{R}^N$,

(3.42)

$$\mathscr{G}^l(A^\varepsilon(x)) = \frac{N}{NK^\varepsilon(x) + (N-1)\bar{\alpha}} = N\left(\frac{\chi^\varepsilon(x)}{NK^1 + (N-1)\bar{\alpha}} + \frac{1-\chi^\varepsilon(x)}{NK^2 + (N-1)\bar{\alpha}}\right),$$

$$\begin{aligned} \mathscr{G}^u(A^\varepsilon(x)) &= N - \frac{N(N-1)\bar{\beta}}{NK^\varepsilon(x) + (N-1)\bar{\beta}} \\ &= N - N(N-1)\bar{\beta}\left(\frac{\chi^\varepsilon(x)}{NK^1 + (N-1)\bar{\beta}} + \frac{1-\chi^\varepsilon(x)}{NK^2 + (N-1)\bar{\beta}}\right). \end{aligned}$$

The functions $\mathscr{G}^l(A^\varepsilon(x))$ and $\mathscr{G}^u(A^\varepsilon(x))$ are bounded in $L_\infty(\mathbb{R}^N)$ and their weak-* limits are explicitly computable; as ε tends to zero,

$$\begin{aligned}
&\mathscr{G}^l(A^\varepsilon) \rightharpoonup N\left(\frac{\theta}{NK^1+(N-1)\bar{\alpha}}+\frac{1-\theta}{NK^2+(N-1)\bar{\alpha}}\right),\\
&\mathscr{G}^u(A^\varepsilon) \rightharpoonup N - N(N-1)\bar{\beta}\left(\frac{\theta}{NK^1+(N-1)\bar{\beta}}+\frac{1-\theta}{NK^2+(N-1)\bar{\beta}}\right),
\end{aligned} \tag{3.43}$$

weak-* in $L_\infty(\mathbb{R}^N)$.

Theorem 3.3 is applicable in view of Lemma 3.1; it yields

$$\begin{aligned}
&\mathscr{G}^l(A^0) \leqq N\left(\frac{\theta}{NK^1+(N-1)\bar{\alpha}}+\frac{1-\theta}{NK^2+(N-1)\bar{\alpha}}\right),\\
&\mathscr{G}^u(A^0) \leqq N - N(N-1)\bar{\beta}\left(\frac{\theta}{NK^1+(N-1)\bar{\beta}}+\frac{1-\theta}{NK^2+(N-1)\bar{\beta}}\right).
\end{aligned} \tag{3.44}$$

The right-hand side of inequalities (3.44) is finite almost everywhere, and thus $\mathscr{G}^l(A^0)$ and $\mathscr{G}^u(A^0)$ are also finite almost everywhere. For future reference, it should be emphasized that the isotropic character of A^0 has *not* been used in establishing (3.44).

If A^0 is isotropic, it is of the form (3.38). Lemma 3.2 implies that

$$\bar{\alpha} \leqq 2\mu^0(x) \leqq \bar{\beta}, \text{ for almost every } x \text{ in } \mathbb{R}^N, \tag{3.45}$$

and

$$\mathscr{G}^l(A^0) = \frac{N}{NK^0+(N-1)\bar{\alpha}}, \quad \mathscr{G}^u(A^0) = N - \frac{N(N-1)\bar{\beta}}{NK^0+(N-1)\bar{\beta}}. \tag{3.46}$$

Replacing $\mathscr{G}^l(A^0)$ and $\mathscr{G}^u(A^0)$ by their expressions in (3.44) yields

$$K(\bar{\alpha}, x) \leqq K^0(x) \leqq K(\bar{\beta}, x) \tag{3.47}$$

for almost every x in $\mathbb{R}^N$, where $K(\tilde{\alpha}, x)$ is defined as

$$\frac{1}{NK(\tilde{\alpha}, x)+(N-1)\tilde{\alpha}} = \frac{\theta(x)}{NK^1+(N-1)\tilde{\alpha}}+\frac{1-\theta(x)}{NK^2+(N-1)\tilde{\alpha}}, \tag{3.48}$$

for any positive real number $\tilde{\alpha}$.

The mapping

$$\tilde{\alpha} \to K(\tilde{\alpha}, x) \tag{3.49}$$

is monotonic increasing on $\mathbb{R}^+$ for almost every x in $\mathbb{R}^N$. Thus, within the admissible range (3.41), the sharpest inequalities implied by (3.47) are obtained for $\bar{\alpha} = 2\mu^1$ and $\bar{\beta} = 2\mu^2$. Since $K^l(x) = K(2\mu^1, x)$ and $K^u(x) = K(2\mu^2, x)$, inequalities (3.39) are proved.

Remark 3.7. Since $K(0, x)$ is the harmonic mean of K^1 and K^2 in proportions $\theta(x)$, $1-\theta(x)$, whereas $\lim_{\tilde{\alpha}\to+\infty} K(\tilde{\alpha}, x)$ is the arithmetic mean of K^1 and K^2 in the same proportions, the inequality (3.47) also yields PAUL's bounds on the bulk modulus, *i.e.*,

$$(3.50)\qquad \left(\frac{\theta(x)}{K^1}+\frac{1-\theta(x)}{K^2}\right)^{-1} \leqq K^0(x) \leqq \theta(x)\,K^1 + (1-\theta(x))\,K^2,$$

for almost every x of $\mathbb{R}^N$.

It was remarked in the proof of Theorem 3.4 that inequalities (3.44) hold true whether or not A^0 is isotropic. This observation is at the root of the following

Corollary 3.1. *Let A^ε be a sequence of elasticity tensors satisfying* (HI). *Let A^0 denote the H-limit of A^ε. Then necessarily*

$$(3.51)\qquad K^l(x) \leqq [\operatorname{tr}((A^0(x))^{-1}\,i)]^{-1} \leqq \frac{\operatorname{tr}(A^0(x)\,i)}{N^2} \leqq K^u(x),$$

for almost every x of $\mathbb{R}^N$, where K^l and K^u are defined in (3.40).

Proof of Corollary 3.1. The tensor field $A^0(x)$ is almost everywhere positive-definite. We denote by $(A^0(x))^{\frac{1}{2}}$ its positive-definite square root in $\mathscr{L}_s(\mathscr{M}_s)$. The Cauchy-Schwarz inequality yields

$$(3.52)\qquad \begin{aligned}[][ii]^2 &= [((A^0(x))^{\frac{1}{2}}\,i)\;((A^0(x))^{-\frac{1}{2}}\,i)]^2\\ &\leqq [((A^0(x))^{\frac{1}{2}}\,i)\;((A^0(x))^{\frac{1}{2}}\,i)]\;[((A^0(x))^{-\frac{1}{2}}\,i)\;((A^0(x))^{-\frac{1}{2}}\,i)]\\ &= [(A^0(x)\,i)\;i]\;[((A^0(x))^{-1}\,i)\;i] = \operatorname{tr}(A^0(x)\,i)\,\operatorname{tr}((A^0(x))^{-1}\,i),\end{aligned}$$

which reads as

$$(3.53)\qquad [\operatorname{tr}((A^0(x))^{-1}\,i)]^{-1} \leqq \frac{\operatorname{tr}(A^0(x)\,i)}{N^2}.$$

Inequalities (3.44) are now specialized to the case $\bar{\alpha} = 2\mu^1$, $\bar{\beta} = 2\mu^2$. By virtue of the definitions of $\mathscr{G}^l$ and of $\mathscr{G}^u$ and with (3.40), we obtain for almost every x in $\mathbb{R}^N$ and for every P in $\mathscr{L}(\mathscr{M}_s)$

$$(3.54)\qquad \mathbb{K}^l(P, A^0(x)\circ P) + 2\operatorname{tr}(Pi) - \frac{N}{NK^l(x) + 2(N-1)\,\mu^1} \leqq 0,$$

$$(3.55)\qquad \mathbb{K}^u(P, A^0(x)\circ P) + 2\operatorname{tr}(A^0(x)(Pi)) - N + \frac{2N(N-1)\,\mu^2}{NK^u(x) + 2(N-1)\,\mu^2} \leqq 0.$$

Inequalities (3.51) result from a proper choice of P in (3.54) and (3.55). Specifically, we choose P to be of the form $\tilde{\tilde{\alpha}}((A^0(x))^{-1}\,i)$ in (3.54) and $\tilde{\tilde{\alpha}}i$ in (3.55), where $\tilde{\tilde{\alpha}}$ is *any* real number. We obtain two polynomials of second degree in $\tilde{\tilde{\alpha}}$ which must remain negative for all $\tilde{\tilde{\alpha}}$'s in $\mathbb{R}$. The ensuing conditions on the discriminants yield the first and last inequalities of (3.51).

Remark 3.8. If specialized to an isotropic A^0, inequalities (3.51) reduce to the bounds (3.39) of Theorem 3.4 on the bulk modulus. In a recent paper KANTOR & BERGMAN [1] give the following bounds on the effective properties of the mixture in prescribed volume fractions of two isotropic phases:

$$K^l \leqq \frac{1}{N^2} A^0_{iikk} \leqq K^u. \tag{3.56}$$

Inequalities (3.56) are a mere rewriting of relations (3.13) and (3.18)† of KANTOR & BERGMAN [1] in our system of notation. The bound (3.51) givs the same upper bound as (3.56) but a better and more "symmetric" lower bound.

3.3. Necessary Conditions on the Shear Modulus

The A^ε's under consideration are still assumed to satisfy (HI). The derivation of bounds on the shear modulus μ^0 of the H-limit of the A^ε's requires the introduction of additional quadratic forms involving the trace operator Tr (*cf.* Section 1). The quadratic forms to be considered are

$$\begin{aligned} &\mathrm{M}^l(P, Q) = \bar{\bar{\alpha}}[N \operatorname{Tr}({}^t\bar{P} \circ \bar{P}) - (\operatorname{Tr} \bar{P})^2] - \operatorname{Tr}({}^t\bar{Q} \circ \bar{P}), \\ &\mathrm{M}^u(P, Q) = \left[\frac{N(N-1)}{2} \operatorname{Tr}({}^t\bar{Q} \circ \bar{Q}) - (\operatorname{Tr} \bar{Q})^2\right] - \frac{N^2(N-1)}{2} \bar{\bar{\beta}} \operatorname{Tr}({}^t\bar{Q} \circ \bar{P}) \end{aligned} \tag{3.57}$$

where

$$\bar{P} = P - \frac{Pi \otimes i}{N}, \quad \bar{Q} = Q - \frac{Qi \otimes i}{N}. \tag{3.58}$$

The associated linear forms are taken to be respectively $2 \operatorname{Tr} \bar{P}$ and $2 \operatorname{Tr} \bar{Q}$ The analogues of Lemmas 3.1 and 3.2 hold with

$$0 \leqq N\bar{\bar{\alpha}} \leqq \inf(NK, 2\mu) \leqq \sup(NK, 2\mu) \leqq N\bar{\bar{\beta}}, \tag{3.59}$$

and an analogue of Theorem 3.4 can be proved, namely

Theorem 3.5. *In the setting of Theorem 3.4. the shear modulus μ^0 associated with A^0 satisfies*

$$\mu^l(x) \leqq \mu^0(x) \leqq \mu^u(x) \text{ for almost every } x \text{ of } \mathbb{R}^N, \tag{3.60}$$

with

$$\begin{aligned} &\frac{1}{2\mu^l(x) + \left(\frac{N(N-3)}{2} + N - 1\right)\underline{K}} \\ &= \frac{\theta(x)}{2\mu^1 + \left(\frac{N(N-3)}{2} + N - 1\right)\underline{K}} + \frac{1-\theta(x)}{2\mu^2 + \left(\frac{N(N-3)}{2} + N - 1\right)\underline{K}} \\ &\frac{1}{2\mu^u(x) + \frac{N^2}{2}\bar{K}} = \frac{\theta(x)}{2\mu^1 + \frac{N^2}{2}\bar{K}} + \frac{1-\theta(x)}{2\mu^2 + \frac{N^2}{2}\bar{K}}, \end{aligned} \tag{3.61}$$

† Note that there are misprints in inequalities (3.18), (3.22) of KANTOR & BERGMAN [1], which give bounds on the "generalized" bulk and shear moduli. As regards the inequalities (3.18) on the bulk modulus, the last $\varkappa^{(1)}$ should read $\varkappa^{(2)}$.

where, by definition,

(3.62) $\quad N\underline{K} = \inf(NK^1, NK^2, 2\mu^1), \qquad N\overline{K} = \sup(NK^1, NK^2, 2\mu^2).$

Remark 3.9. The bounds (3.60), (3.61) do not coincide with the bounds given by HASHIN & SHTRIKMAN [1] (*cf.* (0.2), (0.3)) and the resulting admissible interval is larger and almost always strictly larger. It is however worth pointing out that our bounds are better bounds than the classical bounds obtained by PAUL (*cf.* (0.4), (0.5)).

4. Sufficient Conditions

The proof of the optimality of a set of bounds can be conveniently achieved through explicit computation. In this section we analyze the H-limits of sequences of "multilayered" elastic materials. In Subsection 4.1 we derive an explicit formula for the elastic material resulting from the multiple layering of *any* homogeneous elastic material into another. Thus attention is *not* restricted to isotropic components. The computation is largely inspired by that performed by TARTAR for the case of heat conduction (*cf.* Propositions 3 and 4 of TARTAR [5]). Subsection 4.2 is devoted to the determination of the possible isotropic H-limits obtained by "multilayers" of two isotropic phases, and to the computation of the associated bulk and shear moduli.

4.1. Multiple Layering

Let us consider two *homogeneous but not necessarily isotropic* elastic materials. We construct a mixture of these two materials by layering material 2 with material 1 in a given direction. The elasticity tensor A^0 associated with the resulting material is explicitly computable. Specifically, the following theorem holds:

Theorem 4.1. *Let ξ be an arbitrary non zero vector in $\mathbb{R}^N$ and χ^ε be a sequence of characteristic step functions (defined on $\mathbb{R}$) which converges to the constant θ weak-* in $L_\infty(\mathbb{R})$, with $0 < \theta < 1$. Let A^1 and A^2 be two elements of $\mathcal{M}(\alpha, \beta)$ associated with two homogeneous, not necessarily isotropic elastic materials. Then the sequence*

$$A^\varepsilon(x) = A^1\chi^\varepsilon(x\xi) + A^2(1 - \chi^\varepsilon(x\xi)) \tag{4.1}$$

H-converges to A^ξ given by

$$A^\xi h = A^1 h = A^2 h \quad \textit{for } h \textit{ in } \operatorname{Ker}(A^2 - A^1), \tag{4.2}$$

$$(A^\xi - A^1)^{-1} h = \frac{(A^2 - A^1)^{-1} h}{1 - \theta} + \frac{1}{2}\frac{\theta}{1-\theta}\{[q(\xi) \circ h \circ m(\xi)] + {}^t[q(\xi) \circ h \circ m(\xi)]\}$$
$$\textit{for } h \textit{ in } \operatorname{Ker}(A^2 - A^1)^\perp,$$

where the second-order tensor $q(\xi)$ *is defined by*

$$((q(\xi))^{-1}\,\zeta)\,\eta=\left(A^1\left(\zeta\otimes\frac{\xi}{|\xi|}\right)\right)\left(\eta\otimes\frac{\xi}{|\xi|}\right)\quad \textit{for any } (\zeta,\eta) \textit{ in } \mathbb{R}^N\times\mathbb{R}^N, \tag{4.3}$$

and where

$$m(\xi)=\frac{\xi\otimes\xi}{|\xi|^2}. \tag{4.4}$$

Remark 4.1. For a given non zero ξ in $\mathbb{R}^N$, $q(\xi)$ is a well defined element of $\mathscr{M}_s$. Indeed, since A^1 lies in $\mathscr{M}(\alpha,\beta)$,

$$((q(\xi))^{-1}\,\zeta)\,\zeta=\left(A^1\left(\zeta\otimes\frac{\xi}{|\xi|}\right)\right)\left(\zeta\otimes\frac{\xi}{|\xi|}\right)\geqq\alpha\,|\zeta|^2\quad \text{for any } \zeta \text{ in } \mathbb{R}^N, \tag{4.5}$$

thus $(q(\xi)^{-1})$ is an invertible element of $\mathscr{M}_s$.

Let us postpone for now the proof of Theorem 4.1 and specialize the result to the case of an isotropic A^1. We obtain the following

Corollary 4.1. *In the context of Theorem 4.1, if the material 1 is further assumed to be isotropic* (cf. (2.1)), A^ξ *is given by*

$$A^\xi h=A^1h=A^2h\quad \textit{for } h \textit{ in } \operatorname{Ker}(A^2-A^1), \tag{4.6}$$

$$(A^\xi-A^1)^{-1}\,h=\frac{(A^2-A^1)^{-1}\,h}{1-\theta}$$
$$+\frac{\theta}{1-\theta}\left\{\frac{1}{2\mu^1}(h\circ m(\xi)+m(\xi)\circ h)-\frac{\lambda^1+\mu^1}{\mu^1(\lambda^1+2\mu^1)}\operatorname{tr}({}^t m(\xi)\circ h)m(\xi)\right\}$$
$$\textit{for } h \textit{ in } \operatorname{Ker}(A^2-A^1)^\perp.$$

Proof of Corollary 4.1. The proof is a simple computation of $q(\xi)$ when $A^1=\lambda^1\,i\otimes i+2\mu^1\,I$. We obtain

$$(q(\xi))^{-1}=\mu^1 i+(\lambda^1+\mu^1)\,m(\xi), \tag{4.7}$$

and thus

$$q(\xi)=\frac{1}{\mu^1}\,i-\frac{\lambda^1+\mu^1}{\mu^1(\lambda^1+2\mu^1)}\,m(\xi). \tag{4.8}$$

Remark 4.2. If both materials are isotropic, the tensor A^2-A^1 is invertible if and only if $K^1\neq K^2$ and $\mu^1\neq\mu^2$. Any element h of $\mathscr{M}_s$ can be decomposed into its hydrostatic part, $\frac{\operatorname{tr}h}{N}\,i$, and its deviatoric part $d=h-\frac{\operatorname{tr}h}{N}\,i$.

If $K^1=K^2$, Ker A^2-A^1 is the 1-dimensional subspace of $\mathscr{M}_s$ of all hydrostatic tensors whereas, if $\mu^1=\mu^2$, Ker A^2-A^1 is the $\left(\frac{N(N+1)}{2}-1\right)$-dimensional subspace of $\mathscr{M}_s$ of all purely deviatoric tensors.

We now address the proof of Theorem 4.1.

Proof of Theorem 4.1. Let a^1 and a^2 be two arbitrary elements of $\mathcal{M}_s$ and set

$$e^\varepsilon = \chi^\varepsilon a^1 + (1-\chi^\varepsilon)\, a^2, \qquad s^\varepsilon = A^\varepsilon e^\varepsilon = \chi^\varepsilon A^1 a^1 + (1-\chi^\varepsilon)\, A^2 a^2. \tag{4.9}$$

The tensor field e^ε is a strain tensor if and only if it satisfies the compatibility conditions (3.7), *i.e.*

(4.10)

$$(a^1 - a^2)_{mk}\, \xi_i \xi_l + (a^1 - a^2)_{il}\, \xi_m \xi_k - (a^1 - a^2)_{kl}\, \xi_i \xi_m - (a^1 - a^2)_{im}\, \xi_k \xi_l = 0$$

for all i, k, l, m in $\{1, \ldots, N\}$.

Similarly the divergence of the tensor field s^ε lies in a compact set of $\mathbb{H}^{-1}_{\mathrm{loc}}(\mathbb{R}^N)$ if and only if

$$(A^1 a^1 - A^2 a^2)\, \xi = 0. \tag{4.11}$$

According to Remark 3.2, $(a^1 - a^2, A^1 a^1 - A^2 a^2)$ belongs to Λ, which implies the existence of a vector τ in $\mathbb{R}^N$ such that

$$a^1 = a^2 + \tau \otimes \xi + \xi \otimes \tau. \tag{4.12}$$

Relation (4.11) becomes

$$(A^1(\tau \otimes \xi + \xi \otimes \tau))\, \xi = ((A^2 - A^1)\, a^2)\, \xi. \tag{4.13}$$

By virtue of (4.3), the left-hand side of (4.13) contracted with an arbitrary vector η of $\mathbb{R}^N$ reads as

$$\begin{aligned} ((A^1(\tau \otimes \xi + \xi \otimes \tau))\, \xi)\, \eta &= 2\, |\xi|^2 \left(A^1\left(\tau \otimes \frac{\xi}{|\xi|}\right)\right)\left(\eta \otimes \frac{\xi}{|\xi|}\right) \\ &= 2\, |\xi|^2\, ((q(\xi))^{-1}\, \tau)\, \eta. \end{aligned} \tag{4.14}$$

If h is defined to be

$$h = (A^2 - A^1)\, a^2, \tag{4.15}$$

the relation (4.13) reduces to

$$2\, |\xi|^2\, ((q(\xi))^{-1}\, \tau)\, \eta = (h\xi)\eta \qquad \text{for any } \eta \text{ in } \mathbb{R}^N, \tag{4.16}$$

i.e.

$$\tau = \frac{1}{2\,|\xi|^2}\, q(\xi)\, (h\xi) = \tfrac{1}{2}(q(\xi) \circ h)\, \frac{\xi}{|\xi|^2}. \tag{4.17}$$

We obtain

$$\tau \otimes \xi + \xi \otimes \tau = \tfrac{1}{2}\, \{q(\xi) \circ h \circ m(\xi) + {}^t[q(\xi) \circ h \circ m(\xi)]\}. \tag{4.18}$$

In view of (4.12), (4.15) and (4.17), the second-order tensor a^1 and the vector τ are uniquely determined as functions of a^2, and a^2 can be arbitrarily chosen in $\mathcal{M}_s$.

Theorem 2.1 implies the existence of a H-converging subsequence of A^ε. Let A^ξ denote its H-limit. Under the above choices of a^1 and of τ, the tensors e^ε and s^ε defined in (4.9) are such that, as ε tends to zero,

$$\begin{aligned} &e^\varepsilon \text{ is a strain tensor, } \operatorname{div} s^\varepsilon = 0, \\ &e^\varepsilon \rightharpoonup e^0 = \theta a^1 + (1-\theta)\, a^2, \\ &s^\varepsilon \rightharpoonup s^0 = \theta A^1 a^1 + (1-\theta)\, A^2 a^2, \end{aligned} \tag{4.19}$$

where the convergences are to be understood as weak-* convergences in $L^s_\infty(\mathbb{R}^N)$.

The properties (4.19) meet all the requirements of Definition 2.1 of H-limits. We obtain

$$s^0 = A^\xi e^0, \tag{4.20}$$

or equivalently,

$$(A^\xi - A^1)\,(\theta a^1 + (1-\theta)\, a^2) = (1-\theta)\,(A^2 - A^1)\, a^2. \tag{4.21}$$

Using (4.12), (4.15) and (4.18) in (4.21) yields

$$(A^\xi - A^1)\left[a^2 + \frac{\theta}{2}\{[q(\xi)\circ h\circ m(\xi)] + {}^t[q(\xi)\circ h\circ m(\xi)]\}\right] = (1-\theta)\, h. \tag{4.22}$$

The result is then obtained by decomposing a^2 along $\operatorname{Ker}(A^2 - A^1)$ and $\operatorname{Ker}(A^2 - A^1)^\perp$ in (4.22). If a^2 belongs to $\operatorname{Ker}(A^2 - A^1)$, h is null and

$$(A^\xi - A^1)\, a^2 = 0; \tag{4.23}$$

the first assertion of (4.2) is proved. Let a^2 belong to $\operatorname{Ker}(A^2 - A^1)^\perp$ and $(A^2 - A^1)^{-1}$ denote the inverse mapping of the restriction of $(A^2 - A^1)$ to $\operatorname{Ker}(A^2 - A^1)^\perp$. Then h defined by (4.15) belongs to $\operatorname{Ker}(A^2 - A^1)^\perp$ and (4.22) reads as

$$(A^\xi - A^1)\, S^\xi h = h \tag{4.24}$$

where

$$S^\xi h = \frac{(A^2 - A^1)^{-1}\, h}{1-\theta} + \tfrac{1}{2}\frac{\theta}{1-\theta}\{[q(\xi)\circ h\circ m(\xi)] + {}^t[q(\xi)\circ h\circ m(\xi)]\}. \tag{4.25}$$

In view of (4.4), the positivity of A^2 and (4.3) applied to

$$\zeta = \eta = q(\xi)\left(h\frac{\xi}{|\xi|}\right), \tag{4.26}$$

we obtain

$$\begin{aligned} &(S^\xi h)\,((A_2 - A_1)\,\tfrac{1}{2}\{[q(\xi)\circ h\circ m(\xi)] + {}^t[q(\xi)\circ h\circ m(\xi)]\}) \\ &\qquad \geqq \left(q(\xi)\left(h\frac{\xi}{|\xi|}\right)\right)\left(h\frac{\xi}{|\xi|}\right). \end{aligned} \tag{4.27}$$

If h belongs to Ker S^ξ, (4.27) and the positive-definiteness (4.5) of $q(\xi)$ imply that

$$h\xi = 0; \tag{4.28}$$

then

$$q(\xi) \circ h \circ m(\xi) = 0, \tag{4.29}$$

and (4.25) yields

$$h = 0. \tag{4.30}$$

Thus S^ξ is invertible on Ker $(A^2 - A^1)^\perp$, and the second assertion of (4.2) is proved.

Since (4.2) completely determines A^ξ, the H-limit is independent of the H-converging subsequence and the sequence A^ε itself H-converges to A^ξ.

In view of its specific form, formula (4.2) is very convenient when multiple layering occurs, which is to be contrasted with previous layering formulae (see *e.g.* BACKUS [1], CHRISTENSEN [1] and references therein). The multiple layering process consists in a repeated application of Theorem 4.1. Specifically, Theorem 4.1 is applied to A^1 with volume fraction $\bar\theta^1$ and A^2 with volume fraction $1 - \bar\theta^1$, in the direction ξ^1. Then Theorem 4.1 is applied to A^1 with volume fraction $\bar\theta^2$ and to A^{ξ^1} with volume fraction $1 - \bar\theta^2$, in the direction ξ^2. The procedure is repeated p times. The H-limit A^{ξ^p} is denoted by A^0. We obtain the following

Proposition 4.2. *Let $\xi^1, \ldots, \xi^p$ be p non zero vectors in $\mathbb{R}^N$. Let θ be the volume fraction of material* 1, $1 - \theta$ *the volume fraction of material* 2 $(0 < \theta < 1)$. *The following class $\mathscr{E}^p$ of elastic materials can be achieved through "multiple layering" of material 2 into material 1:*

$\mathscr{E}^p$ is the set of all A^0 in $\mathscr{L}_s(\mathscr{M}_s)$ such that there exist $\theta^1, \ldots, \theta^p$ in (0, 1) *with $\sum_{i=1}^{p} \theta^i = 1$, and such that*

$$\begin{aligned} &A^0 h = A^1 h = A^2 h \quad \text{if } h \text{ lies in Ker}\,(A^2 - A^1), \\ &(A^0 - A^1)^{-1}\, h = \frac{(A^2 - A^1)^{-1}\, h}{1-\theta} + \frac{\theta}{1-\theta} Xh \quad \text{if } h \text{ lies in Ker}\,(A^2 - A^1)^\perp, \end{aligned} \tag{4.31}$$

where

$$Xh = \tfrac{1}{2} \sum_{i=1}^{p} \theta^i \{[q(\xi^i) \circ h \circ m(\xi^i)] + {}^t[q(\xi^i) \circ h \circ m(\xi^i)]\}. \tag{4.32}$$

Proof of Theorem 4.2. As has already been mentioned, the proof consists in a repeated application of Theorem 4.1. The volume fraction $\bar\theta^i$ is chosen in the following manner:

$$\bar{\theta}^0 = 0,$$

$$\theta^i = \frac{(1-\bar{\theta}^0)\dots(1-\bar{\theta}^{i-1})\,\bar{\theta}^i}{\theta}. \tag{4.33}$$

The $\bar{\theta}^i$'s are determined through an induction process once the θ^i's are given.

Remark 4.3. In the context of Corollary 4.1 (*i.e.* if A^1 is isotropic), (4.32) reads as

(4.34)

$$Xh = \sum_{i=1}^{p} \theta^i \left\{\frac{1}{2\mu^1}(h \circ m(\xi^i) + m(\xi^i) \circ h) - \frac{(\lambda^1+\mu^1)}{\mu^1(\lambda^1+2\mu^1)} \operatorname{tr}({}^t m(\xi^i) \circ h)\, m(\xi^i)\right\}.$$

Remark 4.4. An intrinsic characterization of $\mathscr{E}^p$ has yet to be found, *even* in the case when A^1 is isotropic. In this latter case however, all elements of $\mathscr{E}^p$, $p \geqq 1$, must satisfy

$$(A^0 - A^1)^{-1} = \frac{(A^2 - A^1)^{-1}}{1-\theta} + \frac{\theta}{1-\theta} X, \tag{4.35}$$

where X is an element of $\mathscr{L}_s(\mathscr{M}_s)$ such that

$$\operatorname{Tr} X = \frac{(N-1)\lambda^1 + 2N\mu^1}{2\mu^1(\lambda^1 + 2\mu^1)},$$

$$\operatorname{tr}(Xi) = \frac{1}{\lambda^1 + 2\mu^1}. \tag{4.36}$$

4.2. Isotropic Materials Resulting from the Multiple Layering of Two Isotropic Constituents

Both materials are now assumed to be *isotropic*, and for the sake of simplicity we also assume that K^1 is different from K^2 and μ^1 from μ^2, so that $A^2 - A^1$ *is invertible*. We do *not* impose the ordering restriction (0.1).

We examine the conditions under which the elastic tensor A^0 resulting from a "multiple layering" of material 2 into material 1 (*cf.* Theorem 4.2) is isotropic. These conditions will be shown to uniquely determine such an A^0.

Theorem 4.3. *There is at most one isotropic A^0 that can be obtained through the multiple layering process defined in Theorem 4.2.*

Proof of Theorem 4.3. If A^0 is of the form

$$A^0 = \lambda^0 i \otimes i + 2\mu^0 I, \tag{4.37}$$

the elasticity tensor X associated with A^0 through (4.31) has to be isotropic. We set

$$X = -\frac{\bar{\lambda}}{2N\bar{\mu}\bar{K}}\, i \otimes i + \frac{1}{2\bar{\mu}} I, \qquad \bar{K} = \frac{1}{N}(N\bar{\lambda} + 2\bar{\mu}). \tag{4.38}$$

The constants $\bar{\lambda}$, $\bar{\mu}$, $\bar{K}$ are uniquely determined by (4.36) at least if the dimension N is greater than one. We obtain

$$\begin{gathered} \bar{\mu} = \frac{(N(N+1)-2)\,\mu^1(\lambda^1 + 2\mu^1)}{2(N-1)\,(K^1 + 2\mu^1)}, \\ \bar{K} = \lambda^1 + 2\mu^1. \end{gathered} \tag{4.39}$$

We are now in a position to determine A^0 from (4.31). The constants μ^0 and $K^0 = \frac{1}{N}(N\lambda^0 + 2\mu^0)$ are found to satisfy

$$\begin{gathered} \frac{1-\theta}{2(\mu^0 - \mu^1)} = \frac{1}{2(\mu^2 - \mu^1)} + \frac{\theta(N-1)\,(K^1 + 2\mu^1)}{(N(N+1)-2)\,\mu^1(\lambda^1 + 2\mu^1)}, \\ \frac{1-\theta}{K^0 - K^1} = \frac{1}{K^2 - K^1} + \frac{\theta}{\lambda^1 + 2\mu^1}. \end{gathered} \tag{4.40}$$

The resulting μ^0 and K^0 are

$$K^0 = K^1 + \frac{1-\theta}{\dfrac{1}{K^2 - K^1} + \dfrac{N\theta}{NK^1 + 2(N-1)\,\mu^1}}, \tag{4.41}$$

$$\mu^0 = \mu^1 + \frac{1-\theta}{\dfrac{1}{\mu^2 - \mu^1} + \dfrac{2N(N-1)\,\theta(K^1 + 2\mu^1)}{(N(N+1)-2)\,\mu^1\,(NK^1 + 2(N-1)\,\mu^1)}}. \tag{4.42}$$

Remark 4.5. If we specialize the above results to the dimension $N = 3$, we recover the expressions for the lower bounds (0.3) in *both* bulk and shear modulus. Thus the *only* isotropic material that can be achieved through the procedure described in Theorem 4.2 has a K^0 and a μ^0 that coincide with the lower bounds (0.3) given by HASHIN & SHTRIKMAN. Of course permutation of material 2 and material 1 would yield a material whose K^0 and μ^0 coincide with the upper bounds (0.3) given by HASHIN & SHTRIKMAN.

It remains to show that such a material is in effect achieved through "multiple layering". This last step is performed in dimension two or three.

The computation is outlined in the case of dimension three. We place ourselves in $\mathscr{E}^6$ defined in Theorem 4.2 and choose $\xi^1, \ldots, \xi^6$ on S^2, the unit sphere in $\mathbb{R}^3$. The vector ξ^1 is the direction vector associated with the north pole. The vectors $\xi^2, \dots, \xi^6$ are the direction vectors associated with 5 equidistributed points on the circle that results from the intersection of S^2 with a plane located at $\frac{\sqrt{5}}{5}$ above the equatorial plane. Setting

$$\alpha = \frac{\pi}{5}, \quad \beta = \tfrac{1}{2} \operatorname{Arcos} \frac{\sqrt{5}}{5},$$

we obtain

$$\xi^1 = \begin{vmatrix} 0 \\ 0 \\ 1 \end{vmatrix}, \quad \xi^2 = \begin{vmatrix} \sin 2\beta \\ 0 \\ \cos 2\beta \end{vmatrix}, \quad \xi^3 = \begin{vmatrix} \sin 2\beta \cos 2\alpha \\ \sin 2\beta \sin 2\alpha \\ \cos 2\beta \end{vmatrix},$$

(4.43)

$$\xi^4 = \begin{vmatrix} \sin 2\beta \cos 4\alpha \\ \sin 2\beta \sin 4\alpha \\ \cos 2\beta \end{vmatrix}, \quad \xi^5 = \begin{vmatrix} \sin 2\beta \cos 6\alpha \\ \sin 2\beta \sin 6\alpha \\ \cos 2\beta \end{vmatrix}, \quad \xi^6 = \begin{vmatrix} \sin 2\beta \cos 8\alpha \\ \sin 2\beta \sin 8\alpha \\ \cos 2\beta \end{vmatrix}.$$

Note that the six directions ξ^i are those of the northern hemisphere vertices of the regular icosahedron.

The θ^i's which appear in Theorem 4.2 are chosen to be

$$\theta^1 = \dots = \theta^6 = 1/6. \tag{4.44}$$

We set $\zeta^1, \zeta^2, \zeta^3$ to be a canonical orthonormal basis of $\mathbb{R}^3$. The computation of X is addressed in the following orthonormal basis of $\mathcal{M}_s$ for the inner product on $\mathcal{M}_s$ (*cf.* Section 1):

$$e^1 = \zeta^1 \otimes \zeta^1, \quad e^2 = \zeta^2 \otimes \zeta^2, \quad e^3 = \zeta^3 \otimes \zeta^3,$$

(4.45) $$e^4 = \frac{\sqrt{2}}{2}(\zeta^1 \otimes \zeta^2 + \zeta^2 \otimes \zeta^1), \quad e^5 = \frac{\sqrt{2}}{2}(\zeta^1 \otimes \zeta^3 + \zeta^3 \otimes \zeta^1),$$

$$e^6 = \frac{\sqrt{2}}{2}(\zeta^2 \otimes \zeta^3 + \zeta^3 \otimes \zeta^2).$$

The computation is organized as follows: a composition table $e^i \otimes e^j$ is produced; the components of the terms $h \circ m(\xi^i) + m(\xi^i) \circ h$ and $\operatorname{tr}({}^t m(\xi^i) \circ h)\, m(\xi^i)$ are calculated in the basis $e^1, \dots, e^6$. After a tedious computation, we obtain

$$X = -\frac{\lambda^1 + \mu^1}{15\mu^1(\lambda^1 + 2\mu^1)}\, i \otimes i + \frac{3\lambda^1 + 8\mu^1}{15\mu^1(\lambda^1 + 2\mu^1)}\, I, \tag{4.46}$$

which is precisely the X foreseen in (4.38), (4.39) in the three-dimensional case.

The same method applies to the case $N = 2$; three directions have to be used, namely, if ζ^1, ζ^2 is a canonical orthonormal basis of $\mathbb{R}^2$,

$$\zeta^1, \quad \zeta^1 \cos\frac{\pi}{3} + \zeta^2 \sin\frac{\pi}{3}, \quad \zeta^1 \cos\frac{2\pi}{3} + \zeta^2 \sin\frac{2\pi}{3}.$$

We have proved the following

Theorem 4.4. *In the case* $N = 3$, *the* HASHIN & SHTRIKMAN *bounds* (0.3) *on the bulk and shear moduli are simultaneously achieved through a finite number of layering processes.*

Remark 4.6. As was mentioned in the introduction, the attainability of the bounds on the shear modulus given in HASHIN & SHTRIKMAN [1] has long been an open problem. This is established here with the help of Theorem 4.4 through explicit construction. Our multiple layering method differs from the incremental process used by MILTON [1], NORRIS [1] and LURIÉ & CHERKAEV [2].

Remark 4.7. The HASHIN & SHTRIKMAN bounds (0.2), (0.3) were originally derived under the ordering restriction (0.1). A set of bounds was later derived by WALPOLE [1], p. 159, when (0.1) is not satisfied. WALPOLE'S bounds on the bulk modulus coincide with the expressions K^l and K^u given in (0.3), which agrees with the results proved in the present study (Theorem 3.4 and Remark 3.6). WALPOLE'S bounds on the shear modulus however allow for a larger interval of possible macroscopic shear moduli than the expressions μ^l and μ^u given in (0.3).

In this context, the multilayered composite proposed here does not achieve WALPOLE'S bounds, at least as far as shear moduli are concerned. In any case, WALPOLE'S bounds on the shear modulus are not optimal (MILTON & PHAN THIEN [1], p. 325).

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